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NONRENORMALIZATION OF THE CHIRAL ANOMALY IN CHIRAL PERTURBATION THEORY*

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ABSTRACT

It is proved to any finite order in (four-dimensional) chiral perturbation theory that the Wess–Zumino term and the chiral anomaly of the generating functional of connected Green functions are not renormalized. This result is analogous to the Adler–Bardeen nonrenormalization theorem for QCD and shows that the effective low-energy meson theory correctly matches the QCD anomaly. All higher-order counterterms in the chiral Lagrangean are four-dimensional chiral invariants, even if they renormalize graphs with Wess–Zumino vertices. The proof is based on a manifestly chiral invariant perturbation expansion of the generating functional around a background field. The fluctuations of the Wess–Zumino term about the background field are shown to be chirally invariant four-forms. In order to preserve chiral symmetry and avoid ambiguities in the continuation of $\varepsilon^{\mu\nu\rho\sigma}$ to $d \neq 4$, operator regularization is used rather than dimensional regularization. The divergent terms are explicitly computed to one-loop order and compared with the most general sixth-order chiral Lagrangean with an ε tensor and flavor nonsinglet external fields.

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1. Introduction

Spontaneously broken chiral symmetry determines the low-energy strong interactions up to a few constants. These constraints may be conveniently summarized by an effective chiral Lagrangean for the octet of pseudoscalar mesons.[1] In general, the predictions of lowest-order chiral perturbation theory ($O(p^2)$ in the expansion in powers of momenta and meson masses) are in fair or good agreement with experiment. Even though chiral Lagrangeans are not renormalizable, a well-defined and systematic procedure for calculating meson-loop effects[2,3] has been developed; unitarity is thereby restored to successively higher orders in the chiral expansion. The divergences arising in these calculations have to be absorbed by counterterms different from the original Lagrangean, but they are chiral invariants of higher order in the momentum expansion.[4] The corresponding coupling constants have to be determined from experimental data. At the one-loop level, theory and experiment agree remarkably well.[2]

An exception to this picture are the anomalous meson processes like $\pi^0 \to \gamma\gamma$ which are forbidden to $O(p^2)$ and $O(p^4)$ by the naïve Ward identities[5] while higher-order effects appear too small to explain the experimentally observed widths and cross sections. However, anomalous breaking of chiral symmetry by quark loops[6–8] provides a leading amplitude in good agreement with experiment. Wess and Zumino[9] constructed a classical action of $O(p^4)$ for mesons and external gauge fields whose variation under chiral transformations exactly reproduces the fermionic anomaly; it correctly describes the anomalous meson processes to that order. Witten later gave an elegant topological representation of the Wess-Zumino (WZ) action in Euclidian space through the five-dimensional Chern-Simons term.[10] His work triggered the development of differential geometric methods[11–14] that will be extensively used in Section 3 below.

Two questions then arise: Firstly, will quantum corrections and possible tree-level contributions at $O(p^6)$ (not at variance with the Veltman-Sutherland theorem[5]) significantly change the lowest-order amplitudes? Secondly, does the anomaly of the WZ action induce higher-order chiral anomalies through loop effects? The first problem has been studied in some detail in Refs. 15–17; the result is essentially that the higher-order effects are too small to be presently seen in $\pi^0 \to \gamma\gamma$ while quite substantial in η decay, but $\eta - \eta'$ mixing and the lack of knowledge of the singlet decay constant again preclude firm conclusions.

These authors also found by direct computation that the chiral anomaly equation is not renormalized to one-loop order and that the one-loop counterterms induced by the WZ action are chiral invariants. It will be shown in this paper that this result indeed holds to all orders of chiral perturbation theory (ChPT). In view of the topological interpretation of the WZ term and the quantization of its coupling constant, one might consider this outcome to be obvious. However, the theory is not renormalizable; radiative corrections contribute to the effective action through terms with additional derivatives. These latter pieces are not topological in nature, so there is no *a priori* reason why they should respect chiral symmetry – to the contrary. The nonrenormalization theorem is a consequence of the particular structure of the fluctuation terms in an expansion of the WZ action around some background-field configuration (see Section 2). It does not imply that no divergences arise in loop graphs with WZ vertices; rather, these infinite pieces can be absorbed by renormalizing the coefficients of chirally invariant higher-order actions.

A necessary ingredient in the proof is a regularization scheme that preserves chiral symmetry and – in contrast to dimensional regularization – does not lead to ambiguities in connection with the totally antisymmetric ε tensor. In a separate paper,[3] an analytic regularization scheme – termed operator regularization[18] (OR) – is discussed in detail and shown to have all the required properties. Section 3 outlines the computation of the divergent parts of one-loop diagrams with one Wess-Zumino vertex and gives the general chiral Lagrangean of order p^6 in the WZ sector for use in the phenomenological analysis of anomalous meson processes. A preliminary account of the present results appeared in Ref. 19.

For the reader's convenience, the external-field formulation of ChPT[2] is briefly reviewed in the remainder of this section. A unitary matrix field U represents the octet of pseudo-scalar mesons π, K and η; it transforms under local chiral $\mathrm{U}(3)_L \times \mathrm{U}(3)_R$ rotations as

$$U(x) = e^{\frac{i}{3}\Theta(x) - \varphi(x)/f} \rightarrow g_L^\dagger(x) U(x) g_R(x) \, , \tag{1.1}$$

where $\varphi(x) = -i\lambda_a \varphi^a(x)$; the singlet field is necessary for consistency (it may be set to 0 if the symmetry group is restricted to $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$). To lowest order, the constant f coincides with the pseudoscalar decay constant $f_\pi \approx 93\,\mathrm{MeV}$.

External right- and left-handed vector gauge fields $V^\mu_{R,L} = V^\mu \pm A^\mu = -\frac{i}{2}\lambda_a V^{a,\mu}_{R,L}$ are used to construct covariant derivatives:

$$\begin{aligned} \nabla^\mu U &= \partial^\mu U + V^\mu_L U - U V^\mu_R \, , \\ \nabla^\mu U^\dagger &= \partial^\mu U^\dagger + V^\mu_R U^\dagger - U^\dagger V^\mu_L \, . \end{aligned} \tag{1.2}$$

and

$$\nabla^\mu \Theta = \partial^\mu \Theta - 2\,\mathrm{Tr}\,(A^\mu) \, . \tag{1.3}$$

The abbreviations $R^\mu = U^\dagger \nabla^\mu U = -\nabla^\mu U^\dagger U$ and $L^\mu = \nabla^\mu U U^\dagger$ will frequently occur in Sect. 3. In the lowest-order chiral Lagrangean,

$$\mathcal{L}_2 = \frac{f^2}{4}\mathrm{Tr}\left(\nabla_\mu U^\dagger \nabla^\mu U + U^\dagger X + X^\dagger U\right) + h \nabla_\mu \Theta \nabla^\mu \Theta \, , \tag{1.4}$$

the meson mass matrix is contained in the external (Hermitean) mixed scalar–pseudoscalar fields X and $X^\dagger$ which transform like U and $U^\dagger$, respectively. From U, Θ, X, $F^{\mu\nu}_{R,L} = \partial^{[\mu} V^{\nu]}_{R,L} + V^{[\mu}_{R,L} V^{\nu]}_{R,L}$ and their covariant derivatives, chiral Lagrangeans of higher-order in the momentum expansion may be constructed.

In order to correctly match low-energy QCD, the generating functional of connected Green functions of ChPT must have the anomaly[20]

$$\delta_\beta W[V,A,S,P,\Theta] = -\frac{iN_c}{16\pi^2}\,\varepsilon^{\mu\nu\rho\sigma}\int d^4x$$
$$\times \mathrm{Tr}\,\big[\beta(x)\big(F_{\mu\nu}F_{\rho\sigma} + \tfrac{4}{3}\,\nabla_\mu A_\nu \nabla_\rho A_\sigma - \tfrac{2}{3}\,\{F_{\mu\nu},\, A_\rho A_\sigma\}$$
$$+\tfrac{8}{3}\,A_\sigma F_{\mu\nu} A_\rho - \tfrac{4}{3}\,A_\mu A_\nu A_\rho A_\sigma\big)\big] \tag{1.5}$$

under infinitesimal axial transformations $g_R(x) = g_L^\dagger(x) = \exp[\beta(x)]$. The gluonic contribution to the axial U(1) anomaly can be compensated by the shift $\Theta(x) \to \Theta(x) - 2i\,\mathrm{Tr}\,[\beta(x)]$. Note also that Abelian vector and axial-vector fields with nontrivial transformation properties are included here; thus the Abelian anomaly manifests itself not only in the nonconservation of the axial U(1) current, but also in the noninvariance of W. The Wess–Zumino action reproducing the anomaly (1.5) will be discussed in Sect. 2.

