INTERACTION BETWEEN A PUNCH AND AN ARBITRARY CRACK OR INCLUSION IN A TRANSVERSELY ISOTROPIC HALF-SPACE

E. Karapetian
Department of Mathematics & Computer Science
Suffolk University, Boston, MA 02114
USA

S.V. Kalinin
The Center for Nanophase Materials Sciences
Oak Ridge National Laboratory, Oak Ridge, TN 37831
USA

I. Fabrikant
Fabrikant-Research
Montreal, Quebec
Canada

Abstract: We consider the problem of an arbitrary shaped rigid punch pressed against the boundary of a transversely isotropic half-space and interacting with an arbitrary flat crack or inclusion, located in the plane parallel to the boundary. The set of governing integral equations is derived for the most general conditions, namely, presence of both normal and tangential stresses under the punch, as well as general loading of the crack faces. In order to verify correctness of the derivations, two different methods were used to obtain governing integral equations: generalized method of images and utilization of the reciprocal theorem. Both methods gave the same results. Axisymmetric coaxial case of interaction between a rigid inclusion and a flat circular punch both centered along the z-axis is considered as an illustrative example. Most of the final results are presented in terms of elementary functions.

Introduction

Mechanical characteristics of solids and surfaces are relevant in multiple areas of science and engineering applications ranging from structural mechanics to bioengineering to corrosion. The need to probe mechanical behavior of surfaces have spurred the development of multiple characterization techniques ranging from micro and nanoindentation[1] to scanning probe microscopies including Atomic Force Acoustic Microscopy[2-11] and frequency tracking[12] and band excitation[13-17] dynamic probes. Measured in these methods are the tip-surface forces as a function of indentation depth (nanoindentation), or resonance frequency shifts (AFAM) directly related to the tip-surface stiffness.

Interpretation of this data in terms of materials functionalities requires the known functional relationships between the force acting on the probe and measured displacement or resonant frequency shift, i.e. relevant contact mechanics model.

Voluminous and significant research has been published by the authors (with other co-authors) [2, 18-27], where the results of theoretical and experimental investigations were presented on validation of Hertzian type solutions for the cases of nanoindentation and their practical applications to various types of scanning probe microscopy and piezo-response force microscopy. A variety of materials were studied both inorganic and biological[18,19]. The theoretical basis for the research is given[25], where the exact solution in terms of elementary functions was obtained for an arbitrary point force and point source acting on the boundary of a piezoelectric transversely isotropic half-space. Nanoindentation of flat, conical and spherical indenters[24,26] was studied in the cases of normal as well as tangential (frictional) contact. The more complicated case of flat and non-flat indenters of arbitrary planform is presented[27]. The investigation of the weak and strong indentations and their applications to piezo-response force microscopy can be found in[22].

However, these analyses are limited to the surfaces of uniform materials of various symmetries and dissimilar piezoelectric or thermal properties, and generally allow only for the certain deviations of surface geometry from planar. The effect of this topographic cross-talk on SPM imaging is well explored.[18] At the same time, realistic materials can contain below surface imperfections such as cracks, voids, and inclusions. A number of studies has visualized such below-surface objects;[19-26] however, the general analytical theory for these imaging modes is generally missing and the studies are limited to finite element models.[24, 27, 28]

The next section is devoted to formulation of the problem. It is based on fundamental results[39], namely, the main potential functions for a general contact problem and the main potential functions for a crack, located inside a transversely isotropic elastic half-space in a plane, parallel to the boundary. We also use the Green's function due to the action of an arbitrary point force, acting on the boundary of the half-space.

The third section provides the derivation of the governing integral equations of the problem. The procedure is executed in two different ways: the method of images and the use of reciprocal theorem. Both derivations give the same results, which proves their correctness. In the general case of a punch interacting with a crack, we need to solve four integral equations, two of them real and the other two in complex form. In the case of the general punch interacting with an inclusion, the problem reduces to two integral equations.

The last section provides the simplest example: an axisymmetric problem of a flat smooth circular punch interacting with a circular inclusion of different radius. This case is chosen because here we need to solve just one equation, which is solvable exactly and in terms of elementary functions. The integrals involved are very non-trivial, so the necessary details of integration are presented in the Appendix.

2. Formulation of the Problem

We consider a transversely isotropic elastic half-space $z \geq 0$ with plane of isotropy being parallel to the boundary $z = 0$. A rigid punch of general shape is pressed against the boundary, creating the domain of contact S, which might be known in advance or unknown and be defined from the condition that stresses vanish at the boundary of S. In general case, the punch might exert both normal σ and tangential τ stresses on the boundary of the half-space.

Further, there exists an arbitrary flat crack S_c in the plane $z = c$. The crack faces might be subjected to arbitrary normal tractions σ_c, symmetric with respect to the plane of the crack, as well as tangential tractions τ_c, which act anti-symmetrically with respect to the plane of the crack, namely in opposite directions on the crack faces. In the case of a rigid inclusion filling up the crack, we shall have normal discontinuity

$$w_c = \frac{1}{2}\left[w(\rho,\phi,c+0) - w(\rho,\phi,c-0)\right] \qquad \text{for } (\rho,\phi) \subseteq S_c \tag{1}$$

and complex tangential discontinuity

$$u_c = \frac{1}{2}\left\{u_x(\rho,\phi,c+0) - u_x(\rho,\phi,c-0) + i\left[u_y(\rho,\phi,c+0) - u_y(\rho,\phi,c-0)\right]\right\} \text{ for } (\rho,\phi) \subseteq S_c \tag{2}$$

prescribed on the crack faces. From the above consideration?? it is obvious that the punch will interact with the crack (inclusion), so we need to derive governing integral equations describing this interaction and to solve them. It was shown in (Fabrikant, 2010) that the most general solution for any transversely isotropic body can be expressed in terms of three harmonic functions F_k, $k = 1,2,3$ satisfying the following differential equations:

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \gamma_k^2 \frac{\partial^2 F}{\partial z^2} = 0 \tag{3}$$

where γ_k are constants defined in (Fabrikant, 2010) as

$$\gamma_{1,2} = \frac{\sqrt{\left(\sqrt{C_{11}C_{33}} - C_{13}\right)\left(\sqrt{C_{11}C_{33}} + C_{13} + 2C_{44}\right)} \pm \sqrt{\left(\sqrt{C_{11}C_{33}} + C_{13}\right)\left(\sqrt{C_{11}C_{33}} - C_{13} - 2C_{44}\right)}}{2\sqrt{C_{11}C_{44}}} \tag{4}$$

$$\gamma_3 = \sqrt{C_{44}/C_{66}} \tag{5}$$

and C_{11}, C_{13}, C_{33}, C_{44} and C_{66} are the transversely isotropic elastic constants of the material of the half-space. After the functions F_k are defined, the field of displacements can be expressed as follows (Fabrikant, 2010)

$$u = u_x + iu_y = \Lambda\left(F_1 + F_2 + iF_3\right), \qquad w = m_1 \frac{\partial F_1}{\partial z} + m_2 \frac{\partial F_2}{\partial z} \tag{6}$$

Here

$$\Lambda = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}, \tag{7}$$

and the constants m_1 and m_2 are defined as:

$$m_1 = \frac{C_{11}\gamma_1^2 - C_{44}}{C_{13} + C_{44}}, \qquad m_2 = \frac{C_{11}\gamma_2^2 - C_{44}}{C_{13} + C_{44}} \tag{8}$$

The stresses can be defined as follows (Fabrikant, 2010)

$$\sigma_1 = \sigma_x + \sigma_y = 2C_{66}\frac{\partial^2}{\partial z^2}\left\{\left[\gamma_1^2 - (m_1+1)\gamma_3^2\right]F_1 + \left[\gamma_2^2 - (m_2+1)\gamma_3^2\right]F_2\right\} \tag{9}$$

$$\sigma_2 = \sigma_x - \sigma_y + 2i\tau_{xy} = 2C_{66}\Lambda^2\left(F_1 + F_2 + iF_3\right) \tag{10}$$

$$\sigma_{zz} = C_{44}\frac{\partial^2}{\partial z^2}\left[(m_1+1)\gamma_1^2 F_1 + (m_1+1)\gamma_2^2 F_2\right] \tag{11}$$

$$\tau_z = \tau_{zx} + i\tau_{yz} = C_{44}\Lambda\frac{\partial}{\partial z}\left[(m_1+1)F_1 + (m_1+1)F_2 + iF_3\right] \tag{12}$$

In the case of a crack inside a transversely isotropic half-space and free of stresses on the boundary the main potential functions are derived in section 3.10 of (Fabrikant, 2010):

$$\begin{aligned} F_{1c} = &-\frac{\gamma_1}{2\pi(m_1-1)}\left[\Phi(z_1-c_1) + \frac{1}{\gamma_1-\gamma_2}\left(2\gamma_2\Phi(z_1+c_2) - (\gamma_1+\gamma_2)\Phi(z_1+c_1)\right)\right] - \\ &-\frac{1}{4\pi(m_1-1)}\left[\mathrm{X}(z_1-c_1) + \frac{1}{\gamma_1-\gamma_2}\left(-2\gamma_1\mathrm{X}(z_1+c_2) + (\gamma_1+\gamma_2)\mathrm{X}(z_1+c_1)\right)\right] \end{aligned}, \tag{13}$$

$$\begin{aligned} F_{2c} = &-\frac{\gamma_2}{2\pi(m_2-1)}\left[\Phi(z_2-c_2) + \frac{1}{\gamma_1-\gamma_2}\left(-2\gamma_1\Phi(z_2+c_1) + (\gamma_1+\gamma_2)\Phi(z_2+c_2)\right)\right] - \\ &-\frac{1}{4\pi(m_2-1)}\left[\mathrm{X}(z_2-c_2) + \frac{1}{\gamma_1-\gamma_2}\left(2\gamma_2\mathrm{X}(z_2+c_1) - (\gamma_1+\gamma_2)\mathrm{X}(z_2+c_2)\right)\right] \end{aligned}, \tag{14}$$

$$F_{3c}=\frac{i}{4\pi}\left[\Upsilon(z_3-c_3)-\Upsilon(z_3+c_3)\right], \tag{15}$$

Here,

$$\Phi(z)=\iint_{S_c}\frac{w_c dS_c}{R(M,N)}, \qquad z_k=z/\gamma_k, \qquad c_k=c/\gamma_k, \qquad \text{for } k=1,2,3 \tag{16}$$

$$\mathrm{X}(z)=\Lambda\iint_{S_c}\ln[R(M,N)+z]\bar{u}_c dS_c+\bar{\Lambda}\iint_{S_c}\ln[R(M,N)+z]u_c dS_c, \tag{17}$$

$$Y(z)=\Lambda\iint_{S_c}\ln[R(M,N)+z]\bar{u}_c dS_c-\bar{\Lambda}\iint_{S_c}\ln[R(M,N)+z]u_c dS_c, \tag{18}$$

where $R(M,N)$ stands for the distance between two points: M with polar coordinates (ρ,ϕ,z) and N with coordinates $(\rho_0,\phi_0,0)$; the overbar, here and in the text to follow, denotes the complex conjugate value, e.g. $\bar{\Lambda}=\partial/\partial x-i\partial/\partial y$; S_c is the surface of the crack and $dS_c=\rho_0 d\rho_0 d\phi_0$. One can verify that the set of quasi-harmonic functions (13-15) satisfy the differential equation (3), and using (6-12), these set of quasi-harmonic functions (13-15) provides for normal displacement discontinuities on the crack faces to be equal w_c and the tangential displacement discontinuities to be equal u_c; they also give us the half-space boundary $z=0$ free of tractions.

The normal contact problem is usually posed as mixed boundary value problem with normal displacement given inside the domain of contact and the rest of the half-space boundary is presumed to be free of tractions. The tangential contact problem is posed in a similar manner, except that the normal tractions are presumed absent all over the plane $z=0$, while tangential displacements are prescribed inside the domain of contact and tangential tractions vanish outside that domain. In order to accommodate both cases of contact problems, we introduce the following quasi-potential functions (Fabrikant, 2010):

$$F_{1p}=\frac{H\gamma_1}{m_1-1}\left[\frac{1}{2}\gamma_2\left(\Lambda\iint_S\{z_1\ln[R(M_1,N)+z_1]-R(M_1,N)\}\bar{\tau}(\rho_0,\phi_0)\rho_0 d\rho_0 d\phi_0+\right.\right.$$
$$\left.+\bar{\Lambda}\iint_S\{z_1\ln[R(M_1,N)+z_1]-R(M_1,N)\}\tau(\rho_0,\phi_0)\rho_0 d\rho_0 d\phi_0\right)+ \tag{19}$$
$$\left.+\iint_S\ln[R(M_1,N)+z_1]\sigma(\rho_0,\phi_0)\rho_0 d\rho_0 d\phi_0\right]$$

$$F_{2p}=\frac{H\gamma_2}{m_2-1}\left[\frac{1}{2}\gamma_1\left(\Lambda\iint_S\{z_2\ln[R(M_2,N)+z_2]-R(M_2,N)\}\overline{\tau}(\rho_0,\phi_0)\rho_0 d\rho_0 d\phi_0+\right.\right.$$
$$\left.+\overline{\Lambda}\iint_S\{z_2\ln[R(M_2,N)+z_2]-R(M_2,N)\}\tau(\rho_0,\phi_0)\rho_0 d\rho_0 d\phi_0\right)+ \tag{20}$$
$$\left.+\iint_S\ln[R(M_2,N)+z_2]\sigma(\rho_0,\phi_0)\rho_0 d\rho_0 d\phi_0\right]$$

$$F_{3p}=\frac{i\gamma_3}{4\pi C_{44}}\left[\overline{\Lambda}\iint_S\{z_3\ln[R(M_3,N)+z_3]-R(M_3,N)\}\tau(\rho_0,\phi_0)\rho_0 d\rho_0 d\phi_0-\right.$$
$$\left.-\Lambda\iint_S\{z_3\ln[R(M_3,N)+z_3]-R(M_3,N)\}\overline{\tau}(\rho_0,\phi_0)\rho_0 d\rho_0 d\phi_0\right] \tag{21}$$

Here σ and τ are the yet unknown normal and tangential stresses under the punch. The superposition of (13-15) and (19-21) yields the main quasi-potential functions, from which the complete field of displacements and stresses in the whole half-space can be obtained by simple differentiation of

$$F_1=F_{1c}+F_{1p},\qquad F_2=F_{2c}+F_{2p},\qquad F_3=F_{3c}+F_{3p}, \tag{22}$$

according to (6-12). In similar way, the proper differentiation of (22) yields the necessary governing integral equations of the specific problems, which can be posed for the punch-crack (inclusion) configuration. The derivation will be given in the next section.

Yet another approach to the subject problem of this article can be made on the basis of the reciprocal theorem. In order to use it, we need to recall some basic results from (Fabrikant, 2010, 1989). First, the field of displacements in the transversely isotropic half-space due to the action of an arbitrary force on its boundary is given by

$$u(\rho_0,\phi_0,z)=\frac{\gamma_3}{4\pi C_{44}}\left[\frac{T}{R_3}+\frac{q^2\overline{T}}{R_3(R_3+z_3)^2}\right]+\frac{H\gamma_2}{m_2-1}\left[\frac{\gamma_1}{2}\left(-\frac{T}{R_2}+\frac{q^2\overline{T}}{R_2(R_2+z_2)^2}\right)-\frac{Pq}{R_2(R_2+z_2)}\right]+$$
$$+\frac{H\gamma_1}{m_1-1}\left[\frac{\gamma_2}{2}\left(-\frac{T}{R_1}+\frac{q^2\overline{T}}{R_1(R_1+z_1)^2}\right)-\frac{Pq}{R_1(R_1+z_1)}\right] \tag{23}$$

$$w(\rho_0,\phi_0,z)=H\left\{-\frac{1}{2}(T\overline{q}+\overline{T}q)\left[\frac{\gamma_2 m_1}{(m_1-1)R_1(R_1+z_1)}+\frac{\gamma_1 m_2}{(m_2-1)R_2(R_2+z_2)}\right]+\right.$$
$$\left.+P\left[\frac{m_1}{(m_1-1)R_1}+\frac{m_2}{(m_2-1)R_2}\right]\right\} \tag{24}$$

Here, P is the normal component of the applied force, $T = T_x + iT_y$ is the complex tangential component of the applied force; $u = u_x + iu_y$ is the complex tangential displacement. One can notice some difference between the formula (23-24) and formula (2.2.9-2.2.10) from (Fabrikant, 2010); this difference is due to the fact that here the force is applied at the point with cylindrical coordinates $(\rho, \phi, 0)$, while in (Fabrikant, 2010) the force is applied at the point $(\rho_0, \phi_0, 0)$. The remaining notations are:

$$q = \rho e^{i\phi} - \rho_0 e^{i\phi_0}, \qquad H = \frac{(\gamma_1 + \gamma_2)C_{11}}{2\pi(C_{11}C_{33} - C_{13}^2)} \tag{25}$$

$$R_k^2 = q\bar{q} + z_k^2, \qquad \text{for} \qquad k = 1,2,3 \tag{26}$$

We shall also need the following expressions for the stresses

$$\sigma_z(\rho_0, \phi_0, z) = -\frac{1}{2\pi(\gamma_1 - \gamma_2)}\left[\frac{1}{2}\gamma_1\gamma_2(T\bar{q} + \bar{T}q) + Pz\right]\left(\frac{1}{R_1^3} - \frac{1}{R_2^3}\right) \tag{27}$$

$$\begin{aligned}\tau_z(\rho_0, \phi_0, z) = {} & \frac{\gamma_2}{4\pi(\gamma_1 - \gamma_2)}\left[\frac{Tz_1}{R_1^3} - \frac{\bar{T}q^2(2R_1 + z_1)}{R_1^3(R_1 + z_1)^2}\right] - \frac{\gamma_1}{4\pi(\gamma_1 - \gamma_2)}\left[\frac{Tz_2}{R_2^3} - \frac{\bar{T}q^2(2R_2 + z_2)}{R_2^3(R_2 + z_2)^2}\right] - \\ & -\frac{1}{4\pi}\left[\frac{Tz_3}{R_3^3} - \frac{\bar{T}q^2(2R_3 + z_3)}{R_3^3(R_3 + z_3)^2}\right] + \frac{Pq}{2\pi(\gamma_1 - \gamma_2)}\left(\frac{1}{R_1^3} - \frac{1}{R_2^3}\right)\end{aligned} \tag{28}$$

As far as the crack is concerned, we shall need the basic quasi-harmonic functions from Sec. 2.4 and 2.6 of (Fabrikant, 2010)

$$F_{1b} = -\frac{\gamma_1 \Phi(z_1)}{2\pi(m_1 - 1)} - \frac{\mathrm{X}(z_1)}{4\pi(m_1 - 1)} \tag{29}$$

$$F_{2b} = -\frac{\gamma_2 \Phi(z_2)}{2\pi(m_2 - 1)} - \frac{\mathrm{X}(z_2)}{4\pi(m_2 - 1)} \tag{30}$$

$$F_{3b} = \frac{i}{4\pi}\mathrm{Y}(z_3) \tag{31}$$

where Φ, X, and Y are defined by (16-18).

3. Derivation of the Governing Integral Equations

At this stage of derivation we presume the punch to be of arbitrary shape and the domain of contact S in general case initially unknown; the crack (or rigid inclusion) is also presumed to be of arbitrary shape and located in the plane $z = c$. The derivation of the governing integral equation of the contact problem by the first method requires substitution of (22) in (6) while taking $z = 0$. After simplifications we get:

$$u(\rho,\phi,0) = \frac{1}{2}G_1\iint_S \frac{\tau(\rho_0,\phi_0)}{R}dS_0 + \frac{1}{2}G_2\iint_S \frac{q^2\bar{\tau}(\rho_0,\phi_0)}{R^3}dS_0 - H\alpha\iint_S \frac{\sigma(\rho_0,\phi_0)}{\bar{q}}dS_0 + $$
$$+\frac{1}{\pi}\frac{\gamma_1\gamma_2}{\gamma_1-\gamma_2}\Lambda[\Phi(c_2)-\Phi(c_1)] + \frac{1}{2\pi}\Lambda\left[\frac{\gamma_2\Upsilon(c_1)-\gamma_1\Upsilon(c_2)}{\gamma_1-\gamma_2}\right] + \frac{1}{2\pi}\Lambda\Upsilon(c_3) \tag{32}$$

$$w(\rho,\phi,0) = H\left[\frac{1}{2}\alpha\iint_S \frac{q\,\bar{\tau}(\rho_0,\phi_0)+\bar{q}\,\tau(\rho_0,\phi_0)}{R^2}dS_0 + \iint_S \frac{\sigma(\rho_0,\phi_0)}{R}dS_0\right] +$$
$$+\frac{1}{\pi}\left[\frac{\gamma_1\Phi'(c_1)-\gamma_2\Phi'(c_2)}{\gamma_1-\gamma_2}\right] + \frac{1}{2\pi}\left[\frac{\mathrm{X}'(c_2)-\mathrm{X}'(c_1)}{\gamma_1-\gamma_2}\right] \tag{33}$$

Here $dS_0 = \rho_0 d\rho_0 d\phi_0$; we used the property $m_1 m_2 = 1$; the notations Φ' and X' are understood as derivatives with respect to the argument in the parentheses and the following notations were introduced

$$R^2 = q\bar{q}, \qquad G_1 = \beta + \gamma_1\gamma_2 H, \qquad G_2 = \beta - \gamma_1\gamma_2 H,$$
$$\beta = \frac{\gamma_3}{2\pi C_{44}}, \qquad \alpha = \frac{(C_{11}C_{33})^{1/2} - C_{13}}{C_{11}(\gamma_1+\gamma_2)} = \frac{H(\gamma_2 m_1 - \gamma_1)}{m_1 - 1} \tag{34}$$

In the case of an interaction of an arbitrary punch with rigid inclusion, where all the displacement discontinuities on the crack faces are known, the *two* governing integral equations (32-33) are sufficient, because they contain only *two* unknowns: tractions under the punch σ and τ. One can also note that the first line in (32-33) represents the usual integral equations for the bonded punch, while the second line reflects the influence of the crack or inclusion.

In order to confirm the correctness of the equations (32-33), we undertake a derivation using an alternative method, namely, the reciprocity theorem. In order to apply this theorem, we recast the punch-crack configuration as being subjected to two different sets of loading: the first is the actual system, characterized by normal σ and tangential τ tractions under the punch and the displacements discontinuities u_c and w_c on the crack faces; the alternative system consists of a unit normal force applied to the boundary of the half-space at the point $(\rho,\phi,0)$ and normal stress p and complex tangential tractions $t = t_x + it_y$ that are applied to the crack faces in such a way that the crack close up and the whole system behaves like a uniform transversely isotropic half-space, so that the formulae (23-28) can be used to describe the resulting stresses and displacements.

The reciprocal theorem states that if we have two system of forces, acting on the same configuration and corresponding two sets of displacements, then the work of the first system of forces on the second set of displacements shall be equal to the work of the second system of forces on the first set of displacements. In the case of a unit normal force we arrive at the following equation

$$w+2\iint_{S_c} p w_c dS_c + 2\iint_{S_c}\left(t_x u_{xc} + t_y u_{yc}\right) dS_c = \iint_S \sigma w_1 dS_0 + \iint_S \left(t_x u_{x1} + t_y u_{y1}\right) dS_0 \tag{35}$$

Here w_1, u_{x1} and u_{y1} are respectively the normal displacement and components of the tangential displacements on the surface of the half-space due to the action of a unit normal force. The factor 2 appears on the left-hand side of (35) due to the definition of w_c and u_c in (1-2) as half of the relevant displacement discontinuity. We remind also that S is the domain of contact and S_c is the crack domain.

Now we can find from (23-28)

$$p = \frac{c}{2\pi(\gamma_1 - \gamma_2)}\left(\frac{1}{R_{1c}^3} - \frac{1}{R_{2c}^3}\right), \qquad t = t_x + it_y = -\frac{q}{2\pi(\gamma_1 - \gamma_2)}\left(\frac{1}{R_{1c}^3} - \frac{1}{R_{2c}^3}\right) \tag{36}$$

$$w_1 = \frac{H}{R}, \qquad u_1 = u_{x1} + iu_{y1} = \frac{H\alpha}{\overline{q}} \tag{37}$$

and the following new notations were introduced

$$R_{1c} = \sqrt{R^2 + c_1^2}\,, \qquad R_{2c} = \sqrt{R^2 + c_2^2}\,, \tag{38}$$

We also use the identities

$$t_x u_{xc} + t_y u_{yc} = \mathrm{Re}\left(t\overline{u}_c\right), \qquad \tau_x u_{x1} + \tau_y u_{y1} = \mathrm{Re}\left(\tau\overline{u}_1\right) \tag{39}$$

Here Re denotes the real part of a complex expression. Substitution of (36-37) and (39) in (35) yields

$$\begin{aligned} w + \frac{c}{\pi(\gamma_1 - \gamma_2)}\iint_{S_c}\left(\frac{1}{R_{1c}^3} - \frac{1}{R_{2c}^3}\right) w_c dS_c - \frac{1}{\pi(\gamma_1 - \gamma_2)}\mathrm{Re}\iint_{S_c}\left(\frac{1}{R_{1c}^3} - \frac{1}{R_{2c}^3}\right) q\overline{u}_c dS_c = \\ = H\left(\iint_S \frac{\sigma}{R} dS_0 + \alpha\mathrm{Re}\iint_S \frac{\tau}{\overline{q}} dS_0\right) \end{aligned} \tag{40}$$

Comparison of (40) with (33) shows that they are identical.

The alternative derivation of (32) is more involved. We apply a unit concentrated force in $0x$ direction at the point $(\rho,\phi,0)$ and we apply the normal p_x and tangential q_x and q_y tractions at the crack faces in such a way that there is no displacement discontinuities and the whole system behaves like an uncut half-space, so that the formulae (23-28) become applicable. The reciprocal theorem in this case gives

$$u_x + 2\iint_{S_c} p_x w_c dS_c + 2\iint_{S_c} \left(q_x u_{xc} + q_y u_{yc}\right) dS_c = \iint_S \left(\tau_x u_{xT_x} + \tau_y u_{yT_x}\right) dS_0 + \iint_S \sigma w_{T_x} dS_0$$

(41)

Here u_{xT_x} and u_{yT_x} denote the tangential displacements in the $0x$ and $0y$ directions at the point $(\rho_0, \phi_0, 0)$ respectively and w_{T_x} stands for the normal displacement at the same point due to the unit force T_x, applied at the point $(\rho, \phi, 0)$. In a similar manner, we can apply a unit concentrated force in $0y$ direction at the point $(\rho, \phi, 0)$ and we apply the normal p_y and tangential s_x and s_y tractions at the crack faces in such a way that there is no displacement discontinuities and the whole system behaves like an uncut half-space, so that the formulae (23-28) become applicable. The reciprocal theorem in this case gives

$$u_y + 2\iint_{S_c} p_y w_c dS_c + 2\iint_{S_c} \left(s_x u_{xc} + s_y u_{yc}\right) dS_c = \iint_S \left(\tau_x u_{xT_y} + \tau_y u_{yT_y}\right) dS_0 + \iint_S \sigma w_{T_y} dS_0 \qquad (42)$$

The interpretation of the notations in (42) is similar to that of (41). We can find from (23-28)

$$p_x + ip_y = -\frac{\gamma_1 \gamma_2 q}{2\pi(\gamma_1 - \gamma_2)}\left(\frac{1}{R_{1c}^3} - \frac{1}{R_{2c}^3}\right) \qquad (43)$$

$$q_x = -\frac{\gamma_2}{4\pi(\gamma_1 - \gamma_2)}\left[\frac{c_1}{R_{1c}^3} - \mathrm{Re}\frac{q^2(2R_{1c} + c_1)}{R_{1c}^3(R_{1c} + c_1)^2}\right] + \frac{\gamma_1}{4\pi(\gamma_1 - \gamma_2)}\left[\frac{c_2}{R_{2c}^3} - \mathrm{Re}\frac{q^2(2R_{2c} + c_2)}{R_{2c}^3(R_{2c} + c_2)^2}\right] + \\ + \frac{1}{4\pi}\left[\frac{c_3}{R_{3c}^3} - \mathrm{Re}\frac{q^2(2R_{3c} + c_3)}{R_{3c}^3(R_{3c} + c_3)^2}\right] \qquad (44)$$

$$q_y = -\frac{\gamma_2}{4\pi(\gamma_1 - \gamma_2)}\left[-\mathrm{Im}\frac{q^2(2R_{1c} + c_1)}{R_{1c}^3(R_{1c} + c_1)^2}\right] + \frac{\gamma_1}{4\pi(\gamma_1 - \gamma_2)}\left[-\mathrm{Im}\frac{q^2(2R_{2c} + c_2)}{R_{2c}^3(R_{2c} + c_2)^2}\right] + \\ + \frac{1}{4\pi}\left[-\mathrm{Im}\frac{q^2(2R_{3c} + c_3)}{R_{3c}^3(R_{3c} + c_3)^2}\right]$$

Hereafter the notation Im stands for the imaginary part of a complex expression.

$$s_x = -\frac{\gamma_2}{4\pi(\gamma_1-\gamma_2)}\left[-\operatorname{Im}\frac{q^2(2R_{1c}+c_1)}{R_{1c}^3(R_{1c}+c_1)^2}\right]+\frac{\gamma_1}{4\pi(\gamma_1-\gamma_2)}\left[-\operatorname{Im}\frac{q^2(2R_{2c}+c_2)}{R_{2c}^3(R_{2c}+c_2)^2}\right]+ \\ +\frac{1}{4\pi}\left[-\operatorname{Im}\frac{q^2(2R_{3c}+c_3)}{R_{3c}^3(R_{3c}+c_3)^2}\right] \tag{45}$$

$$s_y = -\frac{\gamma_2}{4\pi(\gamma_1-\gamma_2)}\left[\frac{c_1}{R_{1c}^3}+\operatorname{Re}\frac{q^2(2R_{1c}+c_1)}{R_{1c}^3(R_{1c}+c_1)^2}\right]+\frac{\gamma_1}{4\pi(\gamma_1-\gamma_2)}\left[\frac{c_2}{R_{2c}^3}+\operatorname{Re}\frac{q^2(2R_{2c}+c_2)}{R_{2c}^3(R_{2c}+c_2)^2}\right]+ \\ +\frac{1}{4\pi}\left[\frac{c_3}{R_{3c}^3}-\operatorname{Re}\frac{q^2(2R_{3c}+c_3)}{R_{3c}^3(R_{3c}+c_3)^2}\right] \tag{46}$$

Now we need to combine two equations (41-42) into one by multiplying (42) by imaginary unit i and adding the result to (41). We use the following identity.

$$(q_x+is_x)u_{xc}+(q_y+is_y)u_{yc}= \\ =\frac{1}{2}\left[(q_x+is_x-iq_y+s_y)(u_{xc}+iu_{yc})+(q_x+is_x+iq_y-s_y)(u_{xc}-iu_{yc})\right] \tag{47}$$

We also notice that $s_x=q_y$, $u_{xc}+iu_{yc}=u_c$ and $u_{xc}-iu_{yc}=\overline{u}_c$. The transformation of the right-hand sides in (41) and (42) is done in a similar manner. The substitution of (44-46) in (47) and following simplification allows us to arrive at the result of unification of (41) with (42) as follows

$$u-\frac{\gamma_1\gamma_2}{\pi(\gamma_1-\gamma_2)}\iint_{S_c}\left(\frac{1}{R_{1c}^3}-\frac{1}{R_{2c}^3}\right)qw_c dS_c+\iint_{S_c}\left[\frac{1}{\gamma_1-\gamma_2}\left(-\frac{\gamma_2 c_1}{R_{1c}^3}+\frac{\gamma_1 c_2}{R_{2c}^3}\right)+\frac{c_3}{R_{3c}^3}\right]\frac{u_c}{2\pi}dS_c+ \\ +\iint_{S_c}\left[\frac{1}{\gamma_1-\gamma_2}\left(\frac{\gamma_2(2R_{1c}+c_1)}{R_{1c}^3(R_{1c}+c_1)^2}-\frac{\gamma_1(2R_{2c}+c_2)}{R_{2c}^3(R_{2c}+c_2)^2}\right)+\frac{(2R_{3c}+c_3)}{R_{3c}^3(R_{3c}+c_3)^2}\right]\frac{q^2\overline{u}_c}{2\pi}dS_c= \\ =\frac{1}{2}G_1\iint_S\frac{\tau}{R}dS+\frac{1}{2}G_2\iint_S\frac{q^2\overline{\tau}}{R^3}dS-H\alpha\iint_S\frac{q\sigma}{R^2}dS \tag{48}$$

The comparison of (48) with (32) shows that they are identical, which proves correctness of our derivation. In the case of interaction of a punch with inclusion, where u_c and w_c are known