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EXTREMAS OF SU(n) HIGGS POTENTIALS AND SYMMETRY BREAKING PATTERN*

H. Ruegg†

Stanford Linear Accelerator Center
Stanford University, Stanford, California 94305

and

Universite de Geneve, Switzerland

ABSTRACT

It is shown that the most general renormalizable Higgs potential for the adjoint representation ϕ of SU(n) has extremas only if at most two eigenvalues of ϕ are different. This result is used to find the symmetry breaking pattern due to scalar fields in the adjoint and fundamental representations of SU(n).

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†Address after June 30, 1980: Dept. Phys. Theor., 32 Bd. d'Yvoy, CH 1211 Geneve 4, Switzerland.

I. INTRODUCTION

In spontaneously broken gauge theories with elementary scalar fields, the renormalizable Higgs potential V is of degree four. The symmetry breaking pattern is determined by the non-zero vacuum expectation values (VEV) which minimize V.

L. F. Li[1] has solved the minimum problem for Higgs fields belonging to irreducible representations (IR) of SU(n) and O(n). His work needs to be continued in several directions. Scalar fields in more than one IR must be considered. For SU(5), a weakly coupled fundamental IR has been added to the adjoint IR.[2] For SU(n), the fundamental and adjoint IR have been treated in an exact way[3] and for the same group, work has been done for several adjoint IR[4] and several antisymmetric tensors.[5]

In most previous references, terms odd in the adjoint IR have been avoided, by requiring invariance under a discrete symmetry. Apart from simplicity, there is no fundamental reason for dropping them.

In this paper, we study the symmetry breaking pattern for SU(n), the Higgs potential being the most general renormalizable function of the adjoint and fundamental IR, thus including odd terms.

During the process of minimizing the potential, we found a lemma which may be interesting in itself: the most general renormalizable Higgs potential for the adjoint IR ϕ admits extrema only if at most two of its eigenvalues are different. This was known for SU(5)[6] and is stronger than the result quoted in Ref. 7, which refers only to absolute extrema.

We shall prove the lemma in Section II, apply it to the symmetry breaking pattern due to the adjoint IR in Section III and treat the general case, including the fundamental IR, in Section IV. In the appendix, we consider absolute minima and maxima of V.

II. EXTREMUM EIGENVALUES OF THE ADJOINT REPRESENTATION

When the scalar Higgs field ϕ transforms according to the adjoint representation of SU(n), the most general renormalizable potential is

$$V(\phi) = -\frac{1}{2}\mu^2 \mathrm{tr}\phi^2 + \frac{a}{4}(\mathrm{tr}\phi^2)^2 + \frac{b}{2}\mathrm{tr}\phi^4 + d\,\mathrm{tr}\phi^3 \qquad (2.1)$$

The diagonal form of ϕ is

$$\phi_i^j = a_i \delta_i^j \qquad i,j = 1 \ldots n \qquad (2.2)$$

$$\sum_{i=1}^{n} a_i = 0 \qquad (2.3)$$

We prove the following lemma: V admits extremas only if at most two eigenvalues a_i are different.

Proof: A necessary condition for an extremum is that V is extremal with respect to four arbitrarily chosen a_i's, the n-4 others being kept fixed. Hence, we consider

$$F = -\frac{1}{2}\mu^2\varphi + \frac{a}{4}\varphi^2 + \frac{b}{2}\sum_{i=1}^{4} a_i^4 + d\sum_{i=1}^{4} a_i^3 \qquad (2.4)$$

$$\sum_{i=1}^{4} a_i^2 \equiv \varphi \qquad (2.5)$$

$$\sum_{i=1}^{4} a_i = -\sum_{i=5}^{n} a_i \equiv \sigma \qquad (2.6)$$

We choose as independent variables a_1, a_2 and φ, eliminating a_3 and a_4 with the help of (2.5) and (2.6). Then

$$a_{\substack{3\\4}} = -\frac{1}{2}(a_1 + a_2 - \sigma) \pm \frac{1}{2}\left[2\varphi - \sigma^2 - 3a_1^2 - 3a_2^2 - 2a_1a_2 + 2(a_1 + a_2)\sigma\right]^{1/2}$$