Meson-loop effects are calculated in the framework of a saddle-point expansion about a background configuration $U_0(x)$ that solves the classical equations of motion derived from (1.4). Setting $u(x) = U_0^{1/2}(x)$, the fluctuations are parametrized as

$$U(x) = u(x)e^{-\xi(x)/f}u(x) \quad , \qquad \xi(x) = -i\lambda_a\xi^a(x)\;. \tag{1.6}$$

ξ is pseudo-scalar and transforms as

$$\xi'(x) = \tilde{h}^\dagger\big(g_L(x),\, g_R(x),\, \varphi_0(x)\big)\,\xi(x)\,\tilde{h}\big(g_L(x),\, g_R(x),\, \varphi_0(x)\big)\;, \tag{1.7}$$

where $U_0(x) = \exp(-\varphi_0(x)/f)$ and

$$\tilde{h}\big(g_L(x),\, g_R(x),\, \varphi_0(x)\big) = u(x)\,g_R(x)\,u'^\dagger(x) = u^\dagger(x)\,g_L(x)\,u'(x)\;. \tag{1.8}$$

For the purpose of generating graphs in a perturbative expansion, it is convenient to couple $\xi(x)$ to a source $J(x)$ with the same transformation properties as ξ itself.

One defines anti-Hermitean vector and axial-vector fields Γ_μ and Δ_μ, where the former is a connection and the latter transforms homogeneously:

$$\Gamma^\mu = \tfrac{1}{2}\,[u^\dagger, \partial^\mu u] + \tfrac{1}{2}\,uV_R^\mu u^\dagger + \tfrac{1}{2}\,u^\dagger V_L^\mu u \;\longrightarrow\; \tilde{h}^\dagger(\Gamma_\mu + \partial_\mu)\tilde{h}\;, \tag{1.9}$$

$$\begin{aligned} \Delta^\mu &= \tfrac{1}{2}\,\{u^\dagger, \partial^\mu u\} - \tfrac{1}{2}\,uV_R^\mu u^\dagger + \tfrac{1}{2}\,u^\dagger V_L^\mu u \\ &= \tfrac{1}{2}\,u^\dagger \nabla^\mu U_0 u^\dagger = -\tfrac{1}{2}\,u\nabla^\mu U_0^\dagger u \;\longrightarrow\; \tilde{h}^\dagger \Delta_\mu \tilde{h} \end{aligned} \tag{1.10}$$

The covariant derivative of ξ is given by[21,2]

$$d_\mu \xi \;=\; \partial_\mu + [\Gamma_\mu,\, \xi]\;. \tag{1.11}$$

In expanding about U_0, one may use the formulae

$$\begin{aligned} \nabla_\mu U &= u\big(d_\mu e^{-\xi} + \{\Delta_\mu,\, e^{-\xi}\}\big)u\;, \\ \nabla_\mu U^\dagger &= u^\dagger\big(d_\mu e^{\xi} - \{\Delta_\mu,\, e^{\xi}\}\big)u^\dagger\;. \end{aligned} \tag{1.12}$$

Finally, the following quantity will frequently appear in Sec. 3:

$$\mathcal{F}^\pm_{\mu\nu} = uF^R_{\mu\nu}u^\dagger \pm u^\dagger F^L_{\mu\nu}u\;. \tag{1.13}$$

2. Nonrenormalization of the Wess–Zumino Term

The first part of this section proves that all fluctuation terms of the Wess–Zumino (WZ) action[9,11,14] are four-dimensional chiral invariants; only the zeroth-order term in the expansion of $S_{\rm WZ}$ is anomalous. The derivation closely follows the approach of Ref. 14 and uses the language of differential forms whenever useful. The conventions are essentially those of Ref. 11, except for $V = -(i/2)\lambda_a V^a$ instead of $-i\lambda_a V^a$, etc. Next, a naïve argument is given why the anomalous variation of the fully unitarized generating functional of ChPT exactly equals the expression (1.5). This conclusion can be rigorously justified to any finite order in ChPT by using the operator regularization scheme described in Refs. 18 and 3. Renormalization of loop graphs thus never requires chirally noninvariant counterterms even if anomalous vertices from the WZ term are present. This result will be verified to one-loop order in Sect. 3.

2.1. PROPERTIES OF THE WESS–ZUMINO TERM

The WZ term by construction has the anomaly (1.5). If vector-current conservation is imposed, it may be expressed in Euclidian space as

$$S_{\mathrm{WZ}} = K_d \int\limits_{B^{d+1}} \left(1 - \hat{T}(I,U)\right)\left(\omega^0_{d+1}(V_R) - \omega^0_{d+1}(V_L) + dC_d(V_R,V_L)\right) . \tag{2.1}$$

The integration is over the ball B^{d+1} whose boundary is the (compactified) space-time manifold M^d; the fields V_H and U, where $H = R, L$, are assumed to be suitably continued into the interior of B^{d+1}. $\omega^0_{d+1}(V_H)$ is the zeroth Chern–Simons form associated with the $(d/2+1)$-th Chern–Pontryagin density $\mathrm{Tr}\left[F^{d/2+1}(V_H)\right]$. C_d is Bardeen's counterterm[20] guaranteeing vector-current conservation. The operator $\hat{T}(g_R, g_L)$ with $g_H \in G_H$ acts on functionals of V_H as $\hat{T}(g_R,g_L)\, f[V_R,V_L] = f[g_R^{-1}(V_R+d)g_R,\, g_L^{-1}(V_L+d)g_L]$. In $d = 4$, the coefficient K_d takes the value $K_4 = -i/(24\pi^2)$.

Due to the group property

$$\hat{T}(g_R,g_L)\, f[V_R,V_L] = \hat{T}(h_R^{-1}g_R,\, h_L^{-1}g_L)\, f[h_R^{-1}(V_R+d)h_R,\, h_L^{-1}(V_L+d)h_L] \tag{2.2}$$

the action (2.1) may be rewritten as

$$S_{\mathrm{WZ}} \;=\; K_d \int\limits_{M^d} \left(\Lambda_{d+1}(U) + d\alpha_d(V_L,U) + dC_d(V_R,V_L) - dC_d(V_R, U^\dagger(V_L+d)U)\right) . \tag{2.3}$$

All terms containing gauge fields are d-forms, i.e. local polynomials of V_R, V_L in d dimensions; only the pure meson term cannot be written as a d-form and is not polynomial when restricted to d dimensions. An exact d-form is usually added to obtain a more symmetrical appearance of the action, but this is irrelevant in the present context. In four dimensions, the forms appearing in (2.3) are given by

$$\Lambda_5(U) = \frac{1}{10}\,\mathrm{Tr}\left((U^\dagger dU)^5\right)\,, \tag{2.4}$$

$$\begin{aligned}\alpha_4(V_L,U) = -\frac{1}{2}\,\mathrm{Tr}\Big((dUU^\dagger)(V_L dV_L + dV_L V_L + V_L^3) - (dUU^\dagger)^3 V_L \\ - \tfrac{1}{2}\,(dUU^\dagger)V_L(dUU^\dagger)V_L\Big)\,,\end{aligned} \tag{2.5}$$

$$C_4(V_R,V_L) = \frac{1}{2}\,\mathrm{Tr}\left((V_LV_R - V_RV_L)(F_R+F_L) + V_R^3V_L + V_RV_L^3 + \tfrac{1}{2}\,(V_LV_R)^2\right)\,, \tag{2.6}$$

$$\begin{aligned} &C_4(V_R, U^\dagger(V_L+d)U) \\ &= \frac{1}{2}\,\mathrm{Tr}\Big((F_R + U^\dagger F_L U)(U^\dagger V_L U V_R - V_R U^\dagger V_L U) \\ &\qquad + V_R^3 U^\dagger V_L U + V_R U^\dagger V_L^3 U + \tfrac{1}{2}\,(U^\dagger V_L U V_R)^2 \\ &\qquad + U^\dagger dU\big(\{V_R, F_R + U^\dagger F_L U\} + V_R U^\dagger V_L U V_R + U^\dagger V_L U V_R U^\dagger V_L U \\ &\qquad\qquad - V_R U^\dagger V_L^2 U - U^\dagger V_L^2 U V_R - V_R^3\big) \\ &\qquad + (U^\dagger dU)^2 (V_R U^\dagger V_L U - U^\dagger V_L U V_R) + \tfrac{1}{2}\,(U^\dagger dU V_R)^2 - (U^\dagger dU)^3 V_R\Big)\,. \end{aligned} \tag{2.7}$$

