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ZERO RANGE SCATTERING THEORY II.
MINIMAL RELATIVISTIC THREE–PARTICLE EQUATIONS AND THE EFIMOV EFFECT*

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ABSTRACT

We present numerical results obtained from a minimal relativistic model for three distinguishable particles of equal mass driven by a single bound or virtual state for each pair, which is a relativistic generalization of the zero range non-relativistic two particle scattering length model. In contrast to the non-relativistic case, the relativistic kinematics makes the three body equations converge to unique, unitary results. When the binding energy or virtual energy of the two body driving term approaches zero (infinite scattering length) the infinite three particle bound state spectrum (Efimov effect) is rigorously derived with numerical results in quantitative agreement with appropriate non-relativistic calculations.

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1. INTRODUCTION

In the first paper in this series[1] we showed that a consistent 3-particle non-relativistic theory could be constructed using only physical two-particle observables *if* the observables can be represented by an analytic function of the two particle relative energy whose only singularities are the elastic unitarily cut $Im\ t(q^2) = Im\ e^{i\delta_q} sin\delta_q/q = sin^2\delta/q = \sqrt{q^2}|t(q^2)|^2$ and bound state poles. Unfortunately for simplicity both relativistic S-matrix theory and non-relativistic "potential" scattering theory agree that there is an additional "left hand cut" due to "meson exchange" or the "potential"; we demonstrated in ZRST I that the presence of this cut *requires* a non-vanishing "off shell" extension, and hence prevents a consistent unitarity zero range limit from being taken. Fortunately our relativistic models need only elastic scattering cuts and poles in the input.

Most approaches to "zero range" models for three particle scattering start from Thomas' observation[2] that this limit gives infinite binding to non-relativistic models of three particle systems with short range interactions; most subsequent work (ZRST I, Ref. 10) has concentrated on removing this singularity by *ad hoc* modifications. This "divergence" occurs due to "ultraviolet" behavior where the non-relativistic model is undefined. Consequently, the on shell behavior can be factored out by using any reasonable cutoff in this region, and the remaining equations are well defined[3] *provided* that the on-shell amplitude is bounded by $const./q^2$ for large q^2. However, the scattering length model $t(q^2) = [(-1/a) + \sqrt{-q^2}]^{-1}$ still does not converge.

The work reported here started from the observation that relativistic kinematics does make even this model convergent, providing a one-parameter, and in that sense minimal, relativistic model for three distinguishable particles of mass m. This model can be thought of as driven by an "s-channel" pole in the two particle subsystems at a mass μ in the subsystem energy. If $\mu \leq 2m$, this pole corresponds to a bound state, while for $\mu \geq 2m$ it is an s-channel resonance or virtual state. The limit $\mu \to 2m$ corresponds to a "bound state at zero energy" or infinite scattering length. Note that our model is defined *kinematically*; there is no "interaction energy" associated with this "bound state".

This kinematic statement in itself does not completely define the model, primarily because various "off shell extensions" can be proposed. One minimal model, which allows breakup and is manifestly covariant, has been proposed by Lindesay and Markevich;[4] another approach is discussed by Noyes and Lindesay.[5] Yet another way to achieve manifest covariance in a flexible relativistic three body theory is to use 4-velocity conservation rather than 3-momentum conservation; the formalism has been worked out[6,7] If the two particle input contains *only* the pole and no scattering cut, there can be only elastic and rearrangement (inelastic) scattering and no breakup, providing a "confined quantum" model[8] For a single confined quantum and N particles, this provides a minimal model for the three nucleon system[8,9] and a relativistic definition of the "potential" in N-particle systems[11] It turns out that this last approach is the most promising starting point for the inclusion of anti-particles and the investigation of "crossing" in the finite particle number approach most closely related to quantum field theories with Yukawa coupling[12]. The results presented here are merely a starting point for a much richer development.

2. BASIC EQUATIONS

Since the general treatment used in this paper has been fully presented elsewhere[13] we discus here only the results obtained for the specific model in which the momentum of each particle of the interacting pair a_+, a_- (a, a_+, a_- cyclic on 1,2,3) has 3-momentum $\pm \underset{\sim}{q}_a$ related to the invariant energy s_a by

$$|\underset{\sim}{q}_a(s_a)|^2 = \frac{[s_a - (m_{a+} + m_{a-})^2][s_a - (m_{a+} - m_{a-})^2]}{4s_a} \tag{2.1}$$

and a bound or virtual state at mass μ_a. Then the quantity which we call $\alpha_a(\mu_a)$, and which corresponds to $-1/a_a$ where a_a is the non-relativistic scattering length, is given by

$$\alpha_a(\mu_a) = sign(m_{a_+} + m_{a_-} - \mu_a)\{\frac{[\mu_a^2 - (m_{a_+} + m_{a_-})^2][\mu_a^2 - (m_{a_+} - m_{a_-})^2]}{4\mu_a^2}\}^{\frac{1}{2}} \tag{2.2}$$

For the minimal model with all particle masses equal to m and all driving pole term masses equal to μ, $|\underset{\sim}{q}_a(s_a)|^2 = \frac{s_a}{4} - m^2$ and $\alpha_a(\mu) = sign(2m - \mu)|\frac{\mu}{4} - m^2|^{\frac{1}{2}}$. The two body on shell amplitude is then taken to be $t_a^{\pm}(s_a) = g_a^2/D_a^{\pm}(s_a)$ where

$$\frac{1}{D_a^{\pm}(s_a)} = \frac{\sqrt{s_a}\theta(s_a)}{\alpha_a(\mu) - \sqrt{-|q_a(s_a \pm i0^+)|^2}} \tag{2.3}$$

Clearly this model has a pole at $s_a = \mu^2$ and a scattering cut starting at $q^2 = 0$. Here the signs and the branch in the square root have been chosen so that for q^2 small the scattering amplitude is the usual scattering length model with the convention that positive (negative) scattering lengths correspond to a bound (virtual) state pole. The square root in the numerator is introduced because the density of states factor in the two-particle phase space is $\rho_a(s_a) = 4\pi|\underset{\sim}{q}_a(s_a)|/\sqrt{s_a}$. The restriction to positive s_a came from Brayshaw's approach to relativistic three-particle kinematics, which we abandon in later papers in this series.

We use as our basis states the on mass shell relativistic free particle 3-momentum states $|\underset{\sim}{k}_i>, i \in 1,2,3$ with $\epsilon_i = \sqrt{m_i^2 + k_i^2}$ which satisfy the normalization and completeness relations

$$< \underset{\sim}{k}_i'|\underset{\sim}{k}_i >= \epsilon_i\delta^3(\underset{\sim}{k}_i - \underset{\sim}{k}_i'); \quad \int \frac{d^3k_i}{\epsilon_i}|\underset{\sim}{k}_i >< \underset{\sim}{k}_i| = 1 \tag{2.4}$$

We restrict ourselves to the three particle zero momentum frame (*cf.*, Ref. 13 for a covariant treatment of this restriction) leaving two vector degrees of freedom thanks to the restriction $\underset{\sim}{k}_a + \underset{\sim}{k}_{a_+} + \underset{\sim}{k}_{a_-} = 0$. The propagator $R_0(z)^{-1} = \epsilon_a + \epsilon_c + \epsilon_{ac} - z$ where $\epsilon_{ac} = \sqrt{(\underset{\sim}{k}_a + \underset{\sim}{k}_c)^2 + m_{ac}^2}$ and $m_{12} = m_3$, *etc.* The on shell value for z is the invariant magnitude of the total 4-momentum, $M = (\Sigma\epsilon)^2 - (\Sigma\underset{\sim}{k})^2$.

The dynamics for our model are supplied by the Faddeev channel decomposition of the transition operator $\mathbf{T} = \Sigma_{a,b}\mathbf{T}_{ab}$ and the Faddeev equations

$$\mathbf{T}_{ab} - \mathbf{t}_a\delta_{ab} = -\Sigma_c \mathbf{t}_a \bar{\delta}_{ac}\mathbf{R}_0\mathbf{T}_{cb} = -\Sigma_c \mathbf{T}_{ac}\mathbf{R}_0\bar{\delta}_{cb}\mathbf{t}_b \tag{2.5}$$

where $\bar{\delta}_{ac} = 1 - \delta_{ac}$. That the two orders of the equation define the same function uniquely is necessary and sufficient for time reversal invariance; their existence simplifies the proof of flux conservation (unitarity), which Freedman , Lovelace and Namyslowski and independently Kowalski, have shown to be an algebraic consequence for solutions of these equations. As in the non-relativistic case, we define the input by expressing the two particle amplitude in the three particle space, assuming three-momentum conservation for the spectator. Allowing only the total energy of the system to be "off-shell", this reduces the degrees of freedom of the component equations to one vector variable, and allows us to decompose the channel amplitudes:

$$T_{ab} - t_a\delta_{ab} = \frac{g_a}{D_a(s_a)}\hat{W}_{ab}(\underset{\sim}{k}_a, \underset{\sim}{k}_b; z)\frac{g_b}{D_b(s_b)} \tag{2.6}$$

If the "bound" or "virtual" state in the two particle input is to describe precisely two of the three particles and nothing else in this three particle space, the unitarity condition then requires that $g_a^2 = (\frac{1}{2\pi})^2$.

