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FORMAL LINEARIZATION OF LIE ALGEBRAS OF VECTOR FIELDS NEAR AN INVARIANT SUBMANIFOLD*

by

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I. INTRODUCTION

In [5], we showed that the formal aspects of classical problem of linearization of a vector field near a singular point by a change of variables (and its generalization to Lie algebras and groups) suggested by Palais and Smale, have a simple foundation in terms of cohomology of Lie algebras. One will notice in this treatment a strong similarity to the work of Kodaira and Spencer on deformations of geometric structures.

The purpose of this paper is to exploit this similarity more systematically, and to use this to treat two other problems of geometric interest:

(a) Linearization of subalgebras of filtered Lie algebras.

(b) Linearization of Lie algebras of vector fields near an invariant submanifold.

We will show that problem (a) only requires a minor modification of the formalism of [5]. However, problem (b) is not quite so simple, even on a formal, algebraic level. However, we will show that it can be treated as a problem of deformation of Lie algebra homomorphisms, for which there is available a cohomological formalism [4, 7].

II. LINEARIZATION THEOREMS

Let $\underset{\sim}{L}$ be a Lie algebra. A <u>filtration</u> on L is defined by a sequence $\underset{\sim}{L}^1, \underset{\sim}{L}^2, \ldots$ of subalgebras of $\underset{\sim}{L}$ such that:

$$\underset{\sim}{L} = \underset{\sim}{L}^1 \supset \underset{\sim}{L}^2 \supset \cdots \tag{2.1}$$

$$\left[\underset{\sim}{L}^r, \underset{\sim}{L}^s\right] \subset \underset{\sim}{L}^{r+s-1} \tag{2.2}$$

$$\text{for } r, s \geq 1$$

See [2] for a description of the general properties of filtered Lie algebras.

The following problem will be discussed in this section: Let $\underset{\sim}{K}$ be a given subalgebra of $\underset{\sim}{L}$. Can one find an $X \in \underset{\sim}{L}$ such that

$$\operatorname{Exp}(\operatorname{Ad}\underset{\sim}{X})(\underset{\sim}{K}) \cap \underset{\sim}{L}^{2} = (0) \quad ? \tag{2.3}$$

In fact, we will be considering a more restrictive problem here; we will attempt to exhibit X <u>formally</u> (i.e., without discussion of the convergence) as a limit

$$\ldots \operatorname{Exp}(\operatorname{Ad}X_2)\ \operatorname{Exp}(\operatorname{Ad}X_1)$$

where (X_r) is a sequence of elements of L, with each X_r in $\underset{\sim}{L}^r$.

Now $\underset{\sim}{L}^2$ is an ideal in $\underset{\sim}{L}$. Suppose that the homomorphism $\underset{\sim}{L} \to V^1 = \underset{\sim}{L}/\underset{\sim}{L}^2$ splits, i.e., there is a subalgebra $\underset{\sim}{H}$ of $\underset{\sim}{L}$ such that

$$\underset{\sim}{L} = \underset{\sim}{L}^2 + \underset{\sim}{H}, \quad \underset{\sim}{H} \cap \underset{\sim}{L}^2 = (0)$$

(We will not consider the more general case in this paper.)

Let ϕ_1 be the projection map $\underset{\sim}{L} \to \underset{\sim}{H}$. For $r > 1$, define $V^r = \underset{\sim}{L}^r/\underset{\sim}{L}^{r-1}$, and let π_r be the projection map: $\underset{\sim}{L}^r \to V^r$. Notice that ϕ_1 is a homomorphism of $\underset{\sim}{L}$ into $\underset{\sim}{H}$. Notice also that $[\underset{\sim}{K}, \underset{\sim}{L}^r] \subset \underset{\sim}{L}^r$ for each $r \geq 1$, hence $\operatorname{Ad}\underset{\sim}{K}$ passes to the quotient to define a representation, denoted by ϕ_r, of $\underset{\sim}{K}$ by linear transformation in V^r.

