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A Minimum Variance Algorithm for Overdetermined TOA Equations with an Altitude Constraint

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Abstract

We present a direct (non-iterative) method for solving for the location of a radio frequency (RF) emitter, or an RF navigation receiver, using four or more time of arrival (TOA) measurements and an assumed altitude above an ellipsoidal earth. Both the emitter tracking problem and the navigation application are governed by the same equations, but with slightly different interpretations of several variables. We treat the assumed altitude as a soft constraint, with a specified noise level, just as the TOA measurements are handled, with their respective noise levels. With 4 or more TOA measurements and the assumed altitude, the problem is overdetermined and is solved in the weighted least squares sense for the 4 unknowns, the 3-dimensional position and time. We call the new technique the TAQMV (TOA Altitude Quartic Minimum Variance) algorithm, and it achieves the minimum possible error variance for given levels of TOA and altitude estimate noise. The method algebraically produces four solutions, the least-squares solution, and potentially three other low residual solutions, if they exist. In the lightly overdermined cases where multiple local minima in the residual error surface are more likely to occur, this algebraic approach can produce all of the minima even when an iterative approach fails to converge. Algorithm performance in terms of solution error variance and divergence rate for baseline (iterative) and proposed approach are given in tables.

Contents

List of Figures

List of Tables

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Chapter 1

Introduction

In this paper we consider the algebraic solution of overdetermined systems of time of arrival (TOA) equations, plus an equation relating the solution and an assumed altitude. This assumed height above the reference ellipsoid (HAE) may be obtained, for example, from digital terrain elevation data (DTED) after making a preliminary estimate of the location of the device. The relative geolocation strength of the TOAs and the assumed altitude depend on the bandwidth of the RF signals and the quality of the altitude data available. One application where the altitude constraint is particularly helpful is that of geolocating a narrowband emitter. A fundamental concept in radar and radio location is that TOA measurement error is inversely proportional to the root-mean-square (RMS) bandwidth of the RF signal [8]. Therefore locating or tracking the source of a narrowband signal, for example a speech modulated signal, results in relatively noisey TOA measurements. In this case the variance of the HAE estimate may be much lower than the TOA error variances (expressed in distance units), and the location solution error will be greatly reduced by using the HAE data.

Solutions of this geolocation problem in the precisely determined case, where there are three TOAs and an HAE have been given in [13], [7] and [10]. There have been numerous papers discussing the solution of overdetermined systems of TOA equations [2, 11, 17, 19, 4, 3, 16, 9], but a paper by Ho and Chan [6] is the only one we know of discussing the overdeteremined problem when an altitude constraint is included. In that paper the altitude constraint is a hard, or exact constraint, whereas in the present paper we weight the altitude constraint as we weight the TOA equations, with the variances of the data errors. This weighting scheme produces the minimum possible solution error variance given the TOA and altitude data error variances [14]. Another difference in our approach is that [6] uses a spherical earth model, which requires an iterative process to handle an ellipsoidal earth, but this may be combined with the iteration that the use of DTED forces (to compute the location used to access the database). The current state of the art for producing minimum variance solutions to this problem is Gauss-Newton iteration, which we show will diverge occasionally in lightly overdetermined cases. We present an algorithm we call the TOA-Altitude Quartic Minimum Variance (TAQMV) algorithm that produces the minimum variance solutions analytically, eliminating divergence failures.

To facilitate our overview of this paper, we now give the basic governing equations. These equations describe both the navigation application where we solve for the position of a receiver, as well as the emitter tracking application where we solve for the location of a transmitter. In the navigation application the user has knowledge of the positions of several radio frequency (RF) transmitters. In the emitter tracking application the system uses the known position of several RF receivers. In either application the equations governing the signal propagation are the same, and the N system RF element positions are denoted $\mathbf{s}_k, k = 1 \ldots N$, while the position of the device to be located is denoted $\mathbf{x}$. In the navigation application the receiver measures the apparent range between the receiver and each transmitter using its biased clock which produces N pseudoranges, τ_k. In the emitter tracking application the system measures the time of arrival (TOA) of the emitted signal at each of the receivers. Here we denote the kth TOA scaled by the speed of light as τ_k. Choosing the time reference epic to make the τ_k small is generally a good thing to do for numerical stability.

Assuming the signal travels with the speed of light c, we can write

$$||\mathbf{x} - \mathbf{s}_k|| = \tau_k - \tau \text{ for } k = 1 \ldots N, \tag{1.1}$$

where $\tau = ct$ is the range-equivalent receiver clock bias, or alternatively the emitter transmit time. With either interpretation Eqn. (1.1) gives us N equations in 4 unknowns, the 3-dimensional vector $\mathbf{x}$ giving the devices's location, and τ, either the receiver clock bias or the emitter transmit time. The subscripted quantities are known or measured parameters/data. We consider the pseudoranges (or the TOAs) to be perturbed with measurement noise so that Eqn. (1.1) are generally inconsistent. Furthermore we will assume that we can estimate the variance of the measurement noise for each pseudorange/TOA which allows us to calculate the minimum-variance solution to Eqn. (1.1), which we consider to be the most desirable solution of an inconsistent system.

We also have an altitude constraint of the form

$$\mathbf{x}^T \mathbf{K}(h) \mathbf{x} - R_e^2 = 0, \tag{1.2}$$

where $\mathbf{K}$ is a positive definite matrix defining an oblate ellipsoid of revolution

$$\mathbf{K} = \begin{pmatrix} \frac{R_e^2}{(R_e+h)^2} & 0 & 0 \\ 0 & \frac{R_e^2}{(R_e+h)^2} & 0 \\ 0 & 0 & \frac{R_e^2}{(R_p+h)^2} \end{pmatrix} \tag{1.3}$$

where R_e and R_p are the equatorial and polar radii of the earth respectively, and h is the HAE [10]. In applications where the altitude data, h, can be estimated and has lower variance than the TOA data, using the altitude constraint gives a better geolocation estimate than the TOA data alone would give. When it can be assumed that the device is on the surface of the earth, h is the height of the local terrain above the reference ellipsoid. Alternatively, if the device is a known distance above the surface of the earth, the additional height of the device above the local terrain would be included in h. In any event, h is the estimated height of the device above the reference ellipsoid, however that estimate is obtained. Depending on the requirements of the application, h

could be estimated using various sources of altitude data such as a DTED database or an altimeter. We assume that an estimate of the standard deviation of the error in h can also be made, as this quantity, σ_h, is used to weight the altitude equation in the weighted least squares solution.

