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GENERALIZATION OF THE QUARK-CONFINING STRING*

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ABSTRACT

It is natural to generalize the quark-confining string model by introducing additional terms coming from the Riemann curvature tensor. For simplicity, only terms linear in the Riemann tensor are considered. In the absence of quark fields, such terms are trivial (as in two dimensional gravity). In the presence of quark fields, embedding of the string in four dimensional Minkowski space renders such terms non-trivial. We formulate this generalized quark-confining string in the first order form, from which we derive its second order form. The coupling between the quark spin and the string curvature via the connections induces a spin-spin interaction among the quark fields. This generalized version has a dimensionless parameter κ in addition to the quark-gluon coupling e and quark masses. The original version is recovered as a special case (i.e., $\kappa = 0$).

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I. INTRODUCTION

The most expedient approach to the theory of hadron physics is probably via the construction of theoretical models which incorporate, as many as possible, physically desirable features. Following this approach, the quark-confining string (QCS) model[1] is constructed. The physical picture of the QCS is intuitively transparent and its mathematical structure well defined. In fact, the QCS is the only known model which is

(1) relativistic invariant,

(2) gauge invariant,

(3) field theoretic (as opposed to quantum mechanical[2]), and has

(4) manifest quark-confinement.

In addition, the model seems to have the properties of the parton picture, as well as Regge behavior and duality. It has reparametrization invariance, asymptotic linear trajectories and Hagedorn-like spectroscopy. In the non-relativistic limit, it provides a linear potential between colored quarks.

How seriously one should take the QCS depends on quantitative comparisons between the model and experimental data. For example, in the non-relativistic limit, the QCS gives a very reasonable fit[3] to the ψ spectroscopy discovered recently. The crucial question is, of course, how the model fares for the various relativistic corrections, such as hyperfine splittings, spin-orbit couplings, etc. To answer this question, we must first examine the uniqueness and generality of the QCS. Only after this issue is clarified can we use a detailed quantitative comparison with data as a test of the model. In this paper, we address ourselves to this issue. We find that there is a very natural way to generalize the QCS constructed earlier.

There are many ways of looking at the QCS. For the sake of definiteness, let us adopt the following point of view. Consider quantum chromodynamics (QCD) in two dimensional Minkowski space. Here, quark-confinement is explicit. Now, let us extend this model to four dimensional Minkowski space. One such extension gives the standard QCD. However, if we want to retain the explicit quark-confinement feature, the simplest extension to the physical space is the QCS. Here the original two dimensional space-time becomes a two dimensional curved subspace (V_2) embedded in four dimensional Minkowski space. In the original version of the QCS,[1] however, the curvature of the string is determined only by the motion of the quark and gluon fields, and not by the spin of the quarks. For a curved subspace V_2, it is natural to introduce into the model a term due to the Riemann curvature tensor. Contrary to two dimensional gravity, the embedding renders such a term non-trivial when (four dimensional Dirac) quark fields are introduced.

In the presence of a Riemann curvature term in the Lagrangian, consistency requires an interaction between the spin of the quark fields and the curvature of the string via the connections. This automatically induces a spin-spin interaction term among the quark fields themselves. The standard method of deriving this spin-spin interaction term in the model is to start from the first order (Palatini-Cartan) formalism[4] developed in General Relativity.

The generalized QCS is given, in the first order form, by

$$S = \int d^2u \, \sqrt{-g}\left[\frac{1}{\kappa} R(\Gamma) + \bar{\psi}\not{\tau}^{\alpha}(\frac{i}{2}\overleftrightarrow{D}_{\alpha} - e\, B^j_{\alpha}T^j)\psi - \bar{\psi}M\psi - \frac{1}{4}F^2\right] \tag{1.1}$$

where $D_\alpha = \partial_\alpha - \frac{i}{4}\Gamma_{\alpha\mu\nu}\,\tilde{\sigma}^{\mu\nu}$ is the covariant derivative and R is the Riemann scalar. Here the connection $\Gamma_{\alpha\mu\nu}$, the string coordinate X_μ, the quark field ψ (flavor index suppressed) and the gluon field B^j_α are treated as independent variables. κ is a dimensionless parameter analogous to the gravitational constant in two dimensions. Since the equation obtained from the variation of $\Gamma_{\alpha\mu\nu}$ is only a constraint equation, we can solve for $\Gamma_{\alpha\mu\nu}$ and rewrite the action (1.1) in the second order form

$$S = \int d^2u \sqrt{-g}\left[\bar{\psi}\not{\tau}^{\alpha}\left(\frac{i}{2}\overleftrightarrow{\partial}_{\alpha} - e B^j_{\alpha} T^j\right)\psi - \bar{\psi}M\psi - \frac{F^2}{4} + \kappa\, S_{\alpha\beta a} S^{\alpha\beta a}\right] \tag{1.2}$$

Here $S_{\alpha\beta a} = -\frac{i}{8}\bar{\psi}[\not{\tau}_{\alpha},\not{\tau}_{\beta}]\not{n}_a\psi$. The parameter κ, the quark-gluon coupling e and quark masses M_j are the parameters of the model. For $\kappa = 0$, the explicit spin-spin interaction term drops out and the model (1.2) reduces to the QCS constructed earlier.[1] We note that the Riemann scalar term is absent in the second order form (1.2).

This paper is organized as follows: In Section II, we review the (old) QCS, in particular the geometric notations and the parallel transport properties of the quark fields along the string. In Section III, we prove that the connection in the first order formulation in the absence of quark fields is precisely that derived from parallel transport argument. In Section IV, we introduce the generalized QCS (1.1) and derive from it the second order form (1.2). Section V contains some remarks, in particular, on the relation between QCD and the QCS. Some of the detailed derivations are relegated to two appendices.

II. REVIEW

A. Notations

The coordinates of the string X_μ are functions of the two parameters u^α, which provide an arbitrary coordination of the world sheet,

$$X_\mu = X_\mu(u^\alpha) \qquad \mu = 0,1,2,3 \quad \alpha = 0,1 \tag{2.1}$$

The induced metric for the two dimensional (string) subspace is given by

$$g_{\alpha\beta} = \tau_{\alpha\mu}\tau_{\beta}{}^{\mu} = \tau_\alpha \cdot \tau_\beta \tag{2.2}$$

where $\tau_{\alpha\mu} \equiv \frac{\partial X_\mu}{\partial u^\alpha} \equiv X_{\mu|\alpha}$ are the tangent vectors. The quark fields are four component Dirac fields in Minkowski space. They are confined along the string:

$$\psi_i = \psi_i(u^\alpha) \tag{2.3}$$

The original QCS is defined by the action[1]

$$\begin{aligned} S &= \int d^2u\, \mathscr{L}_0 \\ &= \int d^2u\, \sqrt{-g} \left\{ \bar{\psi}\left[\not{\tau}^\alpha\left(\frac{i}{2}\overleftrightarrow{\partial}_\alpha - e\, B^j{}_\alpha T^j\right) - M\right]\psi - \frac{1}{4} F^j{}_{\alpha\beta} F^{j\alpha\beta}\right\} \end{aligned} \tag{2.4}$$