$$\sum_{i=1}^{4} a_i^3 = 3(a_1 + a_2)\left[a_1^2 + a_2^2 - \sigma(a_1 + a_2) + \frac{1}{2}\sigma^2 - \frac{1}{2}\varphi\right] + 3\sigma(a_1a_2 + \frac{1}{2}\varphi) - \frac{1}{2}\sigma^3 \tag{2.7}$$

$$\sum_{i=1}^{4} a_i^4 = a_1^4 + a_2^4 + \frac{1}{2}(\varphi - a_1^2 - a_2^2)^2 + \frac{1}{2}(a_1 + a_2 - \sigma)^2\left[2\varphi - \sigma^2 - 3a_1^2 - 3a_2^2 - 2a_1a_2 + 2(a_1 + a_2)\sigma\right]$$

Hence

$$\frac{\partial F}{\partial a_1} = b(a_2 - \sigma)\left[\varphi - \sigma^2 - 6a_1^2 - 2(a_2 - \sigma)(2a_1 + a_2)\right] + d\left[-\frac{3}{2}(\varphi - \sigma^2) + 9a_1^2 + 6a_1a_2 + 3a_2^2 - 3(2a_1 + a_2)\sigma\right] \tag{2.8}$$

$\frac{\partial F}{\partial a_2}$ is obtained from $\frac{\partial F}{\partial a_1}$ by the substitution $a_1 \leftrightarrow a_2$. The solutions of the equations for stationary points

$$\frac{\partial F}{\partial a_1} = \frac{\partial F}{\partial a_2} = 0 \tag{2.9}$$

are the following:

<u>Solution (2, 1, 1)</u>

$$a_1 = a_2 = \sigma + \frac{3}{2}\frac{d}{b} \equiv \frac{1}{4}\sigma + \frac{3}{2}y$$
$$a_{\substack{3\\4}} = \frac{1}{4}\sigma - \frac{3}{2}y \pm \frac{3}{2}(2y_1^2 - y^2)^{1/2} \tag{2.10}$$

$$y \equiv \frac{d}{b} + \frac{1}{2}\sigma \qquad y_1 \equiv \frac{1}{6}(4\varphi - \sigma^2)^{1/2} \tag{2.11}$$

Solution (3, 1)

$$a_1 = a_2 = a_3 = \frac{1}{4}\left[\sigma \pm 2\sqrt{3}\,y_1\right]$$
$$a_4 = \frac{1}{4}\left[\sigma \mp 6\sqrt{3}\,y_1\right] \tag{2.12}$$

Solution (2, 2)

$$a_1 = a_3 = \frac{1}{4}(\sigma + 6y_1)$$
$$a_2 = a_4 = \frac{1}{4}(\sigma - 6y_1) \tag{2.13}$$

We want to show that solution (2, 1, 1) with three different eigenvalues is not an extremum, that is the matrix of the second derivatives has eigenvalues of opposite sign.

In general, we obtain

$$\frac{\partial^2 F}{\partial a_1^2} = 2b(a_2 - \sigma)\left[-6a_1 - 2(a_2 - \sigma) + 3\frac{d}{b}\right] + 18da_1$$

$$\frac{\partial^2 F}{\partial a_2^2} = 2b(a_1 - \sigma)\left[-6a_2 - 2(a_1 - \sigma) + 3\frac{d}{b}\right] + 18da_2$$

$$\frac{\partial^2 F}{\partial a_1 \partial a_2} = b\Big[(\varphi - 3\sigma^2) - 6a_1^2 - 8a_1a_2 - 6a_2^2 + 8\sigma(a_1 + a_2) + 6\frac{d}{b}(2a_1 + 2a_2 - \sigma)\Big] \tag{2.14}$$

$$\frac{\partial F}{\partial \varphi} = -\frac{1}{2}\mu^2 + \frac{a}{2}\varphi + \frac{b}{2}\Big[\varphi - a_1^2 - a_2^2 + (a_1 + a_2 - \sigma)^2 - 3\frac{d}{b}(a_1 + a_2 - \sigma)\Big]$$

$$\frac{\partial^2 F}{\partial \varphi \partial a_1} = b(a_2 - \sigma) - \frac{3d}{2}$$

