SAND2007-8078J

Minimum Variance Direct Methods for the TOA Geolocation Equations

L.A. Romero *and Jeff Mason †

December 13, 2007

Abstract

We present two methods for solving overdetermined systems of the Time of Arrival (TOA) geolocation equations that achieve the minimum possible variance in all cases, not just when the satellites are at large equal radii. One of these techniques gives two solutions, and the other gives four solutions.

1 Introduction

There have been numerous algorithms proposed for solving overdetermined systems of equations for TOA (Time of Arrival) geolocation. These systems arise when using the Global Positioning System (GPS) or when locating an RF emitter using multiple receivers. Before Bancroft [4] presented a direct (non-iterative) method, these equations were solved iteratively [18, 13] using a variant of the Gauss-Newton method [14] for solving non-linear least squares problems. A drawback of the Gauss-Newton method is that it requires a sufficiently accurate initial guess to achieve convergence. Since Bancroft's paper, alternative direct methods [5, 19, 10, 17] have been presented, but we believe Bancroft's algorithm remains the standard direct method for the TOA equations.

If the noise level is small enough, and the configuration of the satellites is not degenerate, all of the solution techniques mentioned in the previous paragraph yield accurate location estimates [6]. To go beyond this most obvious requirement we suggest the following criteria be used to evaluate the superiority of one technique over another.

Criterion I The algorithm should readily allow a numerically stable computer implementation.

Criterion II The algorithm should achieve the Cramer Rao Lower Bound (CRLB) for the position error variance.

Criterion III The algorithm should produce all of the low residual local minima of the objective function when more than one exists.

Criterion IV The algorithm should be as robust as possible with respect to degenerate geometries.

To illustrate some of the issues involved in evaluating techniques according to these criteria, we first discuss them without regard to any specific set of equations. Suppose we are trying to solve an over-determined system of equations with m equations $\boldsymbol{\gamma}$, and n unknowns $\mathbf{z}$ ($m > n$). We write

$$\boldsymbol{\gamma}(\mathbf{z}, \mathbf{d}) = \mathbf{0} \tag{1.1}$$

where $\mathbf{z}$ is the solution, and $\mathbf{d}$ is an $m \times 1$ vector of data. For the GPS problem, $\mathbf{z}$ is a vector containing the solution variables, the receiver position $\mathbf{x}$ and clock bias τ, while $\mathbf{d}$ is the pseudorange data.

A common way of solving this overdetermined system of equations is to minimize the objective function $P(\mathbf{z}, \mathbf{d}) = \frac{1}{2}\boldsymbol{\gamma}^T(\mathbf{z}, \mathbf{d})\mathbf{W}\boldsymbol{\gamma}(\mathbf{z}, \mathbf{d})$, where $\mathbf{W}$ is a weighting matrix. Requiring that the gradient of $P(\mathbf{z}, \mathbf{d})$ vanishes gives us the equations

*Computational and Applied Mathematics Department,Sandia National Laboratories, Albuquerque NM, 87185
†Radar and Signal Analysis Department, Sandia National Laboratories, Albuquerque NM, 87185

$$\mathbf{F}(\mathbf{z},\mathbf{d}) = \mathbf{\Gamma}^T(\mathbf{z},\mathbf{d})\mathbf{W}\boldsymbol{\gamma}(\mathbf{z},\mathbf{d}) = \mathbf{0} \tag{1.2}$$

where $\mathbf{\Gamma}$ is the Jacobian of $\boldsymbol{\gamma}$. That is, $\mathbf{\Gamma}_{ij} = \frac{\partial \gamma_i}{\partial z_j}$. The equation $\mathbf{F}(\mathbf{z},\mathbf{d}) = \mathbf{0}$ is a system of n equations in n unknowns. We refer to these equations as the Full Least Squares (FLS) equations.

In [15] we showed that the FLS equations for TOA geolocation problems have nine critical points, and we give a procedure for finding all of these. However, the procedure suffers from numerical stability problems that we have not been able to completely eliminate. Hence, the algorithm for finding all roots of the FLS equations fails criterion I (numerical stability).

To avoid this pitfall rather than using the equations (1.2), it may be preferable to solve a system of the form

$$\mathbf{F}_A(\mathbf{z},\mathbf{d}) = \mathbf{\Gamma}_A^T(\mathbf{z},\mathbf{d})\mathbf{W}\boldsymbol{\gamma}(\mathbf{z},\mathbf{d}) = \mathbf{0} \tag{1.3}$$

where $\mathbf{\Gamma}_A$ is in some sense an approximate Jacobian. We now discuss what properties $\mathbf{\Gamma}_A$ must have in order to satisfy criterion II (that the CRLB is achieved).

We suppose that in the absence of noise the data is given by $\mathbf{d}_0$, and that in that case a solution $\mathbf{z}_0$ exactly solves our overdetermined system, giving

$$\boldsymbol{\gamma}(\mathbf{z}_0,\mathbf{d}_0) = \mathbf{0} \tag{1.4}$$

This guarantees we will also have $\mathbf{F}(\mathbf{z}_0,\mathbf{d}_0) = \mathbf{F}_A(\mathbf{z}_0,\mathbf{d}_0) = \mathbf{0}$, no matter what matrix $\mathbf{\Gamma}_A$ we use. Note that $\mathbf{\Gamma}_A$ could be a random $m \times n$ matrix, so there is no shortage of algorithms that produce correct answers for no noise yet are statistically suboptimal in noise. We now assume that the data is given by

$$\mathbf{d} = \mathbf{d}_0 + \delta\mathbf{d} \tag{1.5}$$

where $\delta\mathbf{d}$ is a small perturbation to the data. If the change in the computed solution is $\delta\mathbf{z}$, then in the linear approximation Eqn. (1.2) can be approximated as

$$\frac{\partial \mathbf{F}}{\partial \mathbf{z}}\delta\mathbf{z} = -\frac{\partial \mathbf{F}}{\partial \mathbf{d}}\delta\mathbf{d} \tag{1.6}$$

In the above, all partial derivatives are evaluated at $\mathbf{z} = \mathbf{z}_0$, and $\mathbf{d} = \mathbf{d}_0$. We are also using the notation $\frac{\partial \mathbf{F}}{\partial \mathbf{z}}$ to denote the Jacobian of $\mathbf{F}$ with respect to the variable $\mathbf{z}$, and $\frac{\partial \mathbf{F}}{\partial \mathbf{d}}$ to be the Jacobian with respect to the variable $\mathbf{d}$.

Similarly, Eqn. (1.3) can be approximated as

$$\frac{\partial \mathbf{F}_A}{\partial \mathbf{z}}\delta\mathbf{z} = -\frac{\partial \mathbf{F}_A}{\partial \mathbf{d}}\delta\mathbf{d} \tag{1.7}$$

The statistics of a method can be evaluated by analyzing these linearized equations. The variance of the solution $\delta\mathbf{z}$ of the linearized FLS equations (1.6) are minimized for a particular choice of the weighting matrix that depends on the covariance matrix of the data. In general, the optimal weighting matrix is not known until we know the solution. Strictly speaking Eqn. (1.2) was derived assuming that $\mathbf{W}$ was independent of $\mathbf{z}$, but for small noise, there is little error in assuming that Eqn. (1.2) holds even though it is not the gradient of $P(\mathbf{z})$ when $\mathbf{W}$ depends on $\mathbf{z}$. If we choose the proper weighting, our solution is said to be minimum variance. If the noise is Gaussian, this is equivalent to the solutions satisfying the CRLB. In general [15], the variance of the solution $\delta\mathbf{z}$ to the equations (1.7) will be greater than that for the equations (1.6). However, if we choose our approximate Jacobian so the two sets of linear equations are identical, then they will have the same small noise statistics.

