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Emittance growth of accumulating proton beam in Booster due to multiple passes through the H-minus stripping foil

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In Ref. [1] equations were derived for the transverse emittance growth of a single batch of protons passing turn-by-turn through the H-minus stripping foil in Booster. It was assumed there that all particles in the batch go the same number of turns around the machine and pass through the foil the same number of times. The equations therefore apply to the case in which the duration of the pulse of beam injected into Booster is less than one revolution period. We consider here the case in which the incoming beam consists of a series of batches injected one after another into the machine. Each batch is assumed to contain the same number of particles. The leading edge of an incoming batch is assumed to be exactly one revolution period behind the leading edge of the batch in front of it. The batch length is assumed to be less than one revolution period and is assumed to be the same for all batches. By taking the batch length to be less than but close to one revolution period, we can approximate, as closely as we like, the case of a continuous ribbon of beam injected into the machine over a period of several turns. One then can calculate the emittance growth of the accumulating beam by appropriately summing over the single-batch emittances obtained from the formulae in [1]. The result is

$$\Gamma_n^S = \frac{1}{2}\left(\frac{n+1}{2}\right)\beta\left\langle\phi^2\right\rangle \tag{1}$$

where n is the number of accumulated turns, β is the lattice beta at the H-minus stripping foil, and $\langle\phi^2\rangle$ is the average squared scattering angle of particles passing through the foil. It is shown that if the turn number is sufficiently large and the machine tune sufficiently far from integer and half-integer values, then the emittance growth Γ_n^S can be determined from measurements of the average squared position of the particles in the accumulated beam.

1 Averages over batches of particles

Consider n batches of M particles. We use the subscript b to denote the bth batch. Let V_{bi} be a parameter associated with the ith particle of the bth batch. Then the average of V over the M particles of the bth batch is

$$\langle V_b \rangle = \frac{1}{M} \sum_{i=1}^{M} V_{bi} \tag{2}$$

and the average of V over all particles of the n batches is

$$\langle V \rangle_n = \frac{1}{nM} \sum_{b=1}^{n} \sum_{i=1}^{M} V_{bi}. \tag{3}$$

Thus we have

$$\langle V \rangle_n = \frac{1}{n} \sum_{b=1}^{n} \langle V_b \rangle . \tag{4}$$

We also use the notation $\langle V_{b,m} \rangle$ for the average of V over the M particles of the bth batch after it has gone m turns around the machine.

2 Average squared coordinates of accumulated batches

Consider a batch of M particles launched from a point just upstream of the injection foil over a period of time less than the revolution period. When the leading edge of this batch has passed through the foil and gone around the machine returning to the launching point, another identical batch is launched from the same point. This process continues until n batches have been introduced into the ring. The n accumulated batches lie on top of one another and move together around the machine. When the first batch has made n passes through the foil and its leading edge has returned to the launching point, the second batch will have made $n-1$ passes, the third will have made $n-2$ passes, and so on up to the nth batch, which will have made $n-(n-1)=1$ pass through the foil. It is assumed that the incoming H-minus beam is symmetric and centered on the closed orbit at the foil. Each of the n batches therefore initially satisfies the symmetry conditions defined in [1].

Let $\left\langle X_{b,m}^2 \right\rangle$ be the average X^2 of the particles in the bth batch after m turns around the machine starting from the launching point. For a particle distribution that is initially symmetric, these averages are given by (144) of Ref. [1]. Thus, starting with the average X^2 of the particles in the nth batch after one turn, we have

$$\left\langle X_{n,1}^2 \right\rangle = \left\langle X_0^2 \right\rangle + \frac{1}{2}\left(1 - \mathcal{A}_1\right)\beta^2\left\langle \phi^2 \right\rangle \tag{5}$$

$$\left\langle X_{n-1,2}^2 \right\rangle = \left\langle X_0^2 \right\rangle + \frac{1}{2}\left(2 - \mathcal{A}_2\right)\beta^2\left\langle \phi^2 \right\rangle \tag{6}$$

$$\left\langle X_{n-2,3}^2 \right\rangle = \left\langle X_0^2 \right\rangle + \frac{1}{2}\left(3 - \mathcal{A}_3\right)\beta^2\left\langle \phi^2 \right\rangle \tag{7}$$

and so on up to

$$\left\langle X_{1,n}^2 \right\rangle = \left\langle X_0^2 \right\rangle + \frac{1}{2}\left(n - \mathcal{A}_n\right)\beta^2\left\langle \phi^2 \right\rangle \tag{8}$$

which is the average X^2 of the particles in the first batch after n turns. Here $\langle X_0^2 \rangle$ is the average X^2 of the incoming H-minus beam. Adding these n equations and dividing by n gives the average X^2 of all particles accumulated in the ring over n turns. Thus we have

$$\left\langle X^2 \right\rangle_n = \frac{1}{n}\sum_{b=1}^{n}\left\langle X_{b,n+1-b}^2 \right\rangle = \frac{1}{n}\sum_{m=1}^{n}\left\langle X_m^2 \right\rangle \tag{9}$$

where

$$\left\langle X_m^2 \right\rangle = \left\langle X_0^2 \right\rangle + \frac{1}{2}\left(m - \mathcal{A}_m\right)\beta^2\left\langle \phi^2 \right\rangle . \tag{10}$$

Defining

$$\mathcal{A}_n^S = \sum_{m=1}^{n}\mathcal{A}_m \tag{11}$$

and using

$$1 + 2 + 3 + \cdots + n = \frac{n}{2}\left(n+1\right) \tag{12}$$

we then have

$$\left\langle X^2 \right\rangle_n = \left\langle X_0^2 \right\rangle + \frac{1}{2}\left\{\frac{n+1}{2} - \frac{\mathcal{A}_n^S}{n}\right\}\beta^2\left\langle \phi^2 \right\rangle . \tag{13}$$

Similarly, for the Y coordinate we have

$$\left\langle Y^2 \right\rangle_n = \frac{1}{n} \sum_{b=1}^{n} \left\langle Y_{b,n+1-b}^2 \right\rangle = \frac{1}{n} \sum_{m=1}^{n} \left\langle Y_m^2 \right\rangle \tag{14}$$

where, according to (145) of Ref. [1],

$$\left\langle Y_m^2 \right\rangle = \left\langle Y_0^2 \right\rangle + \frac{1}{2} \left(m + \mathcal{A}_m \right) \beta^2 \left\langle \phi^2 \right\rangle . \tag{15}$$

This gives

$$\left\langle Y^2 \right\rangle_n = \left\langle Y_0^2 \right\rangle + \frac{1}{2} \left\{ \frac{n+1}{2} + \frac{\mathcal{A}_n^S}{n} \right\} \beta^2 \left\langle \phi^2 \right\rangle . \tag{16}$$

If we add equations (13) and (16), the $\mathcal{A}_n^S / n$ terms cancel and we have

$$\left\langle X^2 \right\rangle_n + \left\langle Y^2 \right\rangle_n = \left\langle X_0^2 \right\rangle + \left\langle Y_0^2 \right\rangle + \left(\frac{n+1}{2} \right) \beta^2 \left\langle \phi^2 \right\rangle . \tag{17}$$

3 Average emittance of accumulated batches

Let X_{bi} and Y_{bi} be the coordinates of the ith particle in the bth batch. (These coordinates are as defined in Ref. [1].) Then the emittance of the particle is

$$E_{bi} = \frac{1}{\beta} \left(X_{bi}^2 + Y_{bi}^2 \right) \tag{18}$$

and the average emittance of the bth batch is

$$\left\langle E_b \right\rangle = \frac{1}{M} \sum_{i=1}^{M} E_{bi} . \tag{19}$$