The forms discussed above are maps $f : \mathcal{G}\times\mathcal{S} \to L_2(M)$, $(g(x), V(x)) \mapsto f(g(x), V(x))$ where $\mathcal{S}$ and $\mathcal{G}$ are spaces of gauge field configurations and of maps from M into G, respectively. Chiral transformations of such forms are naturally defined in terms of a nonlinear realization of the chiral group: Let the operator $\mathcal{T}$ be defined by

$$\mathcal{T}(h)f(g,V) := f(h^{-1}g, \hat{T}(h)V)\quad ,\qquad (h\in\mathcal{G})\,. \tag{2.8}$$

It is now very easy to show that $\mathcal{T}$ leaves functions with the special structure $f(g,V) = \hat{T}(g)\tilde{f}(V)$ invariant.

Recall that the anomaly and the WZ action may be constructed geometrically from the Chern–Pontryagin density $\mathcal{P}_{d+2}(V) = \mathrm{Tr}\,(F^{d/2+1})$ in a fictitious $(d+2)$-dimensional space-time[11,14,13]: $\mathcal{P}_{d+2}$ is closed and may thus locally be written as the exterior derivative of a $(d+1)$-form $\omega^0_{d+1}(V)$. Furthermore, since $\mathcal{P}$ is invariant under finite gauge transformations generated by some $\beta(x)$ one infers from

$$0 = \delta^n_\beta \mathcal{P}_{d+2}(V) = \delta^n_\beta d\omega^0_{d+1}(V) = d\delta^n_\beta \omega^0_{d+1}(V) \quad , \qquad n \geq 1 \; , \tag{2.9}$$

that the n-th variation of ω^0_{d+1} may locally be written as the derivative of a d-form,

$$\delta^n_\beta \omega^0_{d+1}(V) = d\omega^n_d(V, \beta) \; , \tag{2.10}$$

which for $n = 1$ is just the anomaly for left-handed or right-handed vector fields.

The change in $S_{\rm WZ}$ under a chiral rotation with parameter $\beta(x)$ may be computed from (2.1):

$$\begin{aligned}
&S_{\rm WZ}[U', V', A'] - S_{\rm WZ}[U, V, A] \\
&\quad = K_d \int\limits_{B^{d+1}} \Big(\mathcal{T}(e^\beta, e^{-\beta}) - 1\Big)\Big(1 - \hat{\mathcal{T}}(1, U)\Big)\Big(\omega^0_{d+1}(V_R) - \omega^0_{d+1}(V_L) + dC_d\Big) \\
&\quad = K_d \int\limits_{B^{d+1}} \Big(\mathcal{T}(e^\beta, e^{-\beta}) - 1\Big)\Big(\omega^0_{d+1}(V_R) - \omega^0_{d+1}(V_L) + dC_d\Big) \\
&\quad = K_d \int\limits_{M^d} \sum_{n=1}^{\infty} \frac{1}{n!}\Big(\omega^n_d(V_R, \beta) - (-1)^n \omega^n_d(V_L, \beta) + \delta^n_\beta C_d(V_L, V_R)\Big) \\
&\quad = \int d^d x \,\mathrm{Tr}\,\Big(\beta(x)\,\Omega\big(V_R(x), V_L(x)\big)\Big) + O(\beta^2) \, .
\end{aligned} \tag{2.11}$$

In the third line, the corollary mentioned after (2.8) has been used, and the fourth line was obtained by expanding $\mathcal{T}(e^\beta, e^{-\beta})$ in powers of β and applying (2.10).

From (2.11) one concludes that

$$\delta^n_\beta \big(S_{\rm WZ}[V_R, V_L, U_2] - S_{\rm WZ}[V_R, V_L, U_1]\big) = 0 \, . \tag{2.12}$$

This corollary has an important consequence for computing quantum corrections: To that end, $S_{\rm WZ}[V_R, V_L, U]$ is expanded in powers of ξ about some configuration

U_0,

$$S_{\mathrm{WZ}}[V_R, V_L, U] = S_{\mathrm{WZ}}[V_R, V_L, U_0] + \sum_{k=1}^{\infty} S_{\mathrm{WZ},k}(V_R, V_L, U_0, \xi) \; ; \tag{2.13}$$

$S_{\mathrm{WZ},n}$ contains the fluctuation field to the n-th power. From (2.12) one concludes immediately that only $S_{\mathrm{WZ}}[V_R, V_L, U_0]$ has anomalous transformation properties whereas all the fluctuation terms are chiral invariants.

Are the fluctuation terms in (2.13) purely four-dimensional expressions, or do they contain five-dimensional parts as well? According to the decomposition (2.3), all parts of S_{WZ} with external fields are local in d dimensions, only the pure meson term lives in $d+1$ dimensions. It is thus sufficient to show that

$$\omega^0_{d+1}(U^\dagger dU) - \omega^0_{d+1}(U_0^\dagger dU_0) = \sum_{n=1}^{\infty} d\omega^n_d(U_0^\dagger dU_0, \zeta) \; ; \tag{2.14}$$

the expansion $U(x) = \exp(-\zeta(x))U_0(x)$ is used for notational simplicity [the results of the following argument can immediately be transcribed for the expansion (1.6)]. The vector field $U^\dagger dU$ may be considered the gauge transform of $U_0^\dagger dU_0$, with ζ playing the role of the generator of the corresponding left-handed rotation:

$$U^\dagger dU = \hat{T}(1, e^{-\zeta})(U_0^\dagger dU_0) \; . \tag{2.15}$$

Substituting this in (2.10), one immediately obtains the desired result (2.14).

The original construction leading to (2.1) assumed that the action does not contain Abelian gauge fields; hence, the generating functional $Z[\Phi]$ is invariant under global $\mathrm{U(1)_V} \times \mathrm{U(1)_A}$ transformations only. (However, the functional measure of the underlying fermionic theory is not invariant!) In the presence of Abelian gauge fields, Z also possesses the Abelian anomaly obtained from (1.5) when $\beta(x) = \beta^0(x)\lambda_0$. The latter is correctly contained in (2.1) if U has a singlet component. One may take $U(x) = e^{i\Theta(x)/3}\tilde{U}(x)$ if η' is not dynamical, and $U(x) =$

$e^{i\varphi^0(x)/3g}\tilde{U}(x)$ if η' is explicitly incorporated. One easily verifies that the singlet field does not contribute in $\Lambda_5(U)$. Furthermore, from $U^\dagger dU = \tilde{U}^\dagger d\tilde{U} + (i/3)d\Theta$ or $U^\dagger dU = \tilde{U}^\dagger d\tilde{U} + (i/3g)d\varphi^0$ one obtains

$$\alpha_4(V_L, U) = \alpha_4(V_L, \tilde{U}) - \tfrac{i}{6}\, d\Theta \,\mathrm{Tr}\left(V_L\big(2dV_L + V_L^2 - (d\tilde{U}\tilde{U}^\dagger)^2 - V_L(d\tilde{U}\tilde{U}^\dagger)\big)\right)\,, \tag{2.16}$$

$$\begin{aligned} C_4(V_R, U^\dagger(V_L + d)U) = \tfrac{i}{3}\, d\Theta \,\mathrm{Tr}\left(- V_R^3 + V_R^2\tilde{U}^\dagger(V_L + d)\tilde{U} - V_R\big(\tilde{U}^\dagger(V_L + d)\tilde{U}\big)^2\right) \\ + C_4\big(V_R, \tilde{U}^\dagger(V_L + d)\tilde{U}\big) \end{aligned} \tag{2.17}$$

and analogous formulas if η' is dynamical. Note that there are no quadratic or higher terms in the singlet field.