Rather than dealing with Eqn. (1.1), which we can call the primitive TOA equations, it can be more convenient to solve the squared TOA equations

$$||\mathbf{x}-\mathbf{s}_k||^2 = (\tau - \tau_k)^2 \quad \text{for } k = 1, N \, . \tag{1.4}$$

The squared equations (1.4) can be simpler to treat algebraicly than the primitive equations, Eqns. (1.1). However, it should be noted that as with the case of the primitive equations [15], [7], [1], it is not possible to weight the squared equations a priori to achieve minimum variance solutions because the weights require the lengths of the RF links. In the algorithm we present in this paper, we will choose the weightings based on a preliminary estimate of $\mathbf{x}$ that allows us to achieve solutions that have the minimum possible variance. This is the approach used in [14].

The following is a brief overview of the sections in this paper. In §2 we introduce the notation and equations that we will be solving throughout the paper. The resulting equations are called the TOA-Altitude Full Least Squares (TAFLS) equations, and they yield minimum variance solutions. However, they can only be solved iteratively using a method such as Gauss-Newton [17, 14]. In §3 we introduce the TOA-Altitude Reduced Least Squares (TARLS) algorithm, that does not give a minimum-variance solution. The TARLS algorithm is the extension of Bancroft's [2] non-minimum-variance algorithm to include a weighted altitude constraint. The TARLS solution is used to set certain parameters in a preferred algorithm that we call the TOA-Altitude Quartic Minimum Variance (TAQMV) algorithm. In §4 we introduce the TAQMV algorithm, and in §5 we show that when we linearize the TAQMV equations we get the same equations as when we linearize the TAFLS equations. This shows that the closed-form TAQMV solution is minimum variance as is the TAFLS solution (when an iterative method converges to this solution). In §6 we tabulate simulation results for the closed-form TAQMV algorithm and for iterative techniques of solution of the 4-TOA plus altitude-constraint geolocation problem. The data in this section shows the TAQMV algorithm working as desired, i.e. producing MV solutions without the occasional divergences of Gauss-Newton. In §7 we give our conclusions. In the Appendix we discuss some of the details of using resultants [18, 10] to find the solutions of two quadratic equations in two unknowns.

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Chapter 2

The TAFLS Equations

In this section we define the TOA-Altitude Full Least Squares (TAFLS) equations. A minimum-variance geolocation algorithm based on these equations could be defined by iteratively solving the equations after specifying the algorithm used to determine the starting point of the iteration [17, 14]. In lightly overdetermined conditions the performance of this algorithm would be highly dependent on the initialization. In Section §6 we will define and characterize two such algorithms that are useful for comparison to our proposed preferred algorithm which is given in §4. In the following two sections we will also make approximations to various terms in the TAFLS equations to obtain our proposed non-iterative minimum-variance algorithm. We now derive the TAFLS equations.

If we introduce the vector

$$\mathbf{z} = \begin{pmatrix} \mathbf{x} \\ \tau \end{pmatrix}, \tag{2.1}$$

we can write the equations (1.4) as

$$\mathbf{S}\mathbf{z} = \mu\mathbf{e} + \mathbf{b}, \tag{2.2}$$

where

$$\mu = \mathbf{z}^T\mathbf{L}\mathbf{z} = \tau^2 - \mathbf{x}^T\mathbf{x}, \tag{2.3}$$

$$\mathbf{L} = \begin{pmatrix} -\mathbf{I} & 0 \\ 0 & 1 \end{pmatrix}, \tag{2.4}$$

where $\mathbf{I}$ is the 3-by-3 identity matrix,

$$\mathbf{S} = \begin{pmatrix} -2\mathbf{s}_1^T & 2\tau_1 \\ -2\mathbf{s}_2^T & 2\tau_2 \\ . & . \\ . & . \\ -2\mathbf{s}_N^T & 2\tau_N \end{pmatrix}, \tag{2.5}$$

$$\mathbf{e}^T = (1, 1, 1, ..., 1), \tag{2.6}$$

and

$$\mathbf{b}^T = (b_1, b_2, ...b_N) \quad b_k = \tau_k^2 - \mathbf{s}_k^T\mathbf{s}_k. \tag{2.7}$$

The altitude constraint can be written as $\chi(\mathbf{z}) = 0$, where

$$\chi(\mathbf{z}) = \mathbf{z}^T \mathbf{Q} \mathbf{z} - R_e^2, \tag{2.8}$$

where

$$\mathbf{Q} = \begin{pmatrix} \mathbf{K} & 0 \\ 0 & 0 \end{pmatrix}, \tag{2.9}$$

where $\mathbf{K}$ is defined in Eqn. (1.3).

Ideally we would like to satisfy the system of equations $\gamma(\mathbf{z}) = \mathbf{0}$ where

$$\gamma(\mathbf{z}) = \mathbf{S}\mathbf{z} - \mu(\mathbf{z})\mathbf{e} - \mathbf{b}, \tag{2.10}$$

along with the equation $\chi(\mathbf{z}) = 0$. When we have four or more satellites, this will give us an over-determined system of equations that we will satisfy in the least squares sense. This motivates the definition of the objective function

$$P(\mathbf{z}) = \frac{1}{2}\gamma^T(\mathbf{z})\mathbf{W}\gamma(\mathbf{z}) + \frac{1}{2}w(\chi(\mathbf{z}))^2, \tag{2.11}$$

where $\mathbf{W}$ is a weighting matrix, and w is a scalar giving the weighting of the altitude constraint. We will see in (2.15) and (2.16) that $\mathbf{W}$ and w are both functions of $\mathbf{z}$.