Thus we have

$$\beta \left\langle E_b \right\rangle = \left\langle X_b^2 \right\rangle + \left\langle Y_b^2 \right\rangle \tag{20}$$

where

$$\left\langle X_b^2 \right\rangle = \frac{1}{M} \sum_{i=1}^{M} X_{bi}^2, \quad \left\langle Y_b^2 \right\rangle = \frac{1}{M} \sum_{i=1}^{M} Y_{bi}^2 . \tag{21}$$

The average emittance of n batches at the end of accumulation in the machine is

$$\left\langle E \right\rangle_n = \frac{1}{n} \sum_{b=1}^{n} \left\langle E_b \right\rangle . \tag{22}$$

Using (20) we then have

$$\beta \langle E \rangle_n = \left\langle X^2 \right\rangle_n + \left\langle Y^2 \right\rangle_n \tag{23}$$

where

$$\left\langle X^2 \right\rangle_n = \frac{1}{n} \sum_{b=1}^{n} \left\langle X_b^2 \right\rangle, \quad \left\langle Y^2 \right\rangle_n = \frac{1}{n} \sum_{b=1}^{n} \left\langle Y_b^2 \right\rangle. \tag{24}$$

Here, according to (17), we have

$$\left\langle X^2 \right\rangle_n + \left\langle Y^2 \right\rangle_n = \left\langle X_0^2 \right\rangle + \left\langle Y_0^2 \right\rangle + \left(\frac{n+1}{2}\right) \beta^2 \left\langle \phi^2 \right\rangle \tag{25}$$

which gives

$$\beta \langle E \rangle_n = \beta \langle E_0 \rangle + \left(\frac{n+1}{2}\right) \beta^2 \left\langle \phi^2 \right\rangle \tag{26}$$

and

$$\langle E \rangle_n - \langle E_0 \rangle = \left(\frac{n+1}{2}\right) \beta \left\langle \phi^2 \right\rangle \tag{27}$$

where

$$\beta \langle E_0 \rangle = \left\langle X_0^2 \right\rangle + \left\langle Y_0^2 \right\rangle. \tag{28}$$

Here $\langle E_0 \rangle$ is the average emittance of an incoming batch of H-minus ions at the injection foil. Equation (27) therefore gives the amount the average emittance grows during the accumulation of the n batches.

We define the emittance growth to be half the average growth given by (27). This choice is convenient for the work in the next section. Thus we take

$$\Gamma_n^S = \frac{1}{2} \left(\frac{n+1}{2}\right) \beta \left\langle \phi^2 \right\rangle \tag{29}$$

to be the amount the emittance grows during the accumulation of the n batches. The superscript S serves as a reminder that the emittance is obtained by summing over batches.

As shown in Ref. [1], we have for a single batch

$$\langle E_n \rangle - \langle E_0 \rangle = n\beta \left\langle \phi^2 \right\rangle \tag{30}$$

which is the growth of the average emittance of the batch over n turns. Here, as for the case of accumulating batches, we define the emittance growth to be half the value given by (30). Thus we take

$$\Gamma_n = \frac{1}{2} n\beta \left\langle \phi^2 \right\rangle \tag{31}$$

to be the amount the emittance of a single batch grows over n turns. Comparing (29) and (31) we see that

$$\Gamma_n^S = \frac{1}{2}\,\Gamma_n + \frac{1}{4}\,\beta\left\langle\phi^2\right\rangle \tag{32}$$

which shows that the amount the emittance grows during the accumulation of n batches is essentially half the amount the emittance of a single batch grows over n turns. This is simply a consequence of the factor of 1/2 on the right side of the identity (12).

4 Emittance from average squared position

In the previous section we obtained emittances from the average X^2 and average Y^2 of the particles. In practice one usually has only measurements of the average X^2. Here we show that the emittance growth

$$\Gamma_n^S = \frac{1}{2}\left(\frac{n+1}{2}\right)\beta\left\langle\phi^2\right\rangle \tag{33}$$

can be determined from these measurements. Let

$$\mathcal{E}_{bi} = \frac{1}{\beta}\,X_{bi}^2 \tag{34}$$

where X_{bi} is the position of the ith particle in the bth batch. The average of $\mathcal{E}$ over the M particles of the bth batch is then

$$\langle\mathcal{E}_b\rangle = \frac{1}{M}\sum_{i=1}^{M}\mathcal{E}_{bi} = \frac{1}{\beta}\left\langle X_b^2\right\rangle \tag{35}$$

where

$$\left\langle X_b^2\right\rangle = \frac{1}{M}\sum_{i=1}^{M}X_{bi}^2. \tag{36}$$

Summing over batches, the average of $\mathcal{E}$ over all particles accumulated over n turns is

$$\langle\mathcal{E}\rangle_n = \frac{1}{n}\sum_{b=1}^{n}\langle\mathcal{E}_b\rangle = \frac{1}{\beta}\left\langle X^2\right\rangle_n \tag{37}$$

where, according to (13),

$$\left\langle X^2\right\rangle_n = \left\langle X_0^2\right\rangle + \frac{1}{2}\left\{\frac{n+1}{2} - \frac{\mathcal{A}_n^S}{n}\right\}\beta^2\left\langle\phi^2\right\rangle. \tag{38}$$

Thus

$$\langle \mathcal{E} \rangle_n = \mathcal{E}_0 + \frac{1}{2} \left\{ \frac{n+1}{2} - \frac{\mathcal{A}_n^S}{n} \right\} \beta \left\langle \phi^2 \right\rangle \tag{39}$$

where

$$\mathcal{E}_0 = \frac{1}{\beta} \left\langle X_0^2 \right\rangle . \tag{40}$$

We then have

$$\langle \mathcal{E} \rangle_n - \mathcal{E}_0 = \Gamma_n^S - \frac{1}{2} \frac{\mathcal{A}_n^S}{n} \beta \left\langle \phi^2 \right\rangle \tag{41}$$

which gives

$$\Gamma_n^S = \langle \mathcal{E} \rangle_n - \mathcal{E}_0 + \frac{1}{2} \frac{\mathcal{A}_n^S}{n} \beta \left\langle \phi^2 \right\rangle \tag{42}$$

Here

$$\langle \mathcal{E} \rangle_n - \mathcal{E}_0 = \frac{1}{\beta} \left\{ \left\langle X^2 \right\rangle_n - \left\langle X_0^2 \right\rangle \right\} \tag{43}$$

where $\langle X^2 \rangle_n$ and $\langle X_0^2 \rangle$ are obtained from measurements. If the term $\mathcal{A}_n^S/n$ on the right side of (42) can be neglected, then

$$\Gamma_n^S = \langle \mathcal{E} \rangle_n - \mathcal{E}_0 \tag{44}$$

and $\langle \mathcal{E} \rangle_n - \mathcal{E}_0$ becomes a legitimate measure of emittance growth.

As shown in Ref. [1], we have for a single batch

$$\mathcal{E}_n - \mathcal{E}_0 = \frac{1}{2} (n - \mathcal{A}_n) \beta \left\langle \phi^2 \right\rangle \tag{45}$$

which we can write as

$$\mathcal{E}_n - \mathcal{E}_0 = \Gamma_n - \frac{1}{2} \mathcal{A}_n \beta \left\langle \phi^2 \right\rangle \tag{46}$$

where

$$\Gamma_n = \frac{1}{2} n\beta \left\langle \phi^2 \right\rangle \tag{47}$$

is the growth of the emittance of the batch over n turns. In this case we see that if the term $\mathcal{A}_n$ on the right side of (46) can be neglected, then

$$\Gamma_n = \mathcal{E}_n - \mathcal{E}_0 \tag{48}$$

and $\mathcal{E}_n - \mathcal{E}_0$ becomes a legitimate measure of emittance growth.