2.2. Anomaly of the unitarized generating functional

Consider the generating functional $W[\Phi]$ of the connected Green functions. The action is written as

$$S = S_{\text{inv.}} + S_{\text{WZ}} = \int d^4x \,\left[\mathcal{L}_{\text{inv.}}(U, V, A, \ldots) + \mathcal{L}_{\text{WZ}}(U, V, A)\right] \tag{2.18}$$

where $S_{\text{inv.}}$ contains all the chirally invariant pieces of the action. Disregarding for the moment the need for symmetry-preserving regularization, the variation of W under a chiral rotation with parameter $\beta(x)$ is

$$\begin{aligned} \delta_\beta W[V, A, \ldots] &= -ie^{-iW}(\delta_\beta e^{iW}) \\ &= -ie^{-iW}\delta_\beta \int [dU]\, e^{iS_{\text{inv.}}[U,V,A,\ldots] + iS_{\text{WZ}}[U,V,A]} \\ &= -ie^{-iW} \int [dU](i\delta_\beta S_{\text{WZ}})\, e^{iS_{\text{inv.}} + iS_{\text{WZ}}} \\ &= \int d^4x \,\mathrm{Tr}\left[\beta(x)\,\Omega(V, A)\right]\,, \end{aligned} \tag{2.19}$$

The last line results because $\delta_\beta S_{\text{WZ}}$ does not depend on U and may thus be pulled in front of the functional integral. In conjunction with (2.12), (2.13) this argument suggests that the anomaly of the unitarized generating functional is given by Bardeen's expression[20] alone without higher-order corrections.

In order to make the foregoing argument rigourous, the generating functional W of connected Green functions needs to be regularized explicitly. This can be achieved most conveniently if the background field expansion method is combined with the modified operator regularization scheme described in Ref. 3: In this way, the anomaly of the WZ term is isolated in the classical action and does not participate in quantum corrections. Operator regularization is ideally adapted to this situation and has the additional most significant advantage over dimensional regularization that antisymmetric ε tensors do not give rise to ambiguities.

In the presence of the WZ term and higher-order actions S_{2n}, $W[\Phi]$ is expressed as

$$
\begin{aligned}
e^{iW[\Phi]} &= e^{i\sum_{k=1}^{\infty} S_{2k}[U_0,\Phi]+S_{\mathrm{WZ}}[U_0,V,A]} \cdot e^{\frac{i}{2}\mathrm{Sp}\,(\ln D)} \cdot e^{\frac{1}{2}\delta(0)\int d^4x\,\mathrm{Tr}\,[\ln g_{ab}(i\delta/\delta J)]} \\
&\quad\times e^{i\sum_{k=1}^{\infty}\sum_{n=1}^{\prime\infty} S_{2k,n}[U_0,\Phi,i\delta/\delta J]} \cdot e^{i\sum_{n=1}^{\infty} S_{\mathrm{WZ},n}[U_0,\Phi,i\delta/\delta J]} \\
&\quad\times e^{-\frac{i}{2}\int d^4x d^4y J^a(x) D^{-1}_{ab}(x,y|U_0,\Phi) J^b(y)}\Big|_{J=0}\;;
\end{aligned}
\tag{2.20}
$$

the notation $\sum_{n=1}^{\prime\infty}$ indicates that the summation starts only at $n=3$ for $k=1$.

The significance of the various terms in (2.20) is as follows: The first factor is the contribution of all tree graphs with at most one vertex of higher order than p^2. All one-loop graphs with vertices from $\mathcal{L}_2$ are summarized in the second factor. Due to the curvature of the target manifold in the nonlinear σ model, the invariant functional Haar measure is not just $d\mu(\xi) = \prod_{x,a} d\xi^a(x)$, but also contains the square root of the determinant of the target-space metric[22–24] ; the third factor expresses these contributions by means of extra vertices[25] that are multiplied by the factor $\delta^{(4)}(0)$. As discussed in more detail in Ref. 3, these vertices are chiral invariants because the chiral transformations are precisely the isometries of the metric. Together with the remaining vertices in the fourth factor, they give rise to tree graphs with more than one higher-order vertex, one-loop graphs with one or more higher-order vertices and all graphs with more than one loop.

This representation of W has the distinct advantage that the objects to be regularized, namely the propagator and the logarithm of its inverse along with the δ function from the target-space metric, appear in an explicitly covariant form. The modified operator regularization scheme guarantees that both the finite and divergent parts of graphs built from covariant vertices and covariant propagators $D^{-1}(x, y|U_0, \Phi)$ are chirally invariant. Moreover, Ref. 3 shows how to regularize the δ function in a covariant way that is based on, and compatible with, the regularization of $\ln D$ and D^{-1}; the regularized expression turns out to be proportional to the divergent part of Sp $\ln D$, but to stay finite in the limit $\varepsilon \to 0$. OR introduces U_0 and Φ dependence not present in the unregularized determinant.

Note that all actions in (2.20) except S_2, $S_{2,n}$, S_{WZ} and $S_{\mathrm{WZ},n}$ contain infinite as well as finite pieces as $\varepsilon \to 0$. The divergent parts of $S_{2k,n}$ cancel subgraph divergences while those in S_{2k} eliminate primitive divergences. The Bogoliubov–Parasiuk–Hepp–Zimmermann subtraction program can thus be implemented in a manifestly invariant way.

The chiral transformation properties of the regulated W may now be determined from Eq. (2.20). $W_2 = S_2[U_0, \Phi]$ is a chiral invariant; W_4 contains the anomaly due to $S_{\mathrm{WZ}}[U_0, V, A]$. All contributions to W_6 containing WZ vertices are chiral invariants because the $S_{\mathrm{WZ},n}$, $n \geq 1$, are invariant; so are the terms from the target-space metric. Accordingly, the counterterms of order p^6 and thus also S_6 and all $S_{6,n}$ can be chosen to be invariant. This reasoning can be repeated for any successive order in the momentum expansion.

The conclusion is that the anomaly of the generating functional of connected Green functions is given by the variation of the Wess–Zumino term alone:

$$\delta_\beta W[\Phi] = \int d^4x \,\mathrm{Tr}\left[\beta(x)\,\Omega\left(V(x), A(x)\right)\right] . \tag{2.21}$$

As in the two-dimensional nonlinear σ model with WZ term, the coupling constant of the WZ term is not renormalized. In contrast to the two-dimensional case, the theory is not renormalizable in four dimensions, but only chirally invariant terms would have appeared in the general higher-order chiral actions S_{2k} even if there were no anomaly.

Three remarks concerning the foregoing argumentation are in order: (i) The proof applies to both the Abelian and non-Abelian anomalies. (ii) Using an appropriate expansion of U about U_0 is crucial: The ξ defined by (1.6) have reasonably simple transformation properties and admit covariant derivatives[21,2] to all orders. This property lets one immediately conclude that each term in the expansion of a chirally invariant action is again a chiral invariant. (iii) The precise relation between the actions $S_{2k}[U_0, \Phi]$ and $S_{2k,n}[U_0, \Phi, \xi]$ is not clear at this point. Evaluation of the divergent pieces in $\mathcal{L}_{4,1}(U_0, \xi)$ showed that they equal the expression obtained by expanding $\mathcal{L}_4(U)$ to first order in ξ, but no general proof for such a relation could be given.[3]

3. One-Loop Counterterms Induced by the Wess-Zumino Action

This section verifies the general statements of the previous section at the level of all $O(p^6)$ one-loop divergences in the Wess-Zumino sector and also gives the corresponding general chiral Lagrangean to that order.