Once we have specified the model to this extent, the two-cluster amplitudes $\hat{W}_{ab}$ [which when clothed with the two-particle factors describe breakup, coalescence and 3-3 scattering as well as elastic and rearrangement (anelastic) scattering] satisfy equations in one vector variable:

$$\hat{W}(\underset{\sim}{k}_a, \underset{\sim}{k}_b; z) - V_{ab}(\underset{\sim}{k}_a, \underset{\sim}{k}_b; z)$$

$$= \Sigma_c \int \frac{d^3k'}{\epsilon'_c} \frac{V_{ac}(\underset{\sim}{k}_a, \underset{\sim}{k}'_c; z)\hat{W}_{cb}(\underset{\sim}{k}'_c, \underset{\sim}{k}_b; z)}{D_c(s_c(k'_c, z))} \tag{2.7}$$

$$= \Sigma_c \int \frac{d^3k'}{\epsilon'_c} \frac{\hat{W}_{ac}(\underset{\sim}{k}'_c, \underset{\sim}{k}_b; z) V_{cb}(\underset{\sim}{k}'_c, \underset{\sim}{k}_b; z)}{D_c(s_c(k'_c, z))}$$

where

$$V_{ab}(\underset{\sim}{k}_a, \underset{\sim}{k}_b; z) = \frac{-\bar{\delta}_{ab}}{\epsilon_{ab}(\epsilon_a + \epsilon_b + \epsilon_{ab} - z)} = V_{ba}(\underset{\sim}{k}_b, \underset{\sim}{k}_a; z) \tag{2.8}$$

That a cluster decomposition of an n-particle problem provides, formally, a "Lippmann-Schwinger" equation with the n-particle propagator becoming the n-1 particle potential and the input scattering amplitude becoming the propagator is a general feature of the AGS approach to n-particle dynamics, as Sandhas has pointed out.[14] In our case, all that is required for this to happen is that the two particle input $t_a \delta^3(\underset{\sim}{k}_a - \underset{\sim}{k}'_a)$ depend, outside of the delta function, only on k_a^2 and z, as it does in the non-relativistic situation we are generalizing. Since, in the frame we are using, the spectator momentum $\underset{\sim}{k}$ is balanced by the total momentum of the invariant energy s carried by the scattering pair $M = \sqrt{s + k^2} + \epsilon_m(k^2)$ and $s = M^2 + m^2 - 2M\epsilon$. Consequently, if we take Brayshaw's point of view that the two-particle s should be positive, the three particle spectator momenta are bounded by 0 and $(M^2 - m^2)/2M$; this is assumed in this paper. Alternatives are cited in the Introduction. We have now completely defined the model.

3. BOUND STATES AND THE RELATIVISTIC EFIMOV EFFECT

Taking $m_a = m$ and $\mu_a = \mu$ for all a, the dynamical regions are schematically represented in Fig. 1. In the region bounded by the vertical line $\mu = 2m$ and the horizontal line $M = 3m$ only three free to three free particle scattering can occur. Above $M = 2\mu + m$, we could have an additional bound state, and above $M = 5m$ we could have pair creation; since this would take us outside the three particle sector, we do not consider these regions here. Between $M = \mu + m$ and $m = 2\mu + m$ but below $M = 3m$ we can have only elastic and rearrangement scattering, while above that line we can have breakup as well. Below $M = \mu + m$

and $M = 3m$ we can only have 3 particle bound states. This region will be investigated for states with total angular momentum $J = 0$.

3.1 Form of the Bound State Equations

The discrete spectrum of the fully interacting system will correspond to eigenstates which satisfy the homogeneous scattering equations. The bound state equations for the amplitude $W_{ab}^{J=0}$ for three equal masses will reduce to three identical equations. The kernel $R_{ab}^{J=0}$ can be calculated from Eq. (2.7), and after a simple integration becomes

$$\begin{aligned} R_{ab}^{J=0}(k_a, k_b;\ M) &\equiv R(k_a, k_b;\ M) = R(k_b, k_a;\ M) \\ &= \frac{|g|^2}{k_a k_b} \log\left(\frac{\sqrt{m^2 + (k_a + k_b)^2} + \epsilon_a + \epsilon_b - M}{\sqrt{m^2 + (k_a - k_b)^2} + \epsilon_a + \epsilon_b - M} \right) \end{aligned} \tag{3.1}$$

Thus, the relation for the reduced amplitude (2.7) can be expressed as a single variable homogeneous integral equation

$$\widehat{W}_{ab}^{J=0}(k_a|k_{bo};\ M) \equiv W_B(k_a;\ M, \mu)$$

$$W_B(k;\ M,\mu) = -4\pi \int_0^{\frac{M^2-m^2}{2M}} dk' \, \frac{k'^2\sqrt{s'}R(k,k';\ M)}{\epsilon'\left[\pm\sqrt{|m^2 - \frac{\mu^2}{4}|} - \sqrt{m^2 - \frac{s'}{4}}\right]} W_B(k'; M, \mu)$$

$$\epsilon' = \sqrt{k'^2 + m^2} \quad , \qquad s' = M^2 + m^2 - 2M\epsilon' \tag{3.2}$$

where the $\pm$ represents the sign of $2m - \mu$. The analytic form of this equation will be examined in the next subsection, and numerical solutions will be presented in the subsection following.

3.2 Mathematical Constraints and Predictions

The solutions of Eq. (3.2) will consist of a discrete set of nondegenerate values (M_j) for a given two-body bound state mass μ. Alternatively, one may obtain a discrete set of two-body bound state masses (μ_r) which produce a three-body bound state of mass M. There will be a maximum value for the two-body bound state mass μ_{max} above which there will be no three-body bound state solutions (this value will be shown to be finite). As has been mentioned, the minimum value for the mass μ_{min} is determined by the threshold for elastic scattering $\mu + m \rightarrow \mu + m$, which occurs at $M = \mu_{min} + m$.

$$\begin{aligned} \alpha(\mu_{min}(M)) &= \alpha(M-m) = \frac{1}{2}\sqrt{(3m-M)(m+M)} \\ &= \sqrt{m^2 - \frac{s(k=0)}{4}} \end{aligned} \tag{3.3}$$

The form of the Eqs. (3.2) allow all parameters to be scaled relative to the finite mass m. In studying the analytic form of the equations, it is convenient to use this scale freedom, along with the symmetry of the equation, to define the following:

$$\begin{aligned} & z \equiv \frac{k}{m}\,, \quad \widetilde{M} \equiv \frac{M}{m}\,, \quad k(\widetilde{M}) \equiv \frac{\widetilde{M}^2 - 1}{2\widetilde{M}} \\ & \lambda_r(\widetilde{M}) \equiv \frac{\alpha(\mu_r(M) - \alpha(\mu_{min}(M))}{\alpha(\mu_{max}(M) - \alpha(\mu_{min}(M))}\,, \quad 0 \leq \lambda_r \leq 1 \\ & \frac{k}{\alpha(\mu_r) - \sqrt{m^2 - \frac{s}{4}}}\sqrt{\frac{\sqrt{s}}{\epsilon}}\, W_B(k;\; M, \mu_r) \equiv V(z;\; \widetilde{M}, \lambda_r) \end{aligned} \tag{3.4}$$

$$h(z;\ \widetilde{M}) \equiv -\left[\frac{\sqrt{m^2 - \frac{s}{4}} - \frac{1}{2}\sqrt{(3m-M)(m+M)}}{\alpha(\mu_{max}(M) - \alpha(\mu_{min}(M))}\right] \geq 0$$

$$U(z,z';\ \widetilde{M}) = U(z',z;\widetilde{M}) = -\frac{4\pi|g|^2 m}{\alpha(\mu_{max}(M)) - \alpha(\mu_{min}(M))}$$

$$\times\sqrt{\frac{\sqrt{ss'}}{\epsilon\epsilon'}}\ \log\ \frac{\sqrt{1+(z+z')^2}+\sqrt{1+z^2}+\sqrt{1+z^2}-\widetilde{M}}{\sqrt{1+(z-z')^2}+\sqrt{1+z^2}+\sqrt{1+z'^2}-\widetilde{M}} \tag{3.5}$$

$$\text{finite for}\quad \widetilde{M} \leq 3$$

With these definitions, Eq. (3.2) can be expressed

$$\lambda_r V(z;\ \widetilde{M},\lambda_r) = \int_0^{k(\widetilde{M})} dz'\ U(z,z';\ \widetilde{M})V(z';\ \widetilde{M},\lambda_r) - h(z;\ \widetilde{M})V(z;\ M,\lambda_r) \tag{3.6}$$

The form (3.6) is particularly useful, since the following relation is seen to be valid

$$(\lambda_r - \lambda_s)\int_0^{k(\widetilde{M})} dz\ V(z;\ \widetilde{M},\lambda_r)V(z;\widetilde{M},\lambda_s) = 0 \tag{3.7}$$

This condition amounts (for non-degenerate eigenvalues) to an orthogonality condition for the functions $V(z;\ \widetilde{M},\lambda_r)$. These functions can be normalized to satisfy the condition

$$\int_0^{k(\widetilde{M})} dz\ V(z;\ \widetilde{M},\lambda_r)V(z;\ \widetilde{M},\lambda_s) = \delta_{rs}\quad r,s = 1,\ldots,N(\widetilde{M}) \tag{3.8}$$

where $N(M)$ is the number of three-body bound states of energy M.