5 Summary of emittance formulae

Here, for easy reference and comparison, we summarize the emittance formulae of the previous sections.

For a **single batch** after n turns we have

$$\langle E_n \rangle - \langle E_0 \rangle = n\beta \left\langle \phi^2 \right\rangle \tag{49}$$

and

$$\mathcal{E}_n - \mathcal{E}_0 = F(n)\, \beta \left\langle \phi^2 \right\rangle \tag{50}$$

where

$$F(n) = \frac{1}{2}\left(n - \mathcal{A}_n\right). \tag{51}$$

The emittance growth is taken to be

$$\Gamma_n = \frac{1}{2}\, n\beta \left\langle \phi^2 \right\rangle. \tag{52}$$

If the term $\mathcal{A}_n$ in (51) is small compared to n, then to a good approximation we have

$$\Gamma_n = \mathcal{E}_n - \mathcal{E}_0 \tag{53}$$

and $\mathcal{E}_n - \mathcal{E}_0$ becomes a legitimate measure of emittance growth.

For the **accumulation** of n batches over n turns we have

$$\langle E \rangle_n - \langle E_0 \rangle = \left(\frac{n+1}{2}\right) \beta \left\langle \phi^2 \right\rangle \tag{54}$$

and

$$\langle \mathcal{E} \rangle_n - \mathcal{E}_0 = F_S(n)\, \beta \left\langle \phi^2 \right\rangle \tag{55}$$

where

$$F_S(n) = \frac{1}{2}\left\{\frac{n+1}{2} - \frac{\mathcal{A}_n^S}{n}\right\}. \tag{56}$$

The emittance growth is taken to be

$$\Gamma_n^S = \frac{1}{2}\left(\frac{n+1}{2}\right) \beta \left\langle \phi^2 \right\rangle. \tag{57}$$

If the term $\mathcal{A}_n/n$ in (56) is small compared to $(n+1)/2$, then to a good approximation we have

$$\Gamma_n^S = \langle \mathcal{E} \rangle_n - \mathcal{E}_0 \tag{58}$$

and $\langle \mathcal{E} \rangle_n - \mathcal{E}_0$ becomes a legitimate measure of emittance growth.

As shown in the **Appendix**, we have

$$\mathcal{A}_n = -\frac{1}{2} + \frac{1}{2}\left(\frac{\mathcal{C}_n - \mathcal{C}_{n+1}}{1 - \mathcal{C}_1}\right) \tag{59}$$

and

$$\frac{\mathcal{A}_n^S}{n} = -\frac{1}{2} + \frac{1}{2n}\left(\frac{\mathcal{C}_1 - \mathcal{C}_{n+1}}{1 - \mathcal{C}_1}\right) \tag{60}$$

where

$$\mathcal{C}_n = \cos 2n\psi, \quad \psi = 2\pi Q \tag{61}$$

and Q is the machine tune. This gives

$$F(n) = \frac{n}{2} + \frac{1}{4} - \frac{1}{4}\left(\frac{\mathcal{C}_n - \mathcal{C}_{n+1}}{1 - \mathcal{C}_1}\right) \tag{62}$$

and

$$F_S(n) = \frac{1}{2}\left(\frac{n+1}{2}\right) + \frac{1}{4} - \frac{1}{4n}\left(\frac{\mathcal{C}_1 - \mathcal{C}_{n+1}}{1 - \mathcal{C}_1}\right) \tag{63}$$

which we can write as

$$F(n) = G(n) + \frac{1}{4} - \frac{1}{4}\left(\frac{\mathcal{C}_n - \mathcal{C}_{n+1}}{1 - \mathcal{C}_1}\right) \tag{64}$$

and

$$F_S(n) = G_S(n) + \frac{1}{4} - \frac{1}{4n}\left(\frac{\mathcal{C}_1 - \mathcal{C}_{n+1}}{1 - \mathcal{C}_1}\right) \tag{65}$$

where

$$G(n) = \frac{n}{2}, \quad G_S(n) = \frac{1}{2}\left(\frac{n+1}{2}\right). \tag{66}$$

Figure 1 shows coefficients $\mathcal{A}_n$ and $\mathcal{A}_n^S/n$ for tune $\underline{Q = 0.82}$. These are given by equations (59) and (60) respectively. Here we see that $\mathcal{A}_n^S/n$ settles very quickly to $-1/2$ as n increases. This will always be the case if the fractional part of the tune is well away from 0, 1/2, and 1. For $\underline{n > 38}$ the coefficient $\mathcal{A}_n^S/n$ is small compared to $(n+1)/2$ as shown in **Figure 2**. $\langle \mathcal{E} \rangle_n - \mathcal{E}_0$ then becomes a legitimate measure of emittance growth (57).

Figure 3 shows coefficients $\mathcal{A}_n$ and $\mathcal{A}_n^S/n$ for tune $\underline{Q = 0.51}$. Here, with the fractional part of the tune very close to 1/2, it takes many turns for $\mathcal{A}_n^S/n$ to settle to $-1/2$. For $\underline{n > 78}$ the coefficient $\mathcal{A}_n^S/n$ is small compared to $(n+1)/2$ as shown in **Figure 4**. $\langle \mathcal{E} \rangle_n - \mathcal{E}_0$ may then be taken as a legitimate measure of emittance growth (57).

Figure 5 shows coefficients $\mathcal{A}_n$ and $\mathcal{A}_n^S/n$ for tune $\underline{Q = 0.54}$. Here the coefficient $\mathcal{A}_n^S/n$ settles to $-1/2$ fairly quickly. For $\underline{n > 37}$ the coefficient $\mathcal{A}_n^S/n$ is small compared to $(n+1)/2$ as shown in **Figure 6**. $\langle \mathcal{E} \rangle_n - \mathcal{E}_0$ may then be taken as a legitimate measure of emittance growth (57).

Figure 7 shows functions $F(n)$ and $F_S(n)$ for tune $\underline{Q = 0.54}$. These are given by equations (51) and (56) respectively. Note that $F_S(n)$ quickly converges to a straight line as n increases. This is because the oscillations of $\mathcal{A}_n^S/n$ die out fairly quickly as shown in the previous two figures.

Figure 8 is the same **Figure 7** but with the lines $G(n)$ and $G_S(n)$ added. These are given by (66). Note that the slope of $G_S(n)$ is half that of $G(n)$.

6 Numbers for emittance of accumulated proton beam in Booster

The analysis carried out in [1] using the code TRIM [2] shows that

$$10^3 \left\langle \phi^2 \right\rangle^{1/2} = 0.0344,\ 0.0297,\ 0.0243,\ 0.0165 \tag{67}$$

for 200 MeV protons incident on 200, 150, 100, and 50 microgram per cm^2 carbon foils, respectively. These numbers give, respectively,

$$10^9 \left\langle \phi^2 \right\rangle = 1.1834,\ 0.8821,\ 0.5905,\ 0.2723 \tag{68}$$

which are plotted as black circles in **Figure 9**. The corresponding fitted line

$$10^9 \left\langle \phi^2 \right\rangle = \mathcal{M}\mathcal{X} \tag{69}$$

is shown in red. Here

$$\mathcal{M} = 1.1834/200 \tag{70}$$

and $\mathcal{X}$ is the surface density of the carbon foil in micrograms per cm^2.