Here $g = \det[g_{\alpha\beta}]$ and $d^2u^\alpha\sqrt{-g}$ is the invariant volume element on the world surface. $\not{\tau}^\alpha \equiv g^{\alpha\beta}\not{\tau}_\beta \equiv \tau^\alpha_\mu \gamma^\mu$ where $g^{\alpha\beta}$ is the inverse of $g_{\alpha\beta}$ and γ^μ are the (4×4) Dirac matrices.

$$\bar{\psi}\not{\tau}^\alpha \overleftrightarrow{\partial}_\alpha \psi = \bar{\psi}\not{\tau}^\alpha \partial_\alpha\psi - \left(\partial_\alpha\bar{\psi}\right)\not{\tau}^\alpha\psi$$

T^j are the matrix generators of SU(3) color group with structure constants f_{ijk}: $[T_j, T_k] = i f_{jk\ell} T_\ell$. $B_{j\beta}(u^\alpha)$ are color gauge fields in the internal coordinate space and

$$F^i_{\alpha\beta} = \partial_\alpha B^i_\beta - \partial_\beta B^i_\alpha + e\, f^{ijk} B^j_\alpha B^k_\beta \tag{2.5}$$

The Dirac equation follows from the action (2.4):

$$\left(i\not{\tau}^\alpha\partial_\alpha + \frac{i}{2\sqrt{-g}}\left(\sqrt{-g}\not{\tau}^\alpha\right)_{|\alpha} - e\not{\tau}^\alpha\, T^j B^j{}_\alpha - M\right)\psi = 0 \tag{2.6}$$

where the flavor index is suppressed. To prepare ourselves for the introduction of Riemann curvature scalar into the Lagrangian, we shall review in detail the parallel transport properties of a quark field along the string.

First, let us consider the parallel transport of a vector quantity $V_\mu(u^\alpha)$ along the string. We can then derive the parallel transport of a Dirac field by demanding that the current $\bar{\psi}\gamma_\mu\psi = J_\mu(u^\alpha)$ parallel transports as a vector quantity. This gives the connection for the Dirac fields. The equivalence of the first and the second order form for this connection is then demonstrated.

Throughout, $\alpha, \beta, \gamma, \delta, \eta = 0,1$ are the internal indices; $\mu, \nu, \rho, \lambda, \sigma = 0,1,2,3$ are the Minkowski indices; $a,b,c = 2,3$ are the indices for the normals (to be defined) and $i,j,k = 1,2,\ldots,8$ are the color SU(3) indices.

B. Parallel Transport of a Vector and a Spinor Quantity

The "Christoffel symbol of the second kind" is a function of the induced metric $g_{\alpha\beta}$, or equivalently

$$\left\{{}_{\alpha}{}^{\gamma}{}_{\beta}\right\} = \tau^{\gamma}\cdot\tau_{\alpha|\beta} \tag{2.7}$$

At each point of the string, two normals $n^{a}_{\mu}(u^{\alpha})$, $a = 2,3$ can be defined such that $\tau_{\alpha}\cdot n_{a} = n_{2}\cdot n_{3} = 0$ and $n^{2}_{a} = n_{a\mu}n_{a}{}^{\mu} = -1$. With these, one can write the flat Minkowski metric $\eta_{\mu\nu}$ in the following form

$$\eta_{\mu\nu} = \tau^{\alpha}_{\mu}\tau_{\alpha\nu} - n^{a}_{\mu}n^{a}_{\nu} \tag{2.8}$$

The covariant derivative of $\tau_{\alpha\mu}$ defines the symmetric curvature tensors $h^{a}_{\alpha\beta}$:

$$\tau^{\mu}_{\alpha||\beta} = \tau^{\mu}_{\alpha|\beta} - \left\{{}_{\alpha}{}^{\gamma}{}_{\beta}\right\}\tau_{\gamma}{}^{\mu} = h^{a}_{\alpha\beta}n^{a\mu} \tag{2.9}$$

and the derivative of the normals introduces the antisymmetric torsion ν^{ab}_{α}:

$$n^{a}_{\mu;\alpha} \equiv n^{a}_{\mu|\alpha} - \nu^{ab}_{\alpha}n^{b}_{\mu} = h^{a}_{\alpha\beta}\tau^{\beta}_{\mu} \tag{2.10}$$

Using Eq. (2.8), any vector quantity V_{μ} can be written as

$$\begin{aligned} V_{\mu} &= V\cdot\tau^{\alpha}\tau_{\alpha\mu} - V\cdot n^{a}n^{a}_{\mu} \\ &= V^{\alpha}\tau_{\alpha\mu} - V^{a}n^{a}_{\mu} = V_{p\mu} + V_{n\mu} \end{aligned} \tag{2.11}$$

where

$$V_{p}\cdot n^{a} = V_{n}\cdot\tau^{\alpha} = 0 \tag{2.12}$$

Varying $V_{p\mu}$ and using Eq. (2.9), we obtain

$$\begin{aligned} \delta V_{p\mu} &= (\delta V^{\alpha})\tau_{\alpha\mu} + V^{\alpha}\delta\tau_{\alpha\mu} \\ &= \left(V^{\alpha}{}_{||\beta}\tau_{\alpha\mu} + V^{\alpha}h^{a}_{\alpha\beta}n^{a}_{\mu}\right)\delta u^{\beta} \end{aligned} \tag{2.13}$$

Varying $V_{n\mu}$ and using Eq. (2.10), we obtain

$$\begin{aligned} \delta V_{n\mu} &= -(\delta V^a)n^a_\mu - V^a \delta n^a_\mu \\ &= -\left\{V^a h^a_{\alpha\beta}\tau^\alpha_\mu + V^a{}_{;\beta} n^a_\mu\right\}\delta u^\beta \end{aligned} \tag{2.14}$$

Since we are considering parallel transportation only, $V^\alpha{}_{||\beta} = V^a{}_{;\beta} = 0$; hence

$$\delta V_\mu = \left[(V\cdot\tau^\alpha)n^a_\mu - (V\cdot n^a)\tau^\alpha_\mu\right] h^a_{\alpha\beta}\delta u^\beta \tag{2.15}$$

Let us now consider the spinor field.[5] An infinitesimal displacement of a spinor field along the string is defined to be

$$\delta\psi = \frac{i}{4}\omega_{\alpha\mu\nu}\sigma^{\mu\nu}\psi\,\delta u^\alpha \tag{2.16}$$

where $\omega_{\alpha\mu\nu}$ is the connection and $\sigma^{\mu\nu} = \frac{i}{2}\left[\gamma^\mu,\gamma^\nu\right]$. Demanding $\bar{\psi}\gamma_\mu\psi$ to transform as a vector quantity, we have

$$\delta\left(\bar{\psi}\gamma_\mu\psi\right) = -\frac{i}{4}\bar{\psi}\left[\sigma^{\lambda\nu},\gamma_\mu\right]\psi\,\omega_{\alpha\lambda\nu}\,\delta u^\alpha \tag{2.17}$$

