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# AMG Smoothers for Maxwell's Equations

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# AMG SMOOTHERS FOR MAXWELL'S EQUATIONS *

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**Abstract.** Algebraic Multigrid (AMG) is used to speed up linear system solves in a wide variety of applications. This paper concentrates on expanding AMG's applicability to important new classes of problems through algorithms that automatically construct advanced smoothing techniques when needed. In particular, we apply AMG to solve Maxwell's equations. These AMG relaxation methods have their roots in smoothing methods used for geometric multigrid methods. Arnold, Falk and Winther developed an overlapping Schwarz smoother for geometric multigrid and Hiptmair developed a distributive relaxation approach. We use this knowledge to construct new smoothing procedures for AMG. We develop adapted overlapping Schwarz smoothers and distributive relaxation for AMG. We use Nédélec's H(curl,Ω)-conforming finite elements to discretize the problem. We present first results regarding the smoothing quality of the developed smoother for AMG.

**Key words.** AMG, Maxwell's equations, Nédélec elements, overlapping Schwarz smoother, distributive relaxation

**1. Introduction.** Algebraic Multigrid (AMG) methods [3] are a central tool in the numerical solution of linear and nonlinear systems that arise from the discretization of partial differential equations (PDEs). In this paper, AMG is applied to solve the 2D **definite Maxwell equation**,

$$\nabla \times \nabla \times u + \beta u = f, \qquad \text{in } \Omega, \tag{1}$$

where $\beta > 0$ is the spatially varying electrical conductivity, $u$ is the unknown electric field to be computed and $f$ is the known right-hand side. The domain, $\Omega$, is an open, bounded and connected Lipschitz domain in $\mathbb{R}^2$, and we impose Dirichlet boundary conditions. Note that Equation (1) involves two 2D-curl operators, the 2D-curl of a vector-valued function, $w = (w_1, w_2)^T : \Omega \to \mathbb{R}^2$, and the 2D-curl of a scalar function $v : \Omega \to \mathbb{R}$, defined as

$$\underline{\text{curl}}\; v := \begin{pmatrix} \partial_2 v \\ -\partial_1 v \end{pmatrix}, \qquad \text{curl }\; w := \partial_1 w_2 - \partial_2 w_1,$$

respectively.

The curl operators give rise to the standard Sobolev space,

$$H(\text{curl}, \Omega) := \{v \in L^2\left(\Omega, \mathbb{R}^2\right) \mid \text{curl}\, v \in L^2\left(\Omega\right)\},$$

where $L^2$ denotes the space of square Lesbesque integrable functions. We therefore discretize our model problem using Nédélec's $H(\text{curl}, \Omega)$-conforming finite elements [7]. The resulting linear system is denoted by

$$Au = f, \tag{2}$$

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with $A = N + P$, and where $N$ is the discrete approximation of the weak form of the curl-curl term in (1), and where $P$ is the discrete approximation to the weak form of the $\beta$ term in (1). The difficulty of using AMG to solve Maxwell's equations results from the kernel of the curl operator. Using the identity

$$\nabla \times (\nabla \Phi) = 0, \tag{3}$$

we see that gradients of scalar functions, $\Phi$, lie within the kernel of the curl operator. Thus, the kernel includes the gradients of all differentiable scalar functions $\Phi$. Since the gradient of a smooth function is smooth and the gradient of an oscillatory function is oscillatory, it is obvious that the kernel contains both smooth and oscillatory functions. For further details, consult [6]. The discrete null-space analogue of (3) is

$$FG = \Theta, \tag{4}$$

where the matrix $G$ is a discrete gradient operator and $\Theta$ denotes the zero matrix. The matrix G is trivial to construct. In particular, each row contains at most two nonzeros (with value $\pm 1$) and corresponds to an edge between two nodes of the associated nodal mesh.