Given any carbon foil surface density $\mathcal{X}$ between 0 and 200 micrograms per cm^2, one can obtain $\langle \phi^2 \rangle$ from equations (69) and (70). This then can be used along with a value for β in any of the equations for emittance or emittance growth given in the previous sections. The emittance growth for the accumulation of n batches over n turns is

$$\Gamma_n^S = \frac{1}{2} \left(\frac{n+1}{2} \right) \beta \left\langle \phi^2 \right\rangle . \tag{71}$$

Taking

$$n = \underline{333 \text{ turns}} \tag{72}$$

and

$$\beta = \underline{5 \text{ m}} \tag{73}$$

then gives emittance growths

$$\Gamma_n^S = \underline{0.4941, \;\; 0.3706, \;\; 0.2470, \;\; 0.1235} \tag{74}$$

micrometers for

$$\mathcal{X} = \underline{200, \;\; 150, \;\; 100, \;\; 50} \tag{75}$$

micrograms per cm^2, respectively.

The corresponding **95 percent emittance** growths associated with Γ_n^S are

$$6\,\Gamma_n^S = \underline{2.9644, \;\; 2.2233, \;\; 1.4822, \;\; 0.7411} \tag{76}$$

micrometers, respectively. These are un-normalized emittance growths.

Normalized emittance growths are obtained by multiplying the un-normalized ones by the relativistic factor $\beta\gamma = 0.68684$ for 200 MeV protons. The **normalized 95 percent emittance** growth is then

$$\left(6\,\Gamma_n^S\right)_N = 0.68684\left(6\,\Gamma_n^S\right) \tag{77}$$

and the numbers in (76) give

$$\left(6\,\Gamma_n^S\right)_N = \underline{2.0361, \;\; 1.5271, \;\; 1.0180, \;\; 0.5090} \tag{78}$$

micrometers, respectively. Note that these numbers can be scaled in accordance with (71) to obtain the emittance growth for any values of n or β.

7 Plots of emittance growth

Using (71) in (77) we have normalized 95 percent emittance growth

$$\left(6\,\Gamma_n^S\right)_N = 0.68684 \times \frac{6}{2}\left(\frac{n+1}{2}\right)\beta\left\langle\phi^2\right\rangle. \tag{79}$$

Equations (69) and (70) give $10^9\left\langle\phi^2\right\rangle = 0.5917$ and 1.1834 for carbon foils with surface densities of 100 and 200 micrograms per cm^2, respectively.

Using these numbers in (79) we obtain plots of $\left(6\,\Gamma_n^S\right)_N$ as a function of n for various values of β. These are shown in **Figures 10** and **11** for surface densities of 100 and 200 micrograms per cm^2, respectively. These are to be compared with Figures 18 and 19 of Ref. [1]. The comparison shows that, as already noted, the amount the emittance grows during the accumulation of n batches is half the growth of the emittnce of a single batch over n turns. As already mentioned, this is simply a consequence of the factor of 1/2 on the right side of the identity (12).

8 Summary

In this note the treatment of Ref. [1] has been extended to include the case of accumulating proton beam in Booster at injection. The emittance growth of the accumulating beam is shown to be

$$\Gamma_n^S = \frac{1}{2}\left(\frac{n+1}{2}\right)\beta\left\langle\phi^2\right\rangle \tag{80}$$

where n is the number of accumulated turns, β is the lattice beta at the H-minus stripping foil, and $\langle\phi^2\rangle$ is the average squared scattering angle of particles passing through the foil. The emittance growth Γ_n^S is essentially half the emittance growth

$$\Gamma_n = \frac{1}{2}\,n\beta\left\langle\phi^2\right\rangle \tag{81}$$

of a single batch of protons which occupies less than one turn and circulates for n turns. This is shown to be simply a consequence of the identity

$$1+2+3+\cdots+n = \frac{n}{2}\,(n+1). \tag{82}$$

In Sections 6 and 7, equation (80) is evaluated to obtain emittance growths for various values of n, β, and $\langle\phi^2\rangle$. The numbers obtained are relevant for the accumulation of polarized protons in Booster at injection.

If the turn number is sufficiently large and the tune sufficiently far from integer and half-integer values, the emittance growth Γ_n^S can be determined from measurements of the average squared position $\langle X^2\rangle$ of the particles in the accumulated beam. This is discussed in Sections 4 and 5, and is particularly relevant for the injection of polarized protons where the Booster operates with both tunes close to 4.5. If, for example, the fractional part of the tune is 0.54 then at least 37 turns need to be accumulated before a measurement of $\langle X^2 \rangle$ gives a legitimate number for Γ_n^S. If the fractional part of the tune is 0.51 then at least 78 turns are needed.

9 Appendix

Let

$$\mathcal{Z} = e^{-2i\psi}, \quad \mathcal{Z}^n = e^{-2in\psi}, \quad \psi = 2\pi Q \tag{83}$$

and

$$\mathcal{F}_n = \frac{\mathcal{Z}\,(1 - \mathcal{Z}^n)}{1 - \mathcal{Z}} \tag{84}$$

where Q is the machine tune. Then

$$\mathcal{F}_n = \mathcal{A}_n + i\mathcal{B}_n \tag{85}$$

where

$$\mathcal{A}_n = \frac{\mathcal{S}_1 \mathcal{S}_n}{2\,(1 - \mathcal{C}_1)} - \frac{1}{2}\,(1 - \mathcal{C}_n) \tag{86}$$

$$\mathcal{B}_n = -\frac{\mathcal{S}_n}{2} - \frac{\mathcal{S}_1\,(1 - \mathcal{C}_n)}{2\,(1 - \mathcal{C}_1)} \tag{87}$$

and

$$\mathcal{C}_n = \cos 2n\psi, \quad \mathcal{S}_n = \sin 2n\psi. \tag{88}$$

From the identity

$$(1 - \mathcal{Z})\left(\mathcal{Z} + \mathcal{Z}^2 + \mathcal{Z}^3 + \cdots + \mathcal{Z}^n\right) = \mathcal{Z} - \mathcal{Z}^{n+1} \tag{89}$$

it follows that

$$\sum_{m=1}^{n} \mathcal{Z}^m = \frac{\mathcal{Z}\,(1 - \mathcal{Z}^n)}{1 - \mathcal{Z}} \tag{90}$$

provided $\mathcal{Z} \neq 1$. Thus

$$\sum_{m=1}^{n} (\mathcal{C}_m - i\mathcal{S}_m) = \mathcal{A}_n + i\mathcal{B}_n \tag{91}$$

and writing

$$\mathcal{C}_n^S = \sum_{m=1}^{n} \mathcal{C}_m, \quad \mathcal{S}_n^S = \sum_{m=1}^{n} \mathcal{S}_m \tag{92}$$

we have

$$\mathcal{C}_n^S = \mathcal{A}_n, \quad \mathcal{S}_n^S = -\mathcal{B}_n. \tag{93}$$

Then writing

$$\mathcal{A}_n^S = \sum_{m=1}^{n} \mathcal{A}_m \tag{94}$$

we have

$$\mathcal{A}_n^S = \frac{\mathcal{S}_1 \mathcal{S}_n^S}{2\left(1-\mathcal{C}_1\right)} - \frac{1}{2}\left(n - \mathcal{C}_n^S\right) \tag{95}$$

and

$$\frac{1}{n}\mathcal{A}_n^S = -\frac{1}{2} + \frac{\mathcal{A}_n}{2n} - \left(\frac{\mathcal{S}_1}{1-\mathcal{C}_1}\right)\frac{\mathcal{B}_n}{2n}. \tag{96}$$