Solution (2, 1, 1)

From (2.10) and (2.14), we get

$$\frac{\partial^2 F}{\partial a_1^2} = \frac{\partial^2 F}{\partial a_2^2} = \frac{\partial^2 F}{\partial\varphi\partial a_1} = \frac{\partial^2 F}{\partial\varphi a_2} = 0 \tag{2.15}$$

$$\frac{\partial^2 F}{\partial a_1 \partial a_2} = 54(y_0^2 - y^2) \tag{2.16}$$

$$y_0 = \frac{1}{6\sqrt{3}} (4\varphi - \sigma^2)^{1/2} = \frac{1}{\sqrt{3}}\ y_1 \tag{2.17}$$

For $y^2 = y_0^2$, $a_1 = a_2 = a_3$ or $a_1 = a_2 = a_4$. This proves the lemma.

In the appendix, we directly compute F/b for the three solutions. We show that the absolute maximum is always of the form (3, 1). For SU(n), this means that at the absolute maximum, n - 1 eigenvalues are equal.

For completeness, we give some of the second derivatives for the two other solutions.

Solution (3, 1)

From (2.12) and (2.14) we find

$$\left(\frac{\partial^2 F}{\partial a_i \partial a_j}\right)_{i,j=1,2} = 18\, b\, y_0(-y_0 \pm y)\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \tag{2.18}$$

The eigenvalues of the matrix are +3 and 1. For b > 0, and $|y| < y_0$ both solutions (±) are maxima. For $|y| > y_0$, one solution becomes a minimum, the other one being the absolute maximum.

Solution (2, 2)

From (2.13) and (2.14), one has

$$\frac{\partial^2 F}{\partial a_1 \partial a_2} = 0$$

$$\frac{\partial^2 F}{\partial a_1^2} = 18\, b\, y_1 (y_1 + y) \qquad (2.19)$$

$$\frac{\partial^2 F}{\partial a_2^2} = 18\, b\, y_1 (y_1 - y)$$

This is a minimum for $b > 0$, unless

$$|y| > y_1 \qquad (2.20)$$

where it is no longer an extremum.

III. SYMMETRY BREAKING DUE TO THE ADJOINT REPRESENTATION

Spontaneous symmetry breaking will occur if the vacuum expectation value (VEV) of the scalar field ϕ minimizes the potential V of Eq. (2.1).

According to the lemma proved in the preceding section, an extremum is attained when at most two eigenvalues of ϕ, say a_1 and a_2, are different. Suppose a_1 occurs n_1 times, a_2 n_2 times, with $n_1 + n_2 = n$. The symmetry breaking pattern, i.e., the subgroup which leaves VEV invariant, will not depend on the norm of ϕ, but on the values of n_1 and n_2 which minimize

$$I = \frac{b}{2}(n_1 a_1^4 + n_2 a_2^4) + d(n_1 a_1^3 + n_2 a_2^3) \qquad (3.1)$$

subject to the constraints

$$n_1 + n_2 = n$$

$$n_1 a_1 + n_2 a_2 = 0 \qquad (3.2)$$

$$n_1 a_1^2 + n_2 a_2^2 = \rho^2$$

Solve (3.2) and get, up to a sign

$$a_1 = \left(\frac{n_2}{nn_1}\right)^{1/2} \rho$$
$$a_2 = -\left(\frac{n_1}{nn_2}\right)^{1/2} \rho \qquad (3.3)$$

so that

$$I = \frac{b\rho^4}{n}\left[\frac{1}{2}(x^2 + 1) - \frac{d}{b\rho}\, n^{1/2} x\right] \qquad (3.4)$$

$$x = \frac{n_1 - n_2}{\sqrt{n_1 n_2}} \qquad (3.5)$$

For $b < 0$, I is a decreasing function of x and the minimum is obtained for the maximum value of $|x|$. The subgroup left after symmetry breaking is

