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Two- and Three-Point Functions in the $D = 1$ Matrix Model*

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ABSTRACT

The critical behavior of the genus-zero two point function in the $D = 1$ matrix model is carefully analyzed for arbitrary embedding-space momentum. It is shown that Kostov's result is valid only for momenta below a (non-universal) cutoff. We extend the large-N WKB treatment to calculate the genus-zero three point function, and elucidate its critical behavior for momenta below the cutoff.

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1. Introduction

The continuum limit of matrix models, achieved by taking N to infinity and the potential to criticality, provides both a simple formulation of two dimensional gravity and a laboratory for studying noncritical strings. This method was used to study 2-d gravity in its pure form or with worldsheet matter, on fixed-genus surfaces[1−3]. The results agree with continuum calculations, performed in the framework of Liouville theory[4,5] [6] Moreover, recently some interesting results on nonperturbative (all-genus) strings in the discrete models were obtained, by employing the double scaling limit[7−13]

Of the matrix models for which one can get exact results, the $D = 1$ model is the most promising for the purposes of string theory, since its worldsheet matter consists of a (single) bosonic field. This field may be interpreted as an embedding dimension[14]. One of the basic results in the genus zero $D = 1$ matrix model is the two point function. In-string terminology, it is the propagator of the tachyon in the single embedding dimension. Kostov[3] has obtained an expression for the two point function as an infinite sum, by performing exactly the $N \to \infty$ limit via WKB techniques. His subsequent evaluation of this sum near criticality, however, lacked precision. In this paper, we complete his treatment rigorously via a Poisson resummation and a careful analysis of the $g \to g_c$ limit. The result is that the g-nonanalytic part of his propagator is indeed dominant, but only for momentum P smaller (in magnitude) than a cutoff. The cutoff depends on the (convex) curvature of the critical potential at its top, and is hence non-universal. We roughly interpret this limitation as follows: when the two points are close enough in target space, the propagator is sensitive to the discreteness of the worldsheet − and thus to the shape of the potential. We also apply the WKB and resummation techniques to the three point function, and obtain its critical behavior in the regime of external momenta smaller than the cutoff.

This paper is organized as follows. In section 2 we review the $D = 1$ matrix model. In section 3 we briefly describe Kostov's derivation of the two-point function, represented as an infinite sum over fermion- hole pairs in a fermi sea of WKB states, and find its critical behavior using Poisson resummation. In section 4 we extend these techniques to find the three-point function near criticality. Finally, in section 5 we restate our conclusions and discuss further work to be done along these lines.

2. Review of Matrix Model

Let us begin by reviewing the $D = 1$ matrix model*. ϕ is a hermitian N by N matrix, $\partial/\partial\phi$ the matrix $\partial/\partial\phi_{ij}$, and $V(\lambda)$ a potential; viewing the embedding dimension as time t, the quantum hamiltonian operator in the configuration basis is

$$H = -\frac{1}{2}tr\Big[\big(\frac{\partial}{\partial\phi}\big)^2\Big] + NtrV(\phi/\sqrt{N}). \tag{1}$$

The precise shape of V should not matter in the continuum limit; the important feature of V is that it should be possible to tune it smoothly between the free $V(\phi) = \frac{\phi^2}{2}$ and some critical potential, $V_c(\phi)$, to be characterized below. We shall assume most of the time the quartic form with one coupling,

$$V(\lambda) = \frac{1}{2}\lambda^2 + g\lambda^4 \tag{2}$$

Although the entire treatment generalizes immediately to a large class of potentials. For simplicity, we shall always take V to be symmetric: $V(\lambda) = V(-\lambda)$. We also define $V(0) = 0$.

The matrix ϕ is decomposed as $U\phi_D U^{-1}$, where the 'angular' variable U is a unitary matrix and ϕ_D the diagonal matrix with elements $\sqrt{N}\lambda_i$, $1 \leq i \leq N$. As long as one computes only correlators of single-time traces, the angular variables are not excited, and hence may be ignored. Let $\Omega_0\{\lambda\}$ be the symmetric wave function of the ground state. The 'radial' (i.e. $\{\lambda_i\}$) part of the kinetic operator

* We will consider here an uncompactified embedding dimension.

in (1) is not separable in the λ variables, but becomes so when the wave function is redefined as

$$\chi_0\{\lambda\} = \Delta\Omega_0\{\lambda\}, \tag{3}$$

with A the Vandermonde determinant of $\{\lambda_i\}$. Now χ_0 is totally antisymmetric, and the 'radial' hamiltonian in the new picture is

$$\left.\begin{aligned} H_f &= \sum_{i=1}^{N} Nh(\lambda_i), \\ h(\lambda) &= -\frac{1}{2}\hbar_f^2\frac{\partial^2}{\partial\lambda^2} + V(\lambda),\ \hbar_f = \frac{1}{N} \end{aligned}\right\} \tag{4}$$

We shall denote eigenvalues of the one-fermion hamiltonian by the letter ϵ. The internal product in the new hilbert space is the usual euclidean one:

$$\langle\chi|\chi'\rangle = \int \chi^*\{\lambda\}\chi'\{\lambda\}\prod_i d\lambda_i \tag{3a}$$

This picture is also adequate to describe all the purely radial excitations χ_a, and corresponds to N identical, noninteracting fermions in a potential V (hence the subscript 'f' of H_f). Furthermore, the effective $\hbar_f$ in (4) is $\frac{1}{N}$, so the large-N limit can be treated via WKB methods.

In this limit, the connected Feynman graphs appearing in the perturbative g expansion of the ground state energy, correspond to genus-zero random surfaces. More precisely, defining the 'free energy' of the matrix model as

$$F_N = \frac{1}{N^2}\langle\chi_0|H_f|\chi_0\rangle = \frac{1}{N^2}E_0, \tag{5}$$

one has in the large-time limit:

$$F_N = \frac{1}{N^2}\lim_{T_0\to\infty}\{\frac{i}{T_0}\ln\int e^{i\int_{-T_0/2}^{T_0/2} Ldt}[d\phi]\} \tag{6}$$

Where L is the lagrangian corresponding to eq. (1). The connected graphs appearing in the g expansion of (6) are weighted by $(-g)^A$, where A is the vertex number†.

† A becomes the discrete surface-area in the dual graph.

The feature which distinguishes the $D = 1$ model from the $D = 0$ one is that the propagators in these graphs are not just a constant, but are rather proportional to $e^{-i|t-t'|}$. If, as universality arguments suggest, this gives the same continuum limit as a gaussian propagator would, then the $D = 1$ matrix model can be said to be a discretization of 2-d Liouville quantum gravity with target space R^1. The free energy $F = \lim_{N\to\infty} F_N$ then becomes the partition sum over genus-zero random surfaces. The continuum limit is achieved by tuning g (or more generally the function *V(X))* from g = 0 to a value where F is singular. Averages of geometrical attributes also become singular, which is what one expects of a continuum limit.