Here

$$\mathcal{A}_n = \frac{\mathcal{S}_1 \mathcal{S}_n}{2\left(1-\mathcal{C}_1\right)} - \frac{1}{2}\left(1-\mathcal{C}_n\right) \tag{97}$$

$$\mathcal{B}_n = -\frac{\mathcal{S}_n}{2} - \frac{\mathcal{S}_1\left(1-\mathcal{C}_n\right)}{2\left(1-\mathcal{C}_1\right)} \tag{98}$$

$$\frac{\mathcal{S}_1 \mathcal{B}_n}{1-\mathcal{C}_1} = -\frac{\mathcal{S}_1 \mathcal{S}_n}{2\left(1-\mathcal{C}_1\right)} - \frac{\mathcal{S}_1^2\left(1-\mathcal{C}_n\right)}{2\left(1-\mathcal{C}_1\right)^2} \tag{99}$$

$$\left(\mathcal{A}_n - \frac{\mathcal{S}_1 \mathcal{B}_n}{1-\mathcal{C}_1}\right) = \frac{\mathcal{S}_1 \mathcal{S}_n}{1-\mathcal{C}_1} + \frac{1-\mathcal{C}_n}{2}\left(\frac{1+\mathcal{C}_1}{1-\mathcal{C}_1} - 1\right) \tag{100}$$

$$\left(\mathcal{A}_n - \frac{\mathcal{S}_1 \mathcal{B}_n}{1-\mathcal{C}_1}\right) = \frac{\mathcal{S}_1 \mathcal{S}_n - \mathcal{C}_1 \mathcal{C}_n + \mathcal{C}_1}{1-\mathcal{C}_1} \tag{101}$$

$$\mathcal{S}_1 \mathcal{S}_n - \mathcal{C}_1 \mathcal{C}_n = -\mathcal{C}_{n+1} \tag{102}$$

and therefore

$$\frac{1}{n}\mathcal{A}_n^S = -\frac{1}{2} + \frac{1}{2n}\left(\frac{\mathcal{C}_1 - \mathcal{C}_{n+1}}{1-\mathcal{C}_1}\right). \tag{103}$$

We also have

$$\mathcal{A}_n = -\frac{1}{2} + \frac{\mathcal{S}_1 \mathcal{S}_n}{2\left(1-\mathcal{C}_1\right)} + \frac{\mathcal{C}_n(1-\mathcal{C}_1)}{2(1-\mathcal{C}_1)} \tag{104}$$

and therefore

$$\mathcal{A}_n = -\frac{1}{2} + \frac{1}{2}\left(\frac{\mathcal{C}_n - \mathcal{C}_{n+1}}{1-\mathcal{C}_1}\right). \tag{105}$$

References

[1] C.J. Gardner, "Emittance Growth due to Multiple Passes through the H-minus Stripping Foil in Booster", C-A/AP/Note 583, March, 2017

[2] J.F. Ziegler, www.srim.org

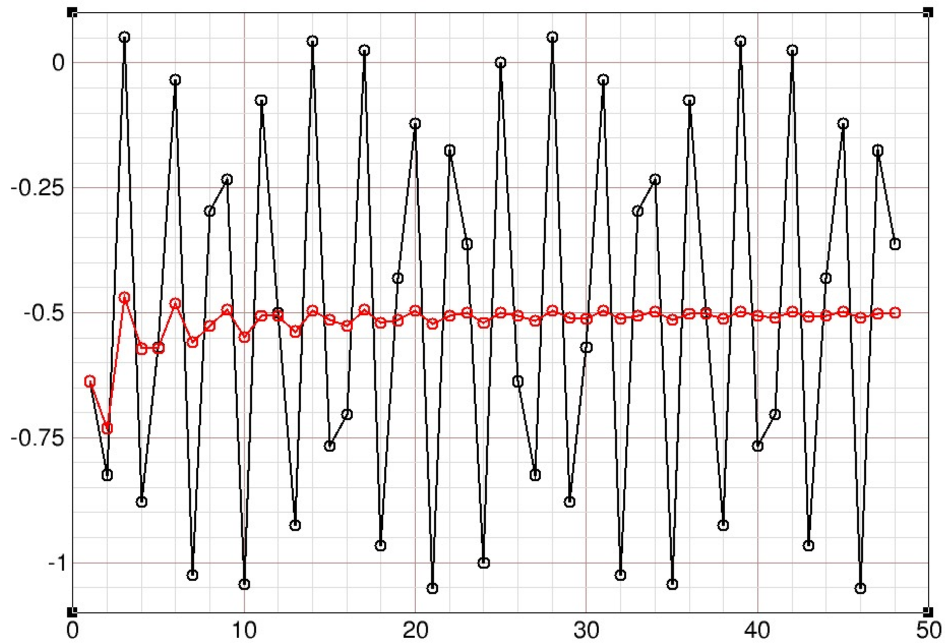

Figure 1: Coefficients $\mathcal{A}_n$ (black circles) and $\mathcal{A}_n^S/n$ (red circles) for tune $\underline{Q = 0.82}$. These are given by equations (59) and (60) respectively. The horizontal axis gives the turn number n. The curves connect the centers of the circles. Here we see that $\mathcal{A}_n^S/n$ settles very quickly to $-1/2$ as n increases. This will always be the case if the fractional part of the tune is well away from 0, 1/2, and 1. For $\underline{n > 38}$ the coefficient $\mathcal{A}_n^S/n$ is small compared to $(n+1)/2$ as shown in **Figure 2**. $\langle \mathcal{E} \rangle_n - \mathcal{E}_0$ then becomes a legitimate measure of emittance growth (57).

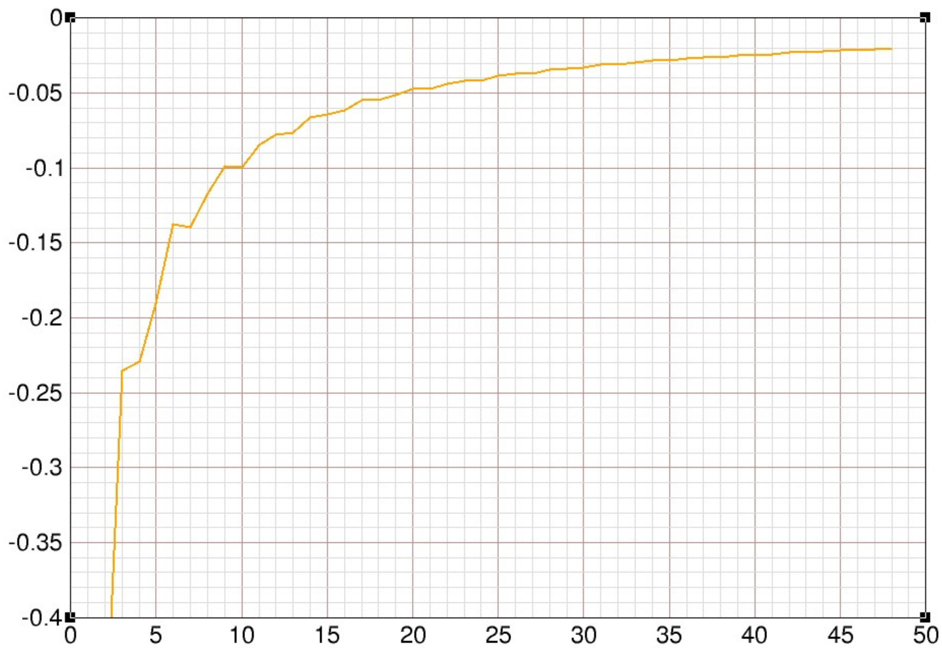

Figure 2: Ratio $(2\mathcal{A}_n^S/n)/(n+1)$ for tune $\underline{Q = 0.82}$. The horizontal axis gives the turn number n. The absolute value of the ratio is less than 2.5% for $\underline{n > 38}$.