$$SU(n - 1) \times U(1) \qquad (3.6)$$

For $b > 0$, I is minimum for

$$x = \frac{dn^{1/2}}{b\rho} \qquad (3.7)$$

Hence, as d/b increases from zero to infinity, x goes through all values allowed by (3.2) and the subgroups are for n even

$$SU(n/2) \times SU(n/2) \times U(1)$$
$$SU(n/2 + 1) \times SU(n/2 - 1) \times U(1) \qquad (3.8)$$
$$\vdots$$
$$SU(n-1) \times U(1)$$

and for n odd:

$$\begin{array}{l} SU\left(\frac{n+1}{2}\right) \times SU\left(\frac{n-1}{2}\right) \times U(1) \\ SU\left(\frac{n+3}{2}\right) \times SU\left(\frac{n-3}{2}\right) \times U(1) \\ \vdots \\ SU\,(n-1) \times U(1) \end{array} \tag{3.9}$$

IV. SYMMETRY BREAKING DUE TO THE ADJOINT AND FUNDAMENTAL REPRESENTATION

We now consider the most general renormalizable Higgs potential function of the scalar fields H belonging to the fundamental representation of SU(n) and ϕ belonging to the adjoint representation.

$$\begin{aligned} V(\phi,H) = &-\frac{1}{2}\mu^2 \operatorname{tr}\phi^2 + \frac{a}{4}(\operatorname{tr}\phi^2)^2 - \frac{1}{2}\nu^2 H^+ H + \frac{\lambda}{4}(H^+H)^2 \\ &+ \alpha \operatorname{tr}\phi^2 H^+H + \frac{b}{2}\operatorname{tr}\phi^4 + d\operatorname{tr}\phi^3 + \beta H^+\phi^2 H + \gamma H^+\phi H \end{aligned} \tag{4.1}$$

The special case $d = \gamma = 0$ has been treated in detail in Ref. 3. As before, if we are only interested in the symmetry breaking pattern, the norms of ϕ and H as well as the phase of H are irrelevant. We again diagonalize ϕ (see Eq. (2.2)). Then we are left with

$$F = \frac{b}{2}\sum_{i=1}^{n} a_i^4 + d\sum_{i=1}^{n} a_i^3 + \sum_{i=1}^{n} H_i^2 a_i (\beta a_i + \gamma) \tag{4.2}$$

$$\sum_{i=1}^{n} a_i = 0 \qquad \sum_{i=1}^{n} a_i^2 = \rho^2 \tag{4.3}$$

We first minimize with respect to H_i. It is easy to see that at the stationary point we can put all H_i equal to zero except one, say H_n

$$H_i = 0 \qquad i = 1 \cdots n-1 \tag{4.4}$$

Then

$$F = \frac{b}{2} \sum_{i=1}^{n} a_i^4 + d \sum_{i=1}^{n} a_i^3 + H^+ H\, a_n (\beta a_n + \gamma) \tag{4.5}$$

We next minimize the first two terms with respect to a_i, $i \neq n$. We again apply the lemma of Section II and find that at most two eigenvalues a_1 and a_2, occurring n_1 and n_2 times, $(n_1 + n_2 = n - 1)$ can be different. Furthermore, from the appendix, we know that for $b < 0$, the absolute minimum is for $n_1 \geq n - 2$. Since $H_n \neq 0$, the subgroup structure after symmetry breaking is now

$$\begin{aligned} &SU(n_1) \times SU(n_2) \times U(1) && n_1 n_2 \neq 0 \\ &SU(n-1) && n_1 n_2 = 0 \\ &n_1 + n_2 = n - 1 && \end{aligned} \tag{4.6}$$

Solving (4.3) one obtains

$$\begin{aligned} a_1 &= \left\{ -a_n \pm \left(\frac{n_2}{n_1}\right)^{1/2} \left[(n-1)\rho^2 - n a_n^2\right]^{1/2} \right\} \frac{1}{n-1} \\ a_2 &= \left\{ -a_n \mp \left(\frac{n_1}{n_2}\right)^{1/2} \left[(n-1)\rho^2 - n a_n^2\right]^{1/2} \right\} \frac{1}{n-1} \end{aligned} \tag{4.7}$$

