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THE BOUND STATES OF WEAKLY-COUPLED LONG-RANGE ONE-DIMENSIONAL QUANTUM HAMILTONIANS*

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ABSTRACT

We study the small λ behavior of the ground state energy, $E(\lambda)$, of the Hamiltonian $-\frac{d^2}{dx^2} + \lambda V(x)$. In particular, if $V(x) \sim -ax^{-2}$ at infinity and if $\int V(x)\, dx < 0$, we prove that

$$\sqrt{-E(\lambda)} = -\left[\frac{1}{2}\lambda + a\,\lambda^2 \ln \lambda\right] \int dx\, V(x) + 0(\lambda^2) \quad .$$

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I. INTRODUCTION

It is well known that a sufficiently shallow square well in three dimensions will not bind. By contrast, in one or two dimensions, there is a special situation, due essentially to an infrared divergence, in which an attractive short-range potential always produces a bound state no matter how small the coupling. For the case of the one-dimensional Hamiltonian

$$H = -\frac{d^2}{dx^2} + \lambda V(x) \quad ,$$

Abarbanel, Callen, and Goldberger [1] derived a formal series for the ground state, $E(\lambda)$, for an attractive V of short range of the form

$$\sqrt{-E(\lambda)} = -\frac{1}{2}\lambda \int dx\, V(x) - \frac{1}{4}\lambda^2 \int dx\, dy\, V(x)\ |x-y|\, V(y) + 0(\lambda^2) \quad . \qquad (1.1)$$

This situation was further studied by Simon [2] who proved that so long as $\int dx\, V(x) \leq 0$, and $\int dx(1+x^2)|V(x)| < \infty$, there is a unique bound state for small λ and its energy is given by (1.1). It was also shown that if $\int dx\, e^{a|x|}\, V(x) < \infty$, then $\sqrt{-E(\lambda)}$ is analytic at $\lambda=0$.

In this note we wish to consider the case where V(x) is of sufficiently long range that

$$\int dx(1+x^2)\ |V(x)| = \infty \quad .$$

There are three cases to consider with

$$V(x) \simeq -ax^{-\beta} \qquad (1.2)$$

as $x \to \infty$.

(A) If $2 < \beta < 3$, then a simple modification of the argument in [2] allows one to prove that (1.1) is still valid.

(B) If $\beta=2$, there is still a unique bound state for small λ so long as $\int dx\, V(x) \leq 0$. However, if this integral is nonzero, then (1.1) is not valid because the λ^2 term is infinite; there is, in fact, a $\lambda^2 \ln\lambda$ term which we explicitly isolate. The situation here is reminiscent of some recent work of Greenlee [3,4] and Harrel [5] who study perturbations of the operator $(-d^2/dx^2)$ by potentials with $x^{-\gamma}$ singularities at the origin on the interval $[0,\alpha]$ or $\left(-\frac{d^2}{dx^2}+x^2\right)$ on the interval $[0,\infty]$ with $\psi(0)=0$ boundary conditions. If $\gamma \geq 3$, then first order perturbation theory is infinite but there is an explicit λ^{-8}, $g=(\gamma-2)^{-1}$ leading term if $\gamma>3$ and a $\lambda\ln\lambda$ leading term if $\gamma=3$. Harrel also considers situations in which the first order term is finite but the second order is infinite; for example, if $\gamma=2.5$, there is a $a\lambda+b\lambda^2\ln\lambda + 0(\lambda^2)$ behavior analogous to what we find in the $\beta=2$ case.