Since the system generates a denumerable set of orthogonal functions, the parameters of the equations can be explored more readily than might otherwise have been the case. To obtain relationships between the parameters, it is advantageous to define functions which sum over the dynamical parameter λ_r

$$\Lambda(z,z';\ \widetilde{M}) \equiv \sum_{r=1}^{N(\widetilde{M})} V(z;\ \widetilde{M},\lambda_r)V(z';\ \widetilde{M},\lambda_r)$$

$$\Delta(z,z';\ \widetilde{M}) \equiv \sum_{r=1}^{N(\widetilde{M})} \lambda_r(\widetilde{M})V(z;\ \widetilde{M},\lambda_r)V(z';\widetilde{M},\lambda_r)$$

$$\Gamma(z,z';\ \widetilde{M}) \equiv \sum_{r=1}^{N(\widetilde{M})} \frac{V(z;\ \widetilde{M},\lambda_r)V(z';\ \widetilde{M},\lambda_r)}{\lambda_r} \tag{3.9}$$

$$Tr\ U^2(\widetilde{M}) \equiv \int_0^{k(\widetilde{M})} dz \int_0^{k(\widetilde{M})} dz'\ U^2(z,z';\ \widetilde{M})$$

These functions are easily related through the integral equation (3.6). Using Eq. (3.8) and simple algebra, the following conditionals are obtained.

$$N(\widetilde{M}) \leq \frac{1}{2}\left[Tr\ U^2(\widetilde{M}) + \sum_{r=1}^{N(\widetilde{M})} \frac{1}{\lambda_r^2(\widetilde{M})}\right]$$

$$\sum_{r=1}^{N(\widetilde{M})} \lambda_r(\widetilde{M}) \leq \frac{1}{2}\ [Tr\ U^2(\widetilde{M}) + 1] \tag{3.10}$$

$$\sum_{r=1}^{N(\widetilde{M})} \lambda_r^2(\widetilde{M}) \leq Tr\ U^2(\widetilde{M})$$

Since $Tr\ U^2(\widetilde{M})$ is always finite within the kinematic region being studied, these equations set finite bounds on the parameters, except for $N(\widetilde{M})$. In addition, for the specific problem at hand, the following is true.

$$\frac{d\lambda_r(\widetilde{M})}{d\widetilde{M}} \geq 0 \tag{3.11}$$

This implies that the bound state trajectories $\lambda_r(\widetilde{M})$ are monotonically increasing, and have one end point along the line $\lambda = 0$, and the other along the three-body continuum threshold $\widetilde{M} = 3$, which are the boundaries of the kinematic region. To obtain an estimate for the number of bound states, the equation for $N(\widetilde{M})$ will be examined. Since $\Lambda(z, z;\ \widetilde{M})$ is a positive semi-definite quantity, the following inequality holds:

$$N(\widetilde{M}) = \int_0^{k(\widetilde{M})} dz\ \Lambda(z, z;\ \widetilde{M}) \geq \int_0^{\xi} dz\ \Lambda(z, z;\ \widetilde{M}) \qquad \text{for any } \xi < k(\widetilde{M}) \tag{3.12}$$

One of the forms for the expression $\Lambda(z, z;\ \widetilde{M})$ can be obtained directly from Eq. (3.6)

$$\Lambda(z, z';\ \widetilde{M}) = \frac{1}{h(z;\ \widetilde{M})}\left[\int_0^{k(\widetilde{M})} dz'' U(z, z'';\ \widetilde{M})\Lambda(z', z'';\ \widetilde{M}) - \Delta(z, z';\ \widetilde{M})\right]$$

$$\Lambda(z, z;\ \widetilde{M}) \equiv \frac{I(z;\ \widetilde{M})}{h(z;\ \widetilde{M})} \xrightarrow[z\to 0]{} 0 \tag{3.13}$$

The behavior of this expression is particularly interesting near the rest energy of the three particles. A binding parameter $\tilde{e}$ will be defined to examine this case:

$$e \equiv 3m - M\ , \qquad \tilde{e} \equiv 3 - \widetilde{M} \tag{3.14}$$

The behavior of $\Lambda(z, z;\ M)$ for small z and $\tilde{e}$ relative to unity (but otherwise arbitrary) is dominated by the factor $h(z; \widetilde{M})$

$$h(z;\ \widetilde{M}) \xrightarrow[\substack{\text{small } z \\ \text{small } \tilde{e}}]{} \left| \frac{m}{\alpha(\mu_{max}(3m)} \right| \left[\sqrt{\frac{3}{4} z^2 + \tilde{e}} - \sqrt{\tilde{e}} \right] \tag{3.15}$$

For small $\tilde{e}$, the factor $k(\widetilde{M}) \rightarrow 4/3$. If one sets the parameter ξ to be small compared to unity, but arbitrary compared to $\tilde{e}$, the expressions (3.12) and (3.13) indicate a scaling behavior of the number of bound states with the parameter $\tilde{e}$:

$$x \equiv \frac{z}{\sqrt{\tilde{e}}}$$

$$N \geq \int\limits_0^{\xi/\sqrt{\tilde{e}}} \sqrt{\tilde{e}}\ dx\ \frac{I(\sqrt{\tilde{e}}\ x;\ \widetilde{M})}{h(\sqrt{\tilde{e}}\ x;\ \widetilde{M})} \xrightarrow[\tilde{e}\rightarrow 0]{} \sqrt{\frac{4}{3}} \left| \frac{\alpha(\mu_{max})}{m} \right| I(\xi;\ \widetilde{M}) \log \frac{\xi}{\sqrt{\tilde{e}}} \tag{3.16}$$

If bound state solutions exist, then the function $I(\xi; \widetilde{M})$ does not identically vanish as a function of ξ or $\widetilde{M}$. Therefore, these equations have at least a logarithmic growth in the number of solutions as the three-body continuum threshold is approached

$$N(e) \geq ((\text{slowly varying non} - \text{zero function}) \times \log\left(\frac{\text{m}}{\text{e}}\right) \qquad \text{as } \frac{e}{m} \rightarrow 0 \tag{3.17}$$

This result is determined by the non-relativistic kinematics, and is consistent with the results obtained by Efimov,[15,16,17] if one relates the scattering length to the two-body binding and associates e as the three-particle binding energy. The actual numerical solutions exhibit the behavior discussed, and will be displayed in the next section.

3.3 NUMERICAL BOUND STATE SOLUTIONS

The bound state trajectories have been calculated, and are consistent with the conditions (3.10), (3.11) and (3.12). The integral equation (3.3) was reduced to a discrete matrix equation using Gaussian quadratures,[18] with Jacobi polynomials as weight functions. Stable solutions for the lowest lying states were obtained using relatively low matrix order (about 8 x 8). The lowest lying states are exhibited in Figs. 2 and 3. Figure 3 is an enlargement of the nonrelativistic region of Fig. 2.

The binding energy of all trajectories remains finite in this model, due to the finite kinematics. The kinematics of all states is nonrelativistic, except for the lowest lying state. Most of the lowest lying trajectory λ_1 is within the relativistic domain of the region, although it lies very close to the threshold for pair-particle scattering ($\mu + m \to \mu + m$). Thus, in terms of the pair-particle scattering kinematics the three-particle bound state trajectory remains nonrelativistic.

The finite binding of all trajectories at all energies differs from various nonrelativistic models (*cf.*, Dodd[19]), for which the lowest lying states may become bound indefinitely. This behavior is exhibited in Fig. 3 by the trajectory λ_1 before the relativistic kinematics become manifest. However, all trajectories have one endpoint on the pair-particle scattering threshold ($M = \mu + m$) and the other endpoint on the three-to-three scattering threshold ($M = 3m$). There is an accumulation of essentially nonrelativistic states in the region $3m - M \to 0, |2m - \mu| \to 0$, consistent with the condition (3.17) and with nonrelativistic models.

4. ELASTIC SCATTERING, REARRANGEMENT AND BREAKUP

The region of Fig. 1 below the three-body breakup threshold ($M < 3m$) for which bound pairs scatter with the third particle, will next be examined. With a given initial condition, there are three possible outcomes for the final state, as indicated in Fig. 4. The first situation represents elastic scattering, and the others represent rearrangement.

The case of particles with identical kinematic and dynamical parameters will be examined in detail.

4.1 Form of Equations

Below three-body breakup threshold, the kernels $R_{ab}^{(J=0)}$ can be expressed in the form given in Eq. (3.1). The scattering equations for the amplitude $\widehat{W}_{ab}^{J=0}$ from Eq. (2.7) can be expressed

$$\widehat{W}_{ab}^{J=0}(k_a|k_{bo};\ M) = -2\pi\bar{\delta}_{ab}R(k_a,k_{bo};M)$$

$$-2\pi\sum_{c}^{\bar{\delta}} ac \int_{0}^{\frac{M^2-m^2}{2M}} dk_c' \frac{k_c'^2\sqrt{s_c'}\,R(k_a,k_c';\ M)}{\epsilon_c'\left[\sqrt{m^2-\frac{\mu^2}{4}}-\sqrt{m^2-\frac{s'}{4}}-10\right]}\widehat{W}_{cb}^{J=0}\ (k_c|k_{bo};\ M) \tag{4.1}$$

Since the masses m and μ are the same for all channels, there will only be two amplitudes; a direct (or elastic) amplitude, and a rearrangement amplitude:

$$\begin{aligned}\widehat{W}_{aa}^{J=0}\ (k_a|k_{ao};\ M) &\equiv W_D(k_a|k_{ao};\ M)\\ \widehat{W}_{a+a}^{j=0}(k_{ao};\ M) = \widehat{W}_{a-a}^{J=0}(k|k_{ao};\ M) &\equiv W_R(k|k_{ao};\ M)\end{aligned} \tag{4.2}$$

Using these amplitudes, the integral equation (3.18) can be discretized into a matrix relation and inverted using elementary linear algebraic techniques

$$\widehat{W}_{ab} = \bar{\delta}_{ab} W_{ab}^{(s)} + \sum_c \bar{\delta}_{ac} K_{ac} \widehat{W}_{cb}$$

$$W_D = 2K(1 - K - 2K^2)^{-1} W^{(s)} \tag{4.3}$$

$$W_R = (1 - K - 2K^2)^{-1} W^{(s)}$$

After inversion, the amplitudes W_D and W_R can be related to physical observables (c. Sec. II-B in Ref.13). The following relations pertain to the specific problem being developed:

$$\langle \Phi_a : \underset{\sim}{k}_a \epsilon_a ;\ \Psi_a(-\underset{\sim}{k}_a \epsilon_{\mu_a}) | A_{ab}^{(+)}(\epsilon_{bo} + \epsilon_{\mu_b o},\ \underset{\sim}{o}) | \Phi_b : \underset{\sim}{k}_{bo} \epsilon_{bo} ;\ \Psi_b(-\underset{\sim}{k}_{bo} \epsilon_{\mu_b o}) \rangle$$

$$= \left[4\mu \sqrt{m^2 - \frac{\mu^2}{4}} \right]^{1/2} \widehat{W}_{ab}(\underset{\sim}{k}_a | \underset{\sim}{k}_b ;\ M) \left[4\mu \sqrt{m^2 - \frac{\mu^2}{4}} \right]^{1/2} \tag{4.4}$$

$$\equiv A_{ab}^{(+)} \ (\underset{\sim}{k}_a | \underset{\sim}{k}_{bo} ;\ \epsilon_{bo} + \epsilon_{\mu_b o},\ \mu)$$