In order to satisfy our first two criteria for a superior method, a reasonable goal is to choose an approximate Jacobian $\mathbf{\Gamma}_A$ so the linearized equations (1.7) are identical to the equations (1.6), but the equations (1.3) are simpler to solve than the equations (1.2). In §2 we will see that the Jacobian for the TOA equations is linear in the solution $\mathbf{z}$, and the Bancroft method is obtained by ignoring this linear term. In our notation, this can be written as

$$\mathbf{\Gamma}_A(\mathbf{z},\mathbf{d}) = \mathbf{\Gamma}(0,\mathbf{d}) \tag{1.8}$$

In §3 we see that the linearized Bancroft equations are not identical to the linearized FLS equations. Thus, Bancroft's method satisfies the first of our criteria, but not the second. The fact that Bancroft's

method does not satisfy the second criterion has been missed by most researchers in the field due to the fact that the method performs well under the conditions that are most common in GPS. In particular, in [15] it was shown that assuming the satellites satisfy the LER (Large Equal Radii) conditions, then the Bancroft method produces solutions whose statistics are close to those from the FLS equations. Here the LER conditions require that the satellites all be at the same radius from the center of the earth, and that they are many earth radii away. For many (but not all) GPS applications, this holds sufficiently, and the statistics from the Bancroft method are almost identical to those from the FLS. In [15] and §6 an example is given where the LER conditions are violated by having four satellites at MEO (Middle Earth Orbit) radius, and one at GEO (Geosynchronous Orbit). In this example, which models the addition of a WAAS satellite, the standard deviation of the Bancroft solution is about 12 times larger than that of the FLS solution.

In this paper we present two methods for solving overdetermined systems of TOA equations. Both of these methods have the property that when we linearize the equations about a zero noise solution, the linearized equations agree with the linearization of the FLS equations. This implies that they will have the same statistical properties as the FLS equations whether or not the LER conditions hold.

The first of our techniques called the MVLS (Minimum Variance Least Squares) solution is describe in §(4 . This uses some non-CRLB technique to get an initial estimate for the position. For example, we could use what we call the RLS (Reduced Least Squares) [19] to find the initial estimate to the overdetermined system, or we could use Bancroft's method. If the RLS method is used, a single estimate for the initial position is given, but if Bancroft's method is used, two estimates are given, and the algorithm splits into two branches. The initial estimate does not have good statistical properties, but it does have the property that in the absence of noise we have $\hat{\underline{z}} = \mathbf{z}_0$. Using this solution we form equations $\mathbf{F}_A(\mathbf{z}, \mathbf{d}) = \mathbf{0}$ as in Eqn. (1.3) where

$$\mathbf{\Gamma}_A(\mathbf{z}, \mathbf{d}) = \mathbf{\Gamma}(\hat{\underline{z}}, \mathbf{d}) \tag{1.9}$$

At this stage we end up with a system of equations that can be solved in a manner almost identical to that used by Bancroft. As with Bancroft's technique, the method gives two solutions. However, it gives solutions that achieve minimum variance even when the LER conditions are not satisfied.

We call the second solution technique the QMVLS (Quartic Minimum Variance Least Squares) technique. This technique yields four solutions rather than two as does Bancroft's or the MVLS method. A natural question is if the additional solutions provided by the QMVLS method ever yield answers that cannot be found by the MVLS or Bancroft technique. This leads us to our third and fourth criteria for judging a solution technique.

The fact that the FLS equations have nine critical points may leave us with the idea that we are missing something if we are not aware of each of these solutions. For example, the Bancroft method gives us two solutions, though in the over-determined case it is likely that one of these solutions will have a much smaller residual than the other. However, it is possible for the Bancroft equations to give two low residual solutions that are widely separated from each other. Though it is not unreasonable to expect that the FLS equation could give three or more widely separated low residual solutions, in section 7 we will see that this is not possible. However, in section §7 we show that it is possible for the FLS equations to have three low residual solutions that are close to each other. This occurs when the FLS equations are singular, which we will see is equivalent to the GDOP (Geometrical Dilution of Precision) being infinite.

In §8 we show that in the singular configurations described in the last paragraph the QMVLS faithfully gives the three low residual solutions given by the FLS equations. The authors are not certain if this is merely an interesting mathematical curiosity, or if it can actually be used to give more robust algorithms near such singularities.

An honest discussion of direct methods for the TOA equations should include some discussion of how Newton's method compares with these techniques. If Newton's method is not given a good initial guess, then it will not compare favorably to a good direct method. However, if we use a non-CRLB method such as Bancroft to give an initial guess to Newtons's method, it will perform extremely well in almost all cases. However, if we are near a singular configuration, then Newton's method may have trouble converging.

We now summarize the rest of this paper. In §2 we introduce the equations and notation that we will use throughout the paper. In §3 we discuss the linearization of the FLS and Bancroft equations. In §4 and §5 we present the MVLS and QMVLS algorithms. In §6 we give examples illustrating the minimum variance properties of the two methods, and compare them to the Bancroft algorithm. In §7 we discuss

how to find singular configurations of satellites, and discuss how many roots we really need out of a good method. In §8 we give numerical examples involving singular situations. In §9 we give our conclusions.

2 Problem Formulation

We begin by reviewing the basic equations for TOA geolocation. The equations describe both the satellite navigation application where the receiver is on the earth and the emitter tracking application where the transmitter is on the earth [12, 11]. Suppose that a receiver attempting to determine its position $\mathbf{x}$ measures N pseudoranges $\{\tau_k\}_{k=1}^N$ to satellites with positions $\{\mathbf{s}_k\}_{k=1}^N$. Assuming the signal travels at light speed c we have,

$$||\mathbf{x} - \mathbf{s}_k|| = \tau_k - \tau, \quad 1 \leq k \leq N \tag{2.1}$$

where $\tau = ct$ is the range equivalent receiver clock bias. Alternatively, from the emitter tracking point of view, Eqn (2.1) equates the distance between the emitter and the kth receiver with the range-equivalent kth TOA minus the emitter transmit time. The N equations (2.1) are used to determine four unknowns, the three dimensional vector $\mathbf{x}$, and τ. At least four satellites are required to determine $\mathbf{x}$ and τ. In practice the pseudoranges (or TOAs) are corrupted with measurement noise and Eqn (2.1) is inconsistent. We will assume that an estimate of the variance of the measurement noise for each pseudorange is available. This allows us to calculate the minimum variance least squares solution to Eqn (2.1), which we consider to be the most desirable solution of an inconsistent over-determined system.