(C) If $1<\beta<2$, then there are infinitely many bound states for any $\lambda>0$ in one and three dimensions. One may ask in this case if there is any difference between the one and three-dimensional case as there is for a short ranged potential $V(x)$. In fact, there is a difference; all the bound state energies in three dimensions and all but the ground state in one dimensions are of order λ^h, $h=2(2-\beta)^{-1}$, as λ approaches zero from above. The ground state energy in one dimension is special in that it is of order λ^2. One still finds that

$$\sqrt{-E(\lambda)} = -\frac{1}{2}\lambda \int dx\, V(x) + 0(\lambda) \quad . \tag{1.3}$$

We shall not consider the case $\beta=1$ or $0<\beta<1$ although on the basis of the work by Greenlee and Harrel and our case (C), there is a natural conjecture: at $\beta=1$, all states but the ground state are of order λ^2, while the ground state is of order $\lambda^2(\ln\lambda)^2$; for $0<\beta<1$, all states are of order λ^h, $h=2(2-\beta)^{-1}$ and the ground state is of order λ^g, $g=2(3-2\beta)^{-1}$.

The outline of this note is as follows. In Section II, we consider, for motivation, the special case $V(x)=-\frac{1}{4}(|x|+d)^{-2}$, which is solvable in terms of Bessel functions. In addition to verifying our general results for $\beta=2$ (see also formula (4.2)) and small λ, we check explicitly a curious behavior at $\lambda=1$; for $\lambda<1$, there is only one bound state, while for $\lambda>1$, there are as infinite number. Such a behavior was proven in general for $\beta=2$ potentials by Simon [6]. In Section III, we consider the cases (A), (C) and certain general features of (B) as defined above, and in Section IV the $\lambda^2 \ln\lambda$ term in case (B) is explicitly isolated.

II. AN EXAMPLE

In this section we shall discuss the potential

$$\lambda V(x) = -\frac{1}{4}\lambda(|x|+d)^{-2} \quad , \tag{2.1}$$

and the solution to the Schrödinger equation

$$\psi''(x) + \left[k^2-\lambda V(x)\right] \psi(x) = 0 \quad .$$

The outgoing wave solution for positive x can be written in terms of Hankel functions:

$$\psi_+(x) = T(k(x+d))^{1/2} H_\nu^{(1)}[k(x+d)] \quad , \tag{2.2}$$

where T is constant and $\nu=\frac{1}{2}(1-\lambda)^{1/2}$. The solution for negative x includes an incident plane wave and a reflected wave at infinity and is written

$$\psi_-(x) = (k(d-x))^{1/2}\left\{H_\nu^{(2)}[k(d-x)] + R\, H_\nu^{(1)}[k(d-x)]\right\} \quad . \tag{2.3}$$

Matching boundary conditions at the origin yields a reflection coefficient, r, and a transmission coefficient, t, of the form

$$r = R\, e^{2iz}\, e^{-i\pi(\nu+1/2)}$$

$$t = T\, e^{2iz}\, e^{i\pi(\nu+1/2)}$$

where $z = kd$ and

$$T = 4i/\pi D(z)$$

$$R = T - H_\nu^{(2)}(z)/H_\nu^{(1)}(z)$$

and

$$D(z) = \frac{d}{dz}\left[z\left(H_\nu^{(1)}(z)\right)\right]^2 \quad . \tag{2.4}$$

The eigenvalue condition is equivalent to the vanishing of D(z) for z pure imaginary and in the upper half plane. Defining z=iy, it becomes

$$y\frac{d}{dy}K_\nu(y) = -\frac{1}{2}K_\nu(y) \quad . \tag{2.5}$$

where $y = d\sqrt{-E(\lambda)}$ and recall that $\nu = \frac{1}{2}(1-\lambda)^{1/2}$ where $K_\nu(y)$ is a modified Bessel function of the second kind related to $H_\nu^{(1)}(z)$ by $K_\nu(y) = \frac{\pi i}{2}\exp(i\pi\nu/2)\,H_\nu^{(1)}(iy)$.