The N independent fermions fill the N lowest-lying levels of the potential V. In the classical $N \to \infty$ limit, the levels form a continuum. The highest filled level of this fermi sea is $\epsilon_F(g)$, the fermi level, and is determined by the condition that the total number of fermions be N. One can continue physical quantities to negative g‡; $g = g_c$ is the value for which $\epsilon_F(g) = V_{max}$, i.e. the fermi sea is on the verge of spilling over (see Fig. 1). We shall denote ϵ_F as ϵ except where confusion might arise. $\lambda_0 = \lambda_0(\epsilon)$ shall denote the turning point: $V(\pm\lambda_0) = \epsilon$.

The Bohr-Sommerfeld semiclassical quantization condition reads

$$\Delta\{\int_{-\lambda_0(\epsilon)}^{\lambda_0(\epsilon)} p_\epsilon(\lambda)d\lambda\} = \frac{\pi}{N} + O(\frac{1}{N^2}) \tag{7.1}$$

where A is a difference between two consecutive levels, ϵ is the energy of an arbitrary level, and

$$p(\lambda) = p_\epsilon(\lambda) = \sqrt{2(\epsilon - V(\lambda))} \tag{7.2}$$

is the classical momentum. Since N levels are filled, summing (7.1) up to the fermi

‡ although the quantum mechanical system is not defined there since $V(\lambda)$ is unbounded from below.

level yields the number-of-particles condition§:

$$\int_{0}^{\lambda_0(\epsilon_F)} p_{\epsilon_F}(\lambda)d\lambda = \frac{\pi}{2}. \tag{8}$$

This condition determines the function $\epsilon_F = \epsilon_F(g)$.

The semiclassical density of states in phase space is given by the measure $dpd\lambda/(2\pi\hbar_f)$, so the free energy is

$$F = \frac{1}{N}\int \frac{dpd\lambda}{2\pi\hbar_f}(\frac{p^2}{2} + V(\lambda))\theta(\epsilon_F - \frac{p^2}{2} - V(\lambda)) = \int_{-\lambda_0(\epsilon)}^{\lambda_0(\epsilon)} \frac{d\lambda}{2\pi}\int_{-p(\lambda)}^{p(\lambda)} dp'(\frac{p'^2}{2} + V(\lambda)) \tag{9}$$

where $\epsilon = \epsilon_F$. Another useful quantity is the classical period of oscillation for a fermion at the fermi level, and the corresponding frequency:

$$\left.\begin{aligned} \omega =& \frac{2\pi}{T} \\ \Delta\epsilon =& \frac{1}{N}\omega(\epsilon) + O(\frac{1}{N^2}) \\ T =& 4t_0 = 2\int_{-\lambda_0}^{\lambda_0} \frac{d\lambda}{p(\lambda)} \end{aligned}\right\} \tag{10}$$

$F(g)$ and $t_0(g)$ are singular at the critical point g = g_c for which $V'(\lambda_0) = 0$. This condition, together with $V(\lambda_0) = \epsilon$ and (8), determines g_c. We denote $V(X) = V_c(\lambda)$ and $\lambda_0 = \lambda_c$ at g = g_c. For a general potential, we require that $V_c(\lambda)$ be convex at $\lambda = \lambda_c$. Near criticality, we denote the location of the local maximum of $V(\lambda; g)$ by $\lambda_m(g)$; $\lambda_m(g_c) = \lambda_0(g_c) = \lambda_c$. (see Fig. 1). The required critical behaviors of various quantities are treated in appendix A.

§ There are corrections to (8) for finite N. Whenever we quote exact semiclassical results with no indication of $\frac{1}{N}$ power corrections, the $N \to \infty$ limit is understood.

3. Two Point Fuction

Following ref. [3], we now consider the euclidean connected two-point function ¶:

$$
\begin{aligned}
G^{(2)}(\tau) &= \lim_{N\to\infty} \langle \Omega_0 | T\{tr\phi^q(-i\tau)tr\phi^q(0)\}|\Omega_0\rangle_c \frac{1}{N^q} \\
&= \lim_{N\to\infty} \langle \chi_0 | T\{\sum_{i=1}^{N} \lambda_i^q(-i\tau) \sum_{j=1}^{N} \lambda_j^q(0)\}|\chi_0\rangle_c
\end{aligned}
\tag{11}
$$

where T here denotes the euclidean time-ordering operation, $|\chi_0\rangle$ is the ground state in the fermion picture and q is an arbitrary even, positive integer. The random surface interpretation of (11) is as a sum over over all surfaces that pass through the two points 0 and τ in euclidean target space.

$G^{(2)}$ is a connected Green's function, and so receives contributions only from intermediate states $\chi \neq \chi_0$. Further, in the fermionic picture the only excitations which contribute are those consisting of a single fermion above the sea plus a single hole in the fermi sea. Expressing the $tr\phi^q$ matrix elements in terms of single-fermion matrix elements and using WKB wave functions, Kostov obtains,

$$
\left.
\begin{aligned}
G^{(2)}(\tau) &= \sum_{n=1}^{\infty} n(f_n)^2 e^{-n\omega|\tau|}, \\
f_n &= \int_0^T \frac{dt}{T} \lambda^q(t)\cos(n\omega t)
\end{aligned}
\right\}
\tag{12}
$$

Here f_n is the matrix element of λ^q between two WKB levels near the fermi level, separated by the energy difference $\frac{n}{N}\omega$. $\lambda(t) = -\lambda(-t)$ is the classical trajectory at the fermi level; thus at $t = 0$ the fermion is at the bottom of the well, $\lambda = 0$. clearly $\lambda(t_0) = \lambda_0$. f_n vanishes for n odd, so only even n values contribute to $G^{(2)}$.

¶ We generalize slightly; ref.[3] treats $q = 4$. The critical behavior of correlators should not, and does not, depend on q as long as q is held fixed in the continuum limit ('microscopic loop').

The euclidean-momentum two point function is the fourier transform of (12):

$$\widetilde{G}^{(2)}(P) = \int_{-\infty}^{\infty} d\tau G^{(2)}(\tau) e^{ip\tau} = \sum_{n\geq 0, even}^{\infty} n(f_n)^2 \frac{2n\omega}{n^2\omega^2 + P^2}, \tag{13.1}$$

and for n even,

$$f_n = \int_{-to}^{t_0} \frac{dt}{2t_0} \lambda^q(t) e^{in\omega t} = \int_0^{t_0} \frac{dt}{t_0} \lambda^q(t) \cos(n\omega t) \tag{13.2}$$

Since $\widetilde{G}^{(2)}$ (P) is symmetric in P, we will henceforth assume $P > 0$.