Instead of working with the Eqns (2.1), it will be convenient to work with the equations obtained by squaring both sides.

$$||\mathbf{x} - \mathbf{s}_k||^2 = (\tau - \tau_k)^2, \quad 1 \leq k \leq N \tag{2.2}$$

If we introduce the vector

$$\mathbf{z} = \begin{pmatrix} \mathbf{x} \\ \tau \end{pmatrix} \tag{2.3}$$

we can write the equations (2.2) as

$$\mathbf{S}\mathbf{z} = \mu\mathbf{e} + \mathbf{b} \tag{2.4}$$

where

$$\mu = \mathbf{z}^T\mathbf{L}\mathbf{z} = \tau^2 - \mathbf{x}^T\mathbf{x}, \tag{2.5}$$

$$\mathbf{L} = \begin{pmatrix} -I & 0 \\ 0 & 1 \end{pmatrix}, \tag{2.6}$$

$$\mathbf{S} = \begin{pmatrix} -2\mathbf{s}_1^T & 2\tau_1 \\ -2\mathbf{s}_2^T & 2\tau_2 \\ . & . \\ . & . \\ -2\mathbf{s}_N^T & 2\tau_N \end{pmatrix} \tag{2.7}$$

$$\mathbf{e}^T = (1, 1, 1, ..., 1), \tag{2.8}$$

and

$$\mathbf{b}^T = (b_1, b_2, ...b_N) \quad b_k = \tau_k^2 - \mathbf{s}_k^T\mathbf{s}_k \tag{2.9}$$

Ideally we would like to satisfy the system of equations

$$\boldsymbol{\gamma}(\mathbf{z}) = \mathbf{S}\mathbf{z} - \mu(\mathbf{z})\mathbf{e} - \mathbf{b} = \mathbf{0} \tag{2.10}$$

When we have more than four satellites, this will give us an over-determined system of equations that we will satisfy in the least squares sense. This motivates the definition of the objective function

$$P(\mathbf{z}) = \frac{1}{2}\boldsymbol{\gamma}^T(\mathbf{z})\mathbf{W}\boldsymbol{\gamma}(\mathbf{z}) \tag{2.11}$$

where $\mathbf{W}$ is a weighting matrix. At any value of $\mathbf{z}$ that minimizes this objective function, we must have $\nabla P(\mathbf{z}) = \mathbf{0}$, which is equivalent to requiring that to first order we have $\delta P = \delta\mathbf{z}^T\nabla P = \mathbf{0}$.

To first order we have

$$\delta\boldsymbol{\gamma} = \mathbf{S}\delta\mathbf{z} - 2\mathbf{e}\left(\mathbf{z}^T\mathbf{L}\delta\mathbf{z}\right) \tag{2.12}$$

It follows that we can write

$$\delta P = \delta \mathbf{z}^T \left(\mathbf{S}^T - 2\mathbf{L}\mathbf{z}\mathbf{e}^T \right) \mathbf{W}\boldsymbol{\gamma}$$

If we require that this vanish for all values of $\delta \mathbf{z}^T$, this gives us what we will refer to as the FLS (Full Least Squares) equations.

Definition 1 (Full Least Squares Equations). *The FLS equations are given by*

$$\boldsymbol{\Gamma}^T(\mathbf{z})\mathbf{W}\boldsymbol{\gamma}(\mathbf{z}) = \mathbf{0} \tag{2.13a}$$

where μ is defined as in Eqn. (2.5), $\boldsymbol{\gamma}(\mathbf{z})$ as in (2.10), and

$$\boldsymbol{\Gamma}(\mathbf{z}) = \mathbf{S} - 2\mathbf{e}\mathbf{z}^T\mathbf{L} \tag{2.13b}$$

Eqn. (2.13) gives four equations for the four unknowns in the vector $\mathbf{z}$. Since the matrix $\boldsymbol{\Gamma}$ depends linearly on $\mathbf{z}$, and the vector $\boldsymbol{\gamma}$ has a quadratic dependence, these equations have a cubic nonlinearity. In [15] we showed that this equation can be converted to a ninth order eigenvalue problem, and hence has nine roots.

The Bancroft algorithm is obtained by ignoring the last term in Eqn. (2.13b).

Definition 2 (The Bancroft Equations). *The Bancroft equations are defined as*

$$\mathbf{S}^T\mathbf{W}\boldsymbol{\gamma} = \mathbf{0} \tag{2.14}$$

where $\boldsymbol{\gamma}$ is defined as in Eqn. (2.10).

In the notation of the last section, this is equivalent to setting $\boldsymbol{\Gamma}_A(\mathbf{z}) = \boldsymbol{\Gamma}(0)$.

Although the Bancroft algorithm has been described elsewhere [4, 15], we give a brief description of the algorithm here, because the MVLS and QMVLS algorithms use it. Using Eqn. (2.10) the solution to Eqn. (2.14) can be written as

$$\mathbf{z} = \mu\boldsymbol{\alpha} + \boldsymbol{\beta} \tag{2.15}$$

where

$$\mathbf{S}^T\mathbf{W}\mathbf{S}\boldsymbol{\alpha} = \mathbf{S}^T\mathbf{W}\mathbf{e} \tag{2.16a}$$

$$\mathbf{S}^T\mathbf{W}\mathbf{S}\boldsymbol{\beta} = \mathbf{S}^T\mathbf{W}\mathbf{b} \tag{2.16b}$$

If we substitute the expression in Eqn. (2.15) into the Eqn. (2.5) defining μ, we get a quadratic equation for μ. The two roots of this quadratic when substituted back into Eqn. (2.15) give us the two roots of the Bancroft algorithm.

We now briefly describe an alternative solution technique that can give an initial (though non-CRLB) estimate of a solution. In particular, we can view μ in Eqn. (2.4) as a new variable that is not a function of $\mathbf{z}$. We can then solve this (possibly over-determined) system of equations for $\mathbf{z}$ and μ. After finding this solution, we then discard μ. Rather than using this algorithm, we use the almost identical algorithm described in [19]. We will call this technique the RLS (Reduced Least Squares) algorithm. If we define the $N \times N$ projection matrix $\mathbf{E}$ as

$$\mathbf{E} = \mathbf{I} - \frac{\mathbf{e}\mathbf{e}^\mathbf{T}}{\mathbf{e}^\mathbf{T}\mathbf{e}} \tag{2.17}$$

it has the properties that $\mathbf{E}\mathbf{e} = \mathbf{0}$, and $\mathbf{E}\mathbf{x} = \mathbf{x}$ if $\mathbf{x}^T\mathbf{e} = 0$. For this reason if we multiply Eqn. (2.4) by $\mathbf{E}$, we get

$$\mathbf{E}\mathbf{S}\mathbf{z} = \mathbf{E}\mathbf{b}, \tag{2.18}$$

thus eliminating μ from the equations. The RLS (reduced Least Squares) solution arises from doing a weighted least squares solution to these equations.

Definition 3 (Reduced Least Squares Equations). *The RLS equations are the linear system*

$$(\mathbf{ES})^{\mathbf{T}}\mathbf{WESz} - (\mathbf{ES})^{\mathbf{T}}\mathbf{WEb} = \mathbf{0} \tag{2.19}$$

These can also be written as

$$\mathbf{\Gamma}_A^T\mathbf{W}\boldsymbol{\gamma}(\mathbf{z}) = \mathbf{0} \tag{2.20}$$

where

$$\mathbf{\Gamma}_A = \mathbf{E}^T\mathbf{WES} \tag{2.21}$$

and $\boldsymbol{\gamma}(\mathbf{z})$ is defined as in Eqn. (2.10).

In practice, it is best to do a QR factorization of $\mathbf{ES}$ in Eqn. (2.19) and factor out the matrix $\mathbf{R}^T$ (which is a 4×4 matrix, such as returned by MATLAB's economy sized QR decomposition) from both sides of this equation. In cases where $(\mathbf{ES})^{\mathbf{T}}\mathbf{WES}$ is poorly conditioned, this will considerably improve the conditioning of the matrices ([7], [9]).