For $0 < \lambda < 1$, there is always one solution to (2.5), as can easily be seen by considering the limiting behavior of both sides of this equation. For small λ. it is convenient to expand $K_\nu(y)$ as

$$K_\nu(y) = \frac{1}{2}\Gamma(\nu)\left(\frac{y}{\nu}\right)^{-\nu}\left[1 + \frac{\Gamma(-\nu)}{\Gamma(\nu)}\left(\frac{y}{2}\right)^{2\nu}\right] + 0(y^{2-\nu})$$

and the eigenvalue condition can be expanded as

$$2\nu = \tanh\left[\nu\;\ln\left(\frac{2}{y}\right)\right] + 0(y) \tag{2.6}$$

or

$$y = \frac{1}{2} - \nu\;\left[1 - \left(\frac{1}{2} - \nu\right)\;\ln 4/y^2\right] + 0\left[\left(\frac{1}{2} - \nu\right)^2\right] \quad .$$

This can be rewritten in the form

$$\sqrt{-E(\lambda)} = -\frac{1}{2}\left[\lambda + 2a\lambda^2\ln\lambda\right]\int dx\,V(x) + 0(\lambda^2) \quad , \tag{2.7}$$

where a=1/4. We shall return in Section IV and derive this expansion for a more general class of potentials that behave as $-ax^{-2}$ for large x.

If λ is larger than one, then one sees that the index of the Bessel function becomes complex. This introduces an oscillatory behavior and profoundly affects the spectrum. If λ is only slightly larger than unity, then a simple expansion is possible for small y. Defining $\nu = i\delta$, then the eigenvalue condition (2.6) becomes

$$2\delta = \tan\left[\delta \ln\left(\frac{2}{y}\right)\right] + 0(y) \quad . \tag{2.8}$$

There are therefore an infinite number of bound states for $\lambda > 1$, and their energies are geometrically related for weak binding.

III. GENERAL CONSIDERATIONS

Throughout this section we shall consider the family of operators H defined earlier with

$$\int dx\, |V(x)| < \infty \quad , \tag{3.1}$$

and will often add the condition

$$\int dx |x|^{\gamma} |V(x)| < \infty \tag{3.2}$$

for some $\gamma>0$.

The discussion in reference [2] assumed eq. (3.2) with $\gamma=2$, and it is our purpose to extend the class of admissible potentials. We note that if $V(x) \sim -ax^{-\beta}$ for large x, and (3.1) holds, then (3.2) will also hold for $\gamma=\beta-1-\epsilon$ for any $\epsilon>0$. Under condition (3.1) two results carry over from reference [2] (propositions 2.1 and 2.2); namely, $E=-\alpha^2$ with $\alpha>0$ is an eigenvalue of H if and only if

$$\det\left[1 + \lambda K_{\alpha}\right] = 0 \tag{3.3}$$

where K_{α} is the integral operator

$$K_{\alpha}(x, y) = \frac{1}{2\alpha} |V(x)|^{1/2} \exp(-\alpha|x-y|)\, V^{1/2}(y) \quad , \tag{3.4}$$

and $V^{1/2}(y) = |V(y)|^{1/2}$ sign $(V(y))$. Moreover,

$$H = -\frac{d^2}{dx^2} + \lambda V(x) \geq c\lambda$$

for some c and all small λ. Thus one needs only look for solutions to (3.3) with $0 < \alpha(\lambda) < (c\lambda)^{1/2}$ for small λ.

It is convenient to decompose K_α into two sets of integral operators:

$$K_\alpha = Q_\alpha + P_\alpha = L_\alpha + M_\alpha \quad .$$

First,

$$\begin{aligned} Q_\alpha(x,y) &= \frac{1}{2\alpha} e^{-\alpha|x|} |V(x)|^{1/2} e^{-\alpha|y|} V^{1/2}(y) \\ P_\alpha(x,y) &= \frac{1}{\alpha} |V(x)|^{1/2} \left[e^{-\alpha|x|_>} \sinh \alpha |x|_<\right] V^{1/2}(y) \quad , \end{aligned} \tag{3.5}$$

where $|x|_< = 0$ if $xy<0$ and $|x|_< = \min |x|, |y|$ otherwise; $|x|_> = \max |x|, |y|$. Second,

$$\begin{aligned} L_\alpha(x,y) &= \frac{1}{2\alpha} |V(x)|^{1/2} V^{1/2}(y) \\ M_\alpha(x,y) &= \frac{1}{2\alpha} |V(x)|^{1/2} \left[e^{-\alpha|x-y|} - 1\right] V^{1/2}(y) \quad . \end{aligned} \tag{3.6}$$

In reference [2], the latter decomposition was used since it results in a simpler implicit equation for α as deduced from eq. (3.3) than does the former decomposition (compare our eq. (3.8) with (9) of ref. [2]).