3.1. Expansion of the Wess-Zumino Action

According to the power counting of ChPT,[4,2] each propagator is $O(p^{-2})$, vertices from $\mathcal{L}_{2k}$ count as $O(p^{2k})$ and each loop adds $O(p^4)$. Three categories of graphs thus contribute to the Wess-Zumino sector of W_6: (i) One-loop graphs with a WZ vertex in the loop; (ii) one-loop graphs with a WZ vertex outside the loop, and (iii) tree graphs with either a sixth-order counterterm, or with a WZ vertex and a fourth-order non-anomalous vertex, or with two WZ vertices. If $U_1(x)$ solves the equations of motion derived from $S_{\leq 4} = S_2 + S_4 + S_{\mathrm{WZ}}$, all graphs in category (iii) are contained either in $S_{\leq 4}[U_1]$, truncated at $O(p^6)$, or in $S_6[U_0]$, where U_0 solves S_2. The graphs in categories (i) and (ii) may be extracted from $\frac{i}{2}\,\mathrm{Sp}\,(\ln D_{\mathrm{reg.}})$. One obtains the differential operator D by expanding $S_{\leq 4}$ to second order in the fluctuation field $\xi(x)$ (see below). Here again, the resulting expression is to be evaluated at U_1 and truncated at $O(p^6)$. As discussed also in Ref. 3, U_1 contains terms to all orders in p because it solves an equation that is not homogeneous in p. The graphs in category (ii) amount to mass and decay-constant renormalization of the meson fields and can be obtained from the calculation in Ref. 2 by substituting U_1 for U_0. Hence it suffices to calculate the divergent pieces of the graphs in category (i).

Finding the second variation of the WZ action (2.3)–(2.7)is tedious because $S_{\rm WZ}$ is not a sum of invariant terms; non-invariant contributions from all pieces of the action must properly combine into a chiral invariant. However, one easily verifies that the second variation of $\Lambda_5(U)$ is indeed a total derivative and may thus be written as a local polynomial of ξ, Γ_μ and Δ_μ. With the convention $\xi = -i\lambda_a\xi^a$, one obtains

$$\mathcal{L}_{\rm WZ,2} = -\tfrac{1}{2}\left(\hat{\mathcal{V}}^\mu_{ab}(\xi^a d_\mu\xi^b - d_\mu\xi^a\xi^b) + \hat{\mathcal{S}}_{ab}\xi^a\xi^b\right). \tag{3.1}$$

$\hat{\mathcal{V}}^\mu_{ab}$ and $\hat{\mathcal{S}}_{ab}$ are matrix fields in the adjoint representation of the group G_V, defined in analogy to $\hat{\Gamma}_\mu$ and $\hat{\sigma}$ (see Ref. 2):

$$\begin{aligned}\hat{\mathcal{V}}^\mu_{ab} = C\,\varepsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}\Big([\lambda_a,\lambda_b](\{\Delta_\nu,\mathcal{F}^+_{\rho\sigma}\} - 8\Delta_\nu\Delta_\rho\Delta_\sigma)&\\ -(\lambda_a\Delta_\nu\lambda_b - \lambda_b\Delta_\nu\lambda_a)(\mathcal{F}^+_{\rho\sigma} - 8\Delta_\rho\Delta_\sigma)\Big)&\,,\end{aligned} \tag{3.2}$$

$$\begin{aligned}\hat{\mathcal{S}}_{ab} = C\,\varepsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}\Big(\{\lambda_a,\lambda_b\}[\mathcal{F}^-_{\mu\nu}, -\tfrac{1}{8}\mathcal{F}^+_{\rho\sigma} + \Delta_\rho\Delta_\sigma]&\\ +(\lambda_a\Delta_\mu\lambda_b + \lambda_b\Delta_\mu\lambda_a)\{\Delta_\nu,\mathcal{F}^-_{\rho\sigma}\}\Big)&\,.\end{aligned} \tag{3.3}$$

where $C := -iN_c/(96\pi^2 f^2)$. Note that these expressions are chirally covariant [compare (1.9), (1.10) and (1.13)]. It is also easy to check that each term is separately parity invariant if one keeps in mind that Δ and $\mathcal{F}^-$ are odd under parity while $\mathcal{F}^+$ is even and that $\varepsilon^{\mu\nu\rho\sigma}$ leads to an additional minus sign.

One may distinguish four different treatments of the flavour-singlet degree of freedom: (i) If it is disregarded altogether, the indices a, b in (3.2) and (3.3) take the values $1,\ldots,8$ and one sets $\mathrm{Tr}\,(R_\mu) = 0$ in (3.15). The construction of $\mathcal{L}_{6,\varepsilon}$ in Sect. 3.3 proceeds on these assumptions. (ii) If the external field $\Theta(x)$ is not zero but η' is not treated as dynamical, or if loops of η' are neglected, the formulae (3.2) and (3.3) allow one to extract the Θ- or $\overline{\varphi}^0$-dependent terms with ease ($\overline{\varphi}^0$ is the singlet component in U_0): Use the fact that the singlet component of Δ_μ is given by $\frac{i}{6}\nabla_\mu\Theta$ and $\frac{i}{6}\nabla_\mu\overline{\varphi}^0$, respectively, to obtain

$$\hat{\mathcal{V}}_{ab}^{(\Theta)\,\mu} = \frac{iC}{6}\,\varepsilon^{\mu\nu\rho\sigma}\nabla_\nu\Theta\,\mathrm{Tr}\left([\lambda_a,\lambda_b]\,\mathcal{F}^+_{\rho\sigma}\right)\,, \tag{3.4}$$

$$\hat{\mathcal{S}}_{ab}^{(\Theta)} = -\frac{iC}{6}\,\varepsilon^{\mu\nu\rho\sigma}\nabla_\mu\Theta\,tr\big([\lambda_a,\Delta_\nu][\lambda_b,\mathcal{F}^-_{\rho\sigma}] + [\lambda_a,\mathcal{F}^-_{\rho\sigma}][\lambda_b,\Delta_\nu]\big)\,. \tag{3.5}$$

If the equations of motion are solved for $S_2 + S_4 + S_{\mathrm{WZ}}$ in the presence of $\Theta(x)$, no linear terms in ξ will appear and the calculation described in the following subsection goes through unchanged. The result (3.15) is also valid for this situation (use $\mathrm{Tr}\,(R_\mu) = i\nabla_\mu\Theta$ or $i\nabla_\mu\overline{\varphi}^0$). (iii) If η' is treated like the octet Goldstone bosons – going beyond the present framework – (3.2) and (3.3) imply that $\hat{\mathcal{V}}^\mu_{00} = \hat{\mathcal{S}}_{00} = 0$. To $O(p^6)$, singlet fluctuations contribute through $\eta - \eta'$ mixing in $\mathcal{L}_2$[2] combined with certain terms in the WZ vertex (3.1):

$$\hat{\mathcal{V}}^\mu_{a0} = -C\cdot\frac{1}{3}\cdot\varepsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}\left(\lambda_a[\Delta_\nu,\mathcal{F}^+_{\rho\sigma}]\right)\,, \tag{3.6}$$

$$\hat{\mathcal{S}}_{a0} = -\frac{C}{4}\cdot\frac{1}{3}\cdot\varepsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}\left(\lambda_a[\mathcal{F}^-_{\mu\nu}\,,\,\mathcal{F}^+_{\rho\sigma} - 2\Delta_\rho\Delta\sigma]\right)\,. \tag{3.7}$$

The factor 1/3 is due to the normalization of the singlet field.

3.2. THE DIVERGENT TERMS OF $\mathcal{L}_6$

The evaluation of the divergent terms in Sp (ln D) in operator regularization is straightforward. One may, however, take a little short-cut here by noting that D has the same structure as the corresponding operator $D^{(2)} = d_\mu d^\mu + \hat{\sigma}$ which is derived from S_2 alone and whose determinant has been calculated previously[2]:

$$
\begin{aligned}
D &= d_\mu d^\mu + \hat{\sigma} + D^{(4)} + (d_\mu \hat{\mathcal{V}}^\mu + 2\hat{\mathcal{V}}_\mu d^\mu + \hat{\mathcal{S}}) \\
&= d'_\mu d'^\mu + \hat{\sigma}' + D^{(4)} + O(p^6) \, ,
\end{aligned}
\tag{3.8}
$$

where $D^{(4)}$ is derived from $\mathcal{L}_4$ and is not relevant for the present purposes. d'_μ is again a covariant derivative because $\hat{\mathcal{V}}_\mu$ is covariant under chiral transformations and $\hat{\sigma}'$ is a chirally covariant scalar:

$$
\begin{aligned}
d'_\mu \xi &= \partial_\mu \xi + [\hat{\Gamma}_\mu + \hat{\mathcal{V}}_\mu \, , \xi] \, , \\
\hat{\sigma}' &= \hat{\sigma} + \hat{\mathcal{S}} \, .
\end{aligned}
\tag{3.9}
$$