We then get for F

$$\begin{aligned} F = &\frac{b}{2(n-1)^3} \Big[n\,(n^2 - 3n + 3)\, a_n^4 + b\, a_n^2 f \\ &+ (1 + x^2)\, f^2 \pm 4 x a_n f^{3/2} \Big] \\ &+ \frac{d}{(n-1)^2} \Big[n(n-2)\, a_n^3 - 3 a_n f \mp x f^{3/2} \Big] \\ &+ H^+ H\, (\beta a_n^2 + \gamma a_n) \end{aligned} \tag{4.8}$$

$$x = \frac{n_1 - n_2}{\sqrt{n_1 n_2}} \; ; \qquad f = (n - 1)\rho^2 - n a_n^2 \tag{4.9}$$

F is a quadratic function of x. For $b < 0$, $n_1 \geq n - 2$. For $\beta < 0$, x and a_n take their largest possible values, hence $n_1 = n - 1$. The subgroup is then

$$b < 0,\ \beta < 0: \quad SU(n - 1) \tag{4.10}$$

For $b < 0$ and $\beta > 0$, we get two possibilities, according to the ratio γ / β:

$$b < 0,\ \beta > 0 \qquad SU(n - 2) \times U(1) \text{ or } SU(n - 1) \tag{4.11}$$

For $b > 0$, the situation is more complicated. However, the special case $\beta = \gamma = 0$ has been dealt with in Section III. $\beta \neq 0$ or $\gamma \neq 0$ with $H_n \neq 0$ decreases the rank of the subgroup by one. The special case $d = \gamma = 0$ was treated in Ref. 3, with the following result: the subgroups are for $b < 0$, $\beta < 0$: $SU(n - 1)$; for $b < 0$, $\beta > 0$: $SU(n - 2) \times U(1)$; for $b > 0$, $\beta > 0$: $SU(n/2) \times SU(n/2 - 1) \times U(1)$, n even, $SU(n - 1)/2 \times SU(n - 1)/2 \times U(1)$, n odd; for $b > 0$, $\beta < 0$, as the ratio β / b increases from 0 to ∞: $SU(n/2) \times SU(n/2 - 1) \times U(1)$, $SU(n/2 + 1) \times SU(n/2 - 2) \times U(1)$, . . . , $SU(n - 1)$, n even, $SU(n + 1)/2 \times SU(n - 3/2) \times U(1)$, $SU(n + 3)/2 \times SU(n - 5)/2 \times U(1)$, . . . , $SU(n - 1)$, n odd.

In the general case, we get the "union" of the special cases, that is, as d, β and γ vary, all possibilities are realized. The results are given in the table.

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TABLE 1

Subgroups of SU(n) which leave invariant the minimum of

$$F = b/2 \sum_{i=1}^{n} a_i^4 + d \sum_{i=1}^{n} a_i^3 + \sum_{i=1}^{n} H_i^2\, a_i(\beta a_i + \gamma) \quad ,$$

where a_i are the eigenvalues of the adjoint IR and H belongs to the fundamental IR of SU(n).

b	d/b	β	γ	Subgroup
< 0	arbitrary	0	0	SU(n - 1) × U(1)
> 0	0	0	0	$SU\left(\frac{n}{2}\right) \times SU\left(\frac{n}{2}\right) \times U(1)^*$
	⋮	0	0	⋮
	∞	0	0	SU(n - 1) × U(1)
	0	0	0	$SU\left(\frac{n+1}{2}\right) \times SU\left(\frac{n-1}{2}\right) \times U(1)^{**}$
	⋮	0	0	⋮
	∞	0	0	SU(n - 1) × U(1)
< 0	arbitrary	< 0	arbitrary	SU(n - 1)
	0	> 0	0	SU(n - 2) × U(1)
	≠ 0	> 0	≠ 0	SU(n - 2) × U(1) or SU(n - 1)
> 0	variable†	variable†	variable†	$SU\left(\frac{n}{2}\right) \times SU\left(\frac{n}{2} - 1\right) \times U(1)^*$
	variable	variable	variable	$SU\left(\frac{n}{2} + 1\right) \times SU\left(\frac{n}{2} - 2\right) \times U(1)^*$
	⋮	⋮	⋮	⋮
	variable	variable	variable	SU(n - 1)
	variable	variable	variable	$SU\left(\frac{n+1}{2}\right) \times SU\left(\frac{n-3}{2}\right) \times U(1)^{**}$
	variable	variable	variable	$SU\left(\frac{n+3}{2}\right) \times SU\left(\frac{n-5}{2}\right) \times U(1)^{**}$
	⋮	⋮	⋮	⋮
				SU(n - 1)