At this point our treatment diverges from that of Kostov. Our method is rigorous and is valid for any potential V which satisfies the above conditions. Upon combining (13.1),(13.2) and employing the symmetry of λ^q to extend the summation to even n values of either sign, we find

$$\begin{aligned}\widetilde{G}^{(2)}(P) &= \sum_{n=1}^{\infty} (f_{2n})^2 \frac{8n^2\omega}{4n^2\omega^2 + P^2} \\ &= \frac{1}{w} \int_{-to}^{t_0} \int_{-to}^{t_0} dt_1 dt_2 \frac{1}{4t_0^2} \lambda^q(t_1) \lambda^q(t_2) \sum_{-CO} e^{2in\omega(t_1 - t_2)} \frac{n^2}{n^2 + \left(\frac{P}{2\omega}\right)^2}\end{aligned} \tag{14}$$

which reduces to

$$\widetilde{G}^{(2)}(P) = \frac{2}{\pi} \int_0^{t_0} dt \lambda^{2q}(t) - \frac{1}{2\pi t_0} \int_{-to}^{t_0} \int_{-to}^{t_0} dt_1 dt_2 \lambda^q(t_1) \lambda^q(t_2) S(t_1 - t_2) \tag{15}$$

where a simple application of Poisson's resummation yields,

$$S(t) \equiv \sum_{-CO}^{\infty} e^{2i\omega nt} \left(\frac{P}{2\omega}\right)^2 \frac{1}{n^2 + \left(\frac{P}{2\omega}\right)^2} = \sum_{m=-\infty}^{\infty} \frac{P\pi}{2\omega} e^{-P|t - \frac{\pi m}{\omega}|}. \tag{16}$$

The last expression is a geometric sum:

$$S(t) = \frac{P\pi}{2\omega} \{e^{-P|t|} + 2\frac{e^{-2Pt_0}}{1 - e^{-2Pt_0}} \cosh Pt\}, \ |t| \leq 2t_0 \tag{17}$$

The last condition is satisfied, since $t = t_1 - t_2$ in (15). Equations (14),(15),(17) suffice to evaluate $\tilde{G}^{(2)}(P)$ *. It is shown in appendix A how to extract the critical behavior of the t integrals.

The results are as follows. Let $P_0 = \sqrt{-V_c''(\lambda_c)}$ (eq. (A.12)), a momentum scale set by the convex curvature of the critical potential at its maximum. We distinguish two cases. For P < P_0, we obtain (restoring explicitly the g-dependence of $\tilde{G}^{(2)}(P)$),

$$
\begin{aligned}
\tilde{G}^{(2)}(P;g) =& \tilde{G}^{(2)}(P;g_c) + O(g-g_c) + O((\lambda_m - \lambda_0)^{1+P/P_0}) \\
& - \frac{4}{\pi} P \frac{e^{-2Pt_0}}{1-e^{-2Pt_0}} [f^{(2)}(P)]^2
\end{aligned}
\tag{18a}
$$

where $f^{(2)}$ is the g-independent function

$$
f^{(2)}(P) = \int_0^\infty dt[\lambda_c^q - \bar{\lambda}(t)] \cosh Pt \tag{18b}
$$

and $\bar{\lambda}(t)$ is the classical trajectory for $\epsilon = \epsilon_c$ and g = g_c †. This integral converges, since $P < P_0$ and (see eq. (A.13)) $\lambda_c - \bar{\lambda}(t) \approx const e^{-P_0 t}$ for $Pt_0 >> 1$.

In (18a), the first term is g independent and hence irrelevant in the continuum limit. The remaining terms are nonanalytic, and singular at g = g_c. Consulting eqs. (A.12) and (A.16) we see that the last term of (18a) is the dominant one, since $P < P_0$. Further, when $P < P_0/(2n+1)$, $n > 0$, the first $(n+1)$ terms in the geometric series $\frac{e^{-2Pt_0}}{1-e^{-2Pt_0}}$ are significant in the continuum limit.

To summarize, for $P < P_0$,

$$
\begin{aligned}
\tilde{G}^{(2)}(P) \approx& (g \text{ independent}) - \frac{4P}{\pi}[f^{(2)}(P)]^2 \frac{e^{-2Pt_0}}{1-e^{-2Pt_0}} \\
=& (g \text{ independent}) - \frac{2P}{\pi}[f^{(2)}(P)]^2 \coth(Pt_0)
\end{aligned}
\tag{19}
$$

which agrees with Kostov's result, and with continuum results [4][5][6]. However, for $P > P_0$, the dominant term in (18a) (apart from the irrelevant first term) is the

* Notice that although P is a *euclidean* momentum, the integration variables t_1 and t_2 are real embedding times.

† $f^{(2)}$ depends on the details of the critical potential through $\bar{\lambda}(t)$.

nonanalytic $O(g - g_c)$ term. Thus eq.(19) is invalid for large momenta[‡]. But this is not a problem since, as pointed out in the introduction, high momenta probe the discrete nature of the random surface and hence one would not expect the correspondence with continuum results to hold in this regime.

4. Three Point Function

Next, we proceed to apply the above techniques to compute the genus-zero connected three-point function. Proper counting of N powers shows that the appropriate quantity is (again we Wick-rotate to euclidean time),

$$\begin{aligned} G^{(3)}(\tau_1, \tau_2, 0) &= \lim_{N\to\infty} \langle \Omega_0 | T\{tr\phi^q(-i\tau_1) tr\phi^q(-i\tau_2) tr\phi^q(0)\} | \Omega_0 \rangle_c N^{1-3q/2} \\ &= \lim_{N\to\infty} \langle \chi_0 | T\{\sum_{i=1}^{N} \lambda_i^q(-i\tau_1) \sum_{j=1}^{N} \lambda_j^q(-i\tau_2) \sum_{k=1}^{N} \lambda_k^q(0)\} | \chi_0 \rangle_c N \end{aligned} \tag{20}$$

We shall first compute $G^{(3)}$ for $\tau_1 > \tau_2 > 0$, and then use symmetries to extend it to the other regions. Note the complication that the connected matrix element is of order $\frac{1}{N}$, so one must handle the WKB approximation more carefully.