At any value of $\mathbf{z}$ that minimizes this objective function, we must have $\nabla P(\mathbf{z}) = 0$, which is equivalent to requiring that to first order we have $\delta P = \delta\mathbf{z}^T \nabla P = 0$ for all $\delta\mathbf{z}$. To first order we have

$$\delta\gamma = \mathbf{S}\delta z - 2\mathbf{e}\left(\mathbf{z}^T \mathbf{L}\delta\mathbf{z}\right), \tag{2.12}$$

and

$$\delta\chi(\mathbf{z}) = 2\mathbf{z}^T \mathbf{Q}\delta\mathbf{z}. \tag{2.13}$$

It follows that we can write

$$\delta P = \delta\mathbf{z}^T\left(\mathbf{S}^T - 2\mathbf{L}\mathbf{z}\mathbf{e}^T\right)\mathbf{W}\gamma + 2w\chi(\mathbf{z})\delta\mathbf{z}^T\mathbf{Q}\mathbf{z}.$$

If we require that this vanish for all values of $\delta\mathbf{z}^T$, this gives us what we will refer to as the TAFLS (TOA Altitude Full Least Squares) equations.

Definition 1 (TOA Altitude Full Least Squares)**.** *The TAFLS equations are given by*

$$\Gamma^T(\mathbf{z})\mathbf{W}\gamma(\mathbf{z}) + 2w\chi(\mathbf{z})\mathbf{Q}\mathbf{z} = \mathbf{0}, \tag{2.14a}$$

where $\chi(\mathbf{z})$ is defined as in Eqn. (2.8), $\gamma(\mathbf{z})$ as in (2.10), and

$$\Gamma(\mathbf{z}) = \mathbf{S} - 2\mathbf{e}\mathbf{z}^T\mathbf{L}. \tag{2.14b}$$

Eqn. (2.14) gives four equations for the four unknowns in the vector $\mathbf{z}$. Since the matrix Γ depends linearly on $\mathbf{z}$, and the vector γ has a quadratic dependence, these equations have a cubic nonlinearity before considering there is an additional non-linearity due to the fact that the weighting matrix depends on the solution $\mathbf{z}$. It is often stated that when solving an overdetermined system of equations in the least squares sense that the equations should be weighted by the inverse of the data covariance matrix. This is true when the derivative of each equation with respect to the data is unity, such as in Eqn. (1.1). In the more general case [15] however the optimal weighting matrix is given by

$$\mathbf{W} = (\mathbf{D}\Sigma\mathbf{D})^{-1}, \tag{2.15}$$

where $\mathbf{D}$ is the matrix of partial derivatives of the equations with respect to the data. For the squared system in Eqn. (1.4) $\mathbf{D}$ is the diagonal matrix with $1/(\tau - \tau_k)$ on the kth diagonal, and Σ is the covariance matrix of the TOAs (or pseudo-ranges), τ_k. Similarly, the scalar analog of Eqn. (2.15) gives the optimal altitude equation weight as

$$w = \frac{1}{\sigma_h^2(\mathbf{z}^T \frac{\partial \mathbf{Q}}{\partial h}\mathbf{z})^2}. \tag{2.16}$$

Recalling that τ is the fourth element of $\mathbf{z}$ we see that the optimal weightings $\mathbf{W}$ and w depend on $\mathbf{z}$, the solution. In the TAQMV equations we use a non-minimum variance solution to estimate $\mathbf{z}$, and set the weights using this estimate of $\mathbf{z}$.

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Chapter 3

The TARLS Algorithm

In this section we present a technique that gives a solution to the overdetermined system of altitude and TOA equations. This solution will not be a minimum variance solution, but it is accurate enough that we can use it either as an initial guess to the Gauss-Newton method, or as input to an algorithm that will be minimum variance. We will call this solution the TARLS (TOA-Altitude Reduced Least Squares) solution.

Referring to (2.2) through (2.9) our overdetermined system of equations can be written

$$\mathbf{S}\mathbf{z} = \mu\mathbf{e} + \mathbf{b}, \tag{3.1}$$

$$\mathbf{z}^T\mathbf{L}\mathbf{z} = \mu, \tag{3.2}$$

$$\mathbf{z}^T\mathbf{Q}\mathbf{z} - R_e^2 = 0. \tag{3.3}$$

Similar to the Bancroft algorithm [2], we will write

$$\mathbf{z} = \mu\alpha + \beta, \tag{3.4}$$

where α and β are the least squares solutions (or if TOA variances vary significantly the weighted least squares solutions) to

$$\mathbf{S}\alpha = \mathbf{e}, \tag{3.5}$$

and

$$\mathbf{S}\beta = \mathbf{b}. \tag{3.6}$$

If we substitute the expression (3.4) into either Eqn. (3.2) or (3.3), we get a quadratic equation for μ. In particular, substituting (3.4) into Eqn. (3.2) gives us the equation

$$p(\mu) = p_2\mu^2 + p_1\mu + p_0 = 0, \tag{3.7}$$

where

$$(p_2, p_1, p_0) = \left(\alpha^T \mathbf{L}\alpha, 2\alpha^T \mathbf{L}\beta - 1, \beta^T \mathbf{L}\beta\right). \tag{3.8}$$

Similarly, substituting the expression for $\mathbf{z}$ from Eqn. (3.4) into Eqn. (3.3) gives us

$$q(\mu) = q_2\mu^2 + q_1\mu + q_0 = 0, \tag{3.9}$$

where

$$(q_2, q_1, q_0) = \left(\alpha^T \mathbf{Q}\alpha, 2\alpha^T \mathbf{Q}\beta, \beta^T \mathbf{Q}\beta - R_e^2\right). \tag{3.10}$$

We now have two quadratic equations to be satisfied. In general we will not be able to satisfy both of these equations. However, we assume that the noise is small enough so that these equations should be nearly consistent. There are two approaches that come to mind. The first approach is to consider these equations as equations for the unknown quantities $\mu_2 = \mu^2$ and $\mu_1 = \mu$, but without requiring that $\mu_2 = \mu_1^2$. This approach has been found to work well most of the time, but it fails when the linear system for μ_2 and μ_1 is nearly singular. This can happen even when there is nothing singular about the particular configuration of satellites. For this reason we prefer the approach that we will now outline.