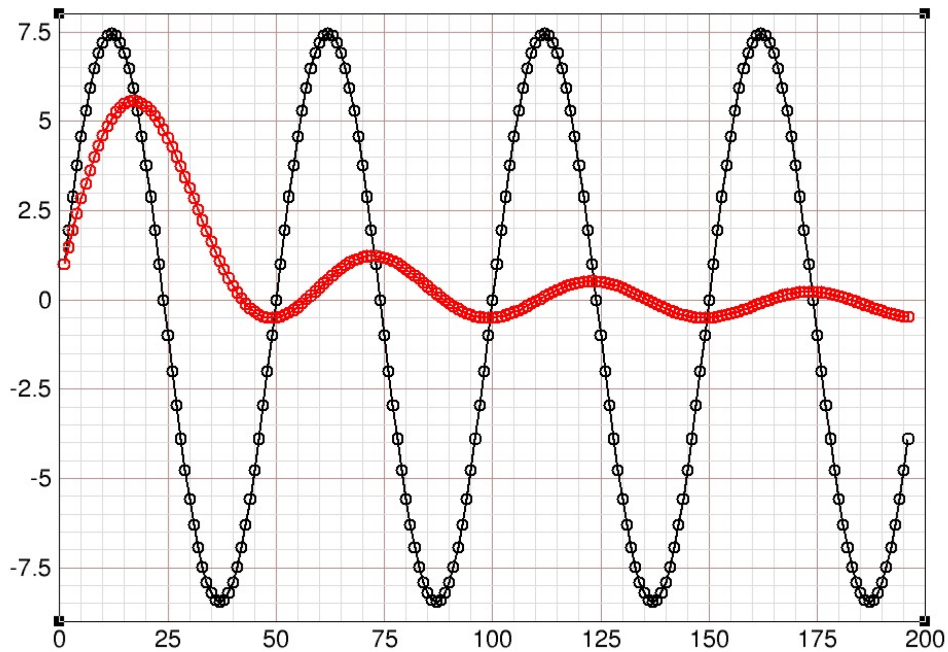

Figure 3: Coefficients $\mathcal{A}_n$ (black circles) and $\mathcal{A}_n^S/n$ (red circles) for tune $\underline{Q = 0.51}$. These are given by equations (59) and (60) respectively. The horizontal axis gives the turn number n. The curves connect the centers of the circles. Here, with the fractional part of the tune very close to 1/2, it takes many turns for $\mathcal{A}_n^S/n$ to settle to $-1/2$. For $\underline{n > 78}$ the coefficient $\mathcal{A}_n^S/n$ is small compared to $(n+1)/2$ as shown in **Figure 4**. $\langle\mathcal{E}\rangle_n - \mathcal{E}_0$ may then be taken as a legitimate measure of emittance growth (57).

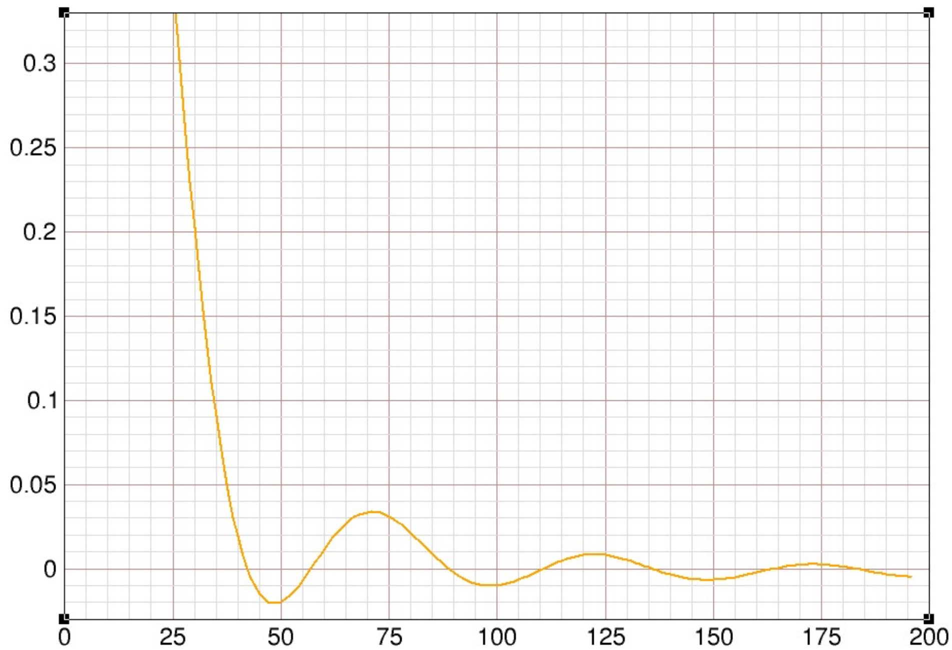

Figure 4: Ratio $(2\mathcal{A}_n^S/n)/(n+1)$ for tune $\underline{Q = 0.51}$. The horizontal axis gives the turn number n. The absolute value of the ratio is less than 2.5% for $\underline{n > 78}$.

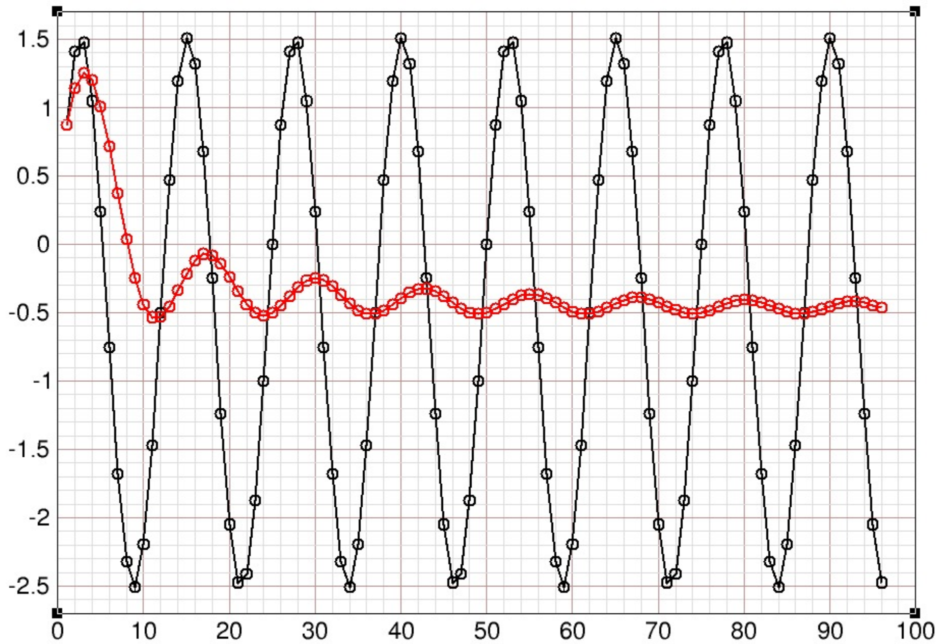

Figure 5: Coefficients $\mathcal{A}_n$ (black circles) and $\mathcal{A}_n^S/n$ (red circles) for tune $\underline{Q = 0.54}$. These are given by equations (59) and (60) respectively. The horizontal axis gives the turn number n. The curves connect the centers of the circles. Here the coefficient $\mathcal{A}_n^S/n$ settles to $-1/2$ fairly quickly. For $\underline{n > 37}$ the coefficient $\mathcal{A}_n^S/n$ is small compared to $(n+1)/2$ as shown in **Figure 6**. $\langle \mathcal{E} \rangle_n - \mathcal{E}_0$ may then be taken as a legitimate measure of emittance growth (57).