$\hat{\mathcal{V}}^2$, which is contained in the operator $d'_\mu d'^\mu$, is of $O(p^6)$ and does not belong to D. However, its effects in the determinant appear only at $O(p^8)$ and can be discarded later. It has been verified that OR and DR give the same expression for the divergent part of $\frac{i}{2}$ Sp (ln D), namely*

$$
\frac{i}{2} \operatorname{Sp}(\ln D) = \frac{1}{16\pi^2 \varepsilon} \int d^4x \, \widehat{\operatorname{Tr}} \left(\tfrac{1}{12} \hat{\Gamma}'_{\mu\nu} \hat{\Gamma}'^{\mu\nu} + \tfrac{1}{2} \hat{\sigma}'^2 \right) \, . \tag{3.10}
$$

$\hat{\Gamma}'_{\mu\nu}$ is the field strength associated with the covariant derivative d':

$$
\hat{\Gamma}'_{\mu\nu} = \partial_{[\mu} (\hat{\Gamma}_{\nu]} + \hat{\mathcal{V}}_{\nu]}) + (\hat{\Gamma}_{[\mu} + \hat{\mathcal{V}}_{[\mu})(\hat{\Gamma}_{\nu]} + \hat{\mathcal{V}}_{\nu]}) \, . \tag{3.11}
$$

Evaluation of the traces proceeds in the usual manner and is tedious. The

---

* The symbol Tr denotes the flavour trace in the fundamental representation while $\widehat{\operatorname{Tr}}$ refers to the adjoint representation. Sp in addition implies the trace over the continuous space-time label.

relation

$$\mathcal{F}^{+}_{\mu\nu} - 2\,\Delta_{[\mu}\Delta_{\nu]} = 2\,\Gamma_{\ddot{\mu}\nu} \tag{3.12}$$

is useful in conjunction with partial integration; besides $R_\mu = U^\dagger \nabla_\mu U$, the following abbreviations serve to make the result more legible:

$$F^{\pm}_{\mu\nu} := F^{R}_{\mu\nu} \pm U^\dagger F^{L}_{\mu\nu} U\ , \tag{3.13}$$

$$H^{\pm}_{\lambda,\mu\nu} := \nabla_\lambda F^{R}_{\mu\nu} \pm U^\dagger \nabla_\lambda F^{L}_{\mu\nu} U\ . \tag{3.14}$$

The final expression for the terms of order p^6 is:

$$\begin{aligned}
T_6 = \frac{1}{\varepsilon}\cdot\frac{1}{16\pi^2}\cdot\frac{-iN_c}{96\pi^2 f^2}\int d^4x\, \varepsilon^{\mu\nu\rho\sigma}\times \\
\times\Big\{ -\frac{N_f}{3}\,\mathrm{Tr}\,\Big(\big(H^{+}{}_{\tau,}{}^{\tau}{}_{\mu} + \tfrac{3}{4}\,[R^\tau, F^{-}_{\tau\mu}] + \tfrac{1}{2}\,[R_\mu, \nabla^\tau R_\tau] - \tfrac{1}{4}\,[R^\tau,\, \nabla_{\{\tau}R_{\mu\}}]\big) \\
* \{R_\nu\,,\, F^{+}_{\rho\sigma} - R_\rho R_\sigma\}\Big)\Big) \\
+ \frac{N_f}{8}\,\mathrm{Tr}\,\big(\,(R^2 - U^\dagger X - X^\dagger U)[F^{-}_{\mu\nu}\,,\, F^{+}_{\rho\sigma} - 2R_\rho R_\sigma]\big) \\
+ \frac{1}{2N_f}\,\mathrm{Tr}\,\big((U^\dagger X + X^\dagger U)[F^{-}_{\mu\nu}\,,\, F^{+}_{\rho\sigma} - R_\rho R_\sigma]\big) \\
- \tfrac{1}{2}\,\mathrm{Tr}\,\big((R^2 - U^\dagger X - X^\dagger U)R_\mu\big)\,\mathrm{Tr}\,\big(R_\nu F^{-}_{\rho\sigma}\big) \\
+ \tfrac{1}{2}\,\mathrm{Tr}\,\big(R_\tau R_\mu\big)\,\mathrm{Tr}\,\big(R^\tau\{R_\nu\,,\, F_{m\rho\sigma}\}\big) \\
+ \tfrac{2}{3}\,\mathrm{Tr}\,\big(H^{+}{}_{\tau,}{}^{\tau}{}_{\mu}\big)\,\mathrm{Tr}\,\big(R_\nu(F^{+}_{\rho\sigma} - R_\rho R_\sigma)\big) \\
+ \tfrac{1}{3}\,\mathrm{Tr}\,\big(F^{+}_{\rho\sigma}\big)\,\mathrm{Tr}\,\Big(R_\mu(H^{+}{}_{\tau,}{}^{\tau}{}_{\nu} + \tfrac{3}{4}\,[R^\tau, F^{-}_{\tau\nu}] \\
- \tfrac{1}{4}\,[R^\tau, \nabla_{\{\tau}R_{\nu\}}] - \tfrac{1}{2}\,[\nabla^\tau R_\tau\,, R_\nu])\Big) \\
- \tfrac{1}{3}\,\mathrm{Tr}\,\big(R_\mu\big)\,\mathrm{Tr}\,\Big(\big(H^{+}{}_{\tau,}{}^{\tau}{}_{\nu} + \tfrac{3}{4}\,[R^\tau, F^{-}_{\tau\nu}] - \tfrac{1}{4}\,[R^\tau, \nabla_{\{\tau}R_{\nu\}}] \\
- \tfrac{1}{2}\,[\nabla^\tau R_\tau\,, R_\nu]\big)\,\big(F^{+}_{\rho\sigma} - 2R_\rho R_\sigma\big)\Big) \\
- \tfrac{1}{4}\,\mathrm{Tr}\,\big(R_\mu\big)\,\mathrm{Tr}\,\big(\{R_\nu\,, F^{-}_{\rho\sigma}\}(R^2 - U^\dagger X - X^\dagger U)\big) \\
- \tfrac{1}{4}\,\mathrm{Tr}\,\big(R_\tau\big)\,\mathrm{Tr}\,\big(R^\tau[F^{-}_{\mu\nu},\, F^{+}_{\rho\sigma} - 2R_\rho R_\sigma]\big)\Big\}\ .
\end{aligned} \tag{3.15}$$

One may verify that the expression (3.15) agrees with the one obtained by Bijnens, Bramon and Cornet[16] if the differences in the normalization conventions are accounted for.

Several properties of the counterterm Lagrangean listed above should be noted: First, it is manifestly a parity and chiral invariant. One may also check that it is Hermitean and symmetric under time reversal (see the following subsection). Second, all of its terms contain the ε tensor and thus never emerge from loops without WZ vertices. They are, however, perfectly legitimate terms that should be included in $\mathcal{L}_6$ even if there were no anomaly. Third, many of these counterterms exist only if there are flavour-singlet external fields $V^{\mu,0}_{L,R}$ or if U contains a singlet component, i.e., if either the QCD vacuum angle Θ is non-zero or η' is a dynamical field. Finally, by far not all admissible $O(p^6)$ terms are required for cancelling divergences; this will be clearly seen in the following susbsection.

3.3. Construction of $\mathcal{L}_6$

For phenomenological applications it is important to know which kind of higher-order tree graphs can contribute to a specific process. As noted above, the one-loop divergences connected with the WZ action do not represent the most general Lagrangean of U and the external fields which is chirally symmetric and invariant under space inversion, time reversal and charge conjugation. Below, the portion of $\mathcal{L}_6$ is listed which contains the Levi-Civita tensor (only such terms may provide corrections to the anomalous processes at this order). Included are only the terms that do not vanish when the flavour-singlet vector and axial-vector fields are switched off and the vacuum angle is set to zero.