3 Linearization of the FLS and Bancroft Equations

In order to determine the statistics of the solutions obtained from a particular solution method, we linearize the equations about the zero noise case, and determine the statistics of the linearized solutions. In Eqn. (2.10) we will suppose that the residual depends on the data $\mathbf{d}$ giving the pseudoranges or TOAs τ_k, ie $\mathbf{d}$ is the fourth column of $\mathbf{S}$. In the absence of noise, we suppose that the data is given by $\mathbf{d}_0$, and that for this value of the data there is a solution $\mathbf{z}_0$ that satisfies the equations exactly. That is, we have

$$\boldsymbol{\gamma}(\mathbf{z}_0, \mathbf{d}_0) = \mathbf{0} \tag{3.1}$$

Here we have included the functional dependence of $\boldsymbol{\gamma}$ on the data $\mathbf{d}$. Ths solution $\mathbf{z}_0$ clearly satisfies the FLS equations (2.13) and the Bancroft equations (2.14) when $\mathbf{d} = \mathbf{d}_0$. We are interested in finding how the solution changes when we add a small amount of noise $\mathbf{n}$ to the data. That is when

$$\mathbf{d} = \mathbf{d}_0 + \mathbf{n} \tag{3.2}$$

We will write the change in the solution as $\boldsymbol{\xi}$. That is , we have

$$\mathbf{z} = \mathbf{z}_0 + \boldsymbol{\xi} \tag{3.3}$$

The linearized FLS equations are obtained by expanding the equations (2.13) about the solution $(\mathbf{z}, \mathbf{d}) = (\mathbf{z}_0, \mathbf{d}_0)$, only keeping terms that are linear in $\boldsymbol{\xi}$ and $\mathbf{n}$. We can write

$$\mathbf{\Gamma}(\mathbf{z}_0 + \boldsymbol{\xi}, \mathbf{d}_0 + \mathbf{n}) = \mathbf{\Gamma}_0 + \delta\mathbf{\Gamma} \tag{3.4}$$

and

$$\boldsymbol{\gamma}(\mathbf{z}_0 + \boldsymbol{\xi}, \mathbf{d}_0 + \mathbf{n}) = \boldsymbol{\gamma}_0 + \delta\boldsymbol{\gamma} \tag{3.5}$$

where

$$\mathbf{\Gamma}_0 = \mathbf{\Gamma}(\mathbf{z}_0, \mathbf{d}_0) \tag{3.6}$$

$$\boldsymbol{\gamma}_0 = \boldsymbol{\gamma}(\mathbf{z}_0, \mathbf{d}_0) = \mathbf{0} \tag{3.7}$$

and $\delta\mathbf{\Gamma}$, and $\delta\boldsymbol{\gamma}$ are corrections to $\mathbf{\Gamma}$ and $\boldsymbol{\gamma}$. We can explicitly compute $\delta\boldsymbol{\gamma}$ to first order using

$$\delta\boldsymbol{\gamma} = \frac{\partial\boldsymbol{\gamma}}{\partial\mathbf{z}}\delta\mathbf{z} + \frac{\partial\boldsymbol{\gamma}}{\partial\mathbf{d}}\delta\mathbf{d} \tag{3.8}$$

giving

$$\delta\boldsymbol{\gamma} = \mathbf{\Gamma}(\mathbf{z}_0, \mathbf{d}_0)\boldsymbol{\xi} - \mathbf{N}\mathbf{n} + ... \tag{3.9}$$

where

$$\mathbf{N} = -\frac{\partial\boldsymbol{\gamma}(\mathbf{z}_0, \mathbf{d}_0)}{\partial\mathbf{d}} \tag{3.10}$$

We could also compute $\delta\mathbf{\Gamma}$ to first order, but as we will see, there is no need to have this term in the linear approximation.

The linearized equations are obtained by keeping the linear terms in the equation

$$(\mathbf{\Gamma}_0 + \delta\mathbf{\Gamma})^T \mathbf{W} (\boldsymbol{\gamma}_0 + \delta\boldsymbol{\gamma}) = \mathbf{0}, \tag{3.11}$$

giving

$$\delta\mathbf{\Gamma}^T\mathbf{W}\boldsymbol{\gamma}_0 + \mathbf{\Gamma}_0^T\mathbf{W}\delta\boldsymbol{\gamma} = \mathbf{\Gamma}_0^T\mathbf{W}\delta\boldsymbol{\gamma} = \mathbf{0} \tag{3.12}$$

Note that we do not need the term $\delta\mathbf{\Gamma}$ in the linear approximation because $\boldsymbol{\gamma}_0 = \mathbf{0}$. Substituting our expression for $\delta\boldsymbol{\gamma}$ from Eqn. (3.9) into this last equation gives us the linearized FLS equations.

Definition 4 (The Linearized FLS Equations). *The linearized FLS equations are given by*

$$\mathbf{\Gamma}_0^T\mathbf{W}\mathbf{\Gamma}_0\boldsymbol{\xi} = \mathbf{\Gamma}_0^T\mathbf{W}\mathbf{N}\mathbf{n} \tag{3.13}$$

All of the alternatives to the FLS equations that we consider will have the form

$$\mathbf{\Gamma}_A^T(\mathbf{z}, \mathbf{d})\mathbf{W}\boldsymbol{\gamma}(\mathbf{z}, \mathbf{d}) = \mathbf{0} \tag{3.14}$$

where $\mathbf{\Gamma}_A$ is defined by the specific alternate procedure. Any equations of this form have the property that $\mathbf{z} = \mathbf{z}_0$ will be a solution to these equations in the absence of noise since $\boldsymbol{\gamma}_0 = \mathbf{0}$. When we linearize this system of equations about $(\mathbf{z}, \mathbf{d}) = (\mathbf{z}_0, \mathbf{d}_0)$ we will once again be able to ignore the first order correction $\delta\mathbf{\Gamma}_A$ to $\mathbf{\Gamma}_A$, since $\boldsymbol{\gamma}_0 = \mathbf{0}$. This gives us the following lemma.

Lemma 3.1. *When we linearize the equations (3.14) about the solution $(\mathbf{z}, \mathbf{d}) = (\mathbf{z}_0, \mathbf{d}_0)$, we get the linearized equations*

$$\mathbf{\Gamma}_{A0}^T\mathbf{W}\mathbf{\Gamma}_0\boldsymbol{\xi} = \mathbf{\Gamma}_{A0}^T\mathbf{W}\mathbf{N}\mathbf{n} \tag{3.15}$$

where

$$\mathbf{\Gamma}_{A0} = \mathbf{\Gamma}_A(\mathbf{z}_0, \mathbf{d}_0) \tag{3.16}$$

When we apply this procedure to the Bancroft equations we get the following.

Definition 5 (Linearized Bancroft Equations). *We define the linearized Bancroft equations as*

$$\mathbf{S}^T\mathbf{W}\mathbf{\Gamma}_0\boldsymbol{\xi} = \mathbf{S}^T\mathbf{W}\mathbf{N}\mathbf{n} \tag{3.17}$$

When solving an overdetermined system of equations $\mathbf{\Gamma}\boldsymbol{\xi} = \mathbf{N}\mathbf{d}$, we usually weight the equations with a matrix $\mathbf{W}$, and then multiply through by $\mathbf{\Gamma}^T$. In [15] we show that in order to minimize the covariance matrix $\mathbf{C} = \langle\boldsymbol{\xi}\boldsymbol{\xi}^T\rangle$ of the output, we need to choose the weighting such that $\mathbf{W} = \left(\mathbf{N}\mathbf{\Sigma}\mathbf{N}^T\right)^{-1}$ where $\mathbf{\Sigma}$ is the covariance matrix of the data $\mathbf{n}$. Furthermore, we could obtain solutions to the overdetermined system by solving the equations $\mathbf{T}^T\mathbf{W}\mathbf{\Gamma}\boldsymbol{\xi} = \mathbf{T}^T\mathbf{W}\mathbf{N}\boldsymbol{\xi}$. That is, we could multiply through by a matrix $\mathbf{T}$ other than $\mathbf{\Gamma}$. However, if we want to minimize the covariance matrix $\mathbf{C}$, we must use $\mathbf{T} = \mathbf{\Gamma}$ [15].

This shows us that in general the Bancroft solutions will not give minimum variance solutions. However, in [15] it was shown that for the case of satellites all at the same radius and far from the center of the Earth (LER conditions), the Bancroft equations come close to giving minimum variance solutions. However, when this restriction is dropped, they can be far from minimum variance.