In addition, the on-shell unitarity of the operator S allows the amplitudes to be directly related to cross sections. Written in terms of $A_{\alpha\beta}$, the unitarity condition can be expressed

$$A_{\alpha\beta}^{(+)}(\vec{P}_{(0)}) - \left[A_{\alpha\beta}^{(+)}(\vec{P}_{(0)}) \right]^{\dagger} =$$

$$= \sum_{\gamma} \left[A_{\gamma\alpha}^{(+)}(\vec{P}_{(0)}) \right]^{\dagger} 2\pi i \delta^4(\vec{P}_{\gamma} - \vec{P}_{(0)}) A_{\gamma\beta}^{(+)}(\vec{P}_{(0)}) \ . \tag{4.5}$$

By examining the "forward" amplitudes and the expression for the total cross section the "optical theorem" follows immediately:

$$\sigma_{\text{total}} = \frac{(2\pi)^3 2Im\langle\Phi_\beta : \text{ initial} \mid A^{(+)}_{\beta\beta}(\vec{P}_{(0)}) \mid \Phi_\beta : \text{ initial}\rangle}{\left[(\vec{k}_{(1)0} \cdot \vec{k}_{(2)0})^2 - (\vec{k}_{(1)0} \cdot \vec{k}_{(1)0})(\vec{k}_{(2)0} \cdot \vec{k}_{(2)0}) \right]^{1/2}} \tag{4.6}$$

In 3-CMS, this can explicitly be expressed:

$$\begin{aligned} \sigma_{\text{ total}} = (2\pi)^3 \, & \frac{2}{k_{bo}M} \\ & \times Im\langle\Phi_b : \underset{\sim}{k}_{bo}\epsilon_{bo}; \ \Psi_b(-\underset{\sim}{k}_{bo}\epsilon_{\mu_b o}) \\ & |A^{(+)}_{bb}(\epsilon_{bo} + \epsilon_{\mu_b o}, \underset{\sim}{o})|\Phi_b : \underset{\sim}{k}_{bo}\epsilon_{bo}; \Psi_b(-\underset{\sim}{k}_{bo}\epsilon_{\mu_b o})\rangle \end{aligned} \tag{4.7}$$

Consider the angular momentum decomposition of the amplitudes. The "partial wave amplitudes" can be related to the calculated quantities using

$$\begin{aligned} A^{(+)}_{ab}(\underset{\sim}{k}_a|\underset{\sim}{k}_{bo}; \ \epsilon_{bo} + \epsilon_{\mu_b o,\mu}) &\equiv \sum_J \frac{2J+1}{4\pi} \\ & \qquad P_J(\hat{k}_a \cdot \hat{k}_{bo}) A^J_{ab}(k_a|k_{bo}; \ \epsilon_{\mu_b o} + \epsilon_{bo}, \mu) \\ A^J_{ab}(k_a|k_b; \epsilon_{bo} + \epsilon_{\mu_b o}, \mu) &= -4\mu\sqrt{m^2 - \frac{\mu^2}{4}} \\ & \qquad \widehat{W}^J_{ab}(k_a|k_{bo}; \ \epsilon_{bo} + \epsilon_{\mu_b o}) \end{aligned} \tag{4.8}$$

The condition (4.5) can be expressed in terms of these angular momentum components

$$Im A^J_{bb}(k_b|k_b; \ M) = \sum_\gamma \frac{\pi k_\gamma}{M} \, |A^J_{\gamma b}(k'_\gamma|k_b; \ M)|^2 \tag{4.9}$$

where

$$k'^2_\gamma = \frac{\left[M^2 - (\mu_\gamma + m_\gamma)^2\right]\left[M^2 - (\mu_\gamma - m_\gamma)^2\right]}{4M^2}$$

This allows the definition of the standard phase shift and absorption parameter in the elastic channel by

$$A^J_{bb}(k_b|k_b;\ M = \frac{M}{2\pi i k_b}\left[\eta^J_b(M,\mu_b)e^{2i\delta^J_b(M,\mu b)} - 1\right] \equiv \frac{M}{\pi k_b}\, f^J_b(M,\mu_b) \qquad (4.10)$$

To calculate these parameters, the singularity structure of the two-body input $D_a^{-1}(s_a)$ must be properly understood. By taking advantage of the relation

$$\frac{1}{x - i\eta} \xrightarrow[\eta\to 0]{} \frac{\mathcal{P}}{x} + i\pi\delta(x) \qquad (4.11)$$

where the symbol $\mathcal{P}$ represents the principal value, the function can be written as follows

$$\frac{1}{D_a(s_a)} = \frac{\mathcal{P}}{D_a(s_a)} - i\pi\left[\frac{4\mu\sqrt{m^2 - \frac{\mu^2}{4}}(M^2 + m^2 - \mu^2)}{M\sqrt{(M^2 + m^2 - \mu^2)^2 - 4m^2M^2}}\right]\delta(k_a - K_a) \qquad (4.12)$$

where

$$K_a = \sqrt{\frac{(M^2 + m^2 - \mu^2)^2 - 4m^2M^2}{4M^2}}$$

The numerical solutions for some of the observables will be examined for various values of μ and M.

4.2 Numerical Results

The phase shifts and inelasticity parameters consistent with Eq. (4.10) have been calculated. The integral equations were discretized using Gaussian quadratures, with Legendre polynomials as weight functions. The functions $f_b^{J=0}(M,\mu_b)$ are plotted (as Argand diagrams) in Fig. 5. The rest energy of the system corresponds to $M = +m$, so that the relative kinetic energy is given by

$$e_K = M - (\mu + m) \qquad (4.13)$$

Figure 6 illustrates the total cross section for $J = 0$ in units of m^2, such that

$$\sigma_{\text{total}} = \sum_J \sigma^J_{\text{tot}} \tag{4.14}$$

It can be noted that the lowest resonance is more sharply peaked as the peak energy approaches the rest energy of the system ($e_K = 0$). In addition, in the regions where the resonance is well defined, it follows a path which is a reflection of the lowest energy three-body bound state about the line $e_K = 0$. From the diagrams, it is apparent that the resonance structure for scattering from the ultra-relativistic bound paris is influenced considerably by the inelastic (*i.e.*,

rearrangement) processes. This structure will be extended into the breakup region ($M > 3m$) in Section III-C.

4.3 Break-up Scattering

The process of breakup can occur if the available center-of-momentum energy is greater than the sum of the rest masses of the constituent particle ($M \geq 3m$). The initial system will be described by a particle b scattering from a bound pair. Figure 7 depicts the possible asymptotic states.

As viewed from the elastic scattering "channel," the possibility of breakup will open an additional inelastic "channel."

4.4 Form of Equations

The particular equations for this process can be obtained in a straightforward way as described in Section III-B of Ref. 13. The amplitudes for breakup from an initial channel b, as well as for three particle-to-three-particle scattering, are summarized below.

$$\langle \Phi_0 : (\underset{\sim}{k}_1\epsilon_1; \underset{\sim}{k}_2\epsilon_2; \underset{\sim}{k}_3\epsilon_3), (M_b, \underset{\sim}{\ell} | A_{ob}^{(+)}(M_b, \underset{\sim}{\ell}) | \Phi_b : \underset{\sim}{k}_{bo}\epsilon_{bo}; \Psi_b(-\underset{\sim}{k}_{bo}\epsilon_{\mu_b o}) \rangle$$

$$= \sum_a -\frac{g_a(\underset{\sim}{k}_{a+}\underset{\sim}{k}_{a-})}{D_a(s_a)} \widehat{W}_{ab}(\underset{\sim}{k}_a | \underset{\sim}{k}_{bo};\ M_b) \left[4\mu\sqrt{m^2 - \frac{\mu^2}{4}} \right]^{1/2}$$

where $\qquad M_B = \epsilon_{bo} + \epsilon_{\mu_b o}$

$$\langle \Phi_0 : (\underset{\sim}{k}_1\epsilon_1; \underset{\sim}{k}_2\epsilon_2; \underset{\sim}{k}_3\epsilon_3), (M, \underset{\sim}{\ell} | A_{\infty}^{(+)}(M, \ell) | \Phi_0 : (\underset{\sim}{k}_{10}\epsilon_{10}; (\underset{\sim}{k}_{20}\epsilon_{20}; \underset{\sim}{k}_{30}\epsilon_{30}), (M, \underset{\sim}{\ell}) \rangle$$

$$= \sum_{ab} -\frac{g_a(\underset{\sim}{k}_{a+}\underset{\sim}{k}_{a-})}{D_a(s_a)} \widehat{W}_{ab}(\underset{\sim}{k}_a | \underset{\sim}{k}_{bo};\ M)\ \frac{g_b(\underset{\sim}{k}_{b+o}\underset{\sim}{k}_{b-o})}{D_b(s_b)} \tag{4.15}$$

where the function D_a is described by Eq. (2.3) and $g_a^2 = (\frac{1}{2\pi})^2$.