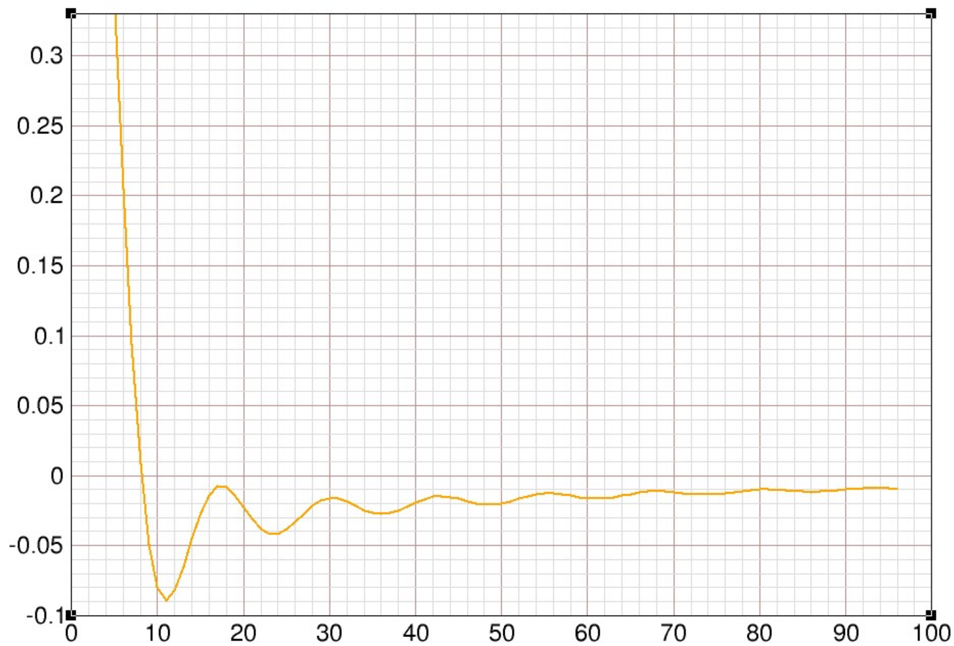

Figure 6: Ratio $(2\mathcal{A}_n^S/n)/(n+1)$ for tune $\underline{Q = 0.54}$. The horizontal axis gives the turn number n. The absolute value of the ratio is less than 2.5% for $\underline{n > 37}$.

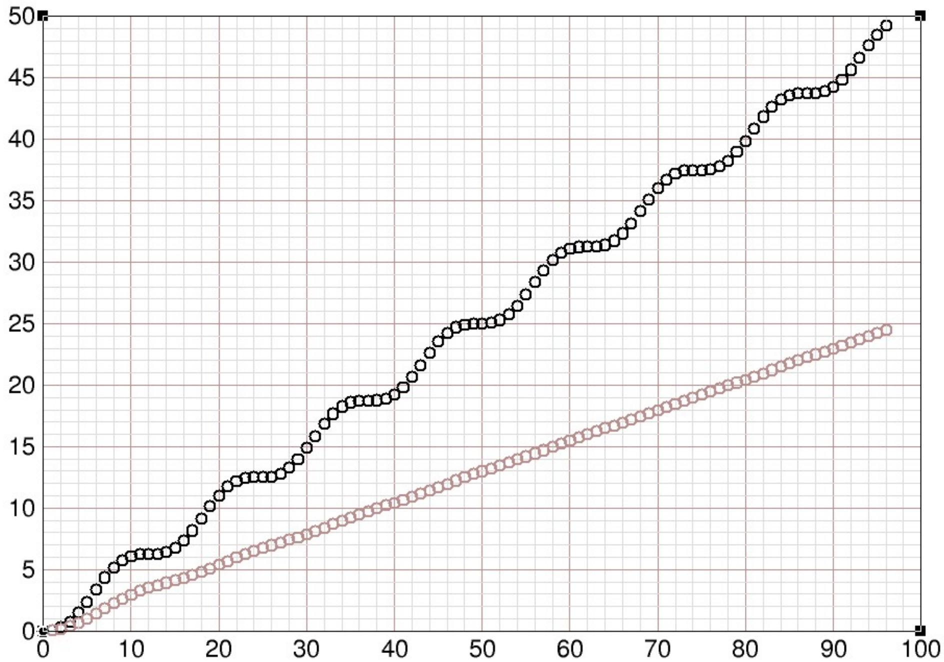

Figure 7: Functions $F(n)$ (black circles) and $F_S(n)$ (brown circles) for tune $Q = 0.54$. These are given by equations (51) and (56) respectively. The horizontal axis gives the turn number n. Note that $F_S(n)$ quickly converges to a straight line as n increases. This is because the oscillations of $\mathcal{A}_n^S/n$ die out fairly quickly as shown in the previous two figures.

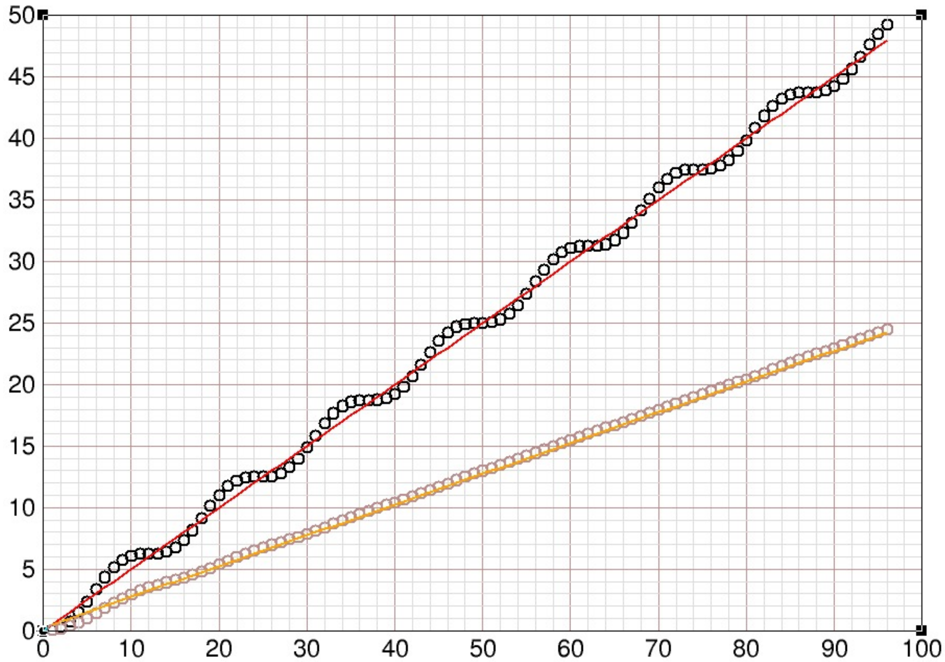

Figure 8: Same as previous figure, but with lines $G(n)$ (red line) and $G_S(n)$ (orange line) added. These are given by (66). Note that the slope of $G_S(n)$ is half that of $G(n)$.