Comparing Eqs. (2.15) and (2.17), we obtain the connection as a function of the curvature tensor

$$\begin{aligned} \omega_{\alpha\mu\nu} &= h^a_{\alpha\beta}\left(\tau^\beta_\mu n^a_\nu - n^a_\mu\tau^\beta_\nu\right) \\ &\equiv h^a_{\alpha\beta}\left[\tau^\beta, n^a\right]_{\mu\nu} \end{aligned} \tag{2.18}$$

where the antisymmetry in μ,ν is explicit. The covariant derivative for spinors is now completely defined

$$\begin{aligned} \partial_\alpha \to D_\alpha &= \partial_\alpha - \frac{i}{4}\omega_{\alpha\mu\nu}\sigma^{\mu\nu} \\ &= \partial_\alpha + \frac{i}{2} h^a_{\alpha\beta} n^a_\mu \tau^\beta_\nu \sigma^{\mu\nu} \end{aligned} \tag{2.19}$$

Contracting D_α with χ^α gives the first two terms in the Dirac equation (2.6).

C. The Riemann Curvature Tensor

The Riemann curvature tensor in V_2 is defined by

$$\tau^{\gamma}_{\mu||\alpha||\beta} - \tau^{\gamma}_{\mu||\beta||\alpha} = R^{\gamma}{}_{\delta\alpha\beta}\tau^{\delta}_{\mu} \tag{2.20}$$

Using Eqs. (2.9) and (2.10), it is straightforward to express $R_{\alpha\beta\gamma\delta}$ in terms of the curvature tensors[6]

$$R_{\alpha\beta\gamma\delta} = h^{a}_{\alpha\gamma}h^{a}_{\delta\beta} - h^{a}_{\alpha\delta}h^{a}_{\gamma\beta} \tag{2.21}$$

which has only one independent component, namely R_{0101}. Eq. (2.21) is called the Gauss-Codazzi equation.

III. RIEMANN CURVATURE TENSOR

In the first order formalism, the Riemann curvature tensor is expressed as a function of the connection $\omega_{\alpha\mu\nu}$.

$$\left[D_{\alpha},D_{\beta}\right] = \left[\partial_{\alpha} - \frac{i}{4}\omega_{\alpha\mu\nu}\,\sigma^{\mu\nu}, \partial_{\beta} - \frac{i}{4}\omega_{\beta\mu\nu}\,\sigma^{\mu\nu}\right] = \frac{i}{4}R_{\mu\nu\alpha\beta}\,\sigma^{\mu\nu} \tag{3.1}$$

or

$$R_{\mu\nu\alpha\beta} = \omega_{\alpha\mu\nu|\beta} - \omega_{\beta\mu\nu|\alpha} + \omega_{\beta\mu}{}^{\sigma}\omega_{\alpha\sigma\nu} - \omega_{\alpha\mu}{}^{\sigma}\omega_{\beta\sigma\nu} \tag{3.2}$$

This Riemann tensor is a mixed tensor: it is an antisymmetric rank two tensor in both the Minkowski space and the internal coordinate space V_2. (The relation of this mixed Riemann curvature tensor with the Riemann tensor introduced in Section II will be given later.) To construct a Lagrangian from this Riemann tensor, we must contract the indices using the dynamical variables available. In the absence of quark and gluon fields, the only variables present are the string variables. Thus, if we consider only terms linear in $R_{\mu\nu\alpha\beta}$ in the Lagrangian, $R_{\mu\nu\alpha\beta}$ must be contracted with the tangent vectors $\tau_{\alpha\mu}$. There are two ways of contraction:

$$R = \tau^{\alpha\mu}\tau^{\beta\nu}R_{\mu\nu\alpha\beta} \tag{3.3}$$

and

$$R^* = U^{\mu\nu\alpha\beta} R_{\mu\nu\alpha\beta} \tag{3.4}$$

where

$$U^{\mu\nu\alpha\beta} = \frac{1}{2}\varepsilon^{\mu\nu\rho\lambda} \tau^{\alpha}_{\rho} \tau^{\beta}_{\lambda} \tag{3.5}$$

In this section, we shall consider the following preliminary model (where the constant C is added solely to render the model non-trivial)

$$\begin{aligned} S &= \int d^2u \sqrt{-g} \left\{ \frac{1}{\kappa_1} R + \frac{1}{\kappa_2} R^* - C \right\} \\ &= \int d^2u \left\{ \mathscr{L}_1 + \mathscr{L}_2 - \sqrt{-g}\, C \right\} \end{aligned} \tag{3.6}$$

Here κ_i are dimensionless parameters, reminiscent of the gravitational coupling constant in two dimensions. The Lagrangian is a function of X_μ and $\omega_{\alpha\mu\nu}$, which are treated as independent variables. Varying with respect to $\omega_\alpha{}^{\mu\nu}$, we obtain

$$\begin{aligned} &\frac{1}{\kappa_1} \left\{ h^{a\beta}{}_{\beta}[\tau^\alpha, n^a]_{\mu\nu} - h^{a\alpha}{}_{\beta}[\tau^\beta, n^a]_{\mu\nu} \right\} - \frac{1}{\kappa_2} e^{\alpha\beta}\, \varepsilon_{ab} h^{b}_{\beta\gamma} [n^a, \tau^\gamma]_{\mu\nu} \\ &+ \omega_{\beta\mu}{}^{\sigma} \left\{ \frac{1}{\kappa_1} [\tau^\alpha, \tau^\beta]_{\sigma\nu} + \frac{1}{\kappa_2} e^{\alpha\beta}\, \varepsilon_{ab} n^a{}_{\nu} n^b{}_{\sigma} \right\} \\ &+ \omega_{\beta\nu}{}^{\sigma} \left\{ \frac{1}{\kappa_1} [\tau^\alpha, \tau^\beta]_{\mu\sigma} - \frac{1}{\kappa_2} e^{\alpha\beta}\, \varepsilon_{ab} n^a{}_{\mu} n^b{}_{\sigma} \right\} = 0 \end{aligned} \tag{3.7}$$

where

$$e_{\alpha\beta} = \sqrt{-g}\, \varepsilon_{\alpha\beta} \qquad \varepsilon_{01} = -\varepsilon^{01} = 1 \tag{3.8}$$

and $U^{\mu\nu\alpha\beta}$ can be written in terms of n^a_μ ($\varepsilon_{23} = \varepsilon^{23} = 1$)

$$U^{\mu\nu\alpha\beta} = -\frac{1}{2} e^{\alpha\beta}\, \varepsilon_{ab} n^{a\mu} n^{b\nu} \tag{3.9}$$

We note that $U^{\mu\nu\alpha\beta} U_{\mu\nu\alpha\beta} = -1$.