The parity transformation properties of the various elementary and composite fields are easily obtained. Under time reversal, $x^\mu \to x'^\mu = -x_\mu$ and $\partial_\mu \to \partial'^\mu = -\partial^\mu$, c-numbers are complex conjugated and the external fields transform like the corresponding quark currents:

$$S(x) \to S(x'), \quad P(x) \to -P(x'), \quad V^\mu(x) \to -V_\mu(x'), \quad A^\mu(x) \to -A_\mu(x') \tag{3.16}$$

(recall that S and P are Hermitean while V and A are anti-Hermitean). It follows that $F^{\pm}$ is $\mathcal{T}$-even. The exponential meson field is even under $\mathcal{T}$ while R_μ is odd:

$$U(x) = e^{\frac{i}{2}\lambda_P \varphi^P(x)} \longrightarrow e^{-\frac{i}{2}\lambda_P^*(-\varphi^P(x'))} = U(x')\ , \tag{3.17}$$

$$R_\mu = U^\dagger(x)\nabla_\mu U(x) \longrightarrow U^\dagger(x')(-\nabla'^\mu)U(x') = -R^\mu(x')\ . \tag{3.18}$$

If a Hermitean expression is separately invariant under $\mathcal{P}$ and $\mathcal{T}$, $\mathcal{CPT}$ invariance automatically guarantees charge conjugation invariance.

Let the terms in the Lagrangean be grouped according to the number of field-strength tensors:

$$\mathcal{L}^{n.s.}_{6,\varepsilon} = \mathcal{L}^{(0)}_{6,\varepsilon} + \mathcal{L}^{(1)}_{6,\varepsilon} + \mathcal{L}^{(2)}_{6,\varepsilon} + \mathcal{L}^{(3)}_{6,\varepsilon}\ . \tag{3.19}$$

The term without field strengths is given by

$$\begin{aligned}\mathcal{L}^{(0)}_{6,\varepsilon} = \frac{i}{f^2}\,\varepsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}\Big(&k_1^{(0)}\,(U^\dagger X - X^\dagger U)R_\mu R_\nu R_\rho R_\sigma \\ &+ k_2^{(0)}\,\nabla_\tau R^\tau R_\mu R_\nu R_\rho R_\sigma \\ &+ k_3^{(0)}\,\nabla_{\{\tau}R_{\mu\}}\,[R^\tau,\, R_\nu R_\rho R_\sigma] \\ &+ k_4^{(0)}\,\nabla_{\{\tau}R_{\mu\}}\,R_\nu[R^\tau,\, R_\rho]R_\sigma\Big)\ .\end{aligned} \tag{3.20}$$

To lowest order in the meson and external fields, these terms contribute to five-meson scattering. Comparing them with the counterterms found earlier, one sees that only $k_2^{(0)}$ and $k_3^{(0)}$ are needed to absorb divergences.

The terms with one field-strength tensor are much more numerous:

$$\begin{aligned}\mathcal{L}^{(1)}_{6,\varepsilon} = \frac{i}{f^2}\,\varepsilon^{\mu\nu\rho\sigma}\Big\{&k_1^{(1)}\mathrm{Tr}\,(F^+_{\mu\nu}\{R_\rho R_\sigma\,,\, U^\dagger X - X^\dagger U\}) \\ &+ k_2^{(1)}\mathrm{Tr}\,(F^+_{\mu\nu}R_\rho R_\sigma)\,\mathrm{Tr}\,(U^\dagger X - X^\dagger U) \\ &+ k_3^{(1)}\mathrm{Tr}\,(F^+_{\mu\nu}R_\rho(U^\dagger X - X^\dagger U)R_\sigma \\ &+ k_4^{(1)}\mathrm{Tr}\,(F^-_{\mu\nu}\,[R_\rho R_\sigma\,,\, U^\dagger X + X^\dagger U]) \\ &+ k_5^{(1)}\mathrm{Tr}\,(F^-_{\mu\nu}R_\rho)\,\mathrm{Tr}\,(R_\sigma(U^\dagger X + X^\dagger U))\end{aligned}$$

$$
\begin{aligned}
&+ k_6^{(1)}\mathrm{Tr}\left(F^-_{\mu\nu}\,[R_\rho R_\sigma\,,\, R_\tau R^\tau]\right)\\
&+ k_7^{(1)}\mathrm{Tr}\left(F^-_{\mu\nu}\,(R_\rho R_\tau R_\sigma R^\tau - R_\tau R_\rho R^\tau R_\sigma)\right)\\
&+ k_8^{(1)}\mathrm{Tr}\left(F^-_{\mu\nu}\,\{R_\rho\,, R_\tau\}\right)\mathrm{Tr}\left(R_\sigma R_\tau\right)\\
&+ k_9^{(1)}\mathrm{Tr}\left(F^-_{\mu\nu} R_\rho\right)\mathrm{Tr}\left(R_\sigma R_\tau R^\tau\right)\\
&+ k_{10}^{(1)}\mathrm{Tr}\left(F^+_{\mu\nu}\,\{R_\rho R_\sigma\,,\, \nabla^\tau R_\tau\}\right)\\
&+ k_{11}^{(1)}\mathrm{Tr}\left(F^+_{\mu\nu} R_\rho \nabla^\tau R_\tau R_\sigma\right)\\
&+ k_{12}^{(1)}\mathrm{Tr}\left(F^-_{\tau\mu}\,[R^\tau,\, R_\nu R_\rho R_\sigma]\right) \qquad\qquad (3.21)\\
&+ k_{13}^{(1)}\mathrm{Tr}\left(F^-_{\tau\mu}\,(R_\nu R^\tau R_\rho R_\sigma - R_\rho R_\sigma R^\tau R_\nu)\right)\\
&+ k_{14}^{(1)}\mathrm{Tr}\left(F^+_{\mu\nu}\,(R_\rho \nabla_{\{\tau} R_{\sigma\}} R^\tau - R^\tau \nabla_{\{\tau} R_{\sigma\}} R_\rho)\right)\\
&+ k_{15}^{(1)}\mathrm{Tr}\left(F^+_{\mu\nu}\,(\nabla_{\{\tau} R_{\rho\}} R_\sigma R^\tau - R^\tau R_\sigma \nabla_{\{\tau} R_{\rho\}})\right)\\
&+ k_{16}^{(1)}\mathrm{Tr}\left(F^+_{\mu\nu}\,(\nabla_{\{\tau} R_{\rho\}} R_\tau R^\sigma - R^\sigma R_\tau \nabla_{\{\tau} R_{\rho\}})\right)\\
&+ k_{17}^{(1)}\mathrm{Tr}\left(F^+{}_\mu{}^\tau \{\nabla_{\{\tau} R_{\nu\}},\, R_\rho R_\sigma\}\right)\\
&+ k_{18}^{(1)}\mathrm{Tr}\left(F^+{}_\mu{}^\tau R^\nu \nabla_{\{\tau} R_{\rho\}} R_\sigma\right) \qquad\qquad \Big\}\,.
\end{aligned}
$$

The constants subjected to renormalization are $k_4^{(1)}$, $k_5^{(1)}$, $k_6^{(1)}$, $k_8^{(1)}$, $k_9^{(1)}$, $k_{10}^{(1)}$, $k_{11}^{(1)}$, $k_{12}^{(1)}$, $k_{14}^{(1)}$, $k_{16}^{(1)}$, $k_{17}^{(1)}$ and $k_{18}^{(1)}$. [The first term in (3.15) can be expressed through the terms proportional to $k_{10}^{(1)}$, $k_{12}^{(1)}$, $k_{17}^{(1)}$, $k_7^{(2)}$ and $k_8^{(2)}$.]