Above breakup threshold, the singularity structure of the noninteracting resolvent must be properly handled. The singularity occurs only for $M > 3$, and within a limited range of the parameters k_a and k_b. This range is given by Eq. (3.34).

$$0 \le k_a \le \sqrt{\frac{(M^2 - 9m^2)(M^2 - m^2)}{4M^2}} \equiv k_{amax}$$

$$k_{bmin} \le k_b \le k_{bmax} \tag{4.16}$$

where

$$k_{b\,min}^{\ max} \equiv \frac{1}{2} \left| k_a \pm (M - \epsilon_0)\sqrt{\frac{M^2 - 3m^2 - 2\epsilon_a M}{M^2 + m^2 - 2\epsilon_a M}} \right| \tag{4.17}$$

The singularity takes the form of Eq. (4.11). Thus the kernel $R_{ab}^{J=0}$ can be expressed

$$R_{ab}^{J=0}(k_a, k_b;\ M) = \frac{g_a^* g_b}{k_a k_b} \Bigg[\pi i \Theta(k_{amax} - k_a)\Theta(k_{bmax} - k_b)\Theta(k_b - k_{bmin})$$

$$+ \mathcal{P} \log \left| \frac{\sqrt{m_{ab}^2 + (k_a + k_b)^2} + \epsilon_a + \epsilon b - M}{\sqrt{m_{ab}^2 + (k_a - k_b)^2} + \epsilon_a + \epsilon_b - M} \right| \Bigg] \tag{4.18}$$

The solutions of (2.7) using the kernel (4.18) will be examined for $J = 0$.

4.5 Numerical Solutions

The numerical treatment of the equations was similar to that developed in Chapter 3. The solutions smoothly matched those below breakup, and required increasing numerical work as the energy increased.

The behavior of the cross sections beyond the resonance regions is demonstrated in Fig. 8. The Argand diagrams exhibited minor variance beyond the regions covered in Fig. 5.

For completeness, the solution for the moderately relativistically bound state $\mu = 1.9m$ is demonstrated in Fig. 9. In this figure the region above and below breakup threshold is demonstrated on the single graphs.

5. CONCLUSIONS

The equations explored define a self-consistent, unitary set of scattering equations which give stable solutions in this model. It should be noted that the equations in the form given are most suited numerically to the relativistic regime, although the non-relativistic models if the parameters involved are related.

The formalism explored in Ref. 13 generates eigenstates of a fully interacting three-body system in terms of boundary states in a covariant way. These states satisfy a type of cluster form invariance if one of the particles does not interact. Integral angular momentum can be included in the formalism in a straightforward way.

Since in the model examined the equations reduce to a single parameter integral equation, the numerical methods involved in this exploration were straightforward. Advanced numerical techniques exist in the literature which allow exploration of the amplitudes involved in a more complex model. However, in order to more reasonably reproduce the high energy phenomenology, the inclusion of particle-antiparticle symmetries and multiparticle processes must be examined in the formalism.

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FIGURE CAPTIONS

Fig. 1. Kinematic regions for three-particle dynamics. Particles are kinematically and dynamically identical, but distinguishable.

Fig. 2. Lowest Lying three-particle bound state trajectories.

Fig. 3. Nonrelativistic region of bound states.

Fig. 4. Elastic and rearrangement scattering.

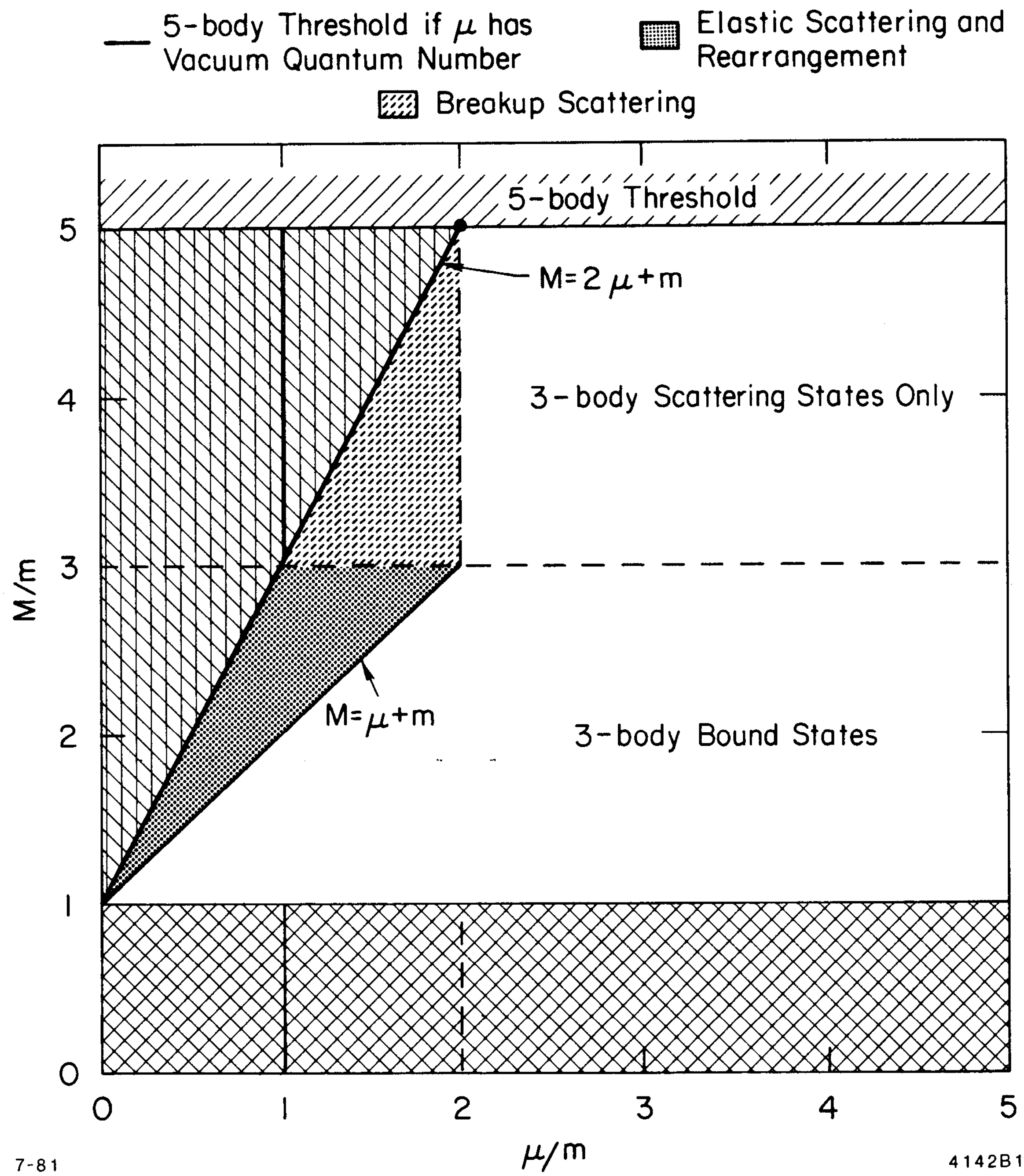
Fig. 5. Calculated Argand plots.

Fig. 6. Total cross sections ($J = 0$).

Fig. 7. Elastic, rearrangement and breakup scattering.

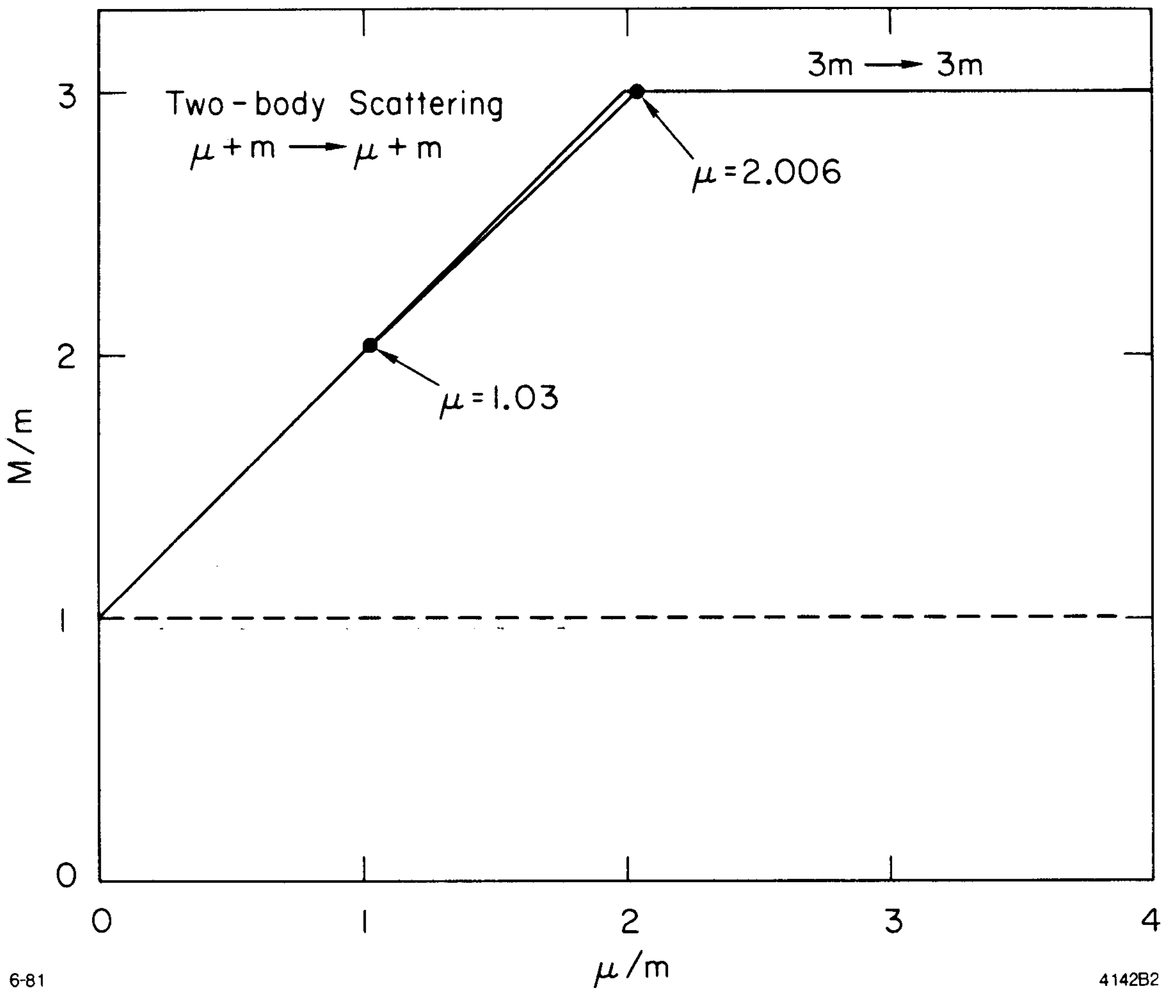
Fig. 8. Behavior of cross sections at breakup threshold.