4 The MVLS Algorithm

The Bancroft equations differ from the FLS equations in that they ignore the term $-2\mathbf{L}\mathbf{z}\mathbf{e}^T\mathbf{W}\boldsymbol{\gamma}$ in Eqn. (2.13a). In the MVLS algorithm, rather than ignoring this term, we will approximate the term $\mathbf{L}\mathbf{z}$ by $\mathbf{L}\underline{\hat{z}}$ where $\underline{\hat{z}}$ is a non-CRLB solution such as the RLS equations in Eqn. (2.19), or the Bancroft equations as in Eqn. (2.14).

Definition 6 (The MVLS Equations). *We define the MVLS equations as*

$$\left(\mathbf{S} - 2\mathbf{e}\underline{\hat{z}}^T\mathbf{L}\right)^T \mathbf{W}\boldsymbol{\gamma} = \mathbf{0} \tag{4.1}$$

where $\underline{\hat{z}}$ satisfies Eqn. (2.19).

The MVLS equations can be solved in a manner almost identical to the Bancroft equations. In particular, if we know μ, we can once again write the solution as in Eqn. (2.15), $\mathbf{z} = \mu\boldsymbol{\alpha} + \boldsymbol{\beta}$, where in this case we have

$$\mathbf{S}_1^T\mathbf{W}\mathbf{S}\boldsymbol{\alpha} = \mathbf{S}_1^T\mathbf{W}\mathbf{e} \tag{4.2}$$

$$\mathbf{S}_1^T\mathbf{W}\mathbf{S}\boldsymbol{\beta} = \mathbf{S}_1^T\mathbf{W}\mathbf{b} \tag{4.3}$$

and

$$\mathbf{S}_1 = \mathbf{S} - 2\mathbf{e}\underline{\hat{z}}^T\mathbf{L} \tag{4.4}$$

If we substitute the expression (2.15) for $\mathbf{z}$ into the equation (2.5) defining μ, we once again get a quadratic equation for μ. This gives us two roots to the MVLS equations.

For optimal numerical robustness, when solving for $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, in Eqn. (4.2) and (4.3), it is best to do a QR factorization of the matrix $\mathbf{S}_1$. That is we write $\mathbf{S}_1 = \mathbf{Q}_1\mathbf{R}_1 = \mathbf{S}_1$, and then factor out the matrix $\mathbf{R}_1^T$ from the left hand side of both sides of Eqn.(4.2) and (4.3).

The next lemma shows that the MVLS algorithm will be CRLB.

Lemma 4.1. *If we linearize the MVLS equations about a zero noise solution, we get the linearized FLS equations as in Eqn. (3.13).*

Proof. In the notation of lemma (3.1) we have $\boldsymbol{\Gamma}_A(\mathbf{z}, \mathbf{d}) = \mathbf{S} - 2\mathbf{e}\underline{\hat{z}}\mathbf{L}$, where $\underline{\hat{z}}$ is a function of the data $\mathbf{d}$. In the absence of noise we have $\underline{\hat{z}} = \mathbf{z}_0$, and hence $\boldsymbol{\Gamma}_{A0} = \boldsymbol{\Gamma}_0$. Hence the linearized equations are the same as the linearized FLS equations. □

As pointed out by Hogg [8] and Aronson [3] and discussed in [15], it is not possible to determine the weighting matrix $\mathbf{W}$ a priori when finding the least squares solution of the Eqn. (2.2). This is because the weighting matrix involves the quantities $1/(\tau - \tau_k)$, which are not known unless we know τ, which is part of our answer. However, in many applications τ is much smaller than τ_k, and the τ_k's are all roughly the same. This assumption is more justifiable when the LER conditions are satisfied. In this case, we have a good guess for the weighting matrix. In the MVLS algorithm the weighting matrix $\mathbf{W}$ can be determined from the initial guess $\underline{\hat{z}}$. In this case, the algorithm remains CRLB even when the quantities $\tau - \tau_k$ are not nearly the same.

5 The QMVLS Algorithm

The Bancroft solution ignores the term $-2\mathbf{L}\mathbf{z}\mathbf{e}^T\mathbf{W}\boldsymbol{\gamma}$ in the FLS equation (2.13a). By doing this, the eventual solution can be written as $\mathbf{z} = \mu\boldsymbol{\alpha} + \boldsymbol{\beta}$ as in Eqn. (2.15), where $\boldsymbol{\alpha}$, and $\boldsymbol{\beta}$ are solutions to Eqns. (2.16) .

In the QMVLS solution we will assume that $\mathbf{z}$ has the form in Eqn. (2.15) when evaluating the term $-2\mathbf{L}\mathbf{z}\mathbf{e}^T\mathbf{W}\boldsymbol{\gamma}$. It should be emphasized that we assume that $\mathbf{z}$ has the form in Eqn. (2.15), but we do not assume that μ has the value given by the Bancroft algorithm.

Definition 7 (The QMVLS Equations). *We define the QMVLS Equations as*

$$\left(\mathbf{S} - 2\mathbf{e}\left(\mu\boldsymbol{\alpha} + \boldsymbol{\beta}\right)^T\mathbf{L}\right)^T\mathbf{W}\boldsymbol{\gamma}(\mathbf{z}) = \mathbf{0} \tag{5.1}$$

where μ is given by Eqn. (2.5) and $\boldsymbol{\gamma}$ is given by Eqn. (2.10), and $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ satisfy Eqns. (2.16).

We have the lemma

Lemma 5.1. *When we linearize the QMVLS equations about a zero noise solution, we get the linearized FLS equations.*

Proof. In the notation of lemma (3.1) we have $\boldsymbol{\Gamma}_A(\mathbf{z}, \mathbf{d}) = \mathbf{S} - 2\mathbf{e}(\mu\boldsymbol{\alpha} + \boldsymbol{\beta})^T\mathbf{L}$. In the absence of noise, we have $\boldsymbol{\Gamma}_A(\mathbf{z}_0, \mathbf{d}_0) = \boldsymbol{\Gamma}_0$. The present lemma now follows directly from lemma (3.1). □

We now show that we can solve the QMVLS equations as defined in Def. (7) by solving a quartic equation. To do this we note that Eqn. (5.1) implies that

$$\mathbf{z} = \mu\boldsymbol{\alpha} + \boldsymbol{\beta} + (\mu\mathbf{p} + \mathbf{q})\,\lambda \tag{5.2}$$

where $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are defined as in Eqns. (2.16), and

$$\mathbf{p} = 2(\mathbf{S}^T\mathbf{W}\mathbf{S})^{-1}\mathbf{L}\boldsymbol{\alpha} \tag{5.3}$$
$$\mathbf{q} = 2(\mathbf{S}^T\mathbf{W}\mathbf{S})^{-1}\mathbf{L}\boldsymbol{\beta} \tag{5.4}$$

and

$$\lambda = \mathbf{e}^T\mathbf{W}\left(\mathbf{S}\mathbf{z} - \mu\mathbf{e} - \mathbf{b}\right) \tag{5.5}$$

For a given value of μ we can solve for λ by substituting our expression (5.2) for $\mathbf{z}$ into our expression (5.5) for λ. This gives us

$$\lambda = \frac{e_1\mu + e_0}{d_1\mu + d_0} \tag{5.6}$$
$$d_0 = 1 - \mathbf{e}^T\mathbf{W}\mathbf{S}\mathbf{q} \tag{5.7}$$
$$d_1 = -\mathbf{e}^T\mathbf{W}\mathbf{S}\mathbf{p} \tag{5.8}$$

$$e_1 = \mathbf{e}^T\mathbf{W}\mathbf{S}\boldsymbol{\alpha} - \mathbf{e}^T\mathbf{W}\mathbf{e} \tag{5.9}$$
$$e_0 = \mathbf{e}^T\mathbf{W}\mathbf{S}\boldsymbol{\beta} - \mathbf{e}^T\mathbf{W}\mathbf{b} \tag{5.10}$$