The classical AMG method fixes the relaxation method, usually, a pointwise smoother, and enforces an efficient interplay with the coarse-grid correction. This procedure is not sufficient for Maxwell's equations, since Maxwell's equations have a large near null space and, therefore, contain high frequency components in the near null space (see Fig. 1 and [6] for details). In order to smooth these oscillatory components and approximate them appropriately on coarser grids, a different relaxation scheme than a simple pointwise smoother is necessary. Efficient relaxation methods for geometric multigrid methods for Maxwell's equations are proposed in [2, 5]. In this paper, we construct AMG smoothing methods based on these relaxation schemes.

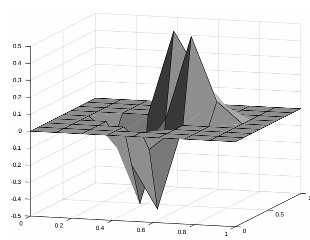

Fig. 1: High frequency component in the near null space for the 2D definite Maxwell equation on a quadrilateral grid.

This paper is structured as follows: Section 2 provides the definition of Nédélec element discretization and we introduce the general idea of smoothing procedures for Maxwell's equations. In Section 3, we describe two AMG smoothing methods for the 2D definite Maxwell equation. Numerical results are presented in Section 4. The paper closes with a conclusion.

**2. Discretization and geometric multigrid for Maxwell's equations.** We discretize the definite Maxwell equation (1) by linear **Nédélec elements**. Assuming that $\Omega$ is discretized by a quadrilateral mesh, $\mathcal{T}$, in 2D, Nédélec elements represent basis functions in $H(\text{curl}, \mathcal{T})$ spaces. We demonstrate the general approach to obtain the system matrix $A$ on the reference element, depicted in Figure 2.

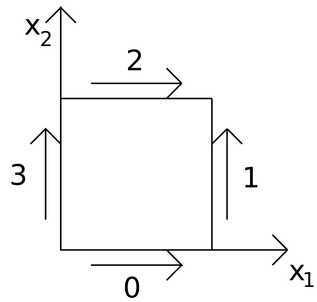


Fig. 2: The four degrees of freedom of 2D Nédélec elements in a reference configuration on a quadrilateral grid.

Let $\eta$ denote the global edge basis functions and let $x = (x_1, x_2)^T \in \Omega$ be a point. The degrees of freedom (dofs), $u$, are related to the edges of the elements,

$$\left\{ \alpha_i(u) = \int_{e_i} t_i \cdot u \, ds, \quad i \in \{1, 2, ...\} \right\}$$

for every edge, $e_i$, in the reference element. There is one dof related to each edge, therefore there are four dofs related to each element. The vector $t_i$ is the tangential unit vector of the edge $e_i$. One has to choose which direction for the unit tangential vectors to use. Our choice is illustated in Figure 2. The reference basis functions of the Nédélec elements are given by the requirement $\alpha_i(\eta_j) = \delta_{ij}$ [1, 7, 9],

$$\eta_0(x) = \begin{pmatrix} 1 - x_2 \\ 0 \end{pmatrix}, \quad \eta_1(x) = \begin{pmatrix} 0 \\ x_1 \end{pmatrix}, \quad \eta_2(x) = \begin{pmatrix} x_2 \\ 0 \end{pmatrix}, \quad \eta_3(x) = \begin{pmatrix} 0 \\ 1 - x_1 \end{pmatrix}.$$

The global Nédélec basis functions are nonzero only in the two elements who share the edge that is related to the basis function. A more detailed treatment can be found in [7, 8].

**2.1. Geometric Multigrid smoothers.** Geometric multigrid with the overlapping Schwarz smoother by Arnold, Falk and Winther (AFW) [2] or with distributive relaxation proposed by Hiptmair [4, 5], is an efficient solver for Maxwell's equations. We therefore construct two AMG smoothers based on these relaxation methods. The goal of both smoothers is to treat the kernel with a special procedure to eliminate the high frequency components. First, we review the idea of the geometric relaxation procedures.

The geometric overlapping Schwarz smoother for Maxwell's equation is a block smoother. The edge points are clustered into small overlapping blocks, $\Omega_i$, (see Figure 3) and we solve the systems $A_i u_i = f_i$, where $i$ indicates the number of the current block, $f_i$ is some right-hand side, $u_i$ is a subvector which contains all unknown edges/dofs of block $\Omega_i$, and $A_i$ is a submatrix of the Maxwell matrix $A$ that belongs to block $\Omega_i$. The number of blocks corresponds to the number of nodal points in the grid. The blocks contain all geometric nearest neighbor edges of the nodal points, depicted in Figure 3.