Fig. 9. Dynamical behavior of three-particle scattering for moderately relativistic system.



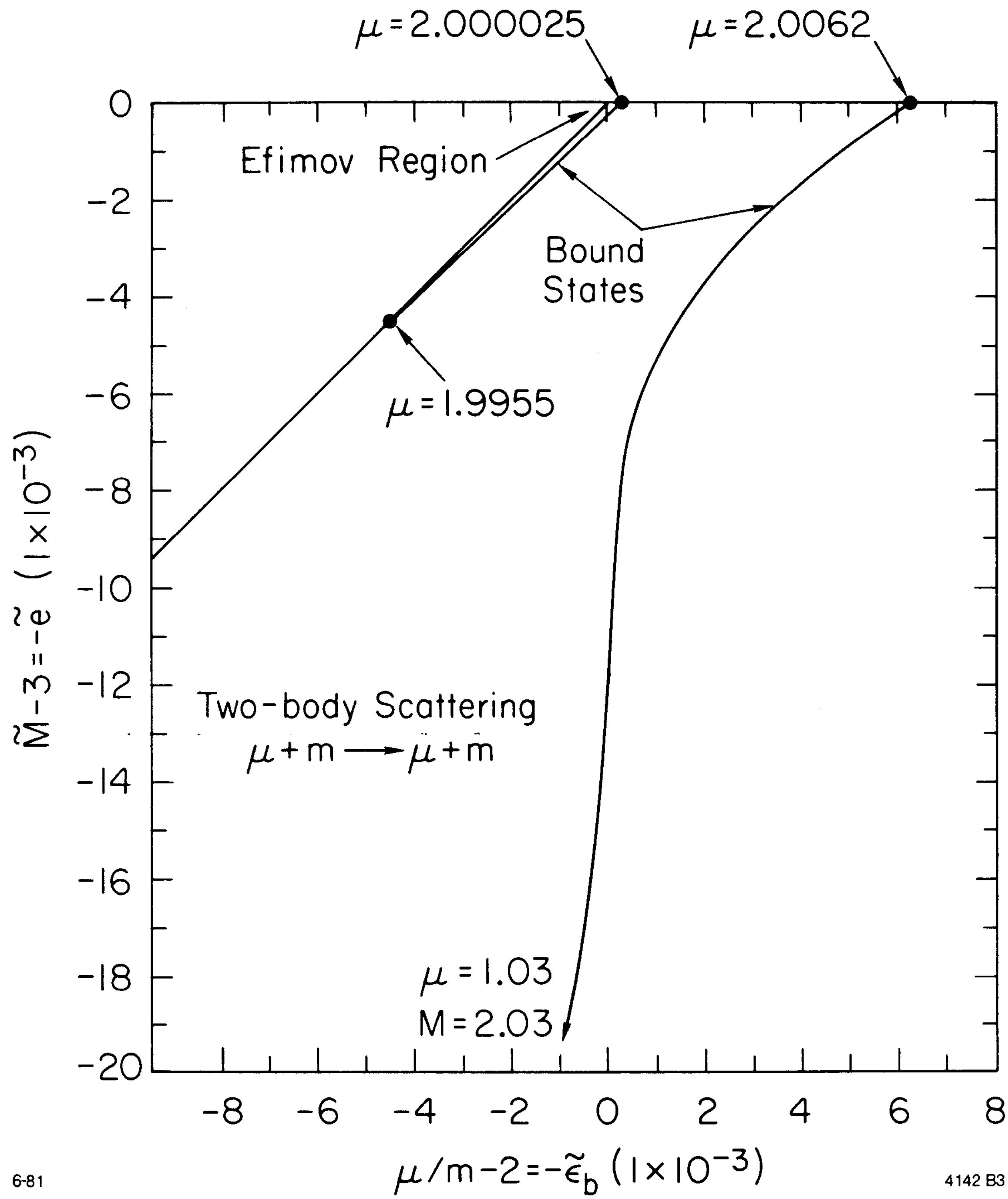
5-body Threshold if μ has Vacuum Quantum Number
Elastic Scattering and Rearrangement
Breakup Scattering
5-body Threshold
M=2μ+m
3-body Scattering States Only
M=μ+m
3-body Bound States
M/m
μ/m
0
1
2
3
4
5
0
1
2
3
4
5
7-81
4142B1

Fig. 1



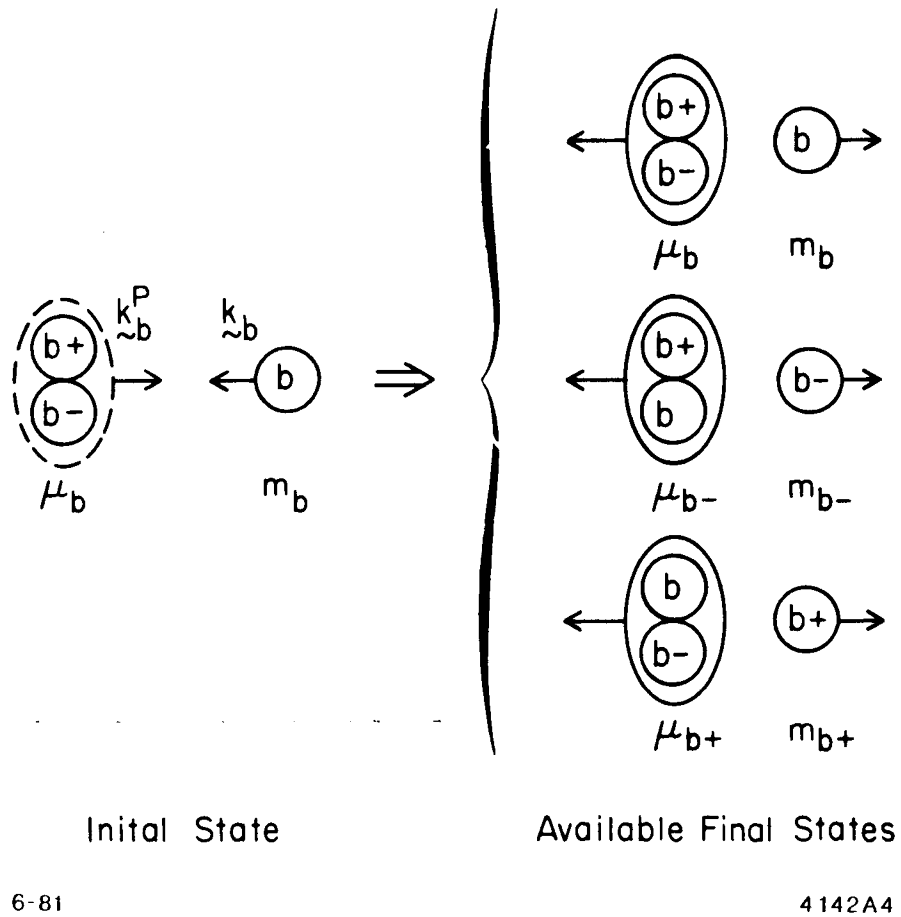
3m → 3m
Two - body Scattering
μ + m → μ + m
μ = 2.006
μ = 1.03
M/m
μ/m
0
1
2
3
4
6-81
4142B2