There are two intermediate states in (20); let us call them χ_1 and χ_2. As before, they must be fermion-hole pairs. There are three types of configurations that contribute:

I) χ_1 has a fermion at level $\beta >$ N (N is the fermi level) and a hole at $\alpha \leq$ N, and $\chi_2 = \chi_1$; II) χ_1 is the same as in (I) and χ_2 has the same hole level but a fermion at $\gamma >$ N, $\gamma \neq \beta$; III) χ_1 is as in (I) and χ_2 has a fermion at level β and a hole at level $\gamma \leq$ N, $\gamma \neq$ cr.

‡ We have checked this explicitly in the large P limit. When both $P >> P_0$ and $g - g_c \to 0$, we find $\tilde{G}^{(2)}(P) \approx \zeta/P^2$ where ζ is independent of P and singular at $g = g_c$.

In terms of single-fermion matrix elements and energies, we thus obtain for $\tau_1 > \tau_2 > 0$,

$$\begin{aligned}
G^{(3)}(\tau_1,\tau_2,0) =& N\sum_{\alpha\leq N}\sum_{\beta>N}|\langle\alpha|\lambda^q|\beta\rangle|^2 e^{-\tau_1 N(\epsilon_\beta-\epsilon_\alpha)}(\langle\beta|\lambda^q|\beta\rangle - \langle\alpha|\lambda^q|\alpha\rangle)\\
&+ N\sum_{\alpha\leq N}\sum_{\substack{\beta,\gamma>N\\ \beta\neq\gamma}} e^{-N\tau_1(\epsilon_\gamma-\epsilon_\alpha)+N\tau_2(\epsilon_\gamma-\epsilon_\beta)}\langle\alpha|\lambda^q|\gamma\rangle\langle\gamma|\lambda^q|\beta\rangle\langle\beta|\lambda^q|\alpha\rangle\\
&- N\sum_{\beta>N}\sum_{\substack{\alpha,\gamma\leq N\\ \alpha\neq\gamma}} e^{-N\tau_1(\epsilon_\beta-\epsilon_\gamma)+N\tau_2(\epsilon_\alpha-\epsilon_\gamma)}\langle\alpha|\lambda^q|\gamma\rangle\langle\gamma|\lambda^q|\beta\rangle\langle\beta|\lambda^q|\alpha\rangle
\end{aligned} \tag{21}$$

In the limit N $\rightarrow \infty$.

Our first task is to evaluate the N $\rightarrow \infty$ limit fore fixed g. To leading WKB approximation, the single-fermion wave functions are*

$$\langle\lambda|\alpha\rangle \approx \frac{1}{\sqrt{p_{\epsilon_\alpha}(\lambda)}}\frac{1}{\sqrt{t_0}}e^{i\pi n/2}\cos[N\int_0^\lambda p_{\epsilon_\alpha}(x)dx + \frac{\pi n}{2}] \tag{22}$$

in the interior of the classically allowed region $p(X) \geq 0$. The transition region near the turning points $\pm\lambda_0$, as well as the classically-forbidden region, contribute to matrix elements only to order $\frac{1}{N}$ †. Thus we obtain, for $\alpha - \beta$ fixed in the N $\rightarrow \infty$ limit,

$$\langle\alpha|\lambda^q|\beta\rangle = \int_{-t_0}^{t_0}\frac{dt}{2t_0}\lambda^q(t)\exp[iN(\epsilon_\alpha-\epsilon_\beta)t] + \frac{1}{N}\mu(N(\epsilon_\alpha-\epsilon_\beta),\frac{\epsilon_\alpha+\epsilon_\beta}{2}) + O(\frac{1}{N^2}) \tag{23a}$$

where the function $\mu(a,\ b)$ is independent of N. Also, from the Bohr-Sommerfeld quantization condition, again assuming $\alpha - \beta$ fixed,

$$N(\epsilon_\alpha-\epsilon_\beta) = (\alpha-\beta)\omega(\frac{\epsilon_\alpha+\epsilon_\beta}{2}) + \frac{1}{N}\nu(\alpha-\beta,\frac{\epsilon_\alpha+\epsilon_\beta}{2}) + O(\frac{1}{N^2}) \tag{23b}$$

where again $\nu(a,\ b)$ is independent of N.

⋆ With a particular, convenient phase choice.

† In addition, there are $O(1/N)$ corrections to eq. (22) itself.

Substituting (23b) into (23a) and using (13b) we obtain,

$$\langle\alpha|\lambda^q|\beta\rangle = f_{\alpha-\beta}(\frac{\epsilon_\alpha+\epsilon_\beta}{2}) + \frac{1}{N}\mu'(\alpha-\beta,\frac{\epsilon_\alpha+\epsilon_\beta}{2}) + O(\frac{1}{N^2}), \qquad (23c)$$

where μ' is a third N-independent function.

In equation (21), $|N-\alpha|$, $|N-\beta|$ and $|N-\gamma|$ may be treated as being O(1) ~~in the~~ $N \to \infty$ limit, since the summands are suppressed exponentially as any of these three integers increases. Thus we may use eqs. (23). For the difference in the first sum, we find

$$\langle\beta|\lambda^q|\beta\rangle - \langle\alpha|\lambda^q|\alpha\rangle = \frac{1}{N}(\beta-\alpha)\frac{\partial}{\partial\epsilon}f_{\alpha-\beta}(\epsilon) + O(\frac{1}{N^2}) \qquad (24)$$

Similar leading-order cancellations occur between the second and third sums of (21). The details appear in appendix B‡. The final result is (still for $\tau_1 > \tau_2 > 0$).

$$\begin{aligned} G^{(3)}(\tau_1,\tau_2,0) =& \sum_{n=1}^{\infty} 4\omega n^2 e^{-2n\omega\tau_1}(f_{2n})^2\frac{\partial}{\partial\epsilon}f_0 \\ &+ \sum_{m,n=1}^{\infty} 4\omega\{mnf_{2m}f_{2n}\frac{\partial}{\partial\epsilon}f_{2m+2n} + n(m+n)f_{2m+2n}f_{2n}\frac{\partial}{\partial\epsilon}f_{2m}\} \\ &\times(e^{-2\omega\tau_1 n-2\omega\tau_2 m} + e^{-2\omega\tau_1(m+n)+2\omega\tau_2 m}) \\ &-8\omega\frac{\partial\omega}{\partial\epsilon}\sum_{m,n=1}^{\infty} mn(m+n)f_{2m}f_{2n}f_{2m+2n}(\tau_2 e^{-2\omega\tau_1 n-2\omega\tau_2 m} \\ &\qquad + (\tau_1-\tau_2)e^{-2\omega\tau_1(m+n)+2\omega\tau_2 m}) \end{aligned} \qquad (25)$$

For any other ordering of τ_i, one uses the fact that $G^{(3)}(\tau_1,\tau_2,\tau_3)$ is symmetric and depends only on $\tau_i - \tau_j$.

