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NEW BOUND ON THE PION'S CHARGE RADIUS*

X. Y. Pham† and A. C. D. Wright‡

Stanford Linear Accelerator Center
Stanford University, Stanford, California 94305

ABSTRACT

A new bound for the pion form factor in the timelike and spacelike regions is derived and evaluated with the help of timelike data. The bound is compared with recent Serpukhov-UCLA data near $t = 0$, and implications for the asymptotic behavior of the form factor and the pion's charge radius are discussed.

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†On leave of absence from Institut de Physique Nucléaire, Division Physique Théorique, Paris, France.

‡National Research Council of Canada Postdoctorate Fellow.

I. INTRODUCTION

In the past few years, several authors [1-9] have discussed dispersive inequalities for the electromagnetic form factor of the pion F_π. In the simplest case,[1-6,8,9] these inequalities provide bounds for $F_\pi(t)$ below its cut ($t \leq t_0 = 4m_\pi^2$) if the modulus $|F_\pi(s)|$ or an upper bound for $|F_\pi(s)|$ is known on the cut ($t_0 \leq s < \infty$). The bounds are particularly sensitive to the behavior of $|F_\pi|$ near the elastic threshold, so that the lack of data in that region hampered attempts to evaluate them numerically.[5] In order to reduce the uncertainty caused by interpolating data on $|F_\pi|$ to the threshold region, Levin and Okubo[7] modified the inequalities, whereby data on both the modulus and phase[10] of the form factor are exploited. Essentially, they found that the preliminary Serpukhov - UCLA[11] value of $r^2_{\pi\,\mathrm{expt}} = (0.80 \pm 0.23)\ \mathrm{fm}^2$ requires a large p-wave $\pi\pi$ phase shift δ_1 just above threshold if the upper bound on r_π^2 is to be satisfied. However, their result depends on assumptions about "reasonable" asymptotic behavior of F_π. Furthermore, recent data[12] on $\delta_1(s)$ show no indication of the phase shift behavior suggested by the analysis of Levin and Okubo, and the latest Serpukhov-UCLA measurement[13] of

$$r^2_{\pi\,\mathrm{expt}} = (0.61 \pm 0.15)\ \mathrm{fm}^2 \tag{1.1}$$

is considerably smaller than the preliminary value quoted above. In light of the data of Refs. 12 and 13, and also new data on $|F_\pi|$ in the threshold region[14] and at high momentum transfer,[15] it is worthwhile to reexamine the problem.

In Sec. II we show how to exploit the available experimental information to bound the modulus of the form factor in the timelike region where it has not yet been measured. This result is then applied to existing methods to obtain bounds in the spacelike region. We evaluate the bounds numerically and compare them to the Serpukhov-UCLA data[13] in Section III. Section IV contains a discussion of our results, and conclusions.

II. THE BOUND

We wish to make maximal use of the available timelike information, which consists of data[14-16] on $|F_\pi(s)|$ for $0.05 \leq s \leq 0.15$ GeV^2 and for $0.34 \leq s \leq 9$ GeV^2, and data[12] on $\delta_1(s)$ for $0.20 \leq s \leq 0.31$ GeV^2. Therefore, with the exception of a small gap for $0.15 \leq s \leq 0.20$ GeV^2, we have information on either the modulus or phase of $F_\pi(s)$ for $4m_\pi^2 \leq s \leq 9$ GeV^2. Notice that nowhere do the currently available phase data overlap the modulus data.

Consequently, to define our mathematical problem we assume that an upper bound $w(s)$ is known for $|F_\pi(s)|$ in the regions $t_0 \leq s \leq t_1$ and $t_2 \leq s < \infty$, and that $\delta_1(s)$ is known for $t_1 \leq s \leq t_2$. $F_\pi(s)$ is assumed to be a real-analytic function in the cut s-plane, with asymptotic behavior bounded by

$$w(s) = 0\,(\exp\,(s^{\alpha})) \qquad (\alpha < \tfrac{1}{2}) \quad . \tag{2.1}$$

To motivate the discussion, we consider first the case of Levin and Okubo,[7] namely, $t_1 = t_0$. Knowledge of $\delta_1(s)$ for $t_0 \leq s \leq t_2$ allows one to construct a function G which has no cut in that region:

$$G(t) \equiv F_\pi(t)/\Omega(t) \quad , \tag{2.2}$$

where

$$\ln \Omega(t) \equiv \frac{t}{\pi} \int_{t_0}^{t_2} \frac{ds\, \delta_1(s)}{s(s-t)} \quad . \tag{2.3}$$

We define another function $\bar{G}$ by[17]

$$\ln \bar{G}(t) \equiv \ln G(t) - \frac{(t_2-t)^{\frac{1}{2}}}{\pi} \int_{t_2}^{\infty} \frac{ds\, \ln\, (w(s)/|\Omega(s)|)}{(s-t)(s-t_2)^{\frac{1}{2}}} \quad . \tag{2.4}$$

$\bar{G}(t)$ is real-analytic in the cut t-plane, with asymptotic behavior similar to that of G(t), and satisfies

$$|\bar{G}(t)| \leq 1 \qquad (t_2 \leq t < \infty) \ . \tag{2.5}$$

We note that (2.5) becomes an equality in the case where $w(s) \equiv |F_\pi(s)|$. Then, by the Phragmen–Lindelöf theorem,[18] $|\bar{G}(t)| \leq 1$ for all t. By mapping the cut plane into the unit disc and employing the Schwartz lemma,[19] the bounds derived by Levin and Okubo are recovered.

To derive our bound, we proceed in an analogous way. Our Omnes-type function Ω is given by

$$\ln \Omega(t) = \frac{t}{\pi} \int_{t_0}^{\infty} \frac{ds\, \delta(s)}{s(s-t)} \ , \tag{2.6}$$

where $\delta(s) = \delta_1(s)$ for $t_1 \leq s \leq t_2$ and is continuous for $t_0 \leq s < \infty$. We choose $\delta(s)$ to be continuous, rather than zero outside (t_1, t_2), to avoid the possibility of zeros or poles in $\Omega(t)$. Our results will be independent of $\delta(s)$ except for the region $t_1 \leq s \leq t_2$, as might be expected.

Then the auxiliary function G(t), defined by

$$G(t) \equiv F_\pi(t)/\Omega(t) \tag{2.7}$$

is real-analytic in the t-plane except for cuts from t_0 to t_1 and from t_2 to ∞. Defining $\bar{G}(t)$ by

$$\ln \bar{G}(t) \equiv \ln G(t) - \frac{1}{\pi}\left[\frac{(t_1-t)(t_2-t)}{t_0-t}\right]^{\frac{1}{2}}$$

$$\times\left\{-\int_{t_0}^{t_1}\frac{ds}{s-t}\ln\left(w(s)/|\Omega(s)|\right)\left[\frac{s-t_0}{(t_1-s)(t_2-s)}\right]^{\frac{1}{2}}\right.$$

$$\left.+\int_{t_2}^{\infty}\frac{ds}{s-t}\ln\left(w(s)/|\Omega(s)|\right)\left[\frac{s-t_0}{(s-t_1)(s-t_2)}\right]^{\frac{1}{2}}\right\} \tag{2.8}$$

we see that $|\bar{G}(t)| \leq 1$ for $t_0 \leq t \leq t_1$ or $t_2 \leq t < \infty$. It is not possible, however, to apply the Phragmen-Lindelöf theorem to $\bar{G}(t)$. The reason is that, in general, $\bar{G}(t)$ as defined in (2.8) has an essential singularity at $t = t_0$. In addition, it is not clear that $|\bar{G}(t)| \leq 1$ for $t_1 \leq t \leq t_2$.

To make use of (2.8), we recall that one obtains dispersive <u>inequalities</u> for $F_\pi(t)$, rather than exact dispersion relations, for two reasons. The most obvious is that an upper bound w(s) for the modulus is employed, rather than the modulus itself. The more fundamental reason is that F_π may have zeros in the complex plane, so that in writing a dispersion relation involving $\ln F_\pi(t)$, the contributions from the discontinuities across the cuts arising from the zeros of F_π must be accounted for. In the case considered by the authors of Refs. 1-9, Cauchy's theorem is written for the function $\ln G(t)/(t_2-t)^{\frac{1}{2}}$ where G(t) is defined in (2.2). It is assumed that the form factor has the representation[1]

$$F_\pi(t) = P_n(t)\exp\left(\frac{t}{\pi}\int_{t_0}^{\infty}\frac{ds\,\delta_1(s)}{s(s-t)}\right), \tag{2.9}$$

where $P_n(t)$ is a polynomial of n-th degree corresponding to the number of zeros of $F_\pi(t)$,

$$P_n(t) = \prod_{j,k,\ell} \left(1 - \frac{t}{a_j}\right)\left(1 - \frac{t}{b_\ell}\right)\left(1 - \frac{t}{b_\ell^*}\right)\left(1 - \frac{t}{c_k}\right) . \qquad (2.10)$$

Here the b_ℓ are complex with Im $b_\ell > 0$, and the a_j and c_k are real with $a_j \leq t_0$ and $c_k \geq t_0$. The resulting dispersion relation is given by (2.4) with $w(s) \equiv |F_\pi(s)|$ and with ln $\bar{G}(t)$ representing the contribution from the zeros of F_π. $\bar{G}(t)$ may be calculated explicitly,[1] and it is found, as expected, that $|\bar{G}(t)| \leq 1$ for all t.

Analogously, to obtain (2.8) in the case where $w(s) \equiv |F_\pi(s)|$, Cauchy's theorem is written for the function

$$\ln G(t) \left[\frac{t_0 - t}{(t_1 - t)(t_2 - t)}\right]^{\frac{1}{2}} .$$

The function $\bar{G}(t)$ in (2.8) then represents the contribution from the zeros of F_π, and is explicitly given by[20] (for real t)

$$\left[\frac{t_0 - t}{(t_1 - t)(t_2 - t)}\right]^{\frac{1}{2}} \ln \bar{G}(t) = \sum_\ell 2 \,\mathrm{Re} \int_{-\infty}^{\mathrm{Re}\, b_\ell} \frac{dx}{s-t} \left[\frac{t_0 - s}{(t_1 - s)(t_2 - s)}\right]^{\frac{1}{2}}$$

$$= \sum_\ell -2 \,\mathrm{Re} \int_{\mathrm{Re}\, b_\ell}^{\infty} \frac{dx}{s-t} \left[\frac{t_0 - s}{(t_1 - s)(t_2 - s)}\right]^{\frac{1}{2}} , \qquad (2.11)$$

where $s = x + i$ Im b_ℓ. Taking $t_1 = 0.2$ GeV^2 and $t_2 = 16\, m_\pi^2$, we have performed a numerical analysis of (2.11) for $t_1 \leq t \leq t_2$, and find that ln $\bar{G}(t)$ is negative for all complex values of b_ℓ. Such an analysis is possible because the integrals in (2.11) reduce to the explicit expression given in Ref. 1 when Im b_ℓ is large enough. Therefore, they need be checked only for b_ℓ in a finite region near t_0.

Taking $t_0 \leq t \leq t_1$, using (2.6) and recalling that $w(s) \geq |F_\pi(s)|$ we therefore find that (2.8) may be cast in the form of a bound for $|F_\pi(t)|$:

$$\ln|F_\pi(t)| \leq w(t)$$

$$= \frac{1}{\pi}\left[\frac{(t-t_1)(t_2-t)}{t-t_0}\right]^{\frac{1}{2}}\left\{I_1(t) + I_2(t) + I_3(t)\right\} \quad (t_1 \leq t \leq t_2), \qquad (2.12)$$

where

$$I_1(t) = \int_{t_0}^{t_1} \frac{ds \ln w(s)}{t-s}\left[\frac{s-t_0}{(t_1-s)(t_2-s)}\right]^{\frac{1}{2}}, \qquad (2.13a)$$

$$I_2(t) = P\int_{t_1}^{t_2} \frac{ds\,\delta_1(s)}{s-t}\left[\frac{s-t_0}{(s-t_1)(t_2-s)}\right]^{\frac{1}{2}}, \qquad (2.13b)$$

$$I_3(t) = \int_{t_2}^{\infty} \frac{ds \ln w(s)}{s-t}\left[\frac{s-t_0}{(s-t_1)(s-t_2)}\right]^{\frac{1}{2}}. \qquad (2.13c)$$

Hence, there exists an upper bound w(t) for the modulus everywhere on the cut, with w(t) given by (2.12) for $t_1 \leq t \leq t_2$.

The final step is to observe that we may apply (2.12) to the usual bounds[5-7] for F_π in the spacelike region ($t < 0$):

$$\exp[I(t)]\left(\frac{\exp[-I(0)] - \eta(t)}{1 - \eta(t)\exp[-I(0)]}\right) \leq F_\pi(t) \leq \exp[I(t)]\left(\frac{\exp[-I(0)] + \eta(t)}{1 + \eta(t)\exp[-I(0)]}\right), \qquad (2.14)$$

where

$$\eta(t) = \frac{(t_0-t)^{\frac{1}{2}} - t_0^{\frac{1}{2}}}{(t_0-t)^{\frac{1}{2}} + t_0^{\frac{1}{2}}}, \qquad (2.15)$$

$$I(t) = \frac{(t_0-t)^{\frac{1}{2}}}{\pi}\int_{t_0}^{\infty} \frac{ds \ln w(s)}{(s-t)(s-t_0)^{\frac{1}{2}}}. \qquad (2.16)$$

The pion's charge radius, defined by

$$\frac{1}{6} r_\pi^2 = F_\pi'(0) , \tag{2.17}$$

satisfies the bounds

$$\frac{-\sinh [I(0)] - I(0)}{2 t_0} + \bar{J} \leq \frac{1}{6} r_\pi^2 \leq \frac{\sinh [I(0)] - I(0)}{2 t_0} + \bar{J} , \tag{2.18}$$

where

$$\bar{J} = \frac{t_0^{\frac{1}{2}}}{\pi} \int_{t_0}^{\infty} \frac{ds \ln w(s)}{s^2 (s-t_0)^{\frac{1}{2}}} . \tag{2.19}$$

With the definition (2.12) of $w(t)$, (2.14) and (2.18) constitute the required bounds on F_π and r_π in terms of the given information about the modulus and phase of F_π on the cut. We do not know whether (2.12) is the strongest possible bound on $|F_\pi(t)|$ in the region $t_1 \leq t \leq t_2$ consistent with our assumptions, but, as we shall see in Section III, when evaluated with the help of timelike data it constrains the modulus in the region .2 GeV^2 to 16 m_π^2 rather strongly.

III. NUMERICAL EVALUATION

The inverse electroproduction data[14] near threshold yield a form factor that appears to be bounded from above by

$$w(s) = 0.79 + 4.71s , \tag{3.1a}$$

and we shall assume that (3.1a) holds for $4 m_\pi^2 \leq s \leq 0.20$ GeV^2. For the ρ-resonance region we employ the fit of Benaksas <u>et al.</u>[16]

$$w(s) = \left| \frac{F_0 m_\rho \Gamma_\rho}{m_\rho^2 - s - i\, m_\rho^2 \Gamma_\rho (p/p_\rho)^3 / s^{\frac{1}{2}}} \right| \quad (16\, m_\pi^2 \leq s \leq 1 \text{ GeV}^2), \tag{3.1b}$$

where $F_0 = 5.83$, $m_\rho = 775.4$ MeV, $\Gamma_\rho = 149.6$ MeV and $p = \frac{1}{2}(s - t_0)^{\frac{1}{2}}$. The form (3.1b) is probably too small for $s \geq 1$ GeV^2, so we take

$$w(s) = 1.5\, s^{-1} \quad (1 \leq s \leq 9 \text{ GeV}^2) . \tag{3.1c}$$

We assume that for $s > 9\ \mathrm{GeV}^2$ the form factor is bounded by a power law

$$w(s) = w(9\ \mathrm{GeV}^2) \times (s/9)^{-n} \qquad (s \geq 9\ \mathrm{GeV}^2)\ , \tag{3.1d}$$

where n is a real number.

The upper bound w(s) is plotted versus the data[14-16] in Fig. 1 (solid lines). There is some disagreement among the modulus data in the ρ-peak region; however, note that Benaksas et al.[16] claim that the earlier Orsay data of Augustin et al.[16] (open circles) are too high by 8% in the cross section due to a systematic error.

For the $\pi\pi$ phase shift, we take an effective range formula[21] for $s < 0.26\ \mathrm{GeV}^2$,

$$(s - t_0)^{\frac{3}{2}} \cot \delta_1(s) = \frac{t_0 s^{\frac{1}{2}}}{a_1 m_\pi^3} + f\, s^{\frac{1}{2}}(s - t_0) \quad (t_0 \leq s \leq 0.26\ \mathrm{GeV}^2)\ , \tag{3.2a}$$

where $a_1 m_\pi^3 = 0.05$ and $f = -3.53$, and use a linear form for $s > 0.26\ \mathrm{GeV}^2$,

$$\delta_1(s) = (-2.6 + 46.2s)\ \mathrm{deg.} \qquad (0.26\ \mathrm{GeV}^2 \leq s \leq 16\ m_\pi^2)\ . \tag{3.2b}$$

Our parametrization for δ_1 is shown with the data[12] in Fig. 2.

By substituting (3.1) and (3.2) in (2.12) and taking $n = 1$, we obtain the upper bound for $|F_\pi(s)|$ in the range $t_1 \leq s \leq t_2$ shown as the dashed line in Fig. 1. Our result is not very sensitive to n because we have chosen a large scale ($9\ \mathrm{GeV}^2$) for asymptotic power-law behavior of the form factor. The bound interpolates $|F_\pi|$ rather smoothly between t_1 and t_2, and rules out any anomalously large behavior of $|F_\pi|$. This might have been anticipated by examining the phase shift data (Fig. 2) which show no evidence for resonance-type behavior of F_π between $0.2\ \mathrm{GeV}^2$ and the inelastic threshold.

Consequently, to evaluate the bounds (2.14) and (2.18) we may, to a good approximation, employ a linear interpolation for w(s) in the region 0.2 $\mathrm{GeV}^2 \leq s \leq 16\ m_\pi^2$. Taking n = 1, we obtain the upper and lower bounds for spacelike momentum transfer shown in Fig. 3 with the Serpukhov-UCLA data.[13] Within the experimental errors, all data points, except the one at 0.0333 GeV^2, are consistent with both bounds, although the data tend to lie along the lower bound.

The experimentalists[13] have extracted a value for the charge radius by fitting their data, including systematic errors, to the form $|F_\pi(t)|^2 = 1/(1-At)^2$; their result is given in (1.1). In Fig. 4 we plot our upper and lower bounds for r_π^2 as a function of the asymptotic power n (solid lines). The requirement that the upper bound exceed the lower bound gives the constraint

$$n \leq 1.98 \quad , \tag{3.3}$$

which may be compared with the value ($n \leq 1.2 \pm 0.3$) obtained by Bonneau et al.[9] The larger value (3.3) is a consequence of our more conservative estimate for w(s) in the threshold region, and our choice of 9 GeV^2 as opposed to 2 GeV^2 for the onset of s^{-n}-type behavior.

For comparison, we also show in Fig. 4 the bounds obtained from the phase-modulus representation[7] with $\delta_1(s)$ given by (3.2) in the region $t_0 \leq s \leq 16\ m_\pi^2$ (dashed lines). We remark that these latter bounds are sensitive to the phase and modulus near $s = t_2$, not near $s = t_0$. Consequently, it is difficult to imagine a form for $\delta_1(s)$ for $s < 0.20\ \mathrm{GeV}^2$ that would alter the dashed lines significantly. In the case of the modulus representation (solid lines) a more accurate determination of $|F_\pi|$ near threshold might allow a considerable strengthening of the bounds.

The upper bound in Fig. 4, which is the one of interest, is weakly dependent on n for n > 0, and is in at best marginal agreement with (1.1) there, although agreement improves as n becomes negative. If the asymptotic behavior (3.1d) really holds true for $s > 9\ \text{GeV}^2$, then (3.3) provides a lower bound on the asymptotic form factor; however, if asymptopia is not reached until some higher momentum transfer, then (3.3) bounds the "average" asymptotic behavior of $|F_\pi|$.[9] In this connection, the recently discovered resonances at 3.105 and 3.695 GeV in e^+e^- annihilation[22] and massive lepton pair production[23] suggest that a scale greater than 3 GeV may apply to the form factor.[24]

IV. DISCUSSION AND CONCLUSION

We have shown that existing data on the phase and modulus of the pion form factor may be used to bound the modulus everywhere on the cut up to $9\ \text{GeV}^2$. This timelike information is then applied to the well-known inequalities[1-9] giving bounds on F_π in the spacelike region.

If we assume the Serpukhov-UCLA value

$$r_\pi^2 = (0.61 \pm 0.15)\ \text{fm}^2 \quad , \tag{1.1}$$

then Fig. 4 shows that the bounds must receive a substantial contribution from momentum transfers greater than $9\ \text{GeV}^2$. The insensitivity of the upper bound for r_π^2 to the large s behavior of the form factor means that $|F_\pi|$ must be anomalously large at high momentum transfer for agreement with (1.1). An alternative hypothesis is that (1.1) is too large, and that no anomalous contribution is needed for consistency between the upper bound and the true charge radius. This idea is supported by the agreement between the spacelike data and the bounds for $n = 1$ (Fig. 3). Furthermore, theoretical studies[25] based on sidewise dispersion relations for the form factor yield values for r_π^2 close to the ρ-dominance value of $0.40\ \text{fm}^2$, in agreement with our bounds. The errors in the Serpukhov-UCLA data, and the uncertainty introduced by the necessity of extrapolating to $t = 0$ suggest that a reliable determination of the pion's charge radius will require further measurements of the form factor at small momentum transfer.

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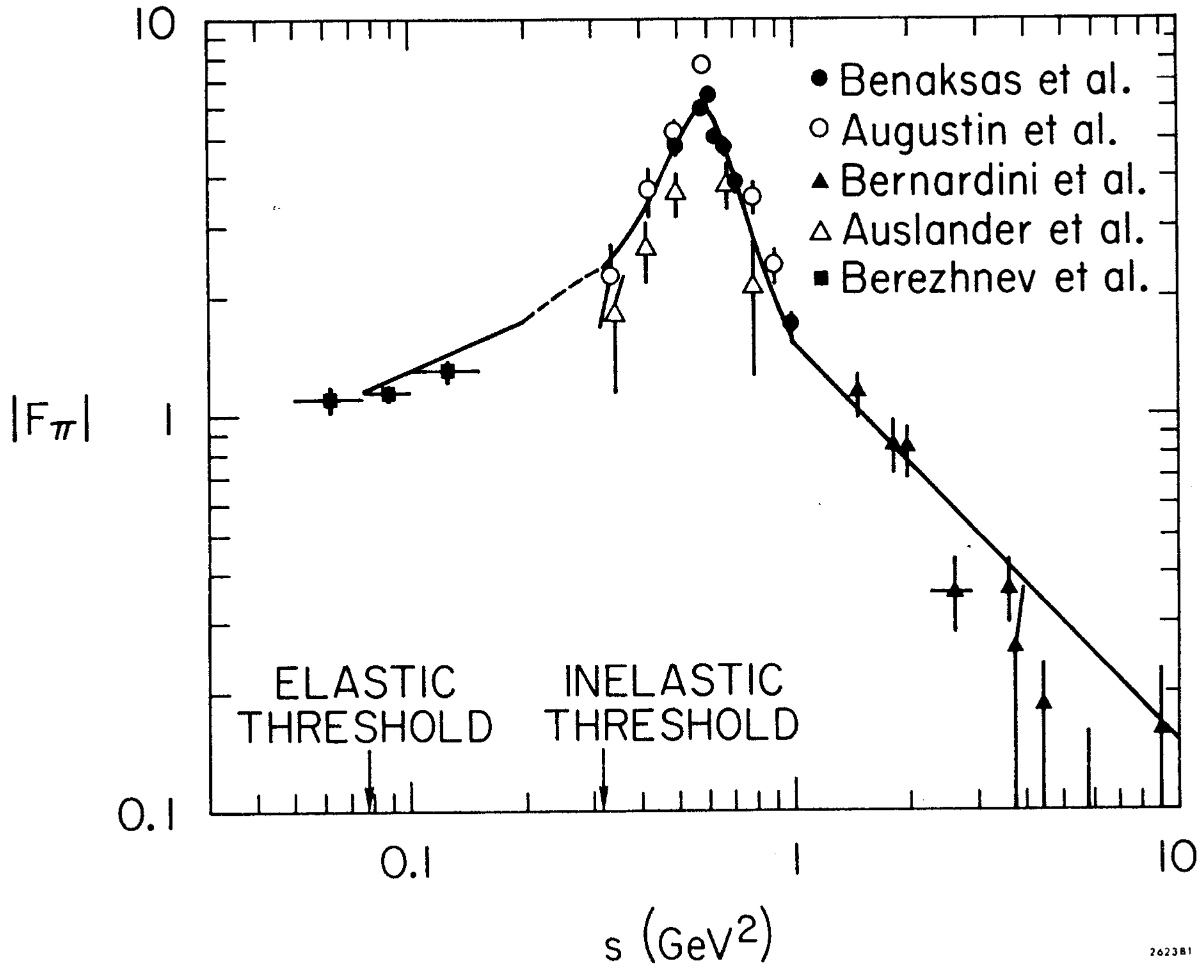
FIGURE CAPTIONS

1. Data for $|F_\pi|$ from Refs. 14-16, plotted versus s. The solid lines are our upper-bound function w defined in Eq. (3.1) and the dashed line is the bound given by Eq. (2.12).

2. The $\pi\pi$ phase shift δ_1 given by Eq. (3.2) (solid line) plotted versus s with the data of Ref. 12.

3. Upper and lower bounds for F_π assuming 1/s asymptotic behavior (solid lines) plotted versus -t. The data are from Ref. 13.

4. Upper and lower bounds for r_π^2 plotted versus the asymptotic power n. The solid and dashed lines are modulus and phase-modulus bounds, respectively, as discussed in the text. The experimental value (1.1) is indicated.



10
1
0.1
|Fπ|
0.1
1
10
s (GeV2)
● Benaksas et al.
○ Augustin et al.
▲ Bernardini et al.
△ Auslander et al.
■ Berezhnev et al.
ELASTIC THRESHOLD
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262381

Figure 1

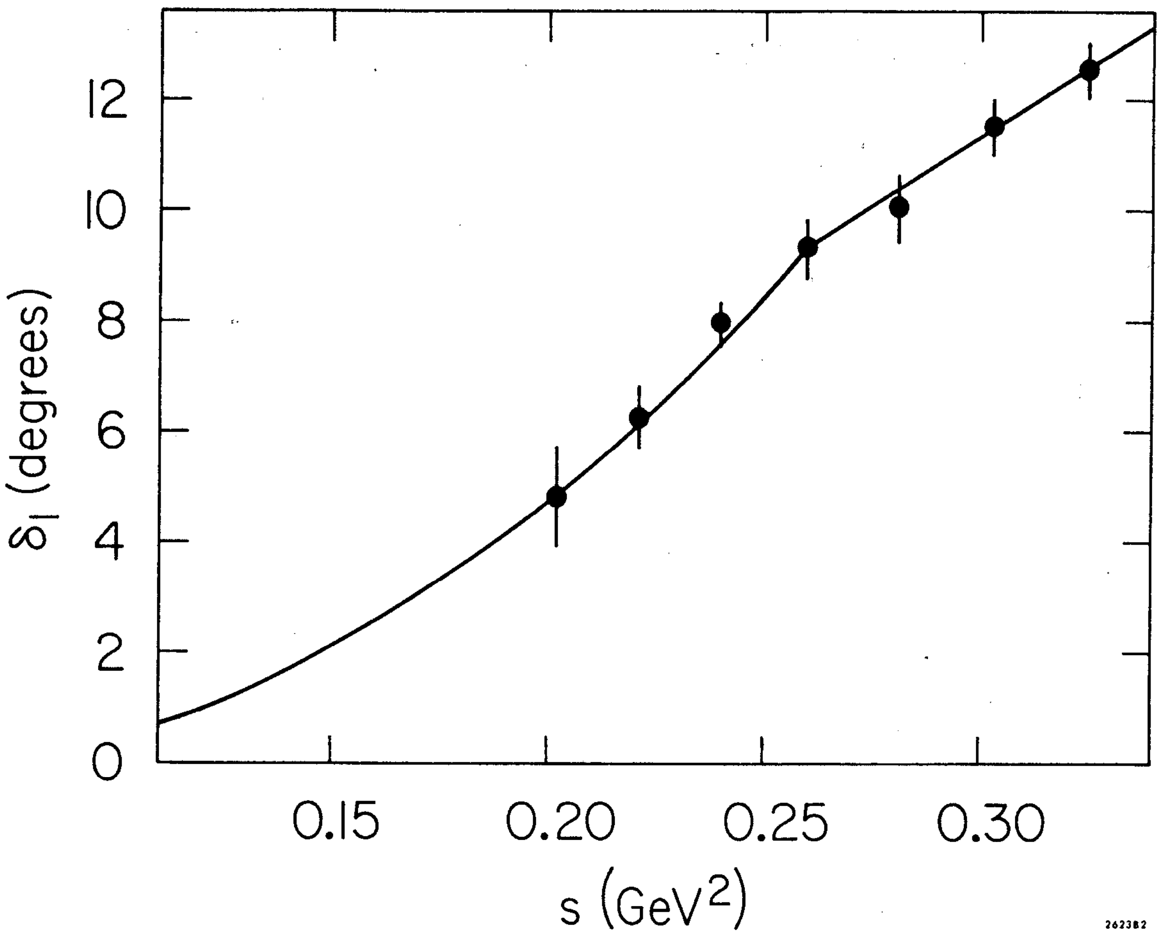

Figure 2

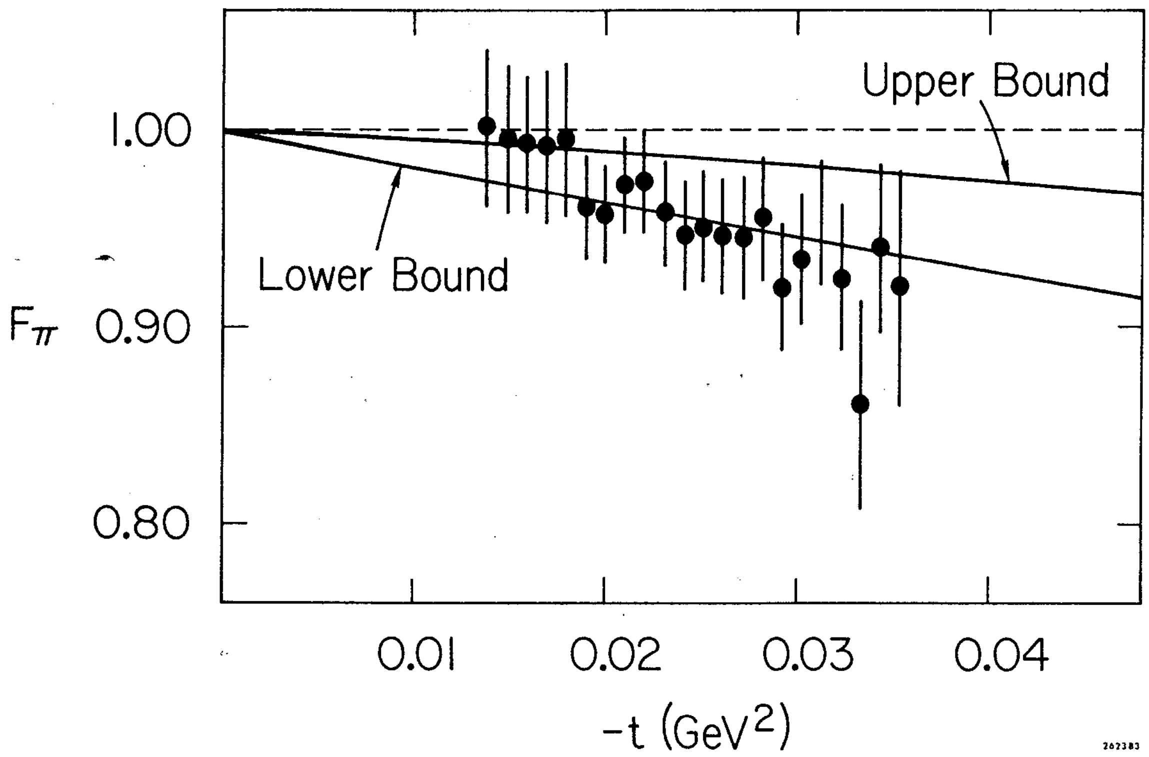

Figure 3

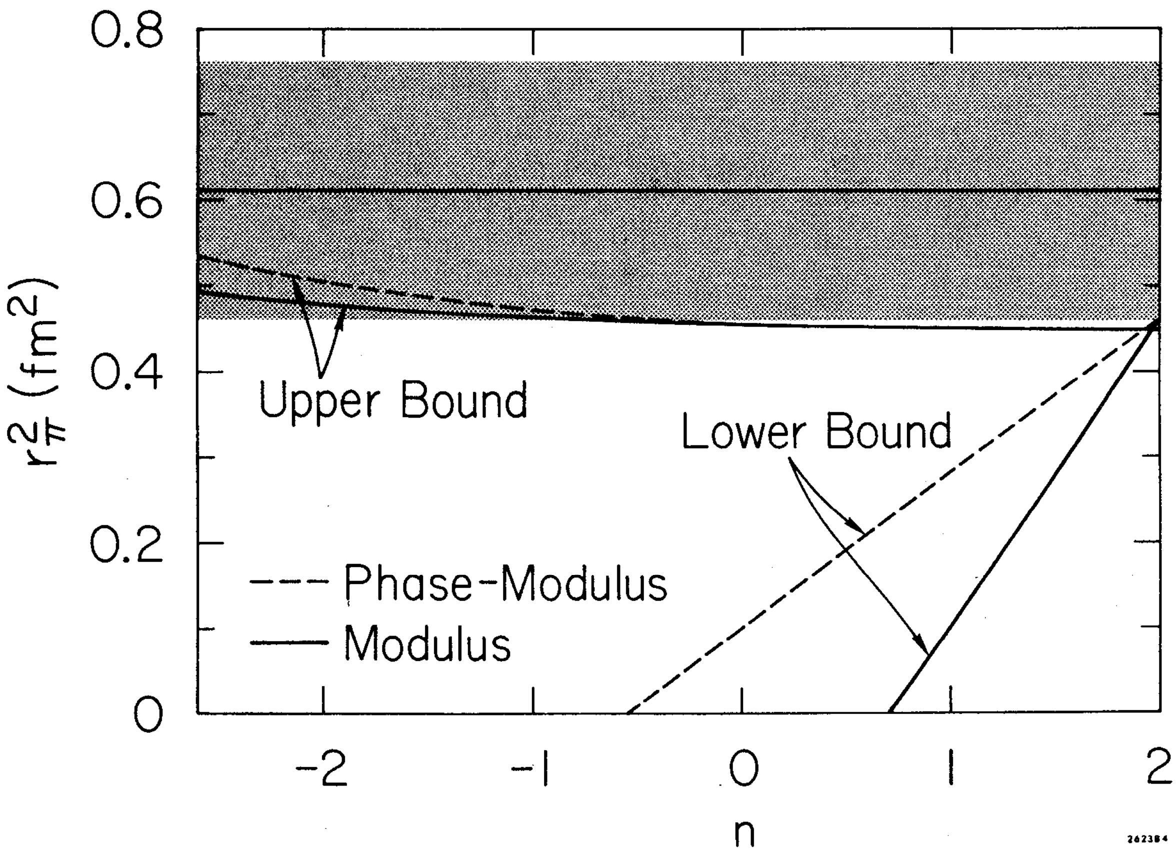

Figure 4