The advantage of the Q_α, P_α pair is that it is more convergent. As $\alpha \to 0$, the factor in brackets of eq. (3.5) approaches $|x|_<$ rather than a factor of $\frac{1}{2}|x-y|$ as in Eq. (3.6). For fixed y, the latter approaches $\frac{1}{2}|x|$ for large x whereas for the former $|x|_< \to 0$ or $|y|$ as $|x| \to \infty$. Therefore P_0 is less singular than M_0.

We emphasize that P_α is very natural—it arises from replacing the Green's function in K_α by the Green's function in which a zero boundary condition is imposed at the origin. The fact that when eq. (3.2) holds with $\gamma=1$, then $\det(1+\lambda P_\alpha) = 0$ has no solutions for λ small is intimately connected with Schwinger's proof [8] of Bargmann's bound [9].

In order to bound P_α independently of α, note that the elementary inequality for $x>0$

$$x^{-1} \sinh x \leq \cosh x \leq e^x$$

leads to

$$|P_\alpha(x,y)| \leq |V(x)|^{1/2} |x|_< |V(y)|^{1/2} \leq |xV(x)|^{1/2} |yV(y)|^{1/2} \tag{3.7}$$

so that by letting

$$P_0(x,y) \equiv |V(x)|^{1/2} |x|_< V^{1/2}(y) \quad ,$$

we have by the dominated convergence theorem

$$\int dxdy \; |P_\alpha - P_0|^2 \to 0$$

as $\alpha \to 0$ so long as (3.2) holds with $\gamma=1$. With this result, one can now mimic the proofs of Theorems (2.4) and (2.5) of ref. [2] and obtain

<u>Theorem 3.1</u> Suppose that (3.1) holds and that (3.2) holds with $\gamma=1$. Then H has at most one negative eigenvalue for λ small and this occurs if and only if $\int dx\, V(x) \leq 0$. If this condition holds, then $\alpha = \sqrt{-E(\lambda)}$ is given by the implicit condition (expand the determinant using the fact that Q_α is a separable integral operator)

$$\alpha = -\tfrac{1}{2}\lambda \left(e^{-\alpha|x|} V^{1/2}, (1+\lambda P_\alpha)^{-1} e^{-\alpha|y|} |V(y)|^{1/2}\right) \tag{3.8}$$

and, in particular, eq. (1.1) holds. This immediately extends the results of ref. [2] from $x^{-3-\epsilon}$ potentials to $x^{-2-\epsilon}$ potentials.

To understand and to anticipate our next result, suppose that $V(x)=V(-x)$ and $V \sim -ax^{-\beta}$ at infinity. If ψ_0 and ψ_1 are two bound states with energies $E_0 < E_1$, then it is possible to find a linear combination $\phi(x)$ on $(0,\infty)$ that vanishes at the origin. Therefore,

$$E_1 \int_0^\infty dx\, \phi^2(x) \geq \int_0^\infty dx \left[(\phi')^2 + \lambda V(x)\, \phi^2(x)\right] \; .$$

As is well known, if $\phi(0)=0$, then [10]

$$\int_0^\infty dx\,(\phi')^2 \geq \frac{1}{4}\int_0^\infty dx\,\phi^2(x)\,x^{-2} \quad ,$$

so that

$$E_1(\lambda) \geq \min\left(\frac{1}{4}x^{-2}+\lambda V(x)\right) \sim -\lambda^g$$

where $g=2(2-\beta)^{-1}$ if $\beta<2$. Thus one expects that all bound states except for the ground state will have energies that behave as λ^g whereas the ground state energy will be $0(\lambda^2)$.