‡ By $\frac{\partial}{\partial\epsilon}$ we mean a partial derivative holding g fixed.

We fourier-transform the amplitude:

$$\widetilde{G}^{(3)}(P,Q) \equiv \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\tau_1 d\tau_2 e^{i(P\tau_1+Q\tau_2)} G^{(3)}(\tau_1,\tau_2,0) = \sum_{perm} \widetilde{G}_0^{(3)}(P,Q). \tag{26}$$

Here P,Q and $-P-Q$ are the euclidean momenta of the three vertex operators, and $\sum_{perm}$ is a sum over all 6 permutations of these three momenta. After some algebra, one obtains

$$\begin{aligned}\widetilde{G}_0^{(3)}(P,Q) =& 2\omega \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} mn f_{2m} f_{2n} \frac{\partial f_{2m+2n}}{\partial\epsilon} \frac{1}{2mw-iQ}\frac{1}{2nw-iP} \\ & -\frac{8}{3}\omega\frac{\partial\omega}{\partial\epsilon} Re \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} mn(m+n) f_{2m} f_{2n} f_{2m+2n} \\ & \times \frac{1}{(2mw-iQ)^2}\frac{1}{2nw-iP}\end{aligned} \tag{27}$$

As in the case of the two-point function, the infinite sums are easy to evaluate by using (12) for f_{2n} and bringing the m, n summations inside the time.integrals. The. basic sums needed are the following two, evaluated via Poisson resummation:

$$F_1(t;P) \equiv \sum_{n=-\infty}^{\infty} \frac{1}{2n\omega - iP} e^{2i\omega nt}\frac{\omega}{\pi i} = sgnP\ e^{-Pt}\Big(\theta(Pt) + \frac{e^{-2|P|t_0}}{1-e^{-2|P|t_0}}\Big), \tag{28a}$$

$$\begin{aligned}\hat{F}_1(t;P) &\equiv \sum_{n=-\infty}^{\infty} \frac{\omega}{\pi}\frac{1}{(2nw-iP)^2} e^{2i\omega nt} = \frac{\partial}{\partial Q}F_1(t;P) \\ &= -tF_1(t;P) - e^{-Pt}2t_0\frac{e^{-2|P|t_0}}{\left(1-e^{-2|P|t_0}\right)^2}\end{aligned} \tag{28b}$$

Another ingredient needed is the representation of the c-derivatives in (27) as time integrals; from (12) it is easy to show

$$\frac{\partial f_{2m}}{de} = -\frac{1}{2}\int_{-\lambda_0}^{\lambda_0} q\lambda^{q-1} e^{2im\omega t(\lambda)} \frac{\partial}{\partial\epsilon}\Big(\frac{t(\lambda)}{t_0}\Big) d\lambda, \tag{29}$$

where $t(\lambda)$ is the inverse function to $\lambda(t)$. From (10), one shows

$$\frac{\partial t_0}{\partial \epsilon} = \frac{1}{\sqrt{2\epsilon}}\frac{1}{V'(\lambda_0)} - \int_0^{\lambda_0} d\lambda \frac{V'(\lambda_0) - V'(\lambda)}{V'(\lambda_0)[p(\lambda)]^3}. \tag{30}$$

From (29)-(30),the critical behavior of these objects can easily be found (see appendix A):

$$\left.\begin{aligned}\frac{1}{t_0}\frac{\partial t_0}{\partial \epsilon} &= \frac{4}{P_0 C_0}\frac{1}{g - g_c} + O\left(\frac{1}{g - g_c}\frac{1}{t_0}\right)\\ \frac{\partial f_{2m}}{\partial \epsilon} &\approx -\frac{1}{2}\frac{\partial}{\partial \epsilon}\left(\frac{1}{t_0}\right)\int_{-\lambda_0}^{\lambda_0} q\lambda^{q-1} e^{2im\omega t(\lambda)} t(\lambda) d\lambda\end{aligned}\right\} \tag{31}$$

The terms neglected in the second equation arise from $\frac{\partial}{\partial \epsilon}$ acting on $t(\lambda)$ in (29), and it is shown in appendix A that these terms contribute to $\widetilde{G}^{(3)}(P, Q)$ a piece of order* $O\left(\frac{1}{\lambda_m - \lambda_0}\right)$.

From eqs. (12), (26)-(29) one thus finds:

$$\begin{aligned}\widetilde{G}^{(3)}(P,Q) = \frac{1}{6\pi}\frac{1}{t_0}\frac{\partial t_0}{\partial \epsilon}\sum_{perm}\int_{-t_0}^{t_0}\int_{-t_0}^{t_0} dt_1 dt_2 \int_{-\lambda_0}^{\lambda_0} d\lambda_3\; q\lambda_3^{q-1}(\lambda_m^q - \lambda_1^q)(\lambda_m^q - \lambda_2^q)\\ \times F_1'(t_2 - t_3; P)[(t_1 + t_2 + t_3)F_1'(t_1 - t_3; Q) + 2F_1(t_1 - t_3; Q)\\ - 4t_0 Q e^{-Q(t_1 - t_3)}\frac{e^{-2|Q|t_0}}{\left(1 - e^{-2|Q|t_0}\right)^2}] + O\left(\frac{1}{\lambda_m - \lambda_0}\right)\end{aligned} \tag{32}$$

where $t_i = t(\lambda_i)$,$\lambda_i = \lambda(t_i)$.

The critical behavior of the triple integral is found by straightforward application of the methods of appendix A. The critical behavior in the small-momenta regime (to be precisely defined below) arises entirely from the explicit t_0 dependence of the integrand of (32) and the $\frac{1}{t_0}\frac{\partial t_0}{\partial \epsilon}$ factor in front of the integral. The

⋆ The critical behavior of $\lambda_m - \lambda_0$ as a function of $g - g_c$ is given by eq. (A.16).