Fig. 2



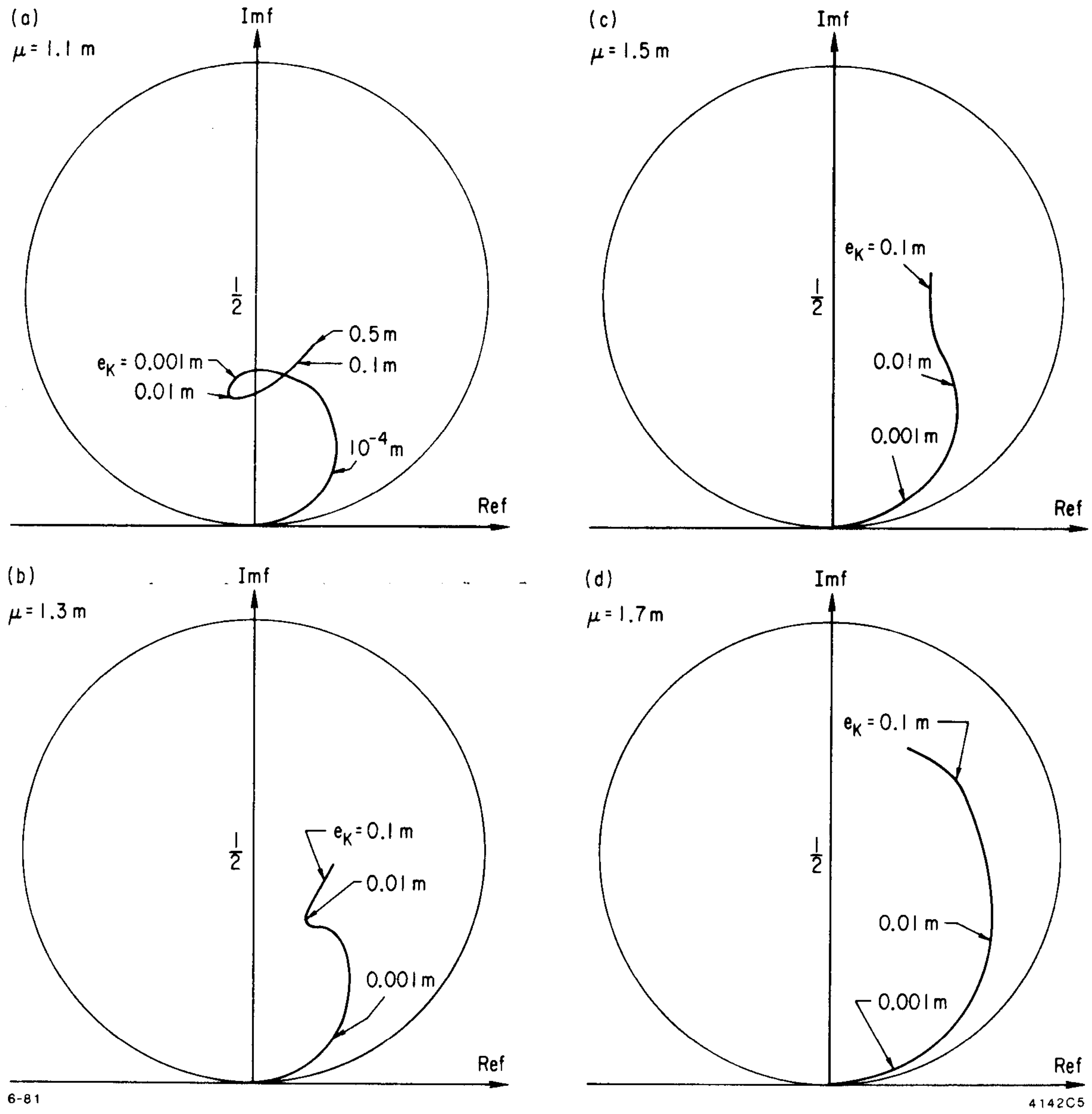
μ=2.000025
μ=2.0062
Efimov Region
Bound States
μ=1.9955
Two-body Scattering
μ+m → μ+m
μ=1.03
M=2.03
0
-2
-4
-6
-8
-10
-12
-14
-16
-18
-20
-8
-6
-4
-2
0
2
4
6
8
$\tilde{M}-3=-\tilde{e}$ (1×10^{-3})
$\mu/m-2=-\tilde{\epsilon}_b$ (1×10^{-3})
6-81
4142 B3

Fig. 3



$\underset{\sim}{k}{}_b^P$
$\underset{\sim}{k}{}_b$
b+
b−
b
μ_b
m_b
Inital State
b+
b−
b
μ_b
m_b
b+
b
b−
μ_{b-}
m_{b-}
b
b−
b+
μ_{b+}
m_{b+}
Available Final States
6-81
4142A4

Fig. 4



(a)
μ = 1.1 m
Imf
1/2
Ref
e_K = 0.001 m
0.01 m
0.5 m
0.1 m
10^-4 m
(c)
μ = 1.5 m
Imf
1/2
Ref
e_K = 0.1 m
0.01 m
0.001 m
(b)
μ = 1.3 m
Imf
1/2
Ref
e_K = 0.1 m
0.01 m
0.001 m
(d)
μ = 1.7 m
Imf
1/2
Ref
e_K = 0.1 m
0.01 m
0.001 m
6-81
4142C5

Fig. 5

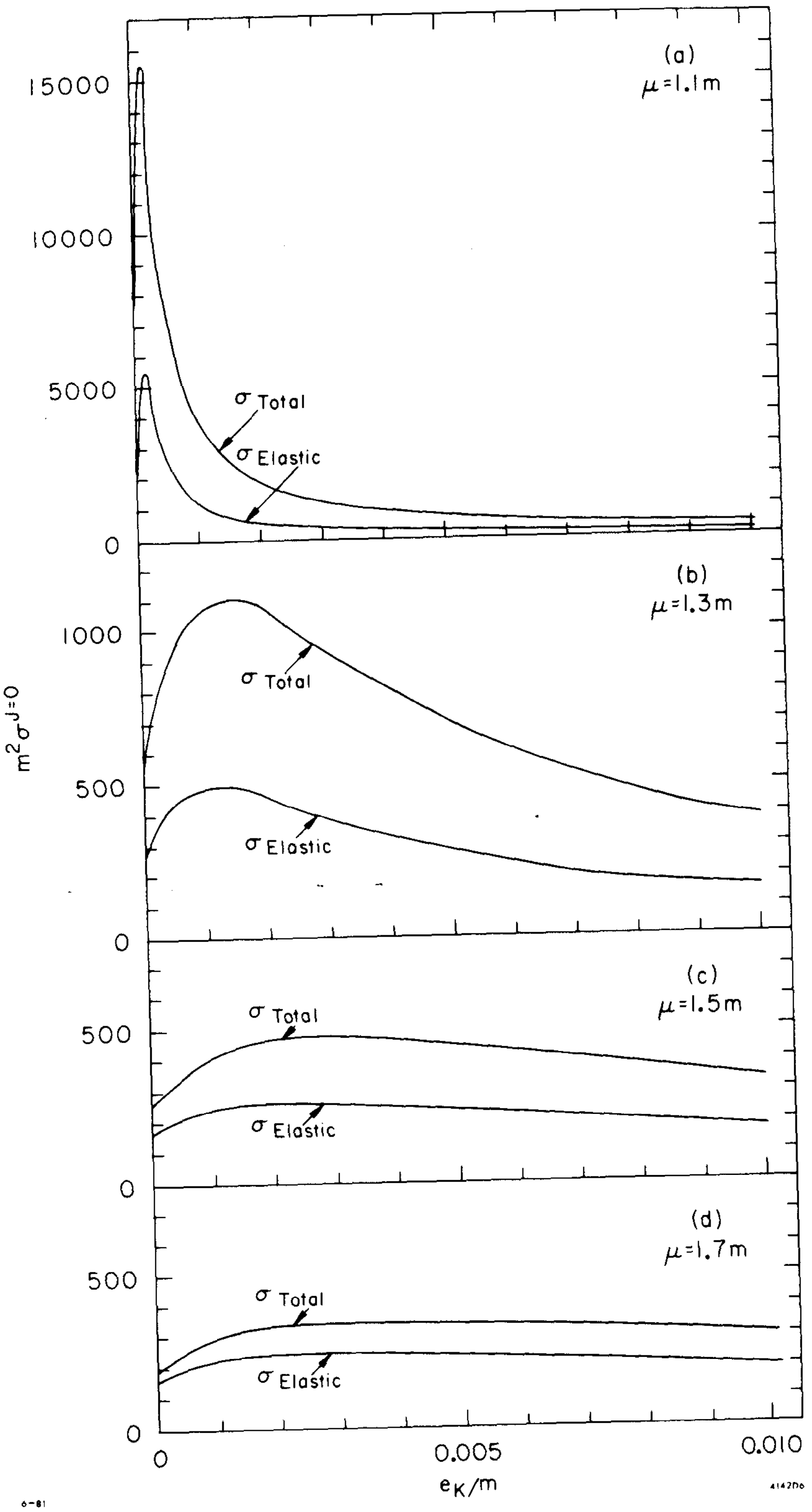

(a)
μ=1.1m
(b)
μ=1.3m
(c)
μ=1.5m
(d)
μ=1.7m
σ Total
σ Elastic
15000
10000
5000
1000
500
0
0.005
0.010
m²σ^J=0
e_K/m
6-81
4142D6