If we substitute the expression (5.6) into the expression (5.2), we get

$$\mathbf{z} = \frac{1}{d_1\mu + d_0}\left(\mathbf{g}_2\mu^2 + \mathbf{g}_1\mu + \mathbf{g}_0\right) \tag{5.11}$$

where

$$\mathbf{g}_2 = e_1\mathbf{p} + d_1\boldsymbol{\alpha} \tag{5.12a}$$
$$\mathbf{g}_1 = e_1\mathbf{q} + e_0\mathbf{p} + d_0\boldsymbol{\alpha} + d_1\boldsymbol{\beta} \tag{5.12b}$$
$$\mathbf{g}_0 = e_0\mathbf{q} + d_0\boldsymbol{\beta} \tag{5.12c}$$

If we use the expression (5.11), the requirement that $\mathbf{z}^T\mathbf{L}\mathbf{z} = \mu$ can be written as

$$(\mathbf{g}_2\mu^2 + \mathbf{g}_1\mu + \mathbf{g}_0)^T\mathbf{L}(\mathbf{g}_2\mu^2 + \mathbf{g}_2\mu + \mathbf{g}_0) = \mu(d_1\mu + d_0)^2 \tag{5.13}$$

which gives us a quartic equation in μ. Once we have determined μ, we can determine $\mathbf{z}$ using Eqn. (5.11).

The Bancroft and MVLS algorithms give two solutions, whereas the QMVLS algorithm gives four solutions. Typically the two solutions from the Bancroft and MVLS algorithms are enough to find the answer we are looking for. However, under degenerate conditions, it is possible for the QMVLS algorithm to find additional solutions that are not found by the other two algorithms. In section §8 we give an example where the other two algorithms in fact miss the best solution.

6 Examples Illustrating the Minimum Variance Properties

In this section we present two simulations illustrating the minimum variance property of the MVLS and QMVLS algorithms. In both simulations we do $10,000$ trials where in each trial we choose 5 randomly placed satellites (subject to radius and visibility constraints), and a randomly selected point on the earth. The TOAs were perturbed with 100 ns (1-σ) Gaussian noise and passed to algorithms which are evaluated using the 50th percentile spherical error probable (SEP50), the 95th percentile error (SEP95) and the location error standard deviation σ. We compare these errors for the FLS, Bancroft, MVLS, and QMVLS algorithms.

In the first simulation the satellites are all chosen to be at the MEO radius, and hence the LER conditions are satisfied. Table 1 shows the data for these trials, and shows that all of the algorithms have very similar statistics. In particular, note that the Bancroft algorithm performs as well as the other three algorithms.

In the second set of trials, four of the satellites are at MEO radius, but the fifth is at GEO. In these trials the LER conditions are not satisfied. Table 2 shows that in this case the MVLS and QMVLS algorithms give statistics almost identical to the FLS algorithm, but the Bancroft algorithm gives significantly different statistics. It should be noted that the difference in the statistics of Bancroft from the other algorithms comes mostly from the tails of the probability distribution. For example, the SEP 50 of the Bancroft solution is only about 10 percent greater than for the other methods, but the SEP 95 is about 4 times bigger, and the standard deviation is 12 times bigger. The median error of Bancroft is not dramatically greater than for the CRLB methods but the 95th percentile error is.

	SEP50	SEP95	σ
FLS	0.067	0.215	0.136
QMVLS	0.067	0.216	0.136
MVLS	0.067	0.216	0.136
Bancroft	0.067	0.216	0.136

Table 1: 50th and 95th percentile spherical error and geolocation error standard deviation σ, all in km, for FLS, QMVLS, MVLS and Bancroft's algorithm for a GPS simulation satisfying the LER conditions. All algorithms perform to the CRLB.

	SEP50	SEP95	σ
FLS	0.071	0.234	0.190
QMVLS	0.071	0.234	0.190
MVLS	0.071	0.234	0.190
Bancroft	0.087	0.803	2.446

Table 2: 50th and 95th percentile spherical error and geolocation error standard deviation σ for FLS, QMVLS, MVLS and Bancroft's algorithm for a GPS+GEO simulation violating the LER conditions. Bancroft's algorithm's standard deviation is a factor of 12 above the others.

7 Singular Configurations of Satellites

In [15] we showed that there are 9 solutions to the FLS equations, i.e. a maximum of 9 critical points of the square error objective function. The QMVLS algorithm gives four solutions, the Bancroft and MVLS algorithms give two, and the RLS algorithm gives one. The question arises, do we gain anything from methods that have more solutions ?

One could argue that we should be able to get close enough to any solution of interest using the RLS algorithm to provide an initial guess for the Gauss-Newton method. Similar reasoning suggests that using the Bancroft algorithm to find two initial guesses for Gauss-Newton should also be sufficient. We will now carefully analyze the justification for these statements, and find that they in fact break down under certain singular configurations of the satellites. In the next section we give an example where the FLS and QMVLS algorithms agree closely with the two solutions produced by Bancroft or MVLS, but the best solution is in fact a third root not found by the Bancroft or MVLS algorithms.

To begin with, we ask the question: what must be true for an over-determined system of TOA equations to have more than one low residual solution ? To answer this question we consider the case where there is no noise, and find the condition for there to be two solutions with zero residual. When the noise is small, these zero residual solutions will get perturbed to low residual solutions. Any zero residual solution $\mathbf{z}$ must satisfy the equation (2.18) exactly. It follows that if there is more than one zero residual solution, then the matrix $\mathbf{ES}$ must have a non-trivial null vector $\boldsymbol{\phi}$, that is, the RLS equations must be singular. Since we are assuming that we have noiseless data, this means that the system in Eqn. (2.18) has at least one solution $\mathbf{z}_1$. Assuming there is only one linearly independent null vector $\boldsymbol{\phi}$ of $\mathbf{ES}$, then the general solution to Eqn. (2.18) can be written as

$$\mathbf{z} = \mathbf{z}_1 + \kappa\boldsymbol{\phi} \tag{7.1}$$

When we require that $\mathbf{z}$ satisfies the condition $\mathbf{z}^T L\mathbf{z} = \mu$, this gives us a quadratic equation for κ, and in this case we see that we expect to see only two zero residual soluitons, as would be found by Bancroft's method.

In a typical situation, we do not expect the matrix $\mathbf{ES}$ to be singular. However, if we have five satellites, it is generic (not atypical) for a system of satellites to pass through a configuration where the matrix $\mathbf{ES}$ is singular. The more satellites we have, the less likely we are to encounter such a configuration.