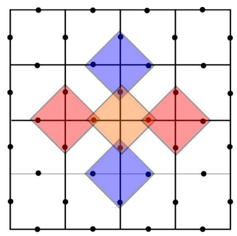

Fig. 3: Geometric idea of grid separation on quadrilateral grids.

Geometric distributive relaxation for Maxwell's equations uses a gradient matrix, $G$, that is constructed such that each row corresponds to an edge between two nodes in the associated nodal mesh. Equivalently, the matrix $G$ can be constructed such that it uses the same blocks, $\Omega_i$, as the overlapping Schwarz smoother. As described above, each block $\Omega_i$, defines a submatrix, $A_i$, of the Maxwell matrix, $A$. The eigenvectors corresponding to the smallest eigenvalue of each submatrix, $A_i$, define the columns of the gradient matrix, $G$. Therefore, the number of columns of the matrix, $G$, equals the number of blocks, $\Omega_i$. The gradient matrix $G$ is applied as a projector for the Maxwell matrix, $A$, defining a new matrix, $M = G^T AG$, and a Gauß-Seidel smoother (GS) is applied to this new matrix $M$.

**3. AMG smoothers.** In contrast to the geometric multigrid method, the geometric structure of the dofs is unknown in the AMG setting. Instead, the dependencies of the dofs are given by the matrix. Therefore, geometric smoothers need to be adapted for AMG. Based on the geometric relaxation methods by AFW [2] and Hiptmair [4, 5], described in Section 2.1, we construct an overlapping Schwarz smoother and a distributive relaxation for AMG.

**3.1. Overlapping Schwarz smoother for AMG.** The idea of the overlapping Schwarz smoother is to seperate the degrees of freedom into overlapping subsets, $\Omega_i$, and solve the corresponding systems, $A_i u_i = f_i$, as described in Section 2.1. Note that the AMG setting influences the choice of the overlapping subsets, $\Omega_i$. We will see that Maxwell's equations require the solution of all systems that contain a near null space component and that correspond to subsets with a diameter smaller than (or equal to) the coarsening factor, $m$. In this paper, the coarsening factor is chosen to be two. The coarse grid handles the larger subdomains and near null space vectors. After smoothing on all of these systems, GS is applied to the system defined by the remaining points, $\Omega_{\text{point}}$.

A simple strategy for choosing the overlapping subsets $\Omega_i$, can be described as follows:

**Algorithm 1** Select $\Omega_i$

```
Set S := ∅, Ω_i := {i}, Ω_point = Ω
for all Ω_i do
    Ω_i := Ω_i ∪ {distance 1 neighbors of i}
    A_i = A(Ω_i, Ω_i)
    if A_i is nearly singular then
        S ← S ∪ {Ω_i}
        Ω_point ← Ω_point − Ω_i
    end if
end for
```

For each $i$, starting with a predefined set, $\Omega_i$ (in the above pseudocode, $\Omega_i$ is chosen such that it contains one edge dof), certain edge dofs are added to the sets, $\Omega_i$, (for example the distance 1 neighbors of the initial edges). Submatrices are constructed from the system matrix, $A$, by restricting it to the colums and rows that belong to the edge dofs in set $\Omega_i$. The last step during the loop is to check if the submatrix includes a near null space component and at the end keep all sets for which the submatrix includes a near null space component.

Remark: The above algorithm only illustrates the general idea for choosing subsets, $\Omega_i$. In particular, considering only distance 1 neighbors is not sufficient to obtain efficient AMG smoothers. An explanation for this is given in Section 3.3 along with a different strategy that does produce good results.