Since we would like to solve both Eqns. (3.7) and (3.9) simultaneously, this suggests trying to minimize the quantity

$$r(\mu) = p^2(\mu) + q^2(\mu). \tag{3.11}$$

The equation

$$\frac{dr}{d\mu} = 0 \tag{3.12}$$

gives us a cubic equation for μ. In order for a root of this equation to be a minimum of $r(\mu)$, we must have

$$\frac{d^2r}{d\mu^2} > 0. \tag{3.13}$$

Taking the roots of the first derive of $r(\mu)$ from a numerical root-finding routine, and checking the sign of the second derivative gives us either one or two minima of (3.11), which we put into (3.4) to compute TARLS solutions. In the vast majority of cases there is only one minimum, but occasionally situations arise where there are two minima. In such cases we take the TARLS solution that satisfies the original over-determined system of equations with the smallest total residual error.

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Chapter 4

The TAQMV Algorithm

The TAFLS equations (2.14) give us four cubic equations (assuming the weighting matrix is known) involving the four dimensional vector $\mathbf{z}$. Rather than solving this system of equations exactly, we now give a method that gives the same linearized equations as this full set of equations when we linearize them about a true, or noiseless, solution. This is similar to the procedure carried out in [14] for the TOA equations. In particular, we will replace the equations (2.14) by the simplified equations

Definition 2 (The TAQMV Equations)**.** *We define the TOA Altitude Quartic Minimum Variance equations as*

$$\Gamma_R^T \mathbf{W}\gamma(\mathbf{z}) + 2w\chi(\mathbf{z})\mathbf{Q}\hat{\mathbf{z}} = \mathbf{0}, \tag{4.1}$$

where

$$\Gamma_R = \mathbf{S} - 2\mathbf{e}\hat{\mathbf{z}}^T\mathbf{L}, \tag{4.2}$$

$\hat{\mathbf{z}}$ is the solution to the TARLS equations from the previous section, $\chi(\mathbf{z})$ is defined as in Eqn. (2.8), $\gamma(\mathbf{z})$ is defined as in Eqn. (2.10), $\mathbf{W}$ is computed using τ from $\hat{\mathbf{z}}$ in (2.15) and w is computed using $\hat{\mathbf{z}}$ in (2.16).

The equation (4.1) is quadratic in $\mathbf{z}$, rather than being cubic as in Eqn. (2.14). Eqn. (4.1) differs from the TAFLS equation in Eqn. (2.14a) in that we use the known quantity $\hat{\mathbf{z}}$ in $\Gamma(\mathbf{z})$ rather than using the unknown $\mathbf{z}$. Similarly, we use $\hat{\mathbf{z}}$ when evaluating $\mathbf{Qz}$ in Eqn. (4.1). We can solve this equation in a way similar to [14] where the Bancroft algorithm [2] was extended with an additional term to give a minimum-variance algorithm. To do this we put (2.10) in to (4.1) and solve for $\mathbf{z}$ in the least squares sense. To express $\mathbf{z}$ in the form in (4.9) we define the quantities α, β, and ξ such that

$$\Gamma_R^T \mathbf{WS}\alpha = \Gamma_R^T \mathbf{We}, \tag{4.3}$$

$$\Gamma_R^T \mathbf{WS}\beta = \Gamma_R^T \mathbf{Wb}, \tag{4.4}$$

and

$$\Gamma_R^T \mathbf{W}\mathbf{S}\xi = -2w\mathbf{Q}\hat{\mathbf{z}}. \tag{4.5}$$

If we only have four satellites, we can factor out the term $\Gamma_R^T \mathbf{W}$ in Eqns. (4.3) and (4.4). Otherwise, it is best to obtain α and β by doing a QR factorization of the matrix Γ_R. In particular, to find

$$\Gamma_R = \mathbf{Q}_0 \mathbf{R}_0. \tag{4.6}$$

If we substitute this into Eqns. (4.3) and (4.4) and factor out the term R_0^T from both sides of these equations we get the equations

$$\mathbf{Q}_0^T \mathbf{W}\mathbf{S}\alpha = \mathbf{Q}_0^T \mathbf{W}\mathbf{e}, \tag{4.7}$$

$$\mathbf{Q}_0^T \mathbf{W}\mathbf{S}\beta = \mathbf{Q}_0^T \mathbf{W}\mathbf{b}. \tag{4.8}$$

When the matrix Γ_R is poorly conditioned, the equations (4.7) and (4.8) will be better conditioned than (4.3) and (4.4) since we have factored out the poorly conditioned matrix R_0 from both sides (see §5.3 in [5]). Unfortunately, we cannot factor out such a term from (4.5) so we address that term below.