result is:

$$\frac{P_0C_0}{4}(g-g_c)\widetilde{G}^{(3)}(P,Q)=\frac{2}{2\pi}t_0\sum_{perm}\frac{e^{-2(|P|+|Q|)t_0}sgnP\;h(P)h(Q)h(-P-Q)}{(1-e^{-2|P|t_0})(1-e^{-2|Q|t_0})^2}$$

$$-\frac{1}{6\pi}\sum_{perm}\frac{e^{-2(|P|+|Q|)t_0}sgnPsgnQ}{(1-e^{-2|P|t_0})(1-e^{-2|Q|t_0})}$$

$$\times[h'(P)h(Q)h(-P-Q)+h(P)h'(Q)h(-P-Q)+h(P)h(Q)h'(-P-Q)]$$

$$+\frac{1}{3\pi}\sum_{perm}\frac{e^{-2|Q|t_0}}{1-e^{-2|Q|t_0}}sgnPsgnQ[-h'(Q)\eta(P,Q)+(2\frac{\partial\eta}{\partial Q}-\frac{\partial\eta}{\partial P})h(Q)]$$

$$+\frac{2}{3\pi}t_0\sum_{perm}\frac{e^{-2|Q|t_0}}{(1-e^{-2|Q|t_0})^2}sgnP\;h(Q)\eta(P,Q)+O((\lambda_m-\lambda_0)t_0) \tag{33}$$

where h and η are the following functions of the momenta (independent of g):

$$h(P)=-2P\int_0^\infty dt(\lambda_c^q-\bar{\lambda}(t)^q)\cosh Pt, \tag{34}$$

$$\eta(P,Q)=\int_{-\lambda_c}^{\lambda_c}\int_{-\lambda_c}^{\lambda_c}d\lambda_1 d\lambda_2 q\lambda_1^{q-1}q\lambda_2^{q-1}\theta(sgnP(\lambda_1-\lambda_2))e^{-Pt_1+(P+Q)t_2} \tag{35}$$

In (35), $t_i=\bar{t}(\lambda_i)$ where $\bar{t}(\lambda)$ is the inverse function to $\bar{\lambda}(t)$.

A few comments are in order here. The expression on the right-hand side of (33) is a continuous function of P and Q, despite appearances. Also, the terms written out dominate over the remainder $O((\lambda_m-\lambda_0)t_0)$ provided at least one of the three numbers $|P|$,$|Q|$,$|P+Q|$ is less than $P_0/2$. As $|P|$ and $|Q|$ decrease, more terms of the type $e^{-2m|P|t_0-2n|Q|t_0}$ become relevant †. Among the terms of (33),

† These terms are obtained by expanding $\frac{1}{1-e^{-2|P|t_0}}$, etc., as geometric series.

the following are dominant‡:

$$\frac{P_0 C_0}{4}(g-g_c)\widetilde{G}^{(3)}(P,Q) \approx \frac{2}{2\pi} t_0 \sum_{perm} sgnP\ h(Q)\eta(P,Q)e^{-2|Q|t_0}. \tag{36}$$

The function ηh is non-universal, as are the constants P_0 and C_0. t_0 depends logarithmically on $(g-g_c)$ (see eqs. (A.12) and (A.16)). The strongest dependence on-momenta is the exponential factors. Thus (36) tells us that $\widetilde{G}^{(3)}(P$, Q) is dominated by the sum of three terms, each of which depends essentially only on one of the three momenta.

When P and Q are set to zero, we obtain from (33)

$$\widetilde{G}^{(3)}(0,0) \approx \frac{8}{\pi\sqrt{P_0}C_0}\frac{1}{t_0^2}\frac{1}{9-g_c}\Big(\int_0^{\lambda_c} dt[\lambda_c^q - \bar{\lambda}^q(t)]\Big)^3. \tag{37}$$

As a check, we computed this quantity via a -different method. A small term $g_q\lambda^q$ is added to *V(X);* differentiating the free energy F three times with respect to g_q at g_q = 0 also gives $\widetilde{G}^{(3)}(0$, 0), and the result agrees with (37).

5. Conclusions

Using a resummation, we have rendered the previous treatment of the critical two-point function in the D = 1 matrix model, both more rigorous and valid for any potential in the simplest universality class (to which the quartic potential belongs). Our result differs from Kostov's for momenta above a non-universal cutoff. This reflects the fact that probing the random surface at small enough embedding distances reveals its potential-dependent, discrete nature. We applied the large-N WKB limit and the Poisson resummation to the three-point function, and found its critical behavior for momenta below the cutoff.

‡ Each of the other terms in (33) is suppressed by a factor $1/t_0$ and/or an exponential factor.

This work can be extended in two obvious directions: to all genera, and to higher n-point functions. The former may be doable by applying the double scaling limit. The extension to n-point functions would yield more information on the full tachyon effective action in noncritical $D = 1$ string theory. It is also of interest to extend the continuum calculations of this effective action, along the lines of ref[6], and compare with matrix-model results. Such a comparison can begin with our eqs. (33) and (36) for the three-point function.

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APPENDIX A

We demonstrate how to obtain the critical behavior of the various time- or X-integrals* appearing in the two- and three-point functions, and list some useful results.

For a potential $V(X) = V(\lambda; g)$ near criticality, we have defined in the text the quantities λ_0,λ_m,$\epsilon = \epsilon_F$ and t_0, all functions of g. Let us also define the convex curvature of $V(X)$ at its maximum λ_m:

$$\alpha = -V''(\lambda_m). \qquad (A.1)$$

At $g = g_c$, $\lambda_m = \lambda_0 = \lambda_c$. We define the critical value of α to be P_0^2:

$$P_0 = \sqrt{\alpha_c} = \sqrt{-V_c''(\lambda_c)} \qquad (A.2)$$

⋆ One can convert between a time integral and its corresponding λ integral by the change of variables $\lambda = X(t)$, the classical trajectory at the fermi level.

where $V_c(\lambda) = V(\lambda; g_c)$ is the critical potential†. We have

$$V(\lambda_m; g) - \epsilon(g) = \frac{1}{2}\alpha(\lambda_m - \lambda_0)^2 + O((\lambda_m - \lambda_0)^2). \qquad (A.3)$$

From eq. (2),

$$\lambda_m - \lambda_0 = O(g - g_c) \qquad (A.4)$$

and we easily obtain from (A.1-4),

$$\epsilon - \epsilon_c = (g - g_c)\lambda_c^4 - \frac{1}{2}P_0^2(\lambda_m - \lambda_0)^2 + O((g - g_c)^2) + O((\lambda_m - \lambda_0)^3) + O((g - g_c)(\lambda_m - \lambda_0)^2) \qquad (A.5)$$

Now, differentiating the number-of-particles condition (eq.(8)) with respect to g with $\epsilon = \epsilon(g)$ yields:

$$\begin{aligned} \epsilon'(g) &= \int_0^{t_0} \frac{dt}{t_0}\lambda^4(t) = \lambda_m^4 - \frac{1}{t_0}\int_0^{t_0} dt(\lambda_m^4 - \lambda^4(t)) \\ &= \lambda_m^4 - \frac{1}{t_0}\int_0^{\lambda_0} \frac{d\lambda}{p(\lambda)}(\lambda_m^4 - \lambda^4), \end{aligned} \qquad (A.6)$$

and also (eq.(10))

$$t_0 = \int_0^{\lambda_0} \frac{d\lambda}{p(\lambda)}, \qquad (A.7)$$

where the classical momentum $p(\lambda)$ is defined in eq. (7.2).