Theorem 3.2 Let V obey (3.1) and (3.2) for some γ, where $0<\gamma<1$; then there is a constant C so that at most one bound state occurs with an energy smaller than $-C\lambda^h$, $h=2(1-\gamma)^{-1}$, for small λ. Such a bound state will exist if $\int dx\,V(x)<0$, and in that case its energy, $E(\lambda)$, is given by eq. (3.8) with $\alpha=\sqrt{-E(\lambda)}$. In particular,

$$\sqrt{-E(\lambda)} = -\frac{1}{2}\lambda\int dx\;V(x) + 0(\lambda^{1+\gamma}) \quad . \tag{3.9}$$

Proof: Since

$$e^{-\alpha|x|_>}\sinh\alpha|x|_< \leq \frac{1}{2}e^{-\alpha(|x|_>-|x|_<)} \leq \frac{1}{2} \quad ,$$

then

$$|P_\alpha(x,y)| \leq \frac{1}{2\alpha}|V(x)|^{1/2}|V(y)|^{1/2} \quad .$$

Recalling the bound on $|P_\alpha|$ given by eq. (3.7), one has for $0<\theta<1$

$$|P_\alpha(x,y)| \leq \left(\frac{1}{2\alpha}\right)^{1-\theta}(xy)^{\theta/2}|V(x)|^{1/2}|V(y)|^{1/2} \quad .$$

Therefore, the Hilbert-Schmidt norm for P_α, choosing $\theta=\gamma$, is bounded by

$$\|P_\alpha(x,y)\|_{HS} \leq (2\alpha)^{\gamma-1}\int dx\,|x|^\gamma\;|V(x)|$$

It now follows that if $\lambda \| P_\alpha \|_{HS} < 1$, or equivalently

$$E(\lambda) < -\frac{1}{4}\left[\lambda \int dx\, |x|\, |V(x)|\right]^h \tag{3.10}$$

where $h = 2(1-\gamma)^{-1}$, then $(1+\lambda P_\alpha)$ is invertible and thus for such α and λ, eq. (3.3) has a solution if and only if (3.8) has a solution with $\alpha>0$. Then $E=-\alpha^2$ is the unique eigenvalue and satisfies the inequality (3.10). The result now follows by mimicking the arguments given in ref. [2].

The $0(\lambda^{1+\gamma})$ error comes from

$$\text{Error} = -\frac{1}{2}\lambda\left(e^{-\alpha|x|}V^{1/2}(x), \left[(1+\lambda P_\alpha)^{-1} - 1\right]|V|^{1/2}e^{-\alpha|y|}\right)$$

$$-\frac{1}{2}\lambda \int dx\, V(x)\left(e^{-\alpha|x|} - 1\right) \quad .$$

The first term in the error is of order $\lambda^2 \| P_\alpha \|_{HS} = 0(\lambda^2 \alpha^{\gamma-1}) = 0(\lambda^{1+\gamma})$ since $\alpha = 0(\lambda)$. By using $(e^{-\alpha|x|}-1) \leq (\alpha|x|)^\gamma$, the second term is also seen to be of order $0(\lambda\alpha^\gamma) = 0(\lambda^{1+\gamma})$.

IV. THE SECOND ORDER TERM FOR β=2 POTENTIALS

In this final section, we will consider potentials that behave as $V \sim -ax^{-2}$ at infinity. For later convenience we will decompose V as

$$V(x) = V_1(x) + V_2(x) \tag{4.1}$$

where

$$V_1(x) = -a(1+x^2)^{-1}$$

and demand that

$$\int dx\, |x|^{1+\delta}\, |V_2(x)| < \infty \tag{4.2}$$

for some $\delta>0$. It will be proved that if $\int dx\, V(x) < 0$, the ground state energy obeys

$$\sqrt{-E(\lambda)} = -\left[\frac{1}{2}\lambda + a\lambda^2 \ln\lambda\right] \int dx\, V(x) + 0(\lambda^2) \quad . \tag{4.3}$$