Assuming we know μ and χ, our solution will be

$$\mathbf{z} = \mu\alpha + \beta + \chi\xi. \tag{4.9}$$

In order to determine the values of χ and μ we substitute the expression (4.9) into our expressions (2.3) and (2.8) defining μ and χ. This will give us a system of two quadratic equations in χ and μ. In particular, the equation (2.3) gives us the equation

$$f(\chi,\mu) = \mu^2 f_2(\chi) + \mu f_1(\chi) + f_0(\chi) = 0, \tag{4.10}$$

where

$$f_k(\chi) = \chi^2 f_{k2} + \chi f_{k1} + f_{k0}, \text{ for } k = 0,1,2, \tag{4.11}$$

$$(f_{22}, f_{21}, f_{20}) = \left(0, 0, \alpha^T \mathbf{L}\alpha\right), \tag{4.12}$$

$$(f_{12}, f_{11}, f_{10}) = \left(0, 2\alpha^T \mathbf{L}\xi, 2\alpha^T \mathbf{L}\beta - 1\right), \tag{4.13}$$

$$(f_{02}, f_{01}, f_{00}) = \left(\xi^T \mathbf{L}\xi, 2\beta^T \mathbf{L}\xi, \beta^T \mathbf{L}\beta\right). \tag{4.14}$$

Similarly, substituting the expression in Eqn. (4.9) into Eqn. (2.8) we get

$$g(\chi, \mu) = \mu^2 g_2(\chi) + \mu g_1(\chi) + g_0(\chi) = 0, \tag{4.15}$$

where

$$g_k(\chi) = g_{k2}\chi^2 + g_{k1}\chi + g_{k0}, \text{ for } k = 0, 1, 2 , \tag{4.16}$$

$$(g_{22}, g_{21}, g_{20}) = \left(0, 0, \alpha^T \mathbf{Q} \alpha\right), \tag{4.17}$$

$$(g_{12}, g_{11}, g_{10}) = \left(0, 2\alpha^T \mathbf{Q} \xi, 2\alpha^T \mathbf{Q} \beta\right), \tag{4.18}$$

$$(g_{02}, g_{01}, g_{00}) = \left(\xi^T \mathbf{Q} \xi, 2\beta^T \mathbf{Q} \xi - 1, \beta^T \mathbf{Q} \beta - R_e^2\right). \tag{4.19}$$

Eqns. (4.10) and (4.15) gives us two quadratic equations in the two unknowns μ and χ. This will have four solutions. In the appendix we show how to solve these equations using the theory of resultants [18, 10]. Once we know μ and χ, we can use Eqn. (4.9) to determine $\mathbf{z}$.

In practice we slightly modify this algorithm to ensure numerical robustness. Since the vector ξ can be large when the matrix Γ_R is poorly conditioned. Rather than using the vector ξ, we use the vector

$$\hat{\xi} = \frac{\xi}{||\xi||}, \tag{4.20}$$

and

$$\hat{\chi} = \chi \, ||\xi|| \tag{4.21}$$

Rather than solving for χ we now solve for $\hat{\chi}$. The equations for $\hat{\chi}$ are identical to the equations for χ , except we replace ξ by $\hat{\xi}$. and we need to replace Eqn. (4.19) by

$$g_0(\hat{\chi}) = g_{02}\hat{\chi}^2 + g_{01}\hat{\chi} + g_{00} = \hat{\chi}^2 \left(\hat{\xi}^T \mathbf{Q} \hat{\xi}\right) + \hat{\chi} \left(2\beta^T \mathbf{Q} \hat{\xi} - \frac{1}{||\xi||}\right) + \left(\beta^T \mathbf{Q} \beta - R_e^2\right). \tag{4.22}$$

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Chapter 5

Proof of TAQMV Minimum Variance

In this section we show that when we linearize the TAFLS and TAQMV equations about a solution with no noise, we get the same equations. This shows us that we will get the same statistics for these two solution techniques for small perturbations, and that the TAQMV algorithm therefore achieves the minimum error variance for small data errors. This method of establishing the minimum variance property was discussed extensively in [14].

We suppose that in the absence of noise there is a solution $\mathbf{z}_0$ that identically satisfies the equations (2.10) and (2.8). That is, we have

$$\gamma(\mathbf{z}_0, \mathbf{d}_0) = \mathbf{0}, \tag{5.1}$$

and

$$\chi(\mathbf{z}_0, h_0) = 0, \tag{5.2}$$

where we have included the functional dependence on the data $\mathbf{d}$, the vector of the time data τ_k, and h, the altitude, with the subscript naught indicating noiseless quantities. We will introduce the matrix

$$\Gamma_0 = \mathbf{S}_0 - 2\mathbf{e}\mathbf{z}_0^T\mathbf{L}, \tag{5.3}$$

where $\mathbf{S}_0$ is (2.5) evaluated with noiseless data. We also introduce the vector

$$\gamma_0 = \gamma(\mathbf{z}_0, \mathbf{d}_0) = \mathbf{0}, \tag{5.4}$$

and the scalar

$$\chi_0 = \chi(\mathbf{z}_0, h_0) = 0. \tag{5.5}$$

We now consider how this solution changes when we perturb the data so that the equations are no longer consistent. We will let $\delta\Gamma$, $\delta\gamma$, $\delta\chi$, $\delta\mathbf{W}$, δw and $\delta\mathbf{z}$ represent the first order changes in Γ, γ, χ, $\mathbf{W}$, w and $\mathbf{z}$, when we add noise to our system. The perturbed TAFLS equations (2.14) can be written as

$$(\Gamma_0 + \delta\Gamma)^T(\mathbf{W}_0 + \delta\mathbf{W})(\gamma_0 + \delta\gamma) + 2(w_0 + \delta w)(\chi_0 + \delta\chi)(\mathbf{Q}_0 + \delta\mathbf{Q})(\mathbf{z}_0 + \delta\mathbf{z}) = 0. \tag{5.6}$$

However, since $\gamma_0 = \mathbf{0}$, and $\chi_0 = 0$, to first order we have

$$\Gamma_0^T \mathbf{W}_0 \delta\gamma + 2 w_0 \delta\chi \mathbf{Q}_0 \mathbf{z}_0 = 0 \tag{5.7}$$

where we have dropped all terms which are higher order than linear in the variations, since these terms involve the product of small terms.