After selecting the sets, $\Omega_i$, we apply the overlapping Schwarz algorithm:

**Algorithm 2** Overlapping Schwarz algorithm

**for** $k = 1, 2, ...$ **do**
    **for** $\Omega_i \in S$ **do**
        $r_k \leftarrow f - Au_k$
        $u_k \leftarrow u_k + I_{\Omega_i} A_i^{-1} I_{\Omega_i}^T r_k$
    **end for**
    $r_k \leftarrow f - Au_k$
    $u_{k+1} = u_k + I_{\Omega_{\text{point}}} A_{\text{point}}^{\sim 1} I_{\Omega_{\text{point}}}^T r_k$
**end for**

Here, $A, A_i, f$ and $u$ are defined as introduced in Section 1 and Section 2.1. The injection matrix with regard to a subdomain, $d$, is indicated by $I_d$ and $d = \Omega_{\text{point}}$ refers to the set of remaining points, which are not included in one of the near null space sets $\Omega_i$. The term $A_{\text{point}}^{\sim 1}$ denotes the approximate solution of the system $A_{\text{point}} u_{\text{point}} = I_{\Omega_{\text{point}}}^T r_k$ by the application of the Gauß-Seidel relaxation method. We use a forward iteration loop for the presmoother and a backward iteration loop for the postsmoother to obtain a symmetric algorithm.

**3.2. Distributive relaxation smoother for AMG.** The distributive relaxation smoother uses the subsets, $\Omega_i$, to construct a matrix, $G$, that is equivalent to the discrete gradient matrix used in the geometric Hiptmair smoother described in Section 2.1. The Gauß-Seidel relaxation method is used to relax all dofs and afterwards the Galerkin matrix, $G^T AG$, is used to relax the near null space components with fine scale support.

**Algorithm 3** Distributive Relaxation algorithm

**for** $\Omega_i \in S$ **do**
  Compute smallest eigenvector $v_i$ of $A_i$
  Add $v_i$ to columns of G
**end for**
**for** $k = 1, 2, ...$ **do**
  $r_k \leftarrow f - Au_k$
  $u_k \leftarrow u_k + A^{\sim 1} r_k$
  $r_k \leftarrow f - Au_k$
  $u_{k+1} = u_k + G(G^T AG)^{\sim 1} G^T r_k$
**end for**

Here, $A, A_i, G, f$ and $u$ are defined as introduced in Section 1 and Section 2.1. The terms $(G^T AG)^{\sim 1}$ and $A^{\sim 1}$ denote the approximate solution by the application of the symmetric Gauß-Seidel relaxation method.

**3.3. Subdomain decision.** The general idea of the AMG smoothers is to eliminate the near null space components that cannot be represented on the coarse grid. This can be done in the following way:

$S = \emptyset$
$K =$ all sets with diameter $\leq$ m
**for** $J \in K$ **do**
  **for all** $J$ that include near null space components **do**
    $S \leftarrow S \cup \{J\}$
  **end for**
**end for**

Since in this paper, we use a coarsening factor of $m = 2$, we inspect all sets with diameter $m = 2$ and check for near null space components. Then, we use one of the two relaxation methods, overlapping Schwarz relaxation or distributive relaxation.

Sufficient subsets are necessary to eliminate near null space components. As already mentioned in Section 3.1, the distance 1 neighbor sets are not sufficient for AMG. In the context of geometric multigrid, nearest neighbors are closest edge dofs to a nodal point in a geometrical sense (see Fig. 3). In the AMG context, we do not know the location of nodal points and edge dofs. Instead, distance 1 neighbors in terms of AMG are the closest edge dof to another edge dof (see Fig. 4). Closest edges to edge $i$ in the AMG context are all edge dofs that have a corresponding matrix entry in row $i$.

Fig. 4: Examples of distance 1 neighbors in AMG context on quadrilateral grids.

The distance 1 neighbor sets in AMG are not sufficient as block smoothing sets $\Omega_i$, since they do not account for all diameter 2 subdomains. We need to establish sets that cover all diameter 2 subdomains which include a near null space component. Accordingly, we construct different sets that inspect all diameter $m = 2$ subdomains and includes near null space components. The algorithm looks as follows and is explained below.