*n even **n odd †see text

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†Address after June 30, 1980: Universite de Geneve, Dept. Phys. Theor., 32 Bd. d'Yvoy, CH 1211 Geneve 4, Switzerland.

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APPENDIX

We compute the value of the function

$$G = \frac{1}{2} \sum_{i=1}^{4} a_i^4 + d/b \sum_{i=1}^{4} a_i^3 \tag{A1}$$

at the points where G is stationary. At these points, only three eigenvalues can be different, and if they are different, they must obey

$$\begin{aligned} a_1 &= a_2 \\ a_1 + a_3 + a_4 &= -3d/2b \end{aligned} \tag{A2}$$

We want to find the conditions for absolute minima and maxima of G. These will be necessary conditions for SU(n) if we do not fix the trace

$$\sum_{i=1}^{4} a_i = \sigma \tag{A3}$$

Furthermore, let φ be the norm

$$\sum_{i=1}^{4} a_i^2 = \varphi \tag{A4}$$

Using (A2) to (A4), we can solve for a_i. We then find the three solutions of Section II. Notice that for solution (3.1), the two possible signs (±) give different solutions.

Then we find

$$\Delta G^{\pm} \equiv G^{\pm}(3,1) - G(2,2) = 81\, y_0^3 \left(\frac{3}{2} y_0 \mp y\right) \tag{A5}$$

so that for $y > 0$, $G^-(3,1)$ is the absolute maximum

$y < 0$, $G^+(3,1)$ is the absolute maximum for $y > \frac{3}{2} y_0$, $G^+(3,1)$ is the absolute minimum

$y < -\frac{3}{2} y_0$, $G^-(3,1)$ is the absolute minimum

otherwise, G(2,2) is the absolute minimum. For completeness, we compute

$$\Delta G \equiv G(2,1,1) - G(2,2) = 9/8\ (y_1^2 - y^2)\ H \tag{A6}$$

$$H = \varphi + 27d^2/b^2 + 15\ \sigma\ d/b + 3/2\ \sigma^2$$

$$H = 0 \text{ for } y = 2/9\,\sigma \pm 1/18\ (7\ \sigma^2 - 12\varphi)^{1/2} > y_1$$

Hence

$$\Delta G \geq 0 \text{ for } y^2 \leq y_1^2 \tag{A7}$$

Remember from Eq. (2.10) that G(2,1,1) is not defined for $y^2 > y_1^2$.

Finally,

$$\delta G^{\pm} \equiv G^{\pm}(3,1) - G(2,1,1)$$

$$= \frac{81}{8}\ (y \mp y_0)\ \left[8y\, y_0^2 - (y \pm y_0)(3y_0^2 + y^2)\right] \tag{A8}$$

one sees that

$$\delta G^{\pm}(-y) = \delta G^{\mp}(y) \tag{A9}$$

$$\delta G^{\mp}(y) \gtrless \delta G^{\pm}(y) \text{ for } y \gtrless 0 \tag{A10}$$

hence

$$\begin{aligned} &\delta G^{(-)}(y) > 0 \qquad 0 \leq y \leq y_1 \\ &\delta G^{(+)}(y) > 0 \qquad -y_1 \leq y \leq 0 \end{aligned} \tag{A11}$$

Of course, (A7) and (A11) agree with the stronger result of the lemma of Section II.