Eq. (3.7) is linear in $\omega_{\alpha\mu\nu}$ and hence is a constraint equation. To solve it, let us project out the various components of $\omega_{\alpha\mu\nu}$, using its antisymmetric property

$$\omega_{\alpha\mu\nu} = -\omega_{\alpha\nu\mu} \tag{3.10}$$

and Eq. (2.8):

$$\omega^{\alpha}{}_{\mu\nu} = \omega^{\alpha\beta\gamma}[\tau_\beta, \tau_\gamma]_{\mu\nu} + \omega^{\alpha ab}[n^a, n^b]_{\mu\nu} + \omega^{\alpha\beta a}[\tau_\beta, n^a]_{\mu\nu} \tag{3.11}$$

Substituting this into Eq. (3.7), we find the $\omega^{\alpha\beta\gamma}$ and $\omega^{\alpha ab}$ terms disappear. Equating the terms for $[\tau_\beta, n^a]_{\mu\nu}$, the following solution emerges

$$\omega^{\beta\gamma a} = h^{a\beta\gamma} \tag{3.12}$$

We make two observations: (1) the solution (3.12) is independent of κ_i, and (2) $\omega^{\alpha\beta\gamma}$ and $\omega^{\alpha ab}$ are not determined. To gain more insight into (2), let us substitute Eq. (3.11) into Eq. (3.2). It is straightforward to obtain

$$\begin{aligned} R_{\mu\nu\alpha\beta} \;=\; & [\tau^\gamma, \tau^\delta]_{\mu\nu} \left\{ R_{\alpha\beta\gamma\delta}\left(\omega_{\alpha\gamma\delta}\right) + h^a_{\alpha\gamma} h^a_{\beta\delta} \right\} \\ & + [n^a, n^b]_{\mu\nu} \left\{ R_{ab\alpha\beta}\left(\omega_{\alpha ab}\right) + h^{a\gamma}_{\beta} h^b_{\alpha\gamma} \right\} \end{aligned} \tag{3.13}$$

where the following Gauss-Codazzi equation is used

$$h^a_{\alpha\beta:\gamma} - h^a_{\alpha\gamma:\beta} = 0 \tag{3.14}$$

The symbol ":" stands for total covariant derivative, e.g.,

$$v^{a\beta}{}_{:\alpha} = v^{a\beta}{}_{|\alpha} + \left\{ {\beta \atop \alpha\gamma} \right\} v^{a\gamma} - \nu^{ab}_{\alpha} v^{b\beta} \tag{3.15}$$

and

$$R_{\alpha\beta\gamma\delta}\left(\omega_{\alpha\beta\gamma}\right) = \omega_{\alpha\gamma\delta||\beta} + 2\,\omega_{\beta\gamma\eta}\,\omega_{\alpha}{}^{\eta}{}_{\delta} - (\alpha \leftrightarrow \beta) \tag{3.16}$$

$$R_{ab\alpha\beta}\left(\omega_{\alpha ab}\right) = \omega_{\alpha ab:\beta} + 2\,\omega_{\beta ac}\,\omega_{\alpha bc} - (\alpha \leftrightarrow \beta) \tag{3.17}$$

Let us now consider the string equation of motion. The string equation is simply $\partial_\alpha \left(\dfrac{\delta \mathscr{L}}{\delta \tau_\alpha{}^\mu} \right) = 0.$

$$\frac{\delta \mathscr{L}}{\delta \tau_\alpha{}^\mu} = -\sqrt{-g} \left[C g^{\alpha\beta} + \frac{2}{\kappa_1} G^{\alpha\beta} + \frac{2}{\kappa_2} G^{*\alpha\beta} \right] \tau_{\beta\mu} \tag{3.18}$$

Here the "Einstein" tensors are defined as

$$G^{\alpha\beta} = R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R \tag{3.19}$$

$$G^{*\alpha\beta} = R^{*\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R^* \tag{3.20}$$

where

$$R_{\alpha\mu} = R_{\mu\nu\alpha\beta}\tau^{\beta\nu}$$

$$R^{\alpha\beta} = R^{\alpha}_{\mu}{}^{\beta\mu}\tau = R^{\alpha}{}_{\gamma}{}^{\beta\gamma} \tag{3.21}$$

and similarly for $R^{*\alpha\beta}$. Since both $R_{\alpha\beta\gamma\delta}$ and $R_{ab\alpha\beta}$ have only one component, namely R_{0101} and R_{2301}, we have

$$G^{\alpha\beta} = G^{*\alpha\beta} = 0 \tag{3.22}$$

Hence the string equation reduces to

$$g^{\alpha\beta}{}_{||\alpha}\tau_{\beta\mu} + g^{\alpha\beta}\tau_{\beta\mu||\alpha} = h^{a\alpha}_{\alpha} n^{a}_{\mu} = 0 \tag{3.23}$$

This means $\omega_{\alpha\beta\gamma}$ and $\omega_{\alpha ab}$ never enter physics. Thus, without loss of generality we can set them to be zero, so that

$$\omega_{\alpha\mu\nu} = h^{a}_{\alpha\beta}[\tau^{\beta}, n^{a}]_{\mu\nu} \tag{3.24}$$

which is identical to that given in Eq. (2.18). (Actually in the parallel transport argument presented in Section II the connection $\omega_{\alpha\mu\nu}$ can also be augmented by arbitrary $\omega_{\alpha\beta\gamma}$ and $\omega_{\alpha ab}$ terms.[5] Since they do not affect physics we have set them equal to zero.)

From Eqs. (3.13) and (3.24), we find

$$R_{\mu\nu\alpha\beta}\left(\omega_{\alpha\mu\nu}\right) = h^{a}_{\alpha\gamma}\, h^{a}_{\beta\delta}\left[\tau^{\gamma},\tau^{\delta}\right]_{\mu\nu} + h^{b}_{\alpha\gamma}\, h^{a\gamma}_{\beta}\left[n^{a},n^{b}\right]_{\mu\nu} \tag{3.25}$$

Projecting onto V_2, the resulting $R_{\gamma\delta\alpha\beta}$ is identical to that given in Eq. (2.21). Hence we have completed our demonstration of the equivalence of the connection obtained from the action in the first order formalism with that obtained by parallel transport argument. For C = 0, the model (3.6) becomes trivial, as is obvious from Eq. (3.22). In the generalized QCS, the constant C will be taken to be identically zero.

IV. THE GENERALIZED QUARK-CONFINING STRING

In this section we shall introduce the Riemann curvature tensor into the original quark-confining string Lagrangian (2.4). We start from the first order form where the connection is treated as an independent variable and then proceed to derive the second order form by solving for the connection. Since we demand that physics should be independent of the choice of parameters u^α, the Lagrangian is invariant under the reparametrization $u^\alpha \to v^\alpha(u^\beta)$. In particular $R^*(u^1,u^0) = -R^*(-u^1,u^0)$ implies the exclusion of R^* in the Lagrangian if this discrete reparametrization is included in the reparametrization invariance. We shall neglect R^* in the following discussion until the end of this section.