We now ask if it is possible to have more than two low residual solutions that are all widely separated from each other. Suppose we write the FLS equations as

$$\mathbf{F}(\mathbf{z}) = \mathbf{\Gamma}^T(\mathbf{z})\mathbf{W}\boldsymbol{\gamma}(\mathbf{z}) = \mathbf{0} \tag{7.2}$$

Then using $\mathbf{S}^T\mathbf{W}\boldsymbol{\gamma} = \mathbf{F}(\mathbf{z}) + 2\mathbf{Lze}^\mathbf{T}\mathbf{W}\boldsymbol{\gamma}$ the Bancroft equations, $\mathbf{S}^T\mathbf{W}\boldsymbol{\gamma} = \mathbf{0}$, can be written as

$$\mathbf{F}(\mathbf{z}) = -2\mathbf{Lze}^T\mathbf{W}\boldsymbol{\gamma}(\mathbf{z}) \tag{7.3}$$

If we have a solution $\mathbf{z}_0$ to the FLS equations that has a low residual ($\boldsymbol{\gamma}(\mathbf{z}_0)$ is small) then for $||\mathbf{z} - \mathbf{z}_0|| << 1$ the Bancroft equations can be considered as a small perturbation to the FLS equations. That is, we can write the Bancroft equations as

$$\mathbf{F}(\mathbf{z}) = \delta\mathbf{f}(\mathbf{z}) \tag{7.4}$$

where $\delta\mathbf{f}(\mathbf{z})$ is small. The implicit function theorem from multi-variable calculus [1] tells us that if we have a system of equations $\mathbf{F}(\mathbf{z}) = 0$, that has a solution $\mathbf{z}_0$, satisfying $\mathbf{F}(\mathbf{z}_0) = 0$, and if the Jacobian $\frac{\partial \mathbf{F}}{\partial \mathbf{z}}(\mathbf{z}_0)$ is non-singular at $\mathbf{z}_0$, then when we perturb this equation by a small amount, there will be a unique solution to the perturbed equations in the neighborhood of $\mathbf{z}_0$ that is close to $\mathbf{z}_0$. This implies that if the Jacobian of all of our small residual solutions of the FLS equations are non-singular, then all of these solutions should be close to solutions of the Bancroft equations. More precisely, we should require that not only is the Jacobian non-singular, but it is sufficiently far away from being singular.

This suggests that if we are to look for configurations of satellites where the simpler methods are inadequate (even after using Gauss-Newton to polish their solutions), we should look for singular configurations of satellites. In particular, we look for configurations where the Jacobian is singular when we linearize about a noiseless solution. Eqn. (3.13) shows that the Jacobian is given by $\mathbf{\Gamma}_0^T\mathbf{W}\mathbf{\Gamma}_0$. If this is singular it means that there must be a vector $\boldsymbol{\phi}$ such that $\mathbf{\Gamma}_0^T\mathbf{W}\mathbf{\Gamma}_0\boldsymbol{\phi} = 0$. and hence $\boldsymbol{\phi}^T\mathbf{\Gamma}_0^T\mathbf{W}\mathbf{\Gamma}_0\boldsymbol{\phi} = 0$. However, since $\mathbf{W}$ is positive definite, this implies that $\mathbf{\Gamma}_0\boldsymbol{\phi} = 0$.

If we linearize $\boldsymbol{\gamma}(\mathbf{z})$ about a solution $\mathbf{z}_0^T = (\mathbf{x}_0^T, \tau_0)$, that satisfies $\boldsymbol{\gamma}(\mathbf{z}_0) = 0$, we find that $\boldsymbol{\gamma}(\mathbf{z}_0 + \delta\mathbf{z}) = \mathbf{\Gamma}_0\delta\mathbf{z}$. If we let $\delta\mathbf{z} = (\delta\mathbf{x}^T, \delta\tau)$, this gives us

$$2\left(\mathbf{x}_0 - \mathbf{s}_k\right)^T\delta\mathbf{x} - 2\left(\tau_0 - \tau_k\right)\delta\tau = \qquad kth \text{ component of } \mathbf{\Gamma}\delta\mathbf{z} \tag{7.5}$$

This expression for $\mathbf{\Gamma}\delta\mathbf{z}$ could also be obtained using the explicit formula for $\mathbf{\Gamma}$ given by Eqn. (2.13b).

If the system is singular, this implies that there is a vector $\boldsymbol{\phi}$ such that $\mathbf{\Gamma}\boldsymbol{\phi} = 0$. If we write

$$\boldsymbol{\phi} = \begin{pmatrix} -\mathbf{p} \\ \alpha \end{pmatrix} \tag{7.6}$$

this implies that we have

$$\left(\mathbf{s}_k - \mathbf{x}_0\right)^T\mathbf{p} = (\tau_0 - \tau_k)\alpha \tag{7.7}$$

This can be written as

$$\mathbf{e}_k^T\mathbf{p} = \alpha \quad \text{for } k = 1, N \tag{7.8}$$

where

$$\mathbf{e}_k = \frac{\mathbf{s}_k - \mathbf{x}_0}{\tau_0 - \tau_k} = \frac{\mathbf{s}_k - \mathbf{x}_0}{\mid \mathbf{x}_0 - \mathbf{s}_k \mid} \tag{7.9}$$

Geometrically, Eqn. (7.8) implies that the satellites all lie on a cone whose vertex is at the receiver $\mathbf{x}$, where $\mathbf{p}$ orients the cone's axis and α is a constant.

A natural question is: how likely are such configurations to occur ? We answer this question by finding how many parameters we would need to adjust to make such a configuration generic. Here we are using generic in a sense that is used in singularity theory [2] . In singularity theory we say a property is generic if it persists under a small perturbation of our equations. Essentially, if a property is generic it means that we expect to see it in typical systems that are not merely mathematically contrived. For example, we say that it is generic for a system of N equations in N unknowns to have isolated solutions. It is easy to find systems of N equations in N unknowns that do not have any solutions (especially if we limit ourselves to real solutions), but if we find a system that has a solution, when we perturb it, we expect to see it still have a solution. On the other hand, if we have systems with $N + 1$ equations in N

unknowns, we can find systems that have solutions, but when we apply arbitrary perturbations to such systems, they no longer have a solution.

The more satellites in our system that are constantly in view, the less likely that we will encounter a singular configuration. With this in mind, we suppose we have five satellites in view, and hence the matrix $\mathbf{\Gamma}$ will be a 5×4 dimensional matrix. By adjusting two parameters we can make it generic to have a non-trivial solution $\mathbf{\Gamma}\boldsymbol{\phi} = 0$, That is, by adjusting two parameters it is not unreasonable to be able to solve the equations $\mathbf{\Gamma}\boldsymbol{\phi} = 0$, $\boldsymbol{\phi}^T\boldsymbol{\phi} = 1$. This will give us 6 equations in the six unknowns $\boldsymbol{\phi}$ and the two parameters we are adjusting.

When applied to GPS systems, this means that at any point in time, it would be generic to have a singular configuration somewhere on the surface of the earth, since the surface of the earth gives us two parameters to adjust. Similarly, if we let time vary, then at some point in time, we might expect to see such a configuration on a one dimensional curve such as the equator.

8 Example at a Singular Configuration

In this section we will show that when the satellites are in a singular configuration the QMVLS technique finds very nearly the three solutions that are found by the FLS technique. We believe that this is somewhat of an academic point, but it does illustrate that the QMVLS algorithm has sacrificed very little over doing the more difficult and less robust FLS solve.

In [15] we showed that there is something special about systems of satellites where they are all at equal radii from the center of the earth. It turns out that this property also effects the equations when they become singular. For this reason, we will choose our example from a case where the satellites are not all at the same radius. In particular, we will suppose that four of the satellites are at the GPS radius, and that the fifth satellite is at a GEO radius. The period of GEO satellites is twice the period of GPS satellites, and hence using Keplers third law of motion, the GEO radius is $2^{2/3}$ times the GPS radius. If we use a length scale based on the radius of the earth, four of the satellites have a radius of 4.16, and the fifth has a radius of $2^{2/3}$ times this.