**Algorithm 4** Select $\Omega_i$

Set $S := \emptyset$, $\Omega_{\text{point}} = \Omega$
**for** $i = 1 : \#\text{dof}$ **do**
  $T =$\{all distance 2 neighbors of $i$\}
  **for** $j \in T$ **do**
    $\Omega_{i,j} = \{i, j\} \cup$ \{common distance 1 neighbors of $i$ and $j$\}
    $A_{i,j} = A(\Omega_{i,j}, \Omega_{i,j})$
    **if** $A_{i,j}$ is nearly singular **and** $\Omega_{i,j} \notin S$ **then**
      $S \leftarrow S \cup \{\Omega_{i,j}\}$
      $\Omega_{\text{point}} \leftarrow \Omega_{\text{point}} - \Omega_{i,j}$
    **end if**
  **end for**
**end for**

In the special case of quadrilateral elements (see Fig. 5), the algorithm above generates the same sets as AFW and Hiptmair in the geometric case. For all degrees of freedom, first the algorithm constructs a set that includes all distance $m = 2$ neighbors (illustrated by crosses in Figure 5). Thereafter, a set, $\Omega_{i,j}$, is constructed that includes the initial dof, $i$, (red rectangle in Fig. 5), a distance 2 neighbor, $j$ and all common distance 1 neighbors of $i$ and $j$. All matrices are checked if they include a near null space and we keep just the sets that include a near null space. So far, we did not find a way to compute these sets in a more efficient way.

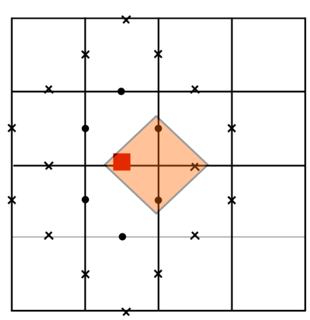

Fig. 5: Example of an AMG subdomain, $\Omega_{ij}$, constructed with Alg. 4. The red box indicates a given initial dof, $i$, crosses indicate distance 2 points of $i$, and dots indicate distance 1 points of $i$.

**4. Numerical Results.** The numerical results in this section show the efficiency of the AMG overlapping Schwarz smoothing method (OSS) and the AMG distributive relaxation method (DR) for the definite Maxwell equation.

**4.1. Cost comparison.** The algorithms in Section 3.1 and Section 3.2 both have a setup phase to create the sets $\Omega_i$. In addition, DR generates the matrix $G$ out of the sets $\Omega_i$. OSS applies a block inverse for each near null space component, so it is a bit more expensive in the solve phase. In contrast DR uses the Gauß-Seidel method to solve the transformed system. OSS has the potential to be cheaper in the setup phase because the actual near null space components do not need to be computed. For example, a few iterations of a pointwise smoother can determine slow convergence and hence the presence of a near null space.

**4.2. Performance.** First, we show numerical results of the smoother by itself without an underlying multigrid method. For this purpose, we apply a Fourier mode smoothing study for OSS and DR as described in Section 3.

The Fourier mode study can be described as follows: The homogenous system $Au = 0$ is iteratively solved with different initial guesses. The system matrix $A$ is generated by using $\beta = 0.1$. The initial guesses, $\sin(k\pi x_1)\sin(l\pi x_2)$, with $l = k = 1, ..., n-2$ are the Fourier modes. Here $x_1, x_2$ denote the coordinates of the unknowns in the grid and $n$ indicates the number of nodal gridpoints in one direction. We set $l = k$ to obtain 2D results.

The results of the Fourier mode study for both smoothers are presented in Figure 6. We indicate results for the 33x33 grid with 2112 dofs. We plot the number of iterations needed to reduce the infinity norm of the the error by 0.001 for every mode $k$. The number of iterations in Figure 6 is between 7 and 338 for DR and between 5 and 663 for OSS.

The initial guesses $\sin(k\pi x_1)\sin(l\pi x_2)$ with small $k$ indicate smooth functions, whereas these initial guesses with large $k$ indicate oscillatory functions. Good smoothers should reduce oscillatory components efficiently.