We now linearize the TAQMV equations, (4.1), in a similar fashion. Adding noise to the data gives

$$(\hat{\Gamma}_0 + \delta\hat{\Gamma})^T (\hat{\mathbf{W}}_0 + \delta\hat{\mathbf{W}})(\gamma_0 + \delta\gamma) + 2(\hat{w}_0 + \delta\hat{w})(\chi_0 + \delta\chi)(\mathbf{Q}_0 + \delta\mathbf{Q})(\hat{\mathbf{z}}_0 + \delta\hat{\mathbf{z}}) = 0 \tag{5.8}$$

where $\hat{\mathbf{z}}$ is the non-minimum variance, but consistent, estimate of $\mathbf{z}$ obtained from the TARLS algorithm, $\hat{\mathbf{z}}_\mathbf{0}$ is the TARLS solution in the absence of noise, which is identical to $\mathbf{z}_\mathbf{0}$, and $\delta\hat{\mathbf{z}}$ is the error in TARLS solution due to the noise in the data. Since $\gamma_0 = \mathbf{0}$, and $\chi_0 = 0$, Eqn. (5.8) is, to first order,

$$\hat{\Gamma}_0^T \hat{\mathbf{W}}_0 \delta\gamma + 2\hat{w}_0 \delta\chi \mathbf{Q}_0 \hat{\mathbf{z}}_0 = 0. \tag{5.9}$$

Comparing Eqns. (5.7) and (5.9) the see that the linearized TAFLS equations, and the linearized TAQMV equations are identical since $\hat{\mathbf{z}}_0 = \mathbf{z}_0$, i.e. the TARLS algorithm produces the solution that satisfies the equations when the equations are consistent. Since the TAFLS solution is minimum variance, and since the linearized TAQMV and the linearized TAFLS equations are the same, then the TAQMV solution is minimum variance [14].

For the sake of concreteness we will expand the two variations in Eqns. (5.7) and (5.9) to show their dependence on the variations in the data and the solution. The quantity $\delta\gamma$ has two contributions. The first arises from changing $\mathbf{z}$ keeping the data fixed, and the second comes from changing the data, and keeping $\mathbf{z}$ fixed. We will write this as

$$\delta\gamma = \Gamma_0 \delta\mathbf{z} + \frac{\partial\gamma}{\partial\mathbf{d}} \delta\mathbf{d}. \tag{5.10}$$

Here the term $\delta\mathbf{d}$ comes from changing the data keeping $\mathbf{z}$ fixed. Similarly, we can expand $\delta\chi$ giving

$$\delta\chi = 2\mathbf{z}_0^T \mathbf{Q}_0 \delta\mathbf{z} + \mathbf{z}_0^T \frac{\partial\mathbf{Q}}{\partial h} \mathbf{z}_0 \delta h. \tag{5.11}$$

Chapter 6

Simulation Results

In this section we give numerical results from simulations that show that the TAQMV algorithm developed in §4 performs similar to the Gauss-Newton iterative solution initialized to the true location (noiseless solution). This ”gold-standard” technique, hereafter denoted ”G/N-true”, computes the minimum variance (least-squares) solution [14]. Since this computed solution uses the true location, it is not an algorithm that can be used in an operational system. Therefore we also compare TAQMV to Gauss-Newton iteration started at the average subpoint of the reporting satellites, hereafter denoted ”G-N/sub”. This is a practical algorithm, which appears to work as well as G/N-true or TAQMV when judged with forgiving metrics, such as the median, 95th, or even 99th percentile error. However using metrics that reflect even infrequent errors, differences in performance are seen. In the following we use RMS error, the square root of the mean square Euclidean (3-D) geolocation error, which heavily penalizes the largest errors. This metric is a very sensitive discriminator of performance between the TAQMV and G-N/true, but gives meaningless numbers for G-N/sub which diverges occasionally, in which case the solution error is arbitrarily large. We say that the error is arbitrarily large in this case since the final error depends on parameters such as the maximum number of allowed iterations, i.e. the error gets increasing worse with each diverging step. In Table 6.1 and Table 6.3 below we tabulate the divergence rate of the TAQMV and G-N/true, and use an astrisk to indicate the large values produced by the G-N/sub algorithm. Note that these divergence rates are the (unitless) fraction of the trials that diverged. They could be interpreted as the probability of divergence.

The TAQMV and two Gauss-Newton variant algorithms were tested with simulated TOAs from 4 satellites plus an assumed HAE. This is a lightly overdetermined situation with just one more measurement than solution variable. This configuration was used to highlight the vulnerability of the iterative technique in the lightly overdetermined case. The iterative approach becomes more robust when the system of equations is more highly overdetermined, and the algebraic algorithm performs equally well in lightly or heavily overdetermined conditions. Therefore we will not present performance data for heavily overdetermined cases where all of the algorithms perform well.

We have found [15, 14] that many algorithms for the solution of systems of space-based geolocation/navigation equations perform markedly better when all the satellite orbital radii are nearly equal, or conversely, perform better when all at the orbital radii are not nearly equal. Constellations of the former type are called LER (large equal radius) and constellations of the latter type are termed non-LER. The TAQMV and the G-N variant algorithms were tested with both types of constellations. Tables 6.1 and 6.2 give results for an LER constellation, while Tables 6.3 and 6.4 give results for a non-LER constellation.

The LER constellation was a nominal GPS constellation of 24 MEO satellites, whereas in the non-LER simulations a single GEO satellite was added to each trial data set. In a navigation setting this could correspond to including a WAAS geosynchronous satellite for example. In the emitter-location application mixing satellites at different orbits, such as GEO and LEO, is very common. At each trial, of 100,000 total trials per noise level, we randomly select the desired number of participating satellites (four) from the satellties in view, while insuring that one GEO is included in the non-LER simulations. The simulation is described in more detail in an appendix of [14].