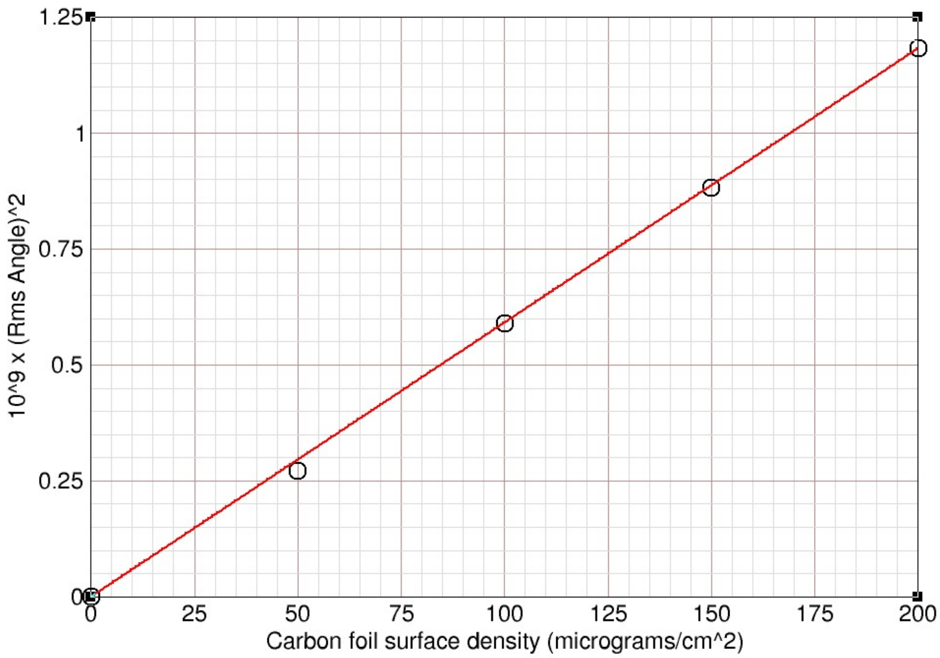

Figure 9: $10^9 \langle \phi^2 \rangle$ versus carbon foil surface density. The horizontal axis gives the surface density in micrograms per cm^2. The black circles show the values of $10^9 \langle \phi^2 \rangle$ for surface densities 200, 150, 100, and 50 micrograms per cm^2. The red line connects the circle at 0 with the circle at 200 micrograms per cm^2. The slope of the line is $\mathcal{M} = 1.1834/200$.

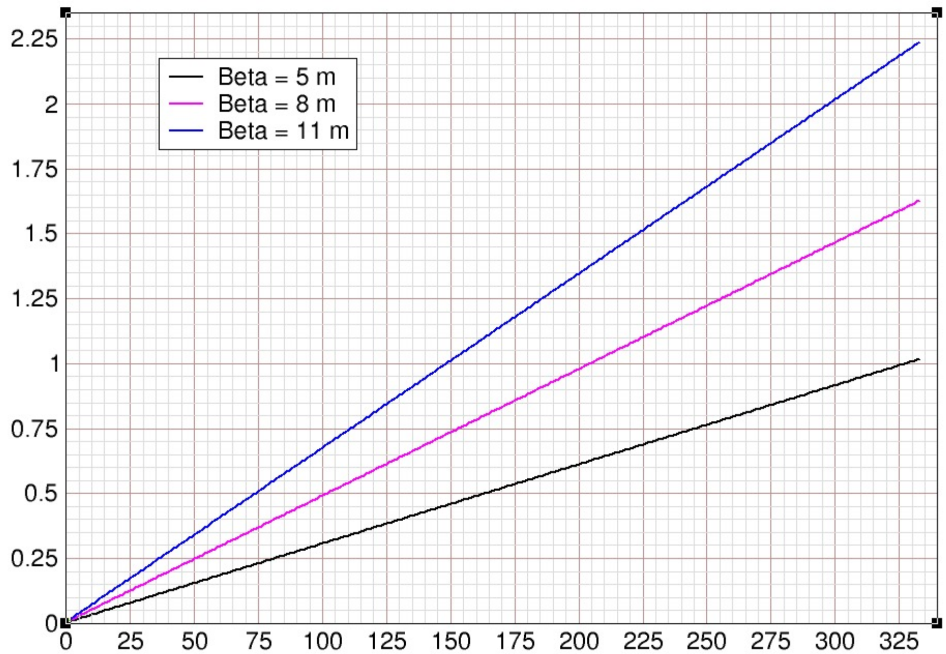

Figure 10: Emittance growth (79) during accumulation of proton beam in Booster with 100 microgram per cm^2 carbon foil. The vertical axis gives the emittance growth in micrometers. The horizontal axis gives the turn number n. The black (lower), magenta (middle), and blue (upper) lines give, respectively, the growth for 5, 8, and 11 m lattice beta at the foil. The revolution period of 200 MeV protons in Booster is 1.1888 microseconds, which gives a time interval of 396 microseconds for the 333 plotted turns.

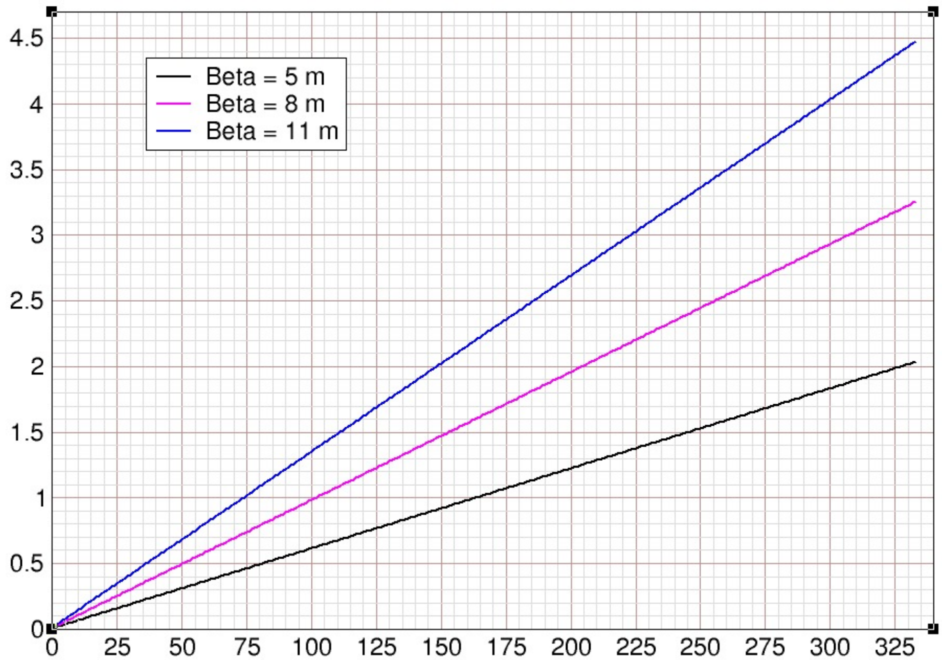

Figure 11: Emittance growth (79) during accumulation of proton beam in Booster with 200 microgram per cm^2 carbon foil. The vertical axis gives the emittance growth in micrometers. The horizontal axis gives the turn number n. The black (lower), magenta (middle), and blue (upper) lines give, respectively, the growth for 5, 8, and 11 m lattice beta at the foil. The revolution period of 200 MeV protons in Booster is 1.1888 microseconds, which gives a time interval of 396 microseconds for the 333 plotted turns.