Let us begin the process of "linearizing" $\underset{\sim}{K}$. For $Y \in K$ define:

$$\omega_2(Y) = \pi_2\big(Y - \phi_1(Y)\big)$$

Consider $\omega_2 : \underset{\sim}{K} \to V^2$ as a 1-cochain of K with respect to the representation ϕ_2 of $\underset{\sim}{K}$ in V^2. (For the notations of Lie algebra cohomology theory that we use, see [4].)

Lemma 2.1 $d\omega_2 = 0$, i.e., ω_2 is a 1-cocycle.

Proof For $Y_1, Y_2 \; X \in \underset{\sim}{K}$,

$$d\omega_2(Y_1, Y_2) = \phi_2(Y_1)\big(\omega_2(Y_2)\big)$$

$$- \phi_2(Y_2)\big(\omega_2(Y_1)\big) - \omega_2\big([Y_1, Y_2]\big)$$

$$= \phi_2(Y_1)\Big(\pi_2\big(Y_2 - \phi_1(Y_2)\big)\Big)$$

$$- \phi_2(Y_2)\Big(\pi_2\big(Y_1 - \phi_1(Y_1)\big)\Big)$$

$$- \pi_2\Big([Y_1, Y_2] - \phi_1\big([Y_1, Y_2]\big)\Big)$$

$$= \pi_2\big([Y_1, Y_2 - \phi_1(Y_2)] - [Y_2, Y_1 - \phi_1(Y_1)]$$

$$- [Y_1, Y_2] + [\phi_1,(Y_1), \phi_1(Y_2)]\big)$$

$$= \pi_2\big([Y_1, Y_2] - [Y_1, \phi_1(Y_2)] + [Y_2, \phi_1(Y_1)] + [\phi_1(Y_1), \phi_1(Y_2)]\big)$$

$$= \pi_2\big([Y_1 - \phi_1(Y_1), Y_2 - \phi_2(Y_2)]\big) = 0,$$ since both $Y_1 - \phi_1(Y_1)$, and $Y_2 - \phi_2(Y_2)$ are in $\underset{\sim}{L}^2$, and $[\underset{\sim}{L}^2, \underset{\sim}{L}^2] \subset \underset{\sim}{L}^3$.

The cohomology class in $H^1(\phi_2)$ determined by ω_2 is the <u>first obstruction</u> to linearizing K. Suppose it is zero, i.e., there is an element $X_2 \in \underset{\sim}{L}^2$ such that:

$$d\pi_2(X_2) = \omega_2, \text{ or}$$

$$\omega_2(Y) = \phi_2(Y)\big(\pi_2, (X_2)\big) \quad \text{for } Y \in \underset{\sim}{K}$$

$$= \pi_2\big([Y, X_2]\big)$$

Then

$$\operatorname{Exp}(\operatorname{Ad} X_2)(Y) = Y + [X_2, Y] + \frac{[X_2, [X_2, Y]]}{2!} + \dots \quad \omega_2\left(\operatorname{Exp}(\operatorname{Ad} X_2)(Y)\right)$$

$$= \pi_2\left(\operatorname{Exp}(\operatorname{Ad} X_2)(Y) - \phi_1\left(\operatorname{Exp}(\operatorname{Ad} X_2)(Y)\right)\right)$$

$$= \pi_2\left(Y + [X_2, Y] + \frac{[X_2, [X_2, Y]]}{2!} + \dots - \phi_1(Y)\right)$$

$$= \pi_2\left(Y - \phi_1(Y)\right) + \pi_2\left([X_2, Y]\right) = 0,$$

$$\text{i.e., } \operatorname{Exp}(\operatorname{Ad} X_2)(Y) - \phi_1(Y) \in \underset{\sim}{L}^3 \text{ for all } Y \in \underset{\sim}{K} \qquad (2.4)$$

Now replace $\underset{\sim}{K}$ by $\underset{\sim}{K}^2 = \operatorname{Exp}(\operatorname{Ad} X_2)(\underset{\sim}{K})$

Then, if Eq. (2.4) is satisfied, we have:

$$Y - \phi_1(Y) \in \underset{\sim}{L}^3 \quad \text{for } Y \in \underset{\sim}{K}^2 \qquad (2.5)$$

Define: $\omega_3(Y) = \pi_3\left(Y - \phi_1(Y)\right)$.

A similar reasoning shows that ω_3, when interpreted as a 1-cochain defined by the representation ϕ_3, is a 1-cocyle. Its cohomology class is the second obstruction to linearizing $\underset{\sim}{K}$. If it is zero, there is an element $X_3 \in L^3$.

$$\omega_3(Y) = \pi_3\left([Y, X_3]\right) \quad \text{for} \quad Y \in \underset{\sim}{K}^2 .$$

Define: $\underset{\sim}{K}^3 = \operatorname{Exp}(\operatorname{Ad} X_3)(\underset{\sim}{K}^2)$.

Notice that, since $X_3 \in \underset{\sim}{L}^3$

$$0 = \omega_2(Y) = \omega_2\left(\operatorname{Exp}(\operatorname{Ad} X_3)(Y)\right) \quad \text{for} \quad Y \in \underset{\sim}{K}^2 .$$

A similar calculation then shows that

$$\omega_3(K^3) = 0$$

We can now continue the process, obtaining a sequence $\underset{\sim}{K} = \underset{\sim}{K}^1, \underset{\sim}{K}^2, \underset{\sim}{K}^3 \ldots$ of subalgebras of L.

Thus, we have proved:

Theorem 2.1 If $H^1(\phi_r) = 0$ for $r = 2,3,\ldots$, then in sequence $\underset{\sim}{K}^1 = \underset{\sim}{K}, \underset{\sim}{K}_2^2, \ldots$ of subalgebras of L. Each K^r is conjugate to K^{r-1} within the subgroup Exp Ad $\underset{\sim}{L}^r$ of the group of inner automorphisms of $\underset{\sim}{L}$. Also, for $Y \in \underset{\sim}{K}^r$,

$$Y - \phi_1(Y) \in \underset{\sim}{L}^{r+1}$$

Notice that we will have succeeded in "linearizing" K, i.e, showing that it is conjugate to a subalgebra of H, if

$$\underset{\sim}{L}^r = 0 \quad \text{for } r \text{ sufficiently large} \tag{2.6}$$

Another hypothesis that will guarantee this linearization is that the "infinite product" .... Exp(Ad X_3) Exp(Ad X_2) converges to an element of the group on inner automorphisms of L. However, there is another more general, and interesting, condition that may be satisfied. Suppose that the "limit" (as explained in [3], chapter 11) of the sequence of subalgebras $\underset{\sim}{K}^1, \underset{\sim}{K}^2, \ldots$ is a subalgebra $\underset{\sim}{K}^\infty$ of $\underset{\sim}{L}$. This means that : Whenever a sequence $Y_1, Y_2, \ldots$, each $Y_r \in \underset{\sim}{K}^r$, converges to Y, then Y belongs to $\underset{\sim}{K}^\infty$. Then

$$Y - \phi_1(Y) = \lim_{r \to \infty} Y_r - \phi_1(Y_r) = 0\,,$$

i.e., the limit algebra K^∞ is a subalgebra of H, hence is "linearized." Now, as explained in [4], there is a close relation between this idea of limit of subalgebras, the Inonu-Wigner "contraction" idea, and the idea of "deformation" of subalgebra, as studied by Kodaira-Spencer, Gernstenhaber, and Nijennuis-Richardson. Thus, we may conjecture that (if the cohomology obstructions vanish) even if the subalgebra itself is not linearizable, one of its contractions is.

III CONSTRUCTION OF FILTERED LIE ALGEBRAS

Let $\underset{\sim}{G}$ be a Lie algebra, and let $\underset{\sim}{L}$ be a subalgebra. We define subspaces $\underset{\sim}{L}^r$, $r = 1,2, \ldots,$ of $\underset{\sim}{L}$, with

$$\underset{\sim}{L} = \underset{\sim}{L}^1 \supset \underset{\sim}{L}^2 \supset \underset{\sim}{L}^3 \ \ldots . \tag{3.1}$$

as follows:

$\underset{\sim}{L}^2$ consists of the elements $X \in \underset{\sim}{L}$ such that $[\underset{\sim}{X}, \underset{\sim}{G}] \subset \underset{\sim}{L}$.

$\underset{\sim}{L}^3$ consists of the elements $X \in \underset{\sim}{L}$ such that $\left[\underset{\sim}{G}, [\underset{\sim}{G}, X]\right] \subset \underset{\sim}{L}$.

In general, define $\underset{\sim}{L}^r$ by induction as the set of elements $X \in \underset{\sim}{L}^{r-1}$ such that:

$$[\underset{\sim}{G}, X] \subset \underset{\sim}{L}^{r-1}$$

<u>Lemma 3.1</u> $[\underset{\sim}{L}^r, \underset{\sim}{L}^s] \subset \underset{\sim}{L}^{r+s-1}$, i.e, the sequence 3.1 forms a filtered Lie algebra.

<u>Proof</u>: Proceed by induction on the total degree $r+s$. Suppose $Y \in \underset{\sim}{G}$. Then

$$\left[Y, [\underset{\sim}{L}^r, \underset{\sim}{L}^s]\right] \subset \left[[Y, \underset{\sim}{L}^r], L^s\right] + \left[\underset{\sim}{L}^r, [Y, \underset{\sim}{L}^s]\right]$$

$$\subset [\underset{\sim}{L}^{r-1}, \underset{\sim}{L}^s] + [\underset{\sim}{L}^r, \underset{\sim}{L}^{s-1}]$$

$$\subset L^{r+s-2}, \text{ by induction hypothesis.}$$

This shows that $\left[\underset{\sim}{G}, [\underset{\sim}{L}^r, \underset{\sim}{L}^s]\right] \subset \underset{\sim}{L}^{r+s-2}$, which shows that $[\underset{\sim}{L}^r, \underset{\sim}{L}^s] \subset \underset{\sim}{L}^{r+s-1}$. Let G and L be connected Lie groups whose Lie algebra is $\underset{\sim}{G}$ and $\underset{\sim}{L}$.

Lemma 3.2 Suppose that $\underset{\sim}{L}$ has no non-zero ideals that are also ideals in $\underset{\sim}{G}$. (Geometrically, this means that G acts almost effectively on G/L, i. e, the set of elements $g \in G$ that acts as the identity on G/L as discrete.) Then, if $\underset{\sim}{L}^{r-1} \neq 0$, $\underset{\sim}{L}^{r-1} \neq \underset{\sim}{L}^{r}$.

Proof: If $\underset{\sim}{L}^{r-1} = \underset{\sim}{L}^{r}$, then $[\underset{\sim}{G}, \underset{\sim}{L}^{r-1}] \subset \underset{\sim}{L}^{r-1}$, i. e, $\underset{\sim}{L}^{r-1}$ is an ideal of G.

Now, let M be the coset space G/L. The action of G on M defines, as usual in Lie group theory, an infinitesimal action of G, i. e., a homomorphism of $\underset{\sim}{G}$ into the Lie algebra (under Jacobi bracket) V(M) of vector fields on M. Each element $X \in \underset{\sim}{G}$ then determines a vector field, i. e., an element of V(M), that we also denote by X. Let p_o be the identity coset. Then,

$$X(p_o) = 0 \quad \text{for} \quad X \in \underset{\sim}{L}.$$

Let V^r, $r = 1, 2, \ldots$ be the set of elements $X \in V(M)$ whose coefficients all vanish to at least the r-th order at p_o.

Lemma 3.3 $\underset{\sim}{L}^r \subset V^r$, for all r.

Proof: Let $(x_1, \ldots x_n) = x$ be a coordinate system for M valid in a neighborhood of p_o with $x(p_o) = 0$. Proceed by induction on r. Since $\underset{\sim}{L}^r \subset L^{r-1}$, we know that $\underset{\sim}{L}^r \subset V^{r-1}$.

Let $X \in \underset{\sim}{L}^r$. About p_o, it can be written in the form

$$X = A_1 \frac{\partial}{\partial x_1} + \ldots + A_n \frac{\partial}{\partial x_n}$$

The coefficients $A_1, \ldots A_n$ vanish to (r-1)-st order at $x = 0$. Since G acts transitively on M, the coordinate vector fields $\partial/\partial x_1, \ldots, \partial/\partial x_n$ can, in a neighborhood of p_o, be written in terms of vector fields of $\underset{\sim}{G}$, i. e.,

$$\frac{\partial}{\partial x_1} = f_1 X_1 + \ldots + f_m X_m, \quad \text{with } X, \ldots, X_m \in \underset{\sim}{G}.$$

Now $[X_1, X], \ldots, [X_m, X] \in \underset{\sim}{L}^{r-1}$, since $X \in \underset{\sim}{L}^r$. Hence also

$$\left[\frac{\partial}{\partial x_1}, X\right] \in I^{t-1}.$$

But, this equals also:

$$\frac{\partial A_1}{\partial x_1} \frac{\partial}{\partial x_1} + \ldots + \frac{\partial A_n}{\partial x_1} \frac{\partial}{\partial x_1} ,$$

which must vanish to order (r-1) at $x = 0$. A similar statement is true for $\partial/\partial x_2, \ldots, \partial/\partial x_n$. This implies that the coefficients $A_1, \ldots A_n$ vanish at least to order r at $x = 0$, i.e., $X \in V^r$, q.e.d.

Conversely, if $\underset{\sim}{G}$ is a Lie algebra of vector fields on a manifold M, if L is the isotropy subalgebra of G at a point $p_0 \in M$, then if we define $\underset{\sim}{L}^r = V^r \cap \underset{\sim}{L}$, this defines a filtration of L, to which we can apply the conjugacy arguments of Section II, and deduce, from the abstract theorem of Section II, the results that under certain conditions, Lie algebras of vector fields $\underset{\sim}{K}$ can be linearized by a change of coordinate (perhaps, if $\underset{\sim}{G}$ is infinite dimensional, requiring a formal power series definition, whose convergence is still unknown) about a common zero point for the elements of $\underset{\sim}{K}$. This returns us to the treatment given in [5].

As an example, suppose that $\underset{\sim}{K}$ is one-dimensional, generated by a single element X. The "cohomology groups" take a very simple form, of course: Suppose ϕ is a representation of $\underset{\sim}{K}$ on a vector space V. Let $\omega: \underset{\sim}{K} \rightarrow V$ be a 1-cochain. It is automatically a 1-cocycle, since $\underset{\sim}{K}$ is one-dimensional. It cobounds if and only if there is a vector $v \in V$ such that

$$\omega(X) = \phi(X)(v) ,$$

i.e., the first cohomology group is zero if and only if $\phi(X)$ maps V onto V, so that if V is finite dimensional, $\phi(X)$ must be one-one.

For example, consider the case where $\underset{\sim}{G}$ is Lie algebra $V(M)$ itself, and L is the subalgebra of those vector fields that vanish at p_o. Suppose $X \in \underset{\sim}{L}$ is of the form

$$X = A_1 \frac{\partial}{\partial x_1} + \dots + A_n \frac{\partial}{\partial x_n}$$

with $A_i(0) = 0$ for $i = 1 \dots, n$.

Suppose that Taylor expansion of A_i about $x = 0$ is of the form:

$$A_i(x) = \sum_{i,j} \lambda_{ij} x_i \frac{\partial}{\partial x_j} + \dots$$

It is readily verified that AdZ acting in V^r/V^{r-1} is one-one if the matrix (λ_{ij}) is diagonalable, and if its eigenvalues have positive real parts. The problem of linearization of X by a change of variable is, of course, a classical problem first considered by Poincaré, and brought to definitive form by S. Sternberg (see [8], and the references quoted there).

IV. FILTRATIONS DEFINED BY SUBMANIFOLDS

First, we will present an algebraic construction, then explain how it applies to a problem (but not the most general) of "linearizing" a Lie algebra of vector fields near an invariant submanifold.

Let F be an algebra over the real numbers, whose elements we denote by f, g, etc. Let V be the Lie algebra of derivations of F. Elements of V will be denoted by X, Y, and the action of $X \in V$ and $f \in F$ by $X(f) \in F$. Let F^1 be a subalgebra of F, and let V^1 be the subalgebra of V consisting of the elements $X \in V(F)$ such that:

$$X(F^1) \subset F^1$$

Define F^r as the subalgebra of polynomials of degree $\geq r$ in the elements of F^1. Then,

$$F^r \cdot F^s \subset F^{r+s}$$

Define: $V^r = \{X \in V: X(F^1) \subset F^r\}$

Then,

$$V^r(F^s) \subset F^{r+s-1}$$

Now, consider $X \in V^r$, $Y \in V^s$, $f \in F^t$

$$\begin{aligned}[X,Y](f) &= X\big(Y(f)\big) - Y\big(X(f)\big) \\ &\in X(F^{s+t-1}) + Y(F^{r+t-1}) \subset F^{r+s+t-2}\end{aligned}$$

Hence, we have proved

<u>Lemma 4.1</u> $[V^r, V^s](F^t) \subset F^{r+s+t-1}$

and

<u>Lemma 4.2</u> $[V^r, V^s] \subset V^{r+s-1}$

Thus $V^1 \supset V^2 \supset \ldots$ forms a filtered Lie algebra to which we can apply the general procedure given in Section II.

The geometric situation that we have in mind can be described as follows: Let M be a manifold, and let $F\big(=F(M)\big)$ be the algebra of C^∞ real valued functions. Then $V\big(=V(M)\big)$ is the Lie algebra of vector fields on M. Suppose F^1 is a subalgebra of F, and suppose N is a submanifold of M defined as the set of points of M where all the functions of F vanish. Then, V^r consists of vector fields that are tangent to N to the r-th order, but does not contain all such vector fields (unless N reduces to a single point). To see what is involved in this point, suppose that $M = R^2$, the Euclidean plane, with x,y the Euclidean coordinate functions. Suppose that F^1 is the subalgebra of F generated by x, so that N is the plane $x = 0$. Suppose $f \in F^r$, then

$$X(x) = a_r x^r + \ldots, \quad \text{i.e.,}$$
$$X = (a_r x^r + \ldots) \frac{\partial}{\partial x} + B \frac{\partial}{\partial y},$$

where B is any function $B(x,y)$, and the coefficients $A_r, \ldots$ are real numbers. Of course, this is not the most general sort of vector field that is tangent to N to the r-th order, since it omits those of the type:

$$X = \left(A_r(y)\, x^r + \ldots\right) \frac{\partial}{\partial x} + B \frac{\partial}{\partial y},$$

but these remarks do give us one type of linearization theorem. For example, if we write

$$X = x^r A(x,y) \frac{\partial}{\partial x} + B \frac{\partial}{\partial y},$$

with $A(0,y) \neq 0$, then we can define

$$X' = \frac{X}{Z} = x^r \frac{\partial}{\partial x} + B' \frac{\partial}{\partial y}.$$

The integral curves of X and X' only differ by a change in parameterization, and we can apply the general theory. A similar remark applies to a single vector field which is tangent to a hypersurface in a general manifold M.

V. CONTRACTION AND DEFORMATION OF LIE ALGEBRA HOMOMORPHISMS

We temporarily leave the problem of linearizing a Lie algebra of vector fields near an invariant submanifold in order to treat a more abstract problem that will be shown later to be relevant.

Suppose $\underset{\sim}{K}$ and $\underset{\sim}{L}$ are Lie algebras, and suppose ϕ, ϕ' are homomorphisms: $\underset{\sim}{K} \to \underset{\sim}{L}$. ϕ and ϕ' are said to be related by a deformation if there is a one-parameter family $\lambda \to \phi_\lambda$ of homomorphisms: $\underset{\sim}{K} \to \underset{\sim}{L}$ such that

(a) $\phi_o = \phi, \phi_1 = \phi'$

(b) ϕ_λ depends analytically on λ.

Then, we can form the Taylor expansion:

$$\phi_\lambda(x) = \sum_{j=0}^{\infty} \theta_j(x)\lambda^j , \quad \text{for } X \in K .$$

Let α be the following representation of $\underset{\sim}{K}$ by linear transformation on L:

$$\alpha(X)(Y) = [\phi(X), Y] , \quad \text{for } X \in \underset{\sim}{K}, \quad Y \in \underset{\sim}{L}$$

Then, the θ_j's are 1-cochains of $\underset{\sim}{K}$ with coefficients in $\underset{\sim}{L}$. The relation of the corresponding cohomology groups and the "triviality" of the deformation has been investigated in [7] and [4], part 4 .

Recall that the deformation $\lambda \to \phi_\lambda$ is said to be <u>trivial</u> if there is a one-parameter family $\lambda \to A_\lambda$ of automorphisms of $\underset{\sim}{L}$ such that

$$\begin{aligned} &\text{(a)}\quad \phi_\lambda(X) = A_\lambda \phi(X) , \quad \text{for } X \in \underset{\sim}{K}, \text{ all } \lambda , \\ &\text{(b)}\quad A_\lambda \text{ depends smoothly on } \lambda . \end{aligned} \tag{5.1}$$

Now (modelling our terminology on that used by Inonu and Wigner in a similar case, the deformation theory of Lie algebra structures) let us say that ϕ is a <u>contraction</u> of ϕ' if:

(a) There is a one-parameter family of homomorphisms ϕ_λ: K → L such that $\phi_o = \phi, \phi_1 = \phi'$.

(b) $\lambda \to \phi_\lambda$ depends continuously on λ for $0 \le \lambda \le 1$.

(c) ϕ_λ depends analytically on λ for $\lambda \neq 0$.

Let us present a set-up which leads to such notions in a very natural way. Suppose in addition that F is a vector space, and $\underset{\sim}{L}$ is a Lie algebra of linear transformation on F, with the bracket in $\underset{\sim}{L}$ given by commutator of linear transformations. Thus, ϕ and ϕ' are representations of $\underset{\sim}{K}$ by linear transformations on F. Suppose that $\lambda \to B_\lambda$ is a one-parameter family of linear transformations: $F \to F$ such that:

(a) B_λ depends analytically on λ, for all λ

(b) B_λ^{-1} exists only for $\lambda \neq 0$.

(c) $A_\lambda(Y) = B_\lambda Y B_\lambda^{-1}$ for $Y \in \underset{\sim}{L}$.

In this case, 5.1 takes the form

$$\phi_\lambda(X) = B_\lambda \phi(X) B_\lambda^{-1} \quad \text{for } X \in \underset{\sim}{K} \tag{5.2}$$

Thus, we have the possibility of the singularity in B_λ^{-1} at $\lambda = 0$ generating nontrivial deformations between ϕ and ϕ' .

Before proceeding further with the general algebraic theory, let us turn to the geometric situation that motivates our work, the linearization problem for Lie algebras of vector fields near invariant submanifolds.

VI. THE LINEARIZATION PROBLEM NEAR AN INVARIANT SUBMANIFOLD

Suppose M is a manifold, $F = F(M)$, the ring of C^∞ real-valued functions on M, $V(M)$ = the derivations of $F(M)$, i.e., the Lie algebra of vector fields on M. Let N be a submanifold of M, and let $\underset{\sim}{K}'$ be a subalgebra of $V(M)$ that is tangent to N. Since we will only be working locally for the moment, suppose (x_i), $1 \leq i, j, \ldots \leq m = \dim M$, is a coordinate system for M such that:

$$x_u = 0, \quad n+1 \leq u, V, \ldots \leq m; \quad \dim \ N = n$$

defines N. (Adopt the summation connection.) Suppose that $\lambda \to B_\lambda$ is the following one-parameter family of linear transformations:

$$F(M) \to F(M),$$

$$B_\lambda\big(f(x_1, \ldots, x_n)\big) = f(x_1, \ldots, x_m, \lambda x_{m+1}, \ldots, \lambda x_n)$$

Let $X \in \underset{\sim}{K}$. Suppose

$$X = f_i \frac{\partial}{\partial x_i} \quad , \text{ i.e.,} \quad X(x_i) = f_i$$

Suppose ϕ_λ is given by 5.2. Then,

$$\phi_\lambda(X)(x_i) = B_\lambda X \, B_\lambda^{-1}(x_i)$$

$$= \begin{cases} \lambda^{-1} B_\lambda(f_i) & \text{for } i > n \\ B_\lambda(f_i) & \text{for } 1 \leq i \leq n \end{cases} \tag{6.1}$$

Suppose now that X is tangent to N, i.e.,

$$f_u(x_1, \ldots, x_n, 0) = 0$$

Then, f_u admits a Taylor expansion of the form:

$$f_u(x_1, \ldots, x_n) = f_{uv}(x_1, \ldots, x_n)\, x_v + f_{uvw}(x_1, \ldots, x_m) x_v x_w$$

$$\phi_\lambda(X)(x_u) = f_{uv}(x_1, \ldots, x_n)\, x_v + \lambda f_{uvw}(x_1, \ldots, x_m)\, x_v x_w$$

$$\phi_\lambda(X)(x_a)\, f_a(x_1, \ldots, x_n, \lambda x_{n+1}, \ldots, \lambda x_m).$$

We see that ϕ_λ, considered as a homomorphism: $\underset{\sim}{K} \to V(M)$, is perfectly analytic at $\lambda = 0$ despite the fact that the transformation B_λ^{-1} used to define it has a pole at $\lambda = 0$. Further,

$$\phi_o(X) = \frac{\partial f_u}{\partial x_v}(x_1, \ldots, x_n, 0)\, x_v \frac{\partial}{\partial x_u} + f_a(x_1, \ldots, x_n, 0) \frac{\partial}{\partial x_a} \quad , \text{ for } X \in \underset{\sim}{K}. \tag{6.2}$$

The subalgebra $\phi_o(\underset{\sim}{K})$ is then the linearization of L. "Linearization" of $\underset{\sim}{K}$ itself is equivalent to proving triviality of the deformation in the neighborhood of $\lambda = 0$, a problem that is solved, in the formal sense at least, by the cohomology theory of [8] and [4], part 4. Now, we turn to the task of freeing this argument from local coordinate systems, thus enabling one to apply to it situations in differential topology, partial differential equations and continuum mechanics. (In the last discipline, one will be interested in seeing how the argument goes for infinite dimensional manifolds.)

Let N be a submanifold of M, and let V(M,N) be the Lie algebra of vector field on M which are tangent to N. Let $\underset{\sim}{K}$ be a subalgebra of V(M,N). Suppose that $\lambda \to \beta_\lambda$ is a one-parameter family of mappings of $M \to M$ such that:

(a) β_λ is a diffeomorphism for $\lambda > 0$, and depends smoothly on λ for $\lambda \geq 0$.

(b) $\beta_\lambda(p) = p$ for $p \in N$.

(c) For each $p \in M$, the curve $\lambda \to \beta_\lambda(p)$ proceeds toward N smoothly and transversally as $\lambda \to 0$. Precisely, $\beta_o(p)$ has a neighborhood with a coordinate system $(x_1, \ldots, x_m)$ having the properties described above.

Now, we can define $\beta_\lambda : F(M) \to F(M)$ as follows:

$$\beta_\lambda(f) = \beta_\lambda^*(f) \qquad \text{for } f \in F(M), \quad \text{i.e.,}$$

$$\beta(f)(p) = f\big(\beta_\lambda(p)\big) \quad \text{for } p \in M .$$

Define $\phi_\lambda : \underset{\sim}{K} \to V(M,N)$ as follows:

$$\phi_\lambda(X)(f) = \beta_\lambda X \, B_\lambda^{-1}(f) \quad \text{for } f \in F(M), \ X \in \underset{\sim}{K} .$$

Then, the local argument given above can be used to show that $\phi_\lambda(X)$ is well-defined and smooth as $\lambda \to 0$. $\phi_o(X)$ can be considered as the "linearization" of the vector field X, relative to the homotopy $\lambda \to \beta_\lambda$ used to retract M into N.

Finally, we might point out the connection between this paper and the remarks of Moser in [6], part 5, which arose from a special case of the general problem.

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