In our example we arrange for the satellites to satisfy the cone condition (7.8). We have been able to encounter such situations by looking at enough GPS configurations, but in this example we have explicitly put the satellites in such a singular configuration. The positions of the satellites are given by

$$\begin{pmatrix} \mathbf{s}_1^T \\ \mathbf{s}_2^T \\ \mathbf{s}_3^T \\ \mathbf{s}_4^T \\ \mathbf{s}_5^T \end{pmatrix} = \begin{pmatrix} (.87299, 1.87293, 6.27191) \\ (2.13601, 0.55425, 3.52644) \\ (2.39458, 0.13330, 3.39908) \\ (2.00895, 0.68732, 3.57732) \\ (2.22186, 0.44388, 3.48882) \end{pmatrix} \tag{8.1}$$

In this example, we have the receiver at the location $\mathbf{x}^T = (0, 0, 1)$. With the satellites and receiver in these positions, we compute the times τ_k, and then add noise to them. We have added random noise at the level of $1.e-5$ earth radii. We then use the Bancroft, QMVLS, and FLS algorithms to find solutions. The particular vector of noise we have used is $1.e-5\mathbf{n}$ where $\mathbf{n}$ is given by

$$\mathbf{n} = \begin{pmatrix} 0.57112950 \\ 0.97354900 \\ 0.43808954 \\ 0.97092952 \\ 0.41250846 \end{pmatrix} \tag{8.2}$$

The Bancroft, QMVLS, and FLS algorithms all find two solutions that are in excellent agreement with each other. We call these $\mathbf{z}_1$ and $\mathbf{z}_2$.

$$\mathbf{z}_1(Banc) = \begin{pmatrix} 0.0165660 \\ -0.0109002 \\ 1.0547863 \\ 0.0498822 \end{pmatrix}, \mathbf{z}_1(QMVLS) = \begin{pmatrix} 0.0167333 \\ -0.0110075 \\ 1.0553338 \\ 0.0503806 \end{pmatrix}, \mathbf{z}_1(FLS) = \begin{pmatrix} 0.0167332 \\ -0.0110075 \\ 1.0553337 \\ 0.0503805 \end{pmatrix}$$

$$\mathbf{z}_2(Banc) = \begin{pmatrix} -0.0166862 \\ 0.0112724 \\ 0.9438322 \\ -0.0511219 \end{pmatrix}, \mathbf{z}_2(QMVLS) = \begin{pmatrix} -0.0165168 \\ 0.0111552 \\ 0.9445073 \\ -0.0505974 \end{pmatrix}, \mathbf{z}_2(FLS) = \begin{pmatrix} -0.0165168 \\ 0.0111552 \\ 0.9445072 \\ -0.0505975 \end{pmatrix}$$

However, there is a third root that Bancroft cannot find. The QMVLS and FLS algorithms both find this root, and give quite good agreement.

$$\mathbf{z}_3(QMVLS) = \begin{pmatrix} -0.0004119 \\ 0.0002131 \\ 0.9986970 \\ -0.0012004 \end{pmatrix}, \mathbf{z}_3(FLS) = \begin{pmatrix} -0.0005043 \\ 0.0002748 \\ 0.9983888 \\ -0.0014812 \end{pmatrix}$$

The QMVLS and FLS algorithms find other roots, but they all have high residuals, and are not in agreement with each other since there is no reason for the QMVLS equations to approximate the FLS equations unless the residuals are small.

It should be noted that the solution $\mathbf{z}_3$ is in fact much more accurate than the solutions $\mathbf{z}_1$ and $\mathbf{z}_2$. The location error for $\mathbf{z}_3$ is $1.7e-3$, while the location errors for the other two solutions are about $5.8e-2$. That is, $\mathbf{z}_3$ is about 30 times more accurate than the other solutions. In [16] we show that it is a general property of non-linear least squares solutions, that near a singular point we will have three solutions. We also show that near such a point, the most accurate solutions in fact has the highest residual. This is the case here, where the solution $\mathbf{z}_3$ has a slightly larger residual than either $\mathbf{z}_1$ or $\mathbf{z}_2$. In [16] we show that the correct solution can in fact be found by choosing the solution that is the least sensitive to perturbations in the data.

9 Conclusions

We have presented two new methods (the MVLS and QMVLS algorithms) for solving overdetermined systems of TOA equations for geolocation. Similar to the Bancroft algorithm, the MVLS algorithm gives two solutions. The QMVLS algorithm gives four solutions. Each of these algorithms can be shown to be minimum variance, even when the satellites are not at equal radii. We have presented a situation where neither of the two solutions coming out of the Bancroft or MVLS algorithms is the best answer, but one of the four roots coming out of the QMVLS algorithm is in close agreement the best answer given by the FLS algorithm.

Acknowledgements

This work was performed at Sandia Laboratories. Sandia is a multiprogram laboratory operated by Sandia Corporation, a Lockheed Martin Company, for the United States Department of Energy under Contract DE-AC04-94AL85000.

References

[1] T.M. Apostol. *Mathematical analysis*. World Student Series Edition, 1974.

[2] VI Arnold. *Catastrophe Theory*. Springer, 1992.

[3] E.A. Aronson. Location algorithms and errors in time-of-arrival systems. Technical Report SAND2001-2892, Sandia National Laboratory, PO Box 5800, Albuquerque, NM 87185-5800, 2001.

[4] S. Bancroft. An algebraic solution of the gps equations. *IEEE Transactions on Aerospace and Electronic Systems*, AES(21):224–232, 1985.

[5] I. Biton, M. Koifman, and IY Bar-Itzhack. Improved direct solution of the Global Positioning System equation. *Journal of Guidance, Control, and Dynamics*, 21(1):45–49, 1998.

[6] E. Choi and D.A. Cicci. Analysis of GPS static positioning problems. *Applied Mathematics and Computation*, 140(1):37–51, 2003.

[7] G.H. Golub, L. Van, and F. Charles. *Matrix Computations*. Johns Hopkins Univ Pr, 1996.

[8] C.J. Hogg. Alternative methods for computing event location in the NUDET detection system. Technical Report SAND2001-2893, Sandia National Laboratory, PO Box 5800, Albuquerque, NM 87185-5800, 2001.

[9] C.L. Lawson and R.J. Hanson. *Solving least squares problems*. Prentice-Hall Englewood Cliffs, NJ, 1974.

[10] J. L. Leva. An alternative closed-form solution to the GPS pseudo-rangee quations. *IEEE Transactions on Aerospace and Electronic Systems*, 32(4):1430–1439, 1996.

[11] Jeff Mason. Algebraic two-satellite TOA/FOA position solution on an ellipsoidal Earth. *Aerospace and Electronic Systems, IEEE Transactions on*, 40(3):1087–1092, 2004.

[12] Jeff Mason and Louis Romero. TOA/FOA geolocation solutions using multivariate resultants. *Navigation(Washington, DC)*, 52(3):163–177, 2005.

[13] PS Noe, KA Myers, and TK Wu. A Navigation Algorithm for the Low-Cost GPS Receiver. *Navigation*, 25(2):258–264, 1978.

[14] W.H. Press et al. *Numerical recipes*. Cambridge University Press New York, 1986.

[15] L.A. Romero, Jeff Mason, and D.M. Day. Analysis and extension of bancroft's method for toa geoloaction problems. *SIAM Sournal of Scientific Comoputing*, 40(3):1087–1092, 2007.

[16] L.A. Romero, J.J. Mason, and D.M. Day. Bifurcations in non-linear least squares optimization problems. *SIAM Journal of Optimization*, 40(3):1087–1092, 2007.

[17] R. Schmidt. Least Squares Range Difference Location. *IEEE Transactions on Aerospace and Electronic Systems*, 32(1):234–242, 1996.

[18] D.J. Torrieri. Statistical Theory of Passive Location Systems. *IEEE Transactions on Aerospace and Electronic Systems*, 20(2):183–198, 1984.

[19] Jonathon D. Wolfe and Jason L. Speyer. Universally convergent statistical solution of pseudo-range equations. *Navigation*, 49(4):183–192, 2002.