To proceed further, we must extract the critical behavior of λ integrals such as $(A.6 - 7)$. We sketch below a convenient method for doing so.

† Actually, it is only one of a universality class of critical potentials; see discussion in text.

For any integral over λ from 0 to λ_0, separate it into $\int_0^{\lambda_1} + \int_{\lambda_1}^{\lambda_0}$, where λ_1 is an arbitrary fixed number satisfying $0 < \lambda_1 < \lambda_0$. The $\int_0^{\lambda_1}$ integral can be expanded as a Taylor series in $\epsilon - \epsilon_c$ and $g - g_c$. In the second piece $\int_{\lambda_1}^{\lambda_0}$, change variable from λ to y, defined as

$$\frac{1}{2}\alpha y^2 = [V(\lambda_m) - V(X)]. \tag{A.8}$$

When y is used as an integration variable, and the integrand expanded simultaneously in powers of $\epsilon - \epsilon_c, g - g_c$ and y, a systematic expansion of the entire $\int_0^{\lambda}$ integral may be obtained. This technique may also be used to determine the behavior of any integral

$$I(\epsilon, g) = \int_0^{\lambda} dx f(x; \epsilon, g) \tag{A.9}$$

when $\lambda \approx \lambda_0(g)$, $g \approx g_c$. By applying the method to

$$t(\lambda) = \int_0^{\lambda} \frac{dx}{\sqrt{2(\epsilon - V(\lambda))}} \tag{A.10}$$

we find, for $\lambda(t) \approx \lambda_0$ and $g \approx g_c$,

$$\lambda_m - A(t) \approx (\lambda_m - \lambda_0) \cosh(P_0(t_0 - t)) \tag{A.lO}$$

with systematically computable corrections. From

$$t_0 = \int_0^{\lambda_0} \frac{d\lambda}{\sqrt{2(\epsilon - V(\lambda))}}, \tag{A.11}$$

we obtain

$$\lambda_m - \lambda_0 = C_1 e^{-P_0 t_0} + O((\lambda_m - \lambda_0)^2) \tag{A.12}$$

where C_1 is a g-independent constant. Like P_0, this constant depends on the particular choice of critical potential V_c, and is thus non-universal.

At $g = g_c$, $\lambda_m = \lambda_0$ so (A.12) implies $t_0 \rightarrow \infty$. for the critical trajectory $\bar{\lambda}(t)$, we find from (A.10) and (A.12),

$$\lambda_c - \bar{\lambda}(t) \approx C_1 e^{-P_0 t} \quad (P_0 t >> 1). \tag{A.13}$$

Analysis of the integral in (A.6) using the y variable yields

$$\left(\frac{\epsilon - \epsilon_c}{g - g_c} - \lambda_c^4\right) = -\frac{C_0}{4t_0} + O\left(\frac{1}{t_0^2}\right), \tag{A.14}$$

where C_0 is another non-universal constant. Thus

$$\epsilon - \epsilon_c = O(g - g_c). \tag{A.15}$$

Using (A.5),(A.12) and (A.14) yields

$$\begin{aligned} 2P_0^2 t_0 (\lambda_m - \lambda_0)^2 + O((\lambda_m - \lambda_0)^2) &= -\, 2P_0(\lambda_m - \lambda_0)^2\, ln(\lambda_m - \lambda_0) \\ &= C_0(g - g_c)\{1 + O(\frac{1}{t_0})\} \end{aligned} \tag{A.16}$$

Hence, up to a logarithmic factor $\lambda_m - \lambda_0$ behaves as $\sqrt{g - g_c}$, so by (A.12), t_0 diverges logarithmically as g $\rightarrow g_c$. We also see from (A.14) that $\epsilon - \epsilon_c$ is a nonanalytic function of g $- \, g_c$, since t_0 is.

The above results enable the determination of the critical behavior of the integrals in eqs. (15) (two point function) and eq.(32) (three point function). As an example, (A.13) implies that the integral $f^{(2)}(P)$ (eq. (17b)) converges for $P < P_0$.

As a final example of these techniques, we show how eqs.(31) were obtained. In eq.(30), $V'(\lambda_0) \approx \alpha(\lambda_m - \lambda_0)$, $\epsilon \approx \epsilon_c$, and by changing the integration variable from λ to y we find

$$\frac{1}{t_0}\frac{\partial t_0}{\partial \epsilon} = O\left(\frac{1}{t_0}\frac{1}{\lambda_m - \lambda_0}\right) + \frac{1}{t_0(\lambda_m - \lambda_0)^2}\frac{1}{P_0^3}\int_\mu^\infty dy \frac{y - \mu}{(y^2 - \mu^2)^{3/2}} \tag{A.17}$$

where μ is the value of y at $\lambda = \lambda_0$. From (A.8) we find

$$\mu = \lambda_m - \lambda_0 + O((\lambda_m - \lambda_0)^2). \qquad (A.17a)$$

The y integral is easily evaluated, and (A.17)-(A.17a) combined with (A.16) yield the first of eqs. (31). The second of eqs. (31) is obtained by starting from eq. (29), and showing that the $\frac{\partial}{\partial\epsilon}t(\lambda)$ contribution to $\widetilde{G}^{(3)}(P,Q)$ is of order $1/(\lambda_m - \lambda_0)$. This reduces to showing that for an even positive integer l,

$$W_l \equiv \int_0^{\lambda_0} l\lambda^{l-1} d\lambda \frac{\partial t(\lambda)}{\partial\epsilon} = o\left(\frac{1}{\lambda_m - \lambda_0}\right). \qquad (A.18)$$

This, in turn, is proven as follows. By differentiating (A.lO) w.r.t. ϵ and integrating (A.18) by parts,

$$W_l = -\int_0^{\lambda_0} (\lambda_0^l - \lambda^l) d\lambda \frac{1}{[p(\lambda)]^3} \qquad (A.19)$$

which is then analyzed using the y integration variable.