In Section III we observe that the connection $\omega_{\alpha\mu\nu}$ connects only to the $[\tau^\beta,n^a]_{\mu\nu}$ part of the quantity $\bar{\psi}\not{\tau}^\alpha\sigma^{\mu\nu}\psi$. Hence, if there is any feedback on the connection, this occurs only in the $[\tau^\beta,n^a]_{\mu\nu}$ components. Therefore, we write the generalized QCS in the following form

$$S = \int d^2u\,\sqrt{-g}\left[\frac{1}{\kappa}R(\Gamma) + \bar{\psi}\not{\tau}^\alpha\left(\frac{i}{2}\overleftrightarrow{\partial}_\alpha - e\,B^i_\alpha T^i\right)\psi - \bar{\psi}M\psi - \frac{F^2}{4} + \Gamma_{\alpha\mu\nu}S^{\alpha\mu\nu}\right] \qquad (1.1)'$$

where

$$S^{\alpha\mu\nu} = \frac{1}{8}\bar{\psi}(\not{\tau}^\alpha\tilde{\sigma}^{\mu\nu} + \tilde{\sigma}^{\mu\nu}\not{\tau}^\alpha)\,\psi = S^{\alpha\beta a}[\tau_\beta,n^a]^{\mu\nu} \qquad (4.1)$$

$\tilde{\sigma}^{\mu\nu}$ is the $[\tau_\beta,n^a]^{\mu\nu}$ projection of $\sigma^{\mu\nu}$

$$\tilde{\sigma}^{\mu\nu} = -i\not{\tau}^\beta\not{n}^a[\tau_\beta,n^a]^{\mu\nu} \qquad (4.2)$$

so that

$$S^{\alpha\beta a} = S^{\alpha\beta a\dagger} = -\frac{i}{8}\bar{\psi}[\not{\tau}^\alpha,\not{\tau}^\beta]\not{n}^a\,\psi \qquad (4.3)$$

From now on, we shall refer to the action (1.1) simply as QCS. The Riemann curvature tensor $R_{\mu\nu\alpha\beta}\left(\Gamma_{\alpha\mu\nu}\right)$ is given by Eq. (3.1) with $\omega_{\alpha\mu\nu}$ replaced by $\Gamma_{\alpha\mu\nu}$. Varying this action with respect to $\Gamma_{\alpha\mu\nu}$, we obtain

$$\Gamma_{\beta\mu}{}^{\sigma}[\tau^\alpha,\tau^\beta]_{\nu\sigma} - \Gamma_{\beta\nu}{}^{\sigma}[\tau^\alpha,\tau^\beta]_{\mu\sigma} - h^{a\beta}_{\beta}[\tau^\alpha,n^a]_{\mu\nu}$$

$$+ h^{a\alpha}_{\beta}[\tau^\beta,n^a]_{\mu\nu} + \kappa S^{\alpha\mu\nu} = 0 \qquad (4.4)$$

Let us decompose $\Gamma_{\alpha\mu\nu}$ into

$$\Gamma_{\alpha\mu\nu} = \omega_{\alpha\mu\nu} + k_{\alpha\mu\nu} \qquad (4.5)$$

where $\omega_{\alpha\mu\nu}$ is given by Eq. (3.24). $k_{\alpha\mu\nu}$ can be written as

$$k_{\alpha\mu\nu} = k_{\alpha\beta\delta}[\tau^\beta,\tau^\delta]_{\mu\nu} + k_{\alpha\beta a}\,[\tau^\beta,n^a]_{\mu\nu} + k_{\alpha ab}[n^a,n^b]_{\mu\nu} \qquad (4.6)$$

Using Eqs. (3.24), (4.1), (4.5) and (4.6), Eq. (4.4) becomes

$$\kappa S^{\beta\alpha a} = k^{\alpha\beta a} - k^{\delta}{}_{\delta}{}^{a} g^{\alpha\beta} \tag{4.7}$$

or

$$k^{\alpha\beta a} = k^{\alpha\beta a\dagger} = \kappa S^{\beta\alpha a} = -\kappa S^{\alpha\beta a} \tag{4.8}$$

where the explicit form (4.3) for $S^{\alpha\beta a}$ is used. We note that $S^{\alpha\beta a}$ has the following property

$$S^{\alpha\gamma a} S_{\beta\gamma}{}^{a} = \frac{1}{2} g^{\alpha}_{\beta} S^{\delta\gamma a} S_{\delta\gamma}{}^{a} \tag{4.9}$$

$k^{\alpha\beta\gamma}$ and $k^{\alpha ab}$ of the connection are left undetermined. It is straightforward to show (see Appendix A) that $k_{\alpha\beta\delta}$ and $k_{\alpha ab}$ do not come into any of the equations of motion. Hence, without loss of generality, we can set them to be zero and obtain

$$k_{\alpha\mu\nu} = -\kappa S_{\alpha\beta a} [\tau^{\beta}, n^{a}]_{\mu\nu} \tag{4.10}$$

Let us make two remarks: (1) For more than one flavor of quarks, we have, for flavor index ℓ

$$S_{\alpha\beta a} = \sum_{\ell} -\frac{i}{8} \bar{\psi}_{\ell} [\not{t}_{\alpha}, \not{t}_{\beta}] \not{n}_{a} \psi_{\ell} .$$

and (2) if, in the first order formalism, we have used instead of $S^{\alpha\mu\nu}$,

$$S^{\alpha\mu\nu} \rightarrow \frac{1}{8} \bar{\psi} (\not{t}^{\alpha} \sigma^{\mu\nu} + \sigma^{\mu\nu} \not{t}^{\alpha}) \psi = Q^{\alpha\mu\nu}$$

then the equation obtained from the variation of $\Gamma^{\alpha\mu\nu}$ implies the vanishing of both $Q^{\alpha\beta\gamma}$ and $Q^{\alpha ab}$ (where $Q^{\alpha\mu\nu}$ is decomposed according to Eq. (3.11)). This in turn implies the vanishing of the current $J_{\alpha} = \bar{\psi} \not{t}_{\alpha} \psi$, which is clearly unacceptable.

Substituting Eqs. (3.24) and (4.10) into the Riemann scalar $R(\Gamma)$, and using Eq. (4.8), it is straightforward to obtain

$$R(\Gamma) = R\left(\omega_{\alpha\mu\nu}\right) - \kappa^{2} S^{\alpha\beta a} S_{\alpha\beta a} \tag{4.11}$$

and

$$\Gamma_{\alpha\mu\nu} S^{\alpha\mu\nu} = 2\kappa S^{\alpha\beta a} S_{\alpha\beta a} + \omega_{\alpha\mu\nu} S^{\alpha\mu\nu} \tag{4.12}$$

Substituting Eqs. (4.11) and (4.12) into the action, we obtain the second order form:

$$S = \int d^2u \sqrt{-g}\left[\frac{1}{\kappa} R(\omega(\tau_{\alpha\mu})) + \bar{\psi}\not{\tau}^{\alpha}\left(\frac{i}{2}\overleftrightarrow{\partial}_{\alpha} - e B_{\alpha}^{i} T^{i}\right)\psi - \bar{\psi} M \psi - \frac{F^2}{4} + \omega_{\alpha\mu\nu} S^{\alpha\mu\nu} + \kappa S^{\alpha\beta a} S_{\alpha\beta a}\right] \quad . \tag{4.13}$$

As shown in Section III, the Riemann scalar $R(\omega)$ does not enter into any of the equations of motion and hence can be dropped from the Lagrangian. The second last term vanishes by symmetry arguments

$$\omega_{\alpha\mu\nu} S^{\alpha\mu\nu} = -2\omega_{\alpha\beta a} S^{\alpha\beta a} = 0$$

so that the action (4.13) reduces to the final second order form (1.2)

$$S = \int d^2u \left(\mathscr{L}_0 + \sqrt{-g}\,\kappa S^2\right) \tag{4.14}$$

where $\mathscr{L}_0$ is the Lagrangian of the QCS with $\kappa = 0$ (see Eq. (2.4)).