At each trial the randomly selected group of four satellites will have a different position dilution of precision (PDOP), which is a metric relating the input and output error levels. A histogram of the PDOP values for the LER and the non-LER trials are shown in Figures 6.1 and 6.2. Here the PDOP is the unitless number obtained by taking the square root of the trace of the position error covariance matrix and dividing that by the standard deviation of TOA/altitude data errors (in units of distance). The data errors were all equal and independent when computing the PDOP histograms, but were not equal in the other simulations where TOA noise varied while the altitude estimate noise was fixed at an arbitrary value of 10 m. Note that this definition of PDOP is a slightly generalized version of the TOA-only PDOP, such as described in [12] since our solution is constrained by an altitude equation as well as the usual TOA equations. In both cases the PDOP gives the scaling of the input errors to output error through the relation $\sigma_p = PDOP \times \sigma_r$, where σ_p is the RMS position error and σ_r is the range and altitude estimate error. Figures 6.1 and 6.2 help illustrate that the simulations tested the algorithms over a wide variety of geometries, as well as noise levels.

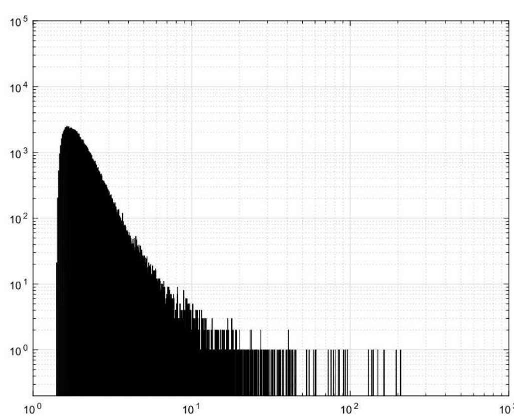

Figure 6.1. PDOP Histogram for the LER simulation.

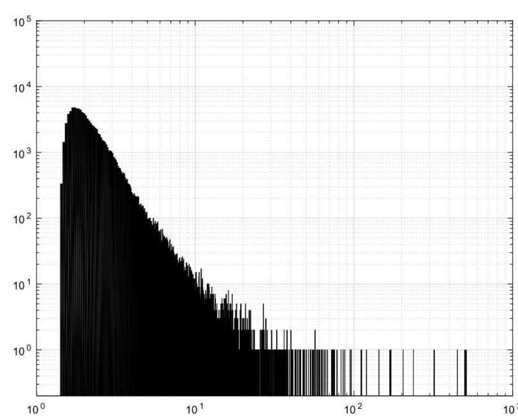

Figure 6.2. PDOP Histogram for the non-LER simulation.

Tables 6.1 through 6.4 give results for 5 different noiselevels, expressed in nanoseconds. The noise levels from 1 to 10,000 nanoseconds represent very small to very large noise levels for most navigation and most emitter tracking systems. In all runs the 1-sigma noise in the assumed altitude (ALT) was 10 meters.

Method	nanoseconds of TOA Noise				
	1	10	100	1000	10000
TAQMV	3.83	13.2	88.2	770	7377
G-N/true	3.83	13.2	78.7	767	6747
G-N/sub-pt	*	*	*	*	*

Table 6.1. Geolocation RMS error, in meters, for the 4-MEO (LER) 4-TOA/ALT problem. * Occasional divergence gave arbitrarily large RMS error for Gauss-Newton initialized with average satellite sub-point.

Method	nanoseconds of TOA Noise				
	1	10	100	1000	10000
TAQMV	0	0	0	0	0
G-N/true	0	0	0	0	0
G-N/sub-pt	0.00001	0.00072	0.00162	0.00167	0.00185

Table 6.2. Divergence rates for the 4-MEO (LER) 4-TOA/ALT problem.

Method	nanoseconds of TOA Noise				
	1	10	100	1000	10000
TAQMV	5.24	18.2	142	1120	9559
G-N/true	4.66	17.2	128	1028	8413
G-N/sub-pt	*	*	*	*	*

Table 6.3. Geolocation RMS error, in meters, for the 3-MEO and 1-GEO (non-LER) 4-TOA/ALT problem. * Occasional divergence gave arbitrarily large RMS error for Gauss-Newton initialized with average satellite sub-point.

Method	nanoseconds of TOA Noise				
	1	10	100	1000	10000
TAQMV	0	0	0	0	0
G-N/true	0	0	0	0	0
G-N/sub-pt	0.00147	0.00171	0.00308	0.00311	0.00343

Table 6.4. Divergence rates for the 3-MEO and 1-GEO (non-LER) 4-TOA/ALT problem.

The very sensitive RMS data presented in the tables show that the TAQMV algorithm performs similar to the G-N/true which computes the minimum variance least-squares solution from the TAFLS equations (2.14). In Table 6.1 we see that for the smallest TOA noise levels the algorithms agree to 3 decimal digits. In Table 6.3 the small difference (5.24 m versus 4.66 m) results from the 10 largest solution errors. Recomputing the RMS metric from the 99,990 smallest 3-D errors gives two digits of agreement. The largest geolocation error in the 10 excluded TAQMV errors here was approximately 500 meters, and for G-N/true was approximately 400 meters. We see that for the rare case where the geometry is so unfavorable as to give hundreds of meters of geolocation error from just 1 ns of TOA error, the approximations in the TAQMV equations (4.1), cause the TAQMV algorithm to perform just noticably worse (e.g. 500 m vs. 400 m). Given the infrequency of significant differences, and that these differences occur at times of exceptionally poor geometry, we conclude the TAQMV algorithm is performing similar to the G/N-true algorithm, which is initialized with the (typically unknown) true location. The G-N/sub algorithm is a realizable algorithm for an operational system but we see that it has a small but non-zero divergence rate, so would not be the algorithm of choice for a high-consequence system.