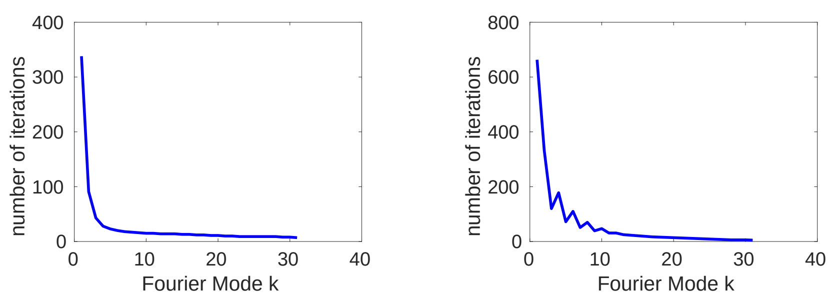


Fig. 6: Number of iterations for different Fourier modes with DR (left) and OSS (right).

In the following, we show numerical results of an AMG two-grid method using the AMG overlapping Schwarz smoothing method and the AMG distributive relaxation method as described in Section 3. The numerical results in this section demonstrate the efficiency of the AMG smoothers for the definite Maxwell equation and we compare both smoothers.

We consider the following model problem: take $\Omega = [0, 1]^2, \beta = 0.1$ and the right-hand-side is set to zero. We use a random initial guess $u_0(x, y)$. The model problem is solved with one pre- and one post smoothing step $(V(1, 1))$. Since it is not the concern of this paper to find an appropriate AMG coarsening, we construct the coarse grid according to the geometric multigrid, see Figure 7. The coarse grid points are new points on the grid and not a subset of the fine grid points. We use a restriction with the weighting,

$$\begin{bmatrix} \frac{1}{4} & & \frac{1}{4} \\ \frac{1}{2} & \bullet & \frac{1}{2} \\ \frac{1}{4} & & \frac{1}{4} \end{bmatrix}, \quad \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ & \bullet & \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{bmatrix},$$

where • denotes the resulting coarse grid point. The left stencil denotes the restriction of an x-edge coarse point and the right stencil of a y-edge coarse point. The interpolation matrix is the transpose of the restriction matrix and we use the Galerkin coarse grid operator.

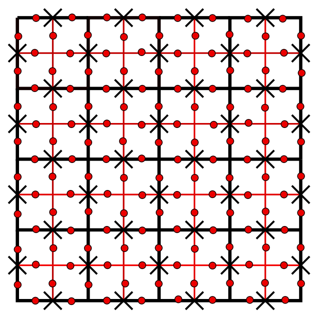

Fig. 7: The coarse grid (black) and the corresponding coarse grid points × resulting from the fine grid (red) and the fine grid points •.

The convergence rate of the AMG two-grid method for different grid sizes and both smoothers is shown in the following. We show the results of three different grid sizes, 17x17 grid with 544 degrees of freedom, 33x33 grid with 2112 dofs and 65x65 grid with 8320 dofs. We use the 2-norm to measure the convergence.

The asymptotic convergence factors of the two-level AMG method with OSS and DR are indicated in Table 1. These results present an asymptotic convergence factor of 0.19 for the two-level method with OSS and an asymptotic convergence factor of 0.14 for the two-level method with DR. We observe h-independent convergence rates. Comparable results can also be found in [6].

| #dofs | OSS | DR |
|---|---|---|
| 544 | 0.19 | 0.13 |
| 2112 | 0.19 | 0.13 |
| 8320 | 0.19 | 0.14 |

Table 1: Convergence factors, $||r_k||_2/||r_{k-1}||_2$, $k \to \infty$, of the two-level AMG with OSS and DR.

The results of the AMG method with DR look more promising. However, the DR uses an expensive symmetric GS method for the entire system $Au = f$, whereas the OSS is symmetrized by forward pre-relaxation and backward post-relaxation. Using this approach for DR yields an asymptotic convergence rate of 0.37.

**5. Conclusion.** We developed AMG based smoothers for the definite Maxwell equation, the overlapping Schwarz smoother and distributive relaxation. The results of the two-grid methods with OSS and DR for the definite Maxwell equation are comparable with results of already known methods for geometric multigrid which can be found in the literature, cf. [6]. Analyzing the smoothers on more general problems is future work.

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