The quark equation of motion obtained from the action (1.2) has an extra piece proportional to κ

$$\left[\not{\tau}^{\alpha}\left(i\partial_{\alpha} - eB_{\alpha}\right) + \frac{i}{2}\not{\tau}^{\alpha}{}_{||\alpha} - M - \frac{i}{2}\kappa\not{\tau}_{\alpha}\not{\tau}_{\beta}\not{n}^{a} S^{\alpha\beta a}\right]\psi = 0 \tag{4.15}$$

The gluon equation is identical to that in the original QCS

$$F_{i}^{\alpha\beta}{}_{||\alpha} + e f_{ijk} B_{j\alpha} F_{k}^{\beta\alpha} = e\bar{\psi}\not{\tau}^{\beta} T_{i}\psi \tag{4.16}$$

The energy momentum tensor of the action (1.2) is given by

$$P_{\mu}^{\alpha} = -\frac{1}{\sqrt{-g}}\frac{\delta\mathscr{L}}{\delta\tau_{\alpha}{}^{\mu}} = T^{\alpha\beta}\tau_{\beta\mu} + V^{\alpha a} n_{\mu}^{a} \tag{4.17}$$

where

$$T^{\alpha\beta} = \bar{\psi}\not{\tau}^{\alpha}\left(\frac{i}{2}\overleftrightarrow{\partial}^{\beta} - eB^{\beta}\right)\psi + \left(\frac{1}{2}E_{j}^{2} + \kappa S^{2}\right) g^{\alpha\beta} \tag{4.18}$$

and

$$V^{\alpha a} = \bar{\psi}\not{n}^{a}\left(\frac{i}{2}\overleftrightarrow{\partial}^{\alpha} - eB^{\alpha}\right)\psi - \kappa Z^{\alpha a} \tag{4.19}$$

The following notations have been used,

$$Z^{\alpha a} = -n^a_\mu \frac{\delta(S^2)}{\delta\tau_{\alpha\mu}}$$

$$= \frac{1}{8}\left[Y_\beta{}^{ab}\left(Y^{\beta\alpha b} - Y^{\alpha\beta b}\right) + Y^{\beta\delta a} Y_{\beta\delta}{}^{\alpha}\right] \tag{4.20}$$

where $Y_{\alpha\beta a} = \frac{1}{2}\bar{\psi}[\not{t}_\alpha \not{t}_\beta] \not{n}_a \psi$, $Y_{\alpha ab} = \bar{\psi}\not{t}_\alpha \not{n}_a \not{n}_b \psi$ and $Y_{\alpha\beta\gamma} = \frac{1}{2}\bar{\psi}[\not{t}_\alpha, \not{t}_\beta]\not{t}_\gamma \psi$. $B^\beta = B^\beta_i T_i$ and $F_{j\alpha\beta} = E_j e_{\alpha\beta}$. Decomposing the string equation $P^\alpha_{\mu||\alpha} = 0$ into tangential components and normal components we obtain

$$T^{\alpha\beta}{}_{||\alpha} + V^{\alpha a} h^{a\beta}_\alpha = 0 \tag{4.21}$$

$$T^{\alpha\beta} h^a_{\alpha\beta} + V^{\alpha a}{}_{:\alpha} = 0 \tag{4.22}$$

It is straightforward to prove that the tangential components (4.21) vanish identically (see Appendix B) and hence are not equations of motion. Their vanishing is, of course, a consequence of (continuous) reparametrization invariance. The proof for the first order form (1.1) is equally straightforward. It is also easy to show that the normal components of the string equation in the first order form are identical to those in the second order form (see Appendix A).

We note that if we include the R^* term in the action (1.1)

$$\frac{1}{\kappa}R(\Gamma) \to \frac{1}{\kappa_1}R(\Gamma) + \frac{1}{\kappa_2}R^*(\Gamma) \quad , \tag{4.23}$$

the action in the second order form becomes

$$S = \int d^2u \left[\mathscr{L}_0 + \frac{\kappa_1 \kappa_2^2}{\kappa_1^2 + \kappa_2^2}\sqrt{-g}\, S^2\right] \tag{4.24}$$

V. DISCUSSIONS

In general, one can add into the QCS Lagrangian any number of interaction terms which preserve Lorentz, gauge and reparametrization invariance. However, the fact that the parameter space is a curved space-time suggests naturally the introduction of the Riemann curvature term into the string Lagrangian. For simplicity, only terms linear in the Riemann tensor are considered. Such terms are trivial in the absence of quark fields; but, in the QCS, such terms are nontrivial. This is because the quarks are four dimensional Dirac fields.

We have shown that the quark-confining string model constructed in this work differs from the earlier version only by a spin-spin interaction term. Among other things this term contributes to the hyperfine splittings in hadron spectroscopy. If we consider the QCS model (1.2) in two dimensions, this extra piece drops out (since there are no normals, we have $-n^a_\mu n^a_\nu = \eta_{\mu\nu} - \tau^\alpha_\mu \tau_{\alpha\nu} = 0$) so that QCD is recovered in two dimensions.

In four dimensions, if we take the point of view that the QCS is a phenomenological model of QCD, we can interpret the color gauge fields in QCD to be parametrized in the QCS as shown in Table 1. The gauge field dynamical degrees of freedom are geometrized. A comparison of their transformations under the gauge and Lorentz groups is shown in Table 2.

Intuitively, gluonic structure in QCD is expected to show up in scattering experiments in the near future. However, according to the QCS picture, there is no such gluonic structure. Experiments will certainly clarify this situation and reveal more on the relation between QCD and the QCS.[7]

The application of the generalized QCS to the ψ spectroscopy, in particular the relativistic effects, is under study.