Chapter 7

Conclusions

In the process of developing our preferred approach to solving the over-determined system of equations arising from geolocation with TOA data and an altitude constraint on an ellipsoidal earth, we define three solution methods. The TAFLS equations presented in §2 give solutions with the minimum mean-square solution error, but require an iterative algorithm for solution, as no direct method has been devised yet. Note we assume zero-mean data errors, and therefore zero-mean solution errors, so mean-square solution error is also called solution error variance. The TARLS equations presented in §3 do not give the minimum variance solution, but allow us to get a consistent solution (correct in the absence of noise) by solving a one-dimensional cubic equation. The third technique, the TAQMV algorithm uses this sub-optimal TARLS solution to define the optimal weighting matrix which requires the geometry of the problem to be known, and to set the term $\mathbf{z}$ in Eqn. (4.2), and in the expression $\mathbf{Qz}$ in Eqn. (4.1) giving a very close approximation to the TAFLS equations. In §4 we show how to algebraically solve the TAQMV equations using resultants as described in the Appendix. In §5 we show that the TAQMV equations give the same solution error variance as the TAFLS equations by linearizing both equations about the zero noise solution. This comparison with a system known to be minimum variance establishes that the TAQMV algorithm is minimum variance as well. In §6 we see that the TAQMV functions robustly over a wide range of simulated geometries and noise levels. The TAQMV therefore reliably gives the optimal solution that the iterative approach gives when it converges. We demonstrate this in Section §6 where we compare the TAQMV solution to the iterative solution of the TAFLS equations using two different initializations. The performance of the TAQMV algorithm is on par TAFLS iteration with the ”omniscient” initialization and better than (never diverges) TAFLS iteration with average sub-point initialization.

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Chapter 8

Appendix

In this appendix we consider how to solve the simultaneous system of bivariate quadratic equations given in Eqn. (4.10) and (4.15). To begin, the theory of one dimensional resultants [18, 10] shows that if we have two polynomials

$$p(\chi) = p_2\chi^2 + p_1\chi + p_0 = 0, \tag{8.1}$$

and

$$q(\chi) = q_2\chi^2 + q_1\chi + q_0 = 0, \tag{8.2}$$

then this system will have a root if and only if the coefficients p_k and q_k satisfy

$$det(\mathbf{R}) = 0, \tag{8.3}$$

where

$$\mathbf{R} = \begin{pmatrix} p_2 & p_1 & p_0 & 0 \\ 0 & p_2 & p_1 & p_0 \\ q_2 & q_1 & q_0 & 0 \\ 0 & q_2 & q_1 & q_0 \end{pmatrix}. \tag{8.4}$$

In our case, this implies that in order to satisfy the equations (4.10) and (4.15) simultaneously, we must have

$$det(\mathbf{R}(\mu)) = det\left(\mu^2\mathbf{F}_2 + \mu\mathbf{F}_1 + \mathbf{F}_0\right) = 0, \tag{8.5}$$

where $\mathbf{R}(\mu)$ is the resultant matrix formed by setting $p(\chi) = f(\chi,\mu)$, and $q(\chi) = g(\chi,\mu)$, where μ is considered as a parameter. In particular, we have

$$\mathbf{F}_k = \begin{pmatrix} f_{k2} & f_{k1} & f_{k0} & 0 \\ 0 & f_{k2} & f_{k1} & f_{k0} \\ g_{k2} & g_{k1} & g_{k0} & 0 \\ 0 & g_{k2} & g_{k1} & g_{k0} \end{pmatrix} \tag{8.6}$$

where f_{kj} and g_{kj} are the coefficients of χ^j in f_k and g_k respectively, which are the coefficients of μ^k in f and g respectively.

We can turn the quadratic eigenvalue problem (8.5) into a linear eigenvalue problem using a block companion matrix. To do this we note that the $det(R(\mu)) = 0$ implies that a vector ϕ_1 exists such that

$$\left(\mu^2 \mathbf{F}_2 + \mu \mathbf{F}_1 + \mathbf{F}_0\right) \phi_1 = 0. \tag{8.7}$$

This can be written as

$$(\mathbf{H} - \mu \mathbf{G})\, \psi = 0, \tag{8.8}$$

where

$$\mathbf{H} = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{F}_0 & \mathbf{0} \end{pmatrix}, \tag{8.9}$$

$$\mathbf{G} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{F}_1 & \mathbf{F}_2 \end{pmatrix}, \tag{8.10}$$

and

$$\psi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}. \tag{8.11}$$

It should be noted that Eqn. (8.8) is an eighth order eigenvalue problem, but Bezout's theorem guarantees us that there are no more than 4 roots to our system of quadratic equations. It turns out that four of the roots to the eigenvalue problem in Eqn. (8.8) are infinite. A careful examination of this problem shows that the matrix $\mathbf{G}$ has a null space of dimension two. One might naively conclude that this would imply that we should only have two infinite eigenvalues rather than four. However, a more careful examination of this problem shows that in fact our eigenvalue problem has one Jordan block of length one and one Jordan block of length three associated with the infinite eigenvalue.

The fact that we are solving such a degenerate eigenvalue problem can lead to computational difficulties. Fortunately, these computational difficulties are only important if we are concerned with the eigenvalues and eigenvectors associated with the infinite eigenvalues. These difficulties show up by eigensolvers (as in MATLAB) occasionally reporting a finite (though very large) eigenvalue where an infinite eigenvalue should be reported. This has little effect on the accuracy of the eigenvalues that we are interested in.

Once we have determined μ, for each value of μ we consider the equations $f(\chi,\mu) = 0$, $g(\chi,\mu) = 0$. These can be viewed as two linear equations for $\chi_2 = \chi^2$ and $\chi_1 = \chi$. We solve these equations to determine χ_1 and hence χ.

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