Fig. 6

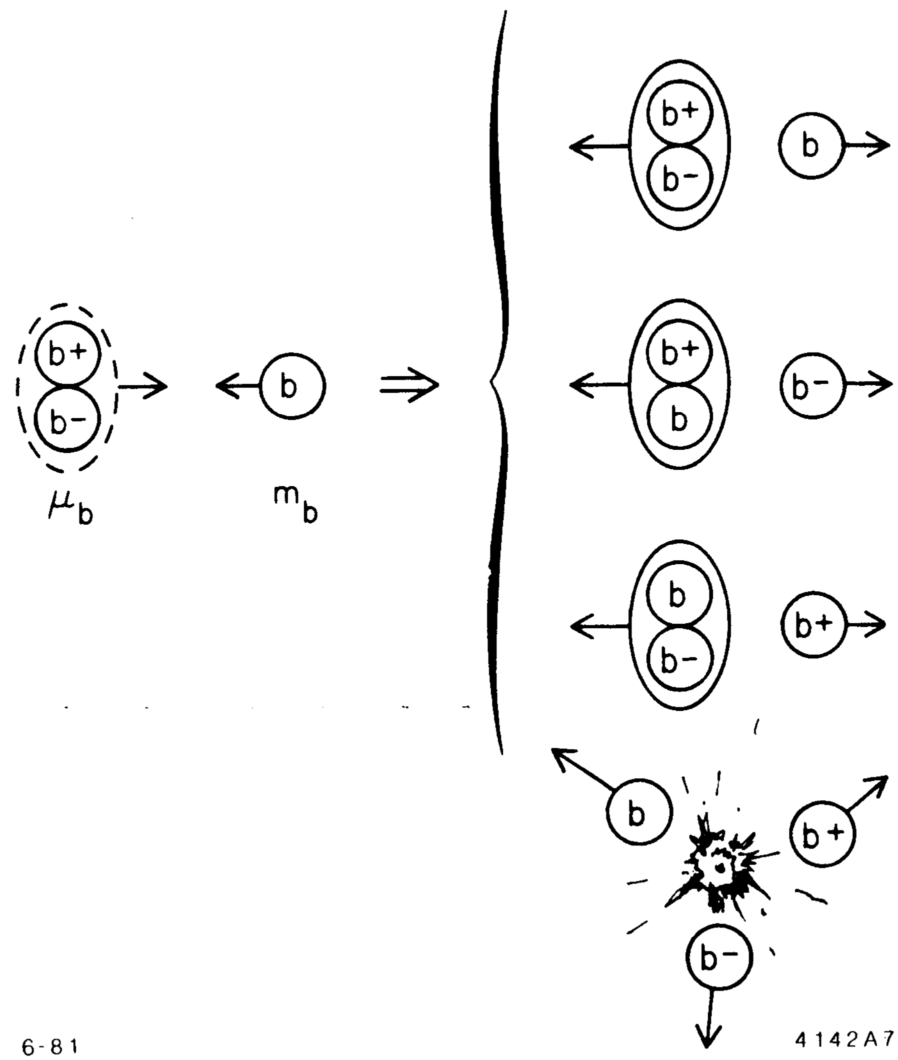


Fig. 7

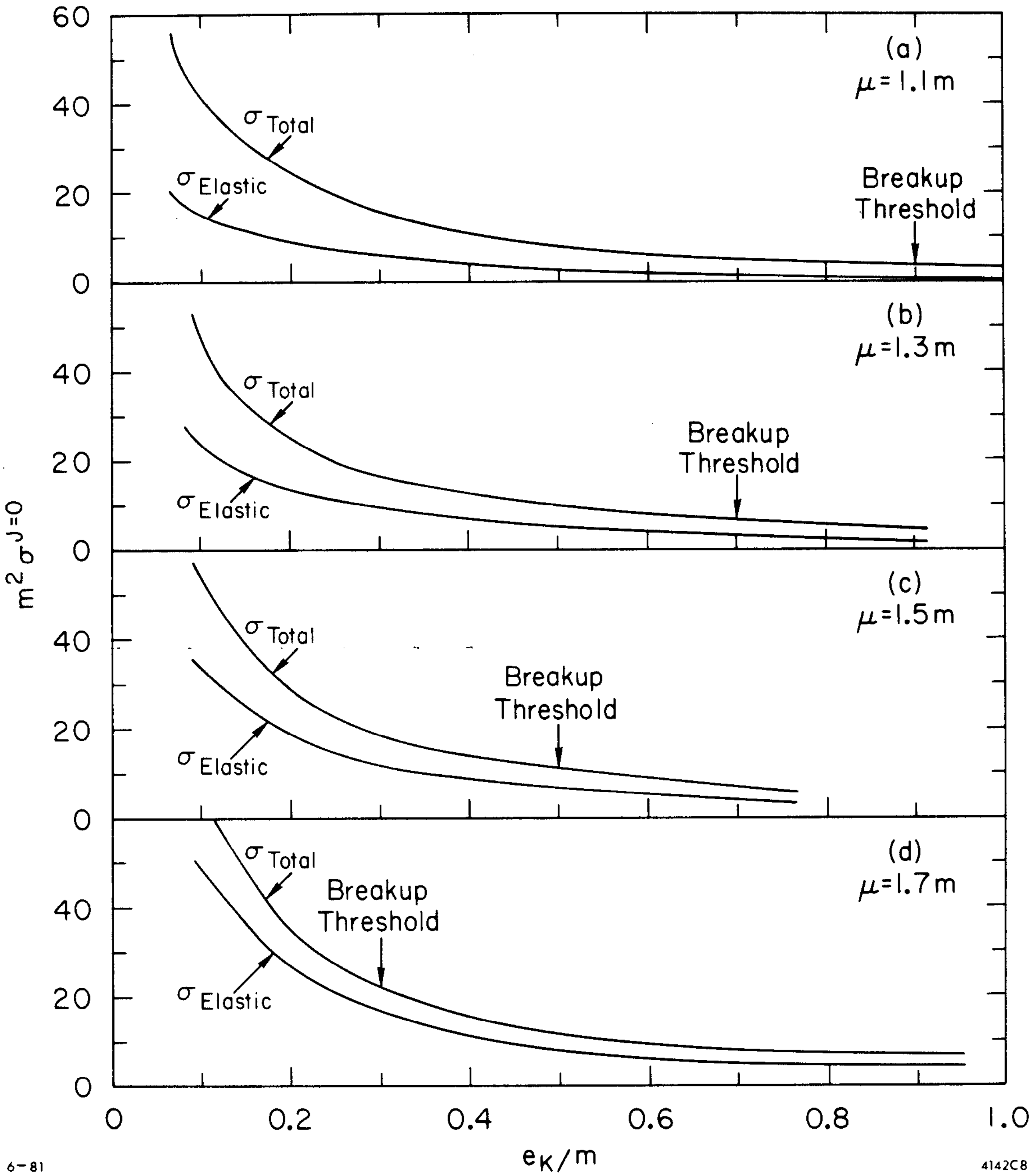

60
40
20
0
(a)
μ=1.1m
σ Total
σ Elastic
Breakup
Threshold
(b)
μ=1.3m
(c)
μ=1.5m
(d)
μ=1.7m
m²σ^J=0
0
0.2
0.4
0.6
0.8
1.0
e_K/m
6−81
4142C8

Fig. 8

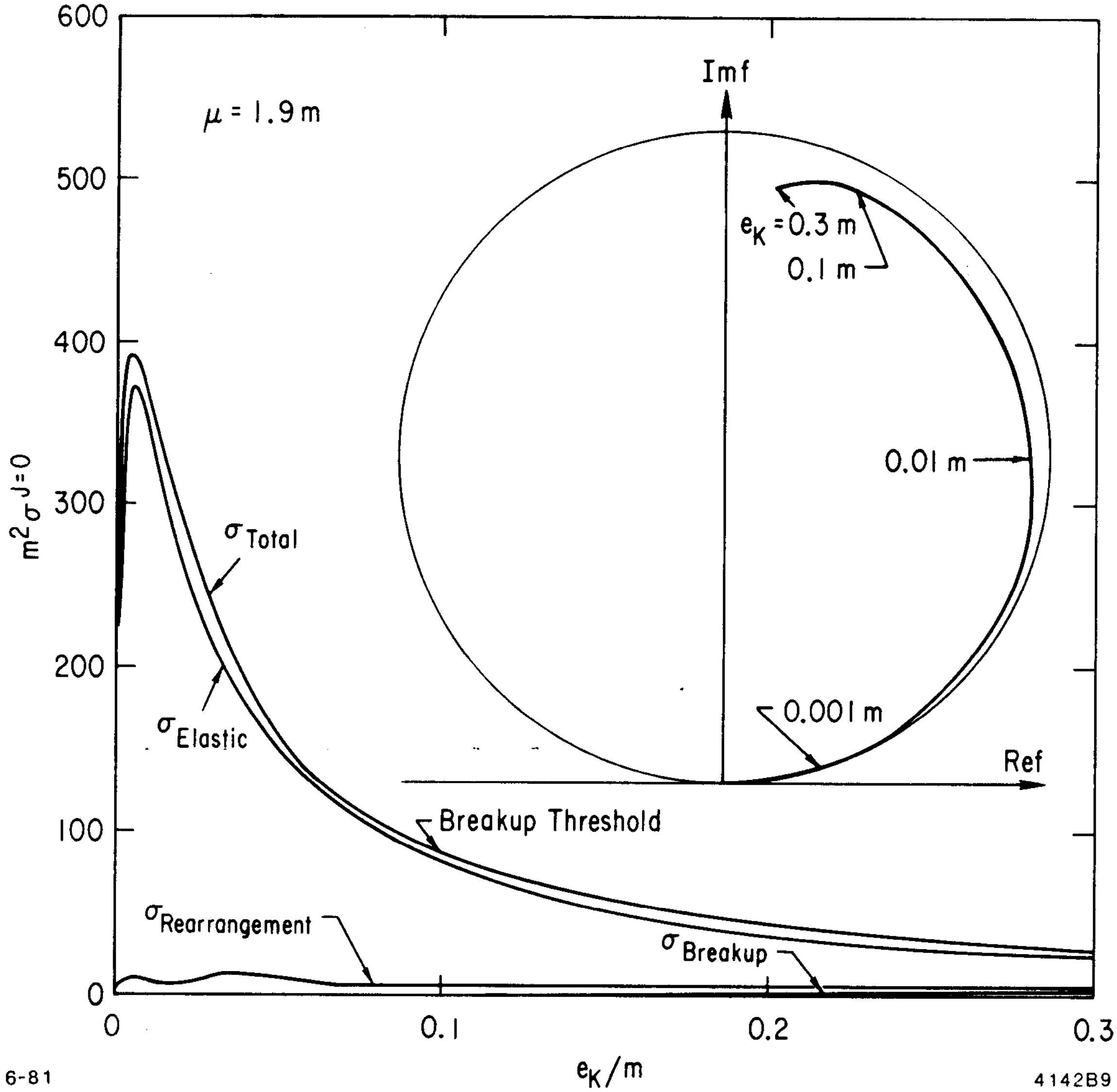

600
500
400
300
200
100
0
0
0.1
0.2
0.3
μ = 1.9 m
Imf
Ref
e_K = 0.3 m
0.1 m
0.01 m
0.001 m
σ_Total
σ_Elastic
Breakup Threshold
σ_Rearrangement
σ_Breakup
m²σ^J=0
e_K/m
6-81
4142B9

Fig. 9