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APPENDIX A

In this appendix we show that the tangent-tangent component, $k_{\alpha\beta\delta}$, and normal-normal component, $k_{\alpha ab}$, of the connection do not appear in the string equation. To prove this we start with the connection $\Gamma_{\alpha\mu\nu} = \omega_{\alpha\mu\nu} + k_{\alpha\mu\nu}$ in its most general form

$$\Gamma_{\alpha\mu\nu} = \Gamma_{\alpha\beta\delta}\left[\tau^{\beta},\tau^{\delta}\right]_{\mu\nu} + \Gamma_{\alpha\beta a}\left[\tau^{\beta},n^{a}\right]_{\mu\nu} + \Gamma_{\alpha ab}\left[n^{a},n^{b}\right]_{\mu\nu} \tag{A.1}$$

The Riemann curvature tensor $R_{\mu\nu\alpha\beta}(\Gamma_{\alpha\mu\nu})$ given by Eq. (3.1) with $\omega_{\alpha\mu\nu}$ replaced by $\Gamma_{\alpha\mu\nu}$ can be written as

$$\begin{aligned} R_{\alpha\beta\mu\nu}(\Gamma_{\alpha\mu\nu}) = & \frac{1}{2}\left[R_{\alpha\beta}{}^{\gamma\delta}(\Gamma_{\beta}{}^{\gamma\alpha}) + R_{\alpha\beta}{}^{\gamma\delta}(\Gamma_{\beta}{}^{\gamma\delta})\right]\left[\tau_{\gamma},\tau_{\delta}\right]_{\mu\nu} \\ & + \frac{1}{2}\left[R_{\alpha\beta}{}^{ab}(\Gamma_{\beta}{}^{\gamma a}) + R_{\alpha\beta}{}^{ab}(\Gamma_{\beta}{}^{ab})\right]\left[n_{a},n_{b}\right]_{\mu\nu} \\ & - R_{\alpha\beta}{}^{\gamma a}(\Gamma_{\alpha\mu\nu})\left[\tau_{\gamma},n_{a}\right]_{\mu\nu} \end{aligned} \tag{A.2}$$

where

$$\begin{aligned} R_{\alpha\beta}{}^{\gamma\delta}(\Gamma_{\beta}{}^{\gamma a}) &= \left[\Gamma_{\alpha}{}^{\gamma a}(2h^{a}{}_{\beta}{}^{\delta} - \Gamma_{\beta}{}^{\delta a})\right] - [\alpha\leftrightarrow\beta] \\ R_{\alpha\beta}{}^{ab}(\Gamma_{\beta}{}^{\gamma a}) &= \left[\Gamma_{\alpha}{}^{\gamma a}(-2h^{b}{}_{\gamma\beta} + \Gamma_{\beta\gamma}{}^{b})\right] - [\alpha\leftrightarrow\beta] \\ R_{\alpha\beta}{}^{\gamma\delta}(\Gamma_{\beta}{}^{\gamma\delta}) &= \left[4\Gamma_{\beta}{}^{\gamma\eta}\,\Gamma_{\alpha\eta}{}^{\delta} + 2\Gamma_{\alpha}{}^{\gamma\delta}{}_{:\beta}\right] - [\alpha\leftrightarrow\beta] \\ R_{\alpha\beta}{}^{ab}(\Gamma_{\beta}{}^{ab}) &= \left[4\Gamma_{\beta}{}^{ac}\,\Gamma_{\alpha}{}^{bc} + 2\Gamma_{\alpha}{}^{ab}{}_{:\beta}\right] - [\alpha\leftrightarrow\beta] \\ R_{\alpha\beta}{}^{\gamma a}(\Gamma_{\alpha\mu\nu}) &= \left[\Gamma_{\beta}{}^{\gamma a}{}_{:\alpha} - 2\Gamma_{\alpha\delta}{}^{\gamma}(\Gamma_{\beta}{}^{\delta a} - h^{a}{}_{\beta}{}^{\delta}) - 2\Gamma_{\alpha}{}^{ab}(\Gamma_{\beta}{}^{\gamma b} - h^{a}{}_{\beta}{}^{\gamma})\right] - [\alpha\leftrightarrow\beta]\,. \end{aligned} \tag{A.3}$$

The quark and gluon equations of motion obtained from the action Eq. (1.1) in the first order form are the same as given in Eq. (4.15) and (4.16) respectively. Varying Eq. (1.1) with respect to $\tau_{\alpha\mu}$ gives

$$-\frac{1}{\sqrt{-g}}\,\frac{\delta\mathscr{L}}{\delta\tau_{\alpha}{}^{\mu}} = \tilde{T}^{\alpha\beta}\tau_{\beta\mu} + \tilde{V}^{\alpha a}n^{a}{}_{\mu} \tag{A.4}$$

where

$$\tilde{T}^{\alpha\beta} = T^{\alpha\beta} - 2h^{a}{}_{\gamma}{}^{\beta} S^{\alpha\gamma a} \tag{A.5}$$

and

$$\tilde{V}^{\alpha a} = \tilde{V}_{(1)}{}^{\alpha a} + \tilde{V}_{(2)}{}^{\alpha a} \tag{A.6}$$

$$\tilde{V}_{(1)}{}^{\alpha a} = V^{\alpha a} + 2S^{\gamma\alpha a}{}_{:\gamma} \tag{A.7}$$

$$\tilde{V}_{(2)}{}^{\alpha a} = 4\left[\left(k_{\gamma\delta}{}^{\gamma} S^{\alpha\delta a} - k^{\alpha}{}_{\delta}{}^{\gamma} S_{\gamma}{}^{\delta a} + k_{\gamma\delta}{}^{\alpha} S^{\gamma\delta a}\right) + \left(k_{\gamma}{}^{ab} S^{\alpha\gamma b} + k_{\gamma}{}^{ab} S^{\gamma\alpha b}\right)\right] \tag{A.8}$$

$T^{\alpha\beta}$ (Eq. (4.18)) and $V^{\alpha a}$ (Eq. (4.19)) are obtained from the action (4.13) in the second order form. In arriving at the above equations we have used Eq. (3.14), (4.5), (4.9), (4.15), (4.20), (A.2) and (A.3). We observe that $k_{\alpha\beta\delta}$ and $k_{\alpha ab}$ appear only in $\tilde{V}_{(2)}{}^{\alpha a}$. But $\tilde{V}_{(2)}{}^{\alpha a}$ vanishes identically

$$\tilde{V}_{(2)}{}^{\alpha a} \equiv 0 \tag{A.9}$$

The proof is straightforward and is facilitated by decomposing $S^{\alpha\gamma a}$ and $k^{\alpha\delta\gamma}$ into the following forms

$$S^{\alpha\gamma a} = -\frac{1}{2} e^{\alpha\gamma} e_{\delta\beta} S^{\delta\beta a} \tag{A.10}$$

$$k^{\alpha\delta\gamma} = -k^{\alpha\gamma\delta} = \frac{1}{2} e^{\delta\gamma}(-e^{\alpha}{}_{\beta} k^{\eta}{}_{\eta}{}^{\beta} + e_{\beta\eta} k^{\beta\eta\alpha}) \tag{A.11}$$

Thus $k_{\alpha\beta\delta}$ and $k_{\alpha ab}$ do not come into the string equations. Since they also do not appear in the Dirac and gluon equations we conclude that they play no physical role in the QCS model and can be ignored without loss of generality.