APPENDIX B

We begin with eq. (21) in the text. In the first sum, we let α = N $- m$, β = N $- m$ + n where the range of (m,n) is $0 \leq m <$ n. In the second sum, we let α = N $- m$,β = N + $l_1 - m$ + 1,γ = N + $l_2 - m$ + 1, with $0 \leq m \leq l_1$,$m \leq l_2$ and $l_1 \neq l_2$. Finally, in the third term, (m, l_1, l_2) have the same range as in the second term, but α = N $- l_1 + m$, β = N + m + 1,γ = N $- l_2$ + m. We thus obtain (for

$\tau_1 > \tau_2 > 0)$,

$$
\begin{aligned}
G^{(3)}(\tau_1, \tau_2, 0) = & \sum_{n=1}^{\infty}\sum_{m=0}^{n-1} e^{-\tau_1 N(\epsilon_{N+n-m}-\epsilon_{N-m})} |\langle N-m|\lambda^q|N-m+n\rangle|^2 \\
& \times N(\langle N-m+n|\lambda^q|N-m+n\rangle - \langle N-m|\lambda^q|N-m\rangle) \\
& + \sum_{l_1=0}^{\infty} \sum_{\substack{0\leq l_2<\infty \\ l_2\neq l_1}} \sum_{m=0}^{min(l_1,l_2)} N\{e^{-\tau_1 N(\epsilon_{N+l_2+1-m}-\epsilon_{N-m})+\tau_2 N(\epsilon_{N+l_2+1-m}-\epsilon_{N+l_1+1-m})} \\
& \times\langle N-m|\lambda^q|N+l_2+1-m\rangle\langle N+l_2+1-m|\lambda^q|N+l_1+1-m\rangle\langle N+l_1+1-m|\lambda^q|N-m\rangle \\
& - e^{-\tau_1 N(\epsilon_{N+m+1}-\epsilon_{N+m-l_2})+\tau_2 N(\epsilon_{N+m-l_1}-\epsilon_{N+m-l_2})} \\
& \times\langle N-l_2+m|\lambda^q|N+m+1\rangle\langle N-l_1+m|\lambda^q|N-l_2+m\rangle\langle N+m+1|\lambda^q|N+m-l_1\rangle\}
\end{aligned}
\qquad (B.1)
$$

The order N^0 cancellation in the first sum was already exhibited in eq. (24). A similar cancellation occurs between the two terms inside curly braces in the second sum of (B.l). In fact, by using eqs. (23a-c) (which is allowed, see comment preceding eq. (24)), we obtain

$$
\begin{aligned}
N(\epsilon_{N+l_2+1-m} - \epsilon_{N-m}) &= (l_2 + 1)\omega + O(\frac{1}{N}), \\
N(\epsilon_{N+l_2+1-m} - \epsilon_{N-m}) - N(\epsilon_{N+m+1} - \epsilon_{N+m-l_2}) &= (l_2 + 1)(l_2 - 2m)\frac{1}{N}\omega\frac{\partial\omega}{\partial\epsilon} \\
& +O(\frac{1}{N^2}), \\
\langle N-m|\lambda^q|N+l_2+1-m\rangle &= f_{l_2+1}(\epsilon) \\
& +O(\frac{1}{N}), \\
\langle N-m|\lambda^q|N+l_2+1-m\rangle - \langle N-l_2+m|\lambda^q|N+m+1\rangle &= (l_2 - 2m)\frac{1}{N}\omega\frac{\partial}{\partial\epsilon}f_{l_2+1} \\
& +O(\frac{1}{N^2}),
\end{aligned}
\qquad (B.2)
$$

and similarly for the other energy differences and matrix elements appearing in (B.l). The factors N in the two sums of (B.l) cancel the l/N factors in the differences, giving a finite N $\rightarrow$ co limit. A further simplification is that in (B.1), the N $\rightarrow \infty$ summands depend on m only through simple factors. In the first sum, there is no m dependence at all, and the m sum is thus just n. In the second sum the following m sums appear:

$$\sum_{m=0}^{min(l_1,l_2)} (l_2 - 2m) = \begin{cases} 0 & \text{if } l_1 > l_2 \geq 0 \\ (l_2 - l_1)(l_1 + 1) & \text{if } l_2 \geq l_1 \geq 0 \end{cases}, \tag{B.3}$$

$$\sum_{m=0}^{min(l_1,l_2)} (l_1 - 2m) = \begin{cases} 0 & \text{if } l_2 > l_1 \geq 0 \\ (l_1 - l_2)(l_2 + 1) & \text{if } l_1 > l_2 \geq 0 \end{cases}, \tag{B.4}$$

$$\sum_{m=0}^{min(l_1,l_2)} (l_1 + l_2 + 1 - 2m) = (l_1 + 1)(l_2 + 1). \tag{B.5}$$

The final expression for the $N \to \infty$ limit of $G^{(3)}(\tau_1, \tau_2, 0)$, given by eq. (25) in the text, is obtained by noting that f_m vanishes for odd subscripts, and changing summation indices in the second sum of (B. 1) from (l_1, l_2) to (m, n), where $l_1 = 2m + 2n - 1$ and $l_2 = 2n - 1$. The range of (m, n) is then $m \geq 1$, $n \geq 1$.

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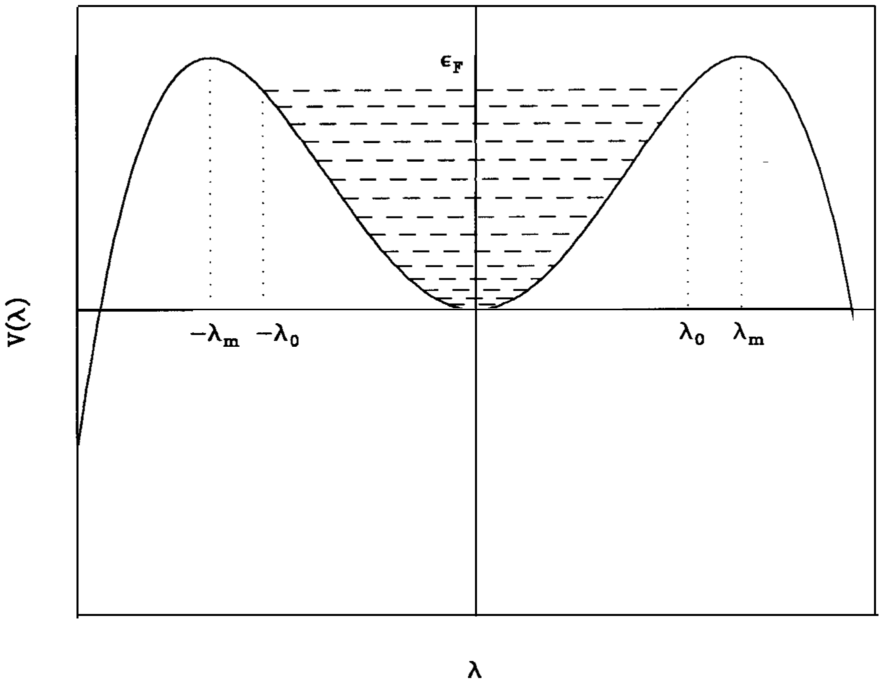

Figure 1. *V(X)* and the Fermi sea near criticality.