Even more terms with two field-strength tensors exist:

$$
\begin{aligned}
\mathcal{L}^{(2)}_{6,\varepsilon} = \frac{i}{f^2}\,\varepsilon^{\mu\nu\rho\sigma}\Big\{ & k_1^{(2)}\mathrm{Tr}\left(F^+_{\mu\nu}F^+_{\rho\sigma}(U^\dagger X - X^\dagger U)\right)\\
&+ k_2^{(2)}\mathrm{Tr}\left(F^+_{\mu\nu}F^+_{\rho\sigma}\right)\mathrm{Tr}\left(U^\dagger X - X^\dagger U\right)\\
&+ k_3^{(2)}\mathrm{Tr}\left(F^-_{\mu\nu}F^-_{\rho\sigma}(U^\dagger X - X^\dagger U)\right)\\
&+ k_4^{(2)}\mathrm{Tr}\left(F^-_{\mu\nu}F^-_{\rho\sigma}\right)\mathrm{Tr}\left(U^\dagger X - X^\dagger U\right)\\
&+ k_5^{(2)}\mathrm{Tr}\left([F^-_{\mu\nu},\, F^+_{\rho\sigma}](U^\dagger X + X^\dagger U)\right) + k_6^{(2)}\mathrm{Tr}\left([F^-_{\mu\nu},\, F^+_{\rho\sigma}]R_\tau R^\tau\right)\\
&+ k_7^{(2)}\mathrm{Tr}\left(\{F^+_{\mu\tau},\, F^-{}_\nu{}^\tau\}R_\rho R_\sigma\right) + k_8^{(2)}\mathrm{Tr}\left(F^+_{\mu\tau}R_\rho F^-{}_\nu{}^\tau R_\sigma\right)\\
&+ k_9^{(2)}\mathrm{Tr}\left(F^+_{\mu\tau}F^-_{\rho\sigma}R^\tau R_\nu - F^-_{\rho\sigma}F^+_{\mu\tau}R_\nu R^\tau\right)\\
&+ k_{10}^{(2)}\mathrm{Tr}\left(F^-_{\mu\tau}F^+_{\rho\sigma}R^\tau R_\nu - F^+_{\rho\sigma}F^-_{\mu\tau}R_\nu R^\tau\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ k_{11}^{(2)}\mathrm{Tr}\left(F^+_{\rho\sigma}F^-_{\mu\tau}R^\tau R_\nu - F^-_{\mu\tau}F^+_{\rho\sigma}R_\nu R^\tau\right) \qquad (3.22)\\
&+ k_{12}^{(2)}\mathrm{Tr}\left(F^-_{\rho\sigma}F^+_{\mu\tau}R^\tau R_\nu - F^+_{\mu\tau}F^-_{\rho\sigma}R_\nu R^\tau\right)\\
&+ k_{13}^{(2)}\mathrm{Tr}\left(F^+_{\mu\tau}R_\nu F^-_{\rho\sigma}R^\tau - F^+_{\mu\tau}R^\tau F^-_{\rho\sigma}R^\nu\right)\\
&+ k_{14}^{(2)}\mathrm{Tr}\left(F^-_{\mu\tau}R_\nu F^+_{\rho\sigma}R^\tau - F^-_{\mu\tau}R^\tau F^+_{\rho\sigma}R^\nu\right)\\
&+ k_{15}^{(2)}\mathrm{Tr}\left(F^+_{\mu\nu}F^+_{\rho\sigma}\nabla^\tau R_\tau\right) + k_{16}^{(2)}\mathrm{Tr}\left(F^-_{\mu\nu}F^-_{\rho\sigma}\nabla^\tau R_\tau\right)\\
&+ k_{17}^{(2)}\mathrm{Tr}\left(\{F^+{}_\mu{}^\tau,\, F^+_{\rho\sigma}\}\nabla_{\{\tau}R_{\nu\}}\right) + k_{18}^{(2)}\mathrm{Tr}\left(\{F^-{}_\mu{}^\tau,\, F^-_{\rho\sigma}\}\nabla_{\{\tau}R_{\nu\}}\right)\\
&+ k_{19}^{(2)}\mathrm{Tr}\left(\{F^+_{\mu\nu},\, H^+_{\tau,\rho}{}^\tau\}R^\sigma\right) + k_{20}^{(2)}\mathrm{Tr}\left(\{F^-_{\mu\nu},\, H^-_{\tau,\rho}{}^\tau\}R^\sigma\right)\\
&+ k_{21}^{(2)}\mathrm{Tr}\left(\{F^+_{\mu\tau},\, H^+_{\rho,\sigma}{}^\tau\}R^\nu\right) + k_{22}^{(2)}\mathrm{Tr}\left(\{F^-_{\mu\tau},\, H^-_{\rho,\sigma}{}^\tau\}R^\nu\right)\Big\}.
\end{aligned}
$$

The one-loop graphs renormalize the couplings $k_5^{(2)}$, $k_6^{(2)}$, $k_7^{(2)}$, $k_8^{(2)}$, $k_{11}^{(2)}$, $k_{14}^{(2)}$ and $k_{19}^{(2)}$.

Finally, there is no renormalization of the single cubic term in $F^\pm$:

$$
\mathcal{L}_{6,\varepsilon}^{(3)} = \frac{i}{f^2}\,\varepsilon^{\mu\nu\rho\sigma}\,k_1^{(3)}\,\mathrm{Tr}\left(F^+_{\mu\nu}\{F^+_{\rho\tau}\,,\,F^-{}_\sigma{}^\tau\}\right). \qquad (3.23)
$$

All 45 constants appearing in (3.20)–(3.23) are real and dimensionless. Note that partial integration and the Bianchi identity $\varepsilon^{\mu\nu\rho\sigma}\nabla_\nu F^{L,R}_{\rho\sigma} = 0$ as well as $\nabla_{[\mu}\nabla_{\nu]}U = F^L_{\mu\nu}U - UF^R_{\mu\nu}$ were used to find the maximum set of independent terms. However, no use was made of the equations of motion. In a calculation to $O(p^6)$, the equations of motion derived from $\mathcal{L}_2$ may be imposed on $U(x)$; for $\Theta(x) = 0$ they are[2]

$$
U^\dagger\nabla^2 U - \nabla^2 U^\dagger U + U^\dagger X - X^\dagger U - \tfrac{1}{3}\,\mathrm{Tr}\,(U^\dagger X - X^\dagger U) = 0\,. \qquad (3.24)
$$

It follows that $k_2^{(0)}$, $k_8^{(1)}$, $k_9^{(1)}$, $k_{15}^{(2)}$ and $k_{16}^{(2)}$ can be absorbed in some of the remaining 40 couplings at the present level of phenomenological analysis.

4. Discussion

The main purpose of Ref. 3 and of the present work was to show that the external-field formulation of ChPT, as given by Gasser and Leutwyler, can be consistently extended to all orders in the momentum expansion. If operator regularization is used, chiral symmetry can be kept explicit at all levels. The formalism thus represents a convenient, systematic and theoretically well-founded method for exploiting the constraints that chiral symmetry imposes on low-energy hadron interactions. The nonrenormalizability of the model does not lead to particular theoretical problems within the perturbative expansion in powers of the external momenta.

The chiral anomaly explicitly breaks chiral symmetry at the tree level within this approach. Quantum effects generally enhance the effect of symmetry breaking terms in the Lagrangean; an example is furnished by the meson mass terms in ChPT itself: At tree level, they lead to the Gell-Mann–Okubo mass formula, but one-loop graphs produce deviations from this simple pattern. This paper has shown how the anomalous chiral Ward identities avoid such a fate by virtue of the special properties of the Wess–Zumino term; ChPT and QCD have the same anomaly to all orders in perturbation theory. The result obtained in this paper may be viewed as the analogue in ChPT of the Adler–Bardeen nonrenormalization theorem.[26]

It should be emphasized again that this nonrenormalization theorem does *not* forbid quantum corrections to anomalous processes like $\pi^0 \to \gamma\gamma$. They produce two kinds of effects: On one hand, the meson masses and decay constants etc. are renormalized as usual; on the other hand, the point couplings due to the WZ term develop form factors. The nonrenormalization theorem for the chiral anomaly asserts, however, that the corrections respect chiral symmetry.

As mentioned in the Introduction, the one-loop corrections to anomalous meson processes have recently been discussed in detail,[15–17] in view of improved experiments at $e^+ - e^-$ colliders. The main difficulty facing a phenomenological analysis is the lack of predictive power due to the host of uncalculable coupling constants, as exemplified by Eqns. (3.20) to (3.23). Clearly, some additional dynamical input is needed at this point. Recently, Ecker, Gasser, Pich and de Rafael[27] showed that vector meson dominance, implemented in a way that respects both chiral symmetry and the power counting of ChPT to order p^4, accounts to a large extent for the values of the couplings in $\mathcal{L}_4$. One may thus expect that analogous estimates[16] of the parameters in $\mathcal{L}_{6,\varepsilon}$ are similarly accurate – a hope that may be put to a test only by significantly more precise and detailed measurements.

On a more theoretical level, this approach leads to two related problems: (i) How can ChPT be extended to non-Goldstone bosons without incurring the problem of double counting? (ii) Do these additional fields directly participate in anomalous interactions? (The long-standing controversy over this point may be traced, e.g., fromRef. 28.) Moreover, the question that was answered for the pure pseudoscalar-meson theory in this paper then arises anew: Is the chiral anomaly still protected from renormalization?

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