The string equation $\partial_\alpha \frac{\delta \mathscr{L}}{\delta \tau_\alpha{}^\mu} = 0$ can be decomposed into tangential and normal components. We obtain

$$\tilde{T}^{\alpha\beta}{}_{||\alpha} + \tilde{V}^{\alpha a} h^a{}_\alpha{}^\beta = 0 \tag{A.12}$$

$$\tilde{T}^{\alpha\beta} h^a{}_{\alpha\beta} + \tilde{V}^{\alpha a}{}_{:\alpha} = 0 \tag{A.13}$$

It is straightforward to show that Eq. (A.12) follows from the Dirac equation (4.15) and the gluon equation (4.16). It is equally simple to show that the normal components Eq. (A.13), obtained here in the first order form, are identical to those in the second order form Eq. (4.22) by using

$$S^{\gamma\alpha a}{}_{:\gamma:\alpha} = S^{\gamma\alpha b}\; h^a{}_{\delta\gamma}\; h^{b\delta}{}_\alpha \tag{A.14}$$

Thus all the equations of motion obtained from the action (1.1) in the first order form are identical to those obtained in the second order form.

APPENDIX B

The tangential components of the string equation $P_\mu{}^\alpha{}_{||\alpha} = 0$ will be shown in this appendix to follow from the Dirac equation (4.15) and the gluon equation (4.16). Since $g^{\alpha\beta}{}_{||\alpha} = 0$, the tangent components (4.21) can be written as

$$T^\alpha{}_{\beta||\alpha} + V^{\alpha a}\, h_{a\alpha\beta} = 0 \tag{B.1}$$

Let us first consider the quark part of $T^\alpha{}_{\beta||\alpha}$; after some rearrangements, we obtain

$$\begin{aligned}\left(\bar\psi \not{\ell}^\alpha \overleftrightarrow{D}_\beta\, \psi\right)_{||\alpha} &\equiv \left[\bar\psi \not{\ell}^\alpha \left(\frac{i}{2} \overleftrightarrow{\partial}_\beta - e\, B_{j\beta}\, T_j\right)\psi\right]_{||\alpha} \\ &= \bar\psi \not{\ell}^\alpha{}_{||\alpha} \overleftrightarrow{D}_\beta\, \psi + (\bar\psi_{|\alpha} \not{\ell}^\alpha)\overleftrightarrow{D}_\beta\, \psi - \frac{i}{2}\bar\psi \not{\ell}^\alpha{}_{||\beta} \overleftrightarrow{\partial}_\alpha \psi \\ &\quad + \bar\psi \overleftrightarrow{D}_\beta\, (\not{\ell}^\alpha\, \partial_\alpha\, \psi) - e\, \bar\psi\, B_{\beta||\alpha} \not{\ell}^\alpha\, \psi\end{aligned} \tag{B.2}$$

where $B_\beta = B_{\beta j} T_j$. Using the Dirac equation (4.15), this becomes

$$\begin{aligned}(\bar\psi \not{\ell}^\alpha \overleftrightarrow{D}_\beta\, \psi)_{||\alpha} &= -\bar\psi \not{\ell}^\alpha{}_{||\beta} \overleftrightarrow{D}_\alpha\, \psi - e\, \bar\psi \not{\ell}^\alpha T_j \psi F_{j\alpha\beta} \\ &\quad + i\kappa \bar\psi (\Delta_{|\beta})\psi\end{aligned} \tag{B.3}$$

where $\Delta = \frac{1}{2} \not{\ell}_\gamma \not{\ell}_\delta \not{n}^a\, S^{\gamma\delta a}$. It is straightforward to demonstrate the following

$$\partial_\beta (S^2) + i\, \bar\psi\, (\Delta_{|\beta})\psi - Z^{\alpha a} h^a{}_{\alpha\beta} = 0 \tag{B.4}$$

Using Eq. (B.3) and (B.4), the left hand side of Eq. (B.1) becomes

$$
\begin{aligned}
&T^{\alpha}{}_{\beta||\alpha} + \bar{\psi}\, \not{e}_a\, \overleftrightarrow{D}^{\alpha}\, \psi\, h_{a\alpha\beta} \\
&\quad = \left(\frac{\vec{F}^2}{4}\, g^{\alpha}{}_{\beta} - F^{j\alpha\delta}\, F^{j}{}_{\beta\delta}\right)_{||\alpha} - e\, \bar{\psi}\gamma^{\alpha}\, T^{j}\, \psi\, F^{j}{}_{\alpha\beta} \\
&\quad = \frac{1}{2}\, F^{j\alpha\delta}\, F^{j}{}_{\alpha\delta||\beta} - F^{j\alpha\delta}\, F^{j}{}_{\beta\delta||\alpha} - e\, f_{jk\ell} B_{k\gamma} F_{\ell}{}^{\alpha\gamma} F_{j\alpha\beta} \qquad \text{(B.5)} \\
&\quad = -\frac{1}{2} F_{j}{}^{\alpha\delta}\left(F_{j\delta\alpha||\beta} + F_{j\beta\delta||\alpha} + F_{j\alpha\beta||\delta}\right) \\
&\quad = 0
\end{aligned}
$$

where the gluon equation (4.16) is used and

$$f_{jki}\, F_{i}{}^{\alpha\gamma}\, F_{j\alpha\beta} = 0 \qquad \text{(B.6)}$$

follows from the fact that $F_{j}{}^{\alpha\beta}$ has only one component:

$$g\, \vec{F}^{01} = \vec{F}_{01} \qquad \text{(B.7)}$$

It is equally straightforward to prove that the tangent components of the string equation in the first order formalism also vanish identically.

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6. Notice the different definition of $R_{\alpha\beta\gamma\delta}$ in ref. [1]. Here $R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta} = -R_{\alpha\beta\delta\gamma}$.
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TABLE CAPTIONS

1. An interpretation of the relation between QCD and the quark-confining string (QCS). For example, the second line should read: the piece in A_μ^i that is responsible for the color electric interaction (which presumably gives quark-confinement) is parametrized by the two dimensional color gauge fields B_α^i.

2. Table for the transformations of the fields under the color, the Lorentz and the reparametrization groups. A blank box implies the field is invariant under the particular transformation. A check mark implies the field transforms under the particular group in the appropriate representation. Rep. stands for reparametrization group.

Table 1

QCD			QCS
$\psi(x_\mu)$		$\longrightarrow$	$\psi(u^\alpha)$
$A^i_\mu(x_\nu)$	color electric interaction	$\longrightarrow$	$B^i_\alpha(u^\beta)$
	gluon degrees of freedom	$\longrightarrow$	$X_\mu(u^\alpha)$ (color independent)
	color magnetic interaction	$\longrightarrow$	κS^2 (color independent spin-spin coupling)

Table 2

Group	QCD		QCS		
	$A^i_\mu(x_\nu)$	$\psi(x_\mu)$	$B^i_\alpha(u^\beta)$	$\psi(u^\alpha)$	$X_\mu(u^\alpha)$
SU(3)	√	√	√	√	
O(3,1)	√	√		√	√
Rep.			√		