To motivate this result, consider the direct expansion of the determinant, eq. (3.3); after some slight manipulations one finds to second order in λ,

$$\alpha = -\frac{1}{2}\lambda \int dx\, V(x) + \frac{\lambda^2}{8\alpha}\int_0^\infty dz\ (1-e^{-2\alpha z})\int_{-\infty}^{\infty} dx\, V(x)\ V(x+z) \quad .$$

The small α limit of the second term depends upon the large z behavior of the convolution integral between two V's. One estimates that

$$\int dx\ V(x)\ V(x+z) \simeq [V(z) + V(-z)] \int dx\ V(x)$$

and for even potentials one finds

$$\int_0^\infty dz\ (1-e^{-2\alpha z})\, V(z) \simeq -2a\alpha\ \ln\alpha + 0(\alpha^2) \quad .$$

Now by noting that $\alpha = 0(\lambda)$, the expansion (4.3) immediately follows.

In order to prove this result, let us return to the eigenvalue condition (3.8) and using Theorem (3.2), where $\gamma = 1-\epsilon$ for $\beta=2$, one has

$$\begin{aligned} \alpha &= -\frac{1}{2}\lambda \int dx\, V(x)\ e^{-2\alpha|x|} + \frac{1}{2}\lambda^2\left(e^{-\alpha|x|}\, V^{1/2}, P_\alpha |V|^{1/2} e^{-\alpha|y|}\right) + 0(\lambda^{3-\epsilon}) \\ &= -\frac{1}{2}\lambda \int dx\, V(x)\ e^{-2\alpha|x|} + \frac{1}{2}\lambda^2\left(V^{1/2}, P_\alpha |V|^{1/2}\right) + 0(\lambda^{3-\epsilon}) \end{aligned} \tag{4.4}$$

Now the second term is most easily estimated by using the relation $P_\alpha = M_\alpha + (L_\alpha - Q_\alpha)$, and one has

$$\begin{aligned} \frac{1}{2}\lambda^2\left(V^{1/2}, (L_\alpha - Q_\alpha)|V|^{1/2}\right) &= -\frac{1}{2}\lambda^2 \int dxdy\, V(x)\, V(y)\ (e^{-\alpha|x|} - 1)/\alpha + 0(\lambda^{3-\epsilon}) \\ &= \lambda \int dx\, V(x)(e^{-\alpha|x|} - 1)\left[\frac{1}{\alpha}(\alpha + 0(\lambda^{2-}\ )\right] + 0(\lambda^{3-\epsilon}) \\ &= \lambda \int dx\, V(x)\ (e^{-\alpha|x|} - 1) + 0(\lambda^{3-\epsilon}) \quad . \end{aligned}$$

Thus eqs. (4.4) achieves the form

$$\alpha = -\frac{1}{2}\lambda\int dx\, V(x) + \frac{1}{2}\lambda^2\left(V^{1/2}, M_\alpha |V|^{1/2}\right)$$
$$+\frac{1}{2}\lambda\int dx\, V(x)\left[2e^{-\alpha|x|} - e^{-2\alpha|x|} - 1\right] + 0(\lambda^{3-\epsilon}) \quad . \tag{4.5}$$

This result shows the advantage of using the P_α, Q_α decomposition, because if one had instead used M_α and L_α directly, the third term in (4.5) might have been missed by assuming that $\lambda^3\left(V^{1/2}, M_\alpha^2, |V|^{1/2}\right)$ is of order $\lambda^{3-\epsilon}$. However, we shall see that this term contributes to order λ^2 only and does not contribute to the $\lambda^2 \ln\lambda$ term that we wish to isolate.

Introducing the fourier transform by

$$\hat{g}(k) = (2\pi)^{-1/2}\int dx\, g(x)\, e^{-ikx} \quad ,$$

then we find that

$$\hat{V}_1(k) = -\frac{1}{2}a\,(2\pi)^{1/2}\, e^{-|k|}$$

and V_2 is continuously differentiable with

$$|\hat{V}_2(k) - \hat{V}_2(k')| \leq c\,|k-k'|^\delta \quad ,$$

and hence by Taylor's theorem with remainder

$$\hat{V}(k) = \hat{V}(0) + \frac{1}{2}a\,(2\pi)^{1/2}|k| + ck + 0(k^{1+\delta}) \quad . \tag{4.6}$$

Now using the fact that the fourier transform of $\exp(-b|x|)$ is $(2\pi)^{-1/2}\,2b(b^2+k^2)^{-1}$, the third term in (4.5) becomes

$$\frac{1}{2}\lambda\int dk\left[\hat{V}(k) - \hat{V}(0)\right](2\pi)^{-1/2}\,4\alpha\left[(k^2+\alpha^2)^{-1} - (k^2+4\alpha^2)^{-1}\right] \quad . \tag{4.7}$$

The contribution to the integral in (4.7) outside the region $(-1<k<1)$ is easily seen to be of order $\lambda\alpha^3 = 0(\lambda^4)$. From this region itself, one sees that the ck term in (4.6) contributes zero and the $0(k^{1+\delta})$ term contributes of order $\lambda\,\alpha^{1+1/2\delta} = 0(\lambda^{2+1/2\delta})$. Finally the $|k|$ term yields (neglecting $0(\lambda^4)$ terms) $\lambda a\,\alpha \ln 4 = 0(\lambda^2)$. As claimed, this contributes a term of order λ^2 to α.

The second term in (4.5) is

$$\frac{\lambda^2}{4\alpha}\int dx\,dy\;V(x)\;V(y)\;(e^{-\alpha|x-y|}-1) \quad ,$$

which can be written as

$$\frac{1}{2}\lambda^2\int dk\,(k^2+\alpha^2)^{-1}\left[|\hat{V}(k)|^2 - |\hat{V}(0)|^2\right] \quad . \tag{4.8}$$

Using the expansion (4.6), we have

$$\begin{aligned}|\hat{V}(k)|^2 - |\hat{V}(0)|^2 &= 2\,\mathrm{Re}\left[\hat{V}(0)\,(\hat{V}(k)-\hat{V}(0))\right] + |\hat{V}(k)-\hat{V}(0)|^2\\ &= a(2\pi)^{1/2}\hat{V}(0)|k| + c'k + 0(k^{1+\delta}) \quad .\end{aligned}$$

Thus eq. (4.8) is estimated to be

$$= \lambda^2\,a(2\pi)^{1/2}\,\hat{V}(0)\int_0^1 dk\,k(k^2+\alpha^2)^{-1} + 0(\lambda^{2+\delta})$$

$$= -a\,\lambda^2\ln\lambda\int dx\,V(x) + 0(\lambda^2) \quad .$$

This then proves eq. (4.3).

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REFERENCES

1. H. D. I. Abarbanel, C. Callen, and M. L. Goldberger (unpublished).

2. B. Simon, Ann. Phys. (N.Y.) 97 (1976) 279.

3. W. Greenlee, Bull. A.M.S. 82 (1976) 341-344.

4. W. Greenlee, Proc. Flagstaff Conference on Perturbation Theory (1976).

5. E. Harrel III, Princeton University thesis (1976).

6. B. Simon, Helv. Phys. Acta 43 (1970) 607-630.

7. G. N. Watson, Theory of Bessel Functions, Cambridge University Press, New York, 1922.

8. J. Schwinger, Proc. Nat. Acad. Sci. 47 (1961) 122-129.

9. V. Bargmann, Proc. Nat. Acad. Sci. 38 (1952) 961-966.

10. For a discussion of generalized versions of this inequality see V. Bargmann, Helv. Phys. Acta. 45 (1972) 249-257.