

CROSS-ORDER RELATIONS IN $\mathcal{N} = 4$ SUPERSYMMETRIC GAUGE THEORIES

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The anti-de Sitter/conformal field theory duality conjecture raises the question of how the perturbative expansion in the conformal field theory can resum to a simple function. We exhibit a relation between the one-loop and two-loop amplitudes whose generalization to higher-point and higher-loop amplitudes would answer this question. We also provide evidence for the first of these generalizations.

1. Introduction

Thirty years ago, shortly after the discovery of asymptotic freedom, 't Hooft¹ studied the large-order behavior of gauge theories. He used the double-line notation, studying the theory in the limit of a large number of colors N_c . In this limit, the insertion of a gluon in a diagram, corresponding to increasing the number of loops by one, costs a power of g^2 but gains a power of N_c . It is thus natural to study the theory in the regime where $g^2 N_c$ is held fixed. If this coupling is of order unity, higher-loop contributions are just as important as lower-order ones. We can think of this as filling in a mesh for any given process, so that we obtain a string worldsheet.

This connection was formulated into a duality conjecture by Maldacena². In its most studied form, the anti-de Sitter/conformal field

*Talk presented at the 3rd International Symposium on Quantum Theory and Symmetries, Cincinnati, OH, Sept. 10–14, 2003

theory conjecture posits a duality between $\mathcal{N} = 4$ supersymmetric gauge theory (the conformal field theory) at strong coupling and large number of colors, and the appropriate limit of string theory (perturbative supergravity) on an anti-de Sitter background. This duality has passed numerous tests³ on quantities protected by supersymmetry, and more recently⁵ on unprotected quantities allowing another expansion parameter (the large-‘spin’ limit of BMN⁴ operators).

From the viewpoint of ‘t Hooft’s original work, the AdS/CFT duality gives rise to another puzzle: how can the entirety of the complicated perturbative expansion of a quantum field theory be simple enough to be expressed at fixed order in another field theory? The computation we summarize here suggests that part of the explanation may lie in special, unexpected relations between different orders of perturbation theory in the $\mathcal{N} = 4$ gauge theory. Eden et al.⁶ have alluded to such relationships.

The operator Green functions that have been studied in ref. 3 are off shell. In contrast to our intuition that manifestly Lorentz- and supersymmetry-invariant calculations should be much harder with increasing numbers of supersymmetries, the opposite is in fact true for on-shell quantities, in particular scattering amplitudes. Indeed, the $\mathcal{N} = 4$ calculations^{7,8,9,10,11} have been pushed further earlier than the corresponding ones in QCD. Such calculations thus allow us to probe further into the perturbative expansion.

Most of these calculations were done using the unitarity-based method first described in ref. 9. To compute a one-loop amplitude, one computes tree amplitudes corresponding to the various cuts of the desired amplitude. In general, one must do these calculations with the cut-crossing legs taken in $D = 4 - 2\epsilon$ dimensions, in order to avoid subtraction ambiguities¹² in the reconstruction of the full loop amplitude from its cuts. At one loop, only two-particle cuts arise. For higher-loop amplitudes, we also compute their various cuts. The amplitudes on either side of the cut now include loop amplitudes, and for an l -loop amplitude, we must consider up to $(l + 1)$ -particle cuts. In practice, it suffices to compute the cuts of these amplitudes as well, which are again tree amplitudes. For the four-point function at two loops, for example, we have to consider three-particle cuts, with five-point tree amplitudes on either side of the cut, and two-particle “double cuts”, each involving a product of three tree-level four-point amplitudes. The construction of the cuts for the $\mathcal{N} = 4$ amplitude is surprisingly simple.¹⁰

Combining the cuts yields the integrand of a Feynman integral for the entire amplitude. In order to complete such a calculation, we also need

to perform the integrals. At one loop, this is a straightforward task, and a general method for doing so in dimensional regularization was written down a decade ago.¹³ At two loops, the inability to compute multi-scale integrals was a barrier to completing such calculations. Thanks to recent work on two-loop integrals^{18,19} and general methods of tensor reductions at higher loops,²⁰ the barrier has been surmounted. Many two-loop amplitudes relevant to experiments have since been computed.²¹

The leading-color (planar) contributions to the two-loop amplitude in the $\mathcal{N} = 4$ theory are given entirely¹⁰ in terms of the planar double box,¹⁸

$$-stA_4^{\text{tree}} \left\{ s \begin{array}{c} 1 \text{---} 4 \\ | \quad | \\ 2 \text{---} 3 \end{array} + t \begin{array}{c} 1 \text{---} 4 \\ | \quad | \\ \hline | \quad | \\ 2 \text{---} 3 \end{array} \right\}. \quad (1)$$

(The $\mathcal{N} = 4$ supersymmetry relates all non-vanishing four-point amplitudes to each other order by order.)

Define the ratio $M_n^{(L)}(\epsilon) \equiv A_n^{(L)}/A_n^{\text{tree}}$ of the l -loop leading-color amplitude to the tree-level one. The explicit computation from eqn. (1) then reveals that

$$M_4^{(2)}(\epsilon) = \frac{1}{2} \left(M_4^{(1)}(\epsilon) \right)^2 + f(\epsilon) M_4^{(1)}(2\epsilon) - \frac{5}{4} \zeta_4 + \mathcal{O}(\epsilon), \quad (2)$$

where $f(\epsilon) \equiv (\psi(1-\epsilon) - \psi(1))/\epsilon = -(\zeta_2 + \zeta_3\epsilon + \zeta_4\epsilon^2 + \dots)$. The relation between different loop orders expressed in this equation requires the use of polylogarithmic identities, and involves a non-trivial cancellation of terms between the two contributing planar double-box integrals. Note that terms through $\mathcal{O}(\epsilon^2)$ in the one-loop amplitude contribute to the $\mathcal{O}(\epsilon^0)$ terms on the right-hand side, since they can multiply the $1/\epsilon^2$ terms.

We may naturally expect a similar relation to hold between the n -point amplitudes $M_n^{(2)}$ and $M_n^{(1)}$. We can test such a generalization by studying the collinear limits of the amplitudes. The behavior of gauge-theory amplitudes in such limits may be described by splitting amplitudes whose ratio to the tree-level splitting amplitude we denote by $r_S^{(L)}$. The one- and two-loop amplitudes factorize as follows,²²

$$M_n^{(1)}(\epsilon) \rightarrow M_{n-1}^{(1)}(\epsilon) + r_S^{(1)}(\epsilon), \quad (3)$$

$$M_n^{(2)}(\epsilon) \rightarrow M_{n-1}^{(2)}(\epsilon) + r_S^{(1)}(\epsilon)M_{n-1}^{(1)}(\epsilon) + r_S^{(2)}(\epsilon). \quad (4)$$

For the conjecture (2) to hold for n -point amplitudes as well, $r_S^{(2)}$ must be related to $r_S^{(1)}$. To see what relation is required, we can use the n -point

analog of eqn. (2) to rewrite the two-loop quantities in eqn. (4) in terms of one-loop quantities and then use the one-loop collinear limit (3). Using the method of ref. 24, we find²³ that

$$r_S^{(2)}(\epsilon) = \frac{1}{2} (r_S^{(1)})^2 + f(\epsilon) r_S^{(1)}(2\epsilon), \quad (5)$$

which is exactly what is required for the generalization of eqn. (2) to hold. This does not prove it, because of possible terms which are finite in the collinear limits; but it provides strong evidence that it is correct.

The form of the $\mathcal{N} = 4$ four-point integrand is known at three loops¹⁰; only two integrals are needed to complete the computation. One of these has been computed recently²⁵ through $\mathcal{O}(\epsilon^0)$. The relative simplicity and regular structure of the integrand suggest that a similar identity to eqn. (2) may hold at higher loops. Another indication comes from the structure of infrared-singular terms, as given by Catani¹⁴ at two loops and Sterman and Tejeda-Yeomans¹⁵ at three loops. The structure at two loops is an excellent guide to full relation (2) between one- and two-loop amplitudes. If the same holds true at higher loops, we expect a relation of the form,

$$M_4^{(L)}(\epsilon; s, t) = \frac{1}{L!} [M_4^{(1)}(\epsilon; s, t)]^L + \text{lower powers of } M_4^{(1)}(m\epsilon; s, t), \quad (6)$$

where the form and coefficient of the leading term are essentially determined by leading-log resummation.

We can also use the known structure of the infrared-singular terms, which must cancel against real emission contributions in any ‘physical’ quantity, to isolate the remaining finite terms; for these, the two-loop relation based on eqn. (2) is $F_4^{(2)} = \frac{1}{2} [F_4^{(1)}]^2 - \zeta_2 F_4^{(1)} - \frac{21}{8} \zeta_4$, and a higher-loop relation would then take the form,

$$F_4^{(L)}(s, t) = \text{Polynomial}(F_4^{(1)}(s, t)) = \frac{1}{L!} [F_4^{(1)}(s, t)]^L + \text{lower powers}. \quad (7)$$

It is worth noting that eqn. (2) does *not* hold beyond order ϵ^0 . That is, the relation only holds as $D \rightarrow 4$, where the theory is conformal. The relation also appears to be special to the planar or leading- N_c contribution, as expected from the Maldacena conjecture.

Several open questions merit further study. It would be desirable, and is feasible, to compute the two-loop five-point amplitude explicitly in order to verify the generalization of eqn. (2). Likewise, knowledge of the three-loop four-point amplitude would clarify the structure of higher-loop generalizations. It would be interesting to identify a symmetry, presumably

related to superconformal invariance, underlying the relation. Witten’s new formulation²⁶ in terms of topological string theory may offer new insight.

Finally, it would be nice to connect the on-shell $\mathcal{N} = 4$ amplitudes to off-shell investigations of the duality conjecture. In this respect, it is worth noting that the tree-level collinear splitting amplitudes are closely related to the leading-order Altarelli–Parisi kernel, whose Mellin moments are the one-loop anomalous dimensions of leading-twist operators. Similarly, the next-to-leading order Altarelli–Parisi kernel can be computed²⁷ from the one-loop splitting amplitudes (and other ingredients), and we expect that the two-loop splitting amplitude can be used to compute next-to-next-to-leading order anomalous dimensions in the $\mathcal{N} = 4$ gauge theory.

There are also implications for the higher-loop structure of $\mathcal{N} = 8$ supergravity. Against prior expectations, this theory was argued¹¹ to be ultraviolet finite through four loops, based on the cut-by-cut absence of ultraviolet divergences in the four-point amplitude. At five loops, finiteness would require non-trivial cancellations between different contributions. The work outlined in this talk shows that such cancellations occur in maximally supersymmetric gauge theories. Given the close connection between the integrands of maximally supersymmetric gauge and gravity theories, as used in ref. 11, it is quite possible that non-trivial cancellations would take place in the gravitational theory.

Acknowledgments

We thank M. Staudacher for helpful discussions, and the Kavli Institute of Theoretical Physics, Santa Barbara, for its hospitality. This research was partially supported by the US Department of Energy under contracts DE-FG03-91ER40662 and DE-AC03-76SF00515, and partially by the National Science Foundation under Grant No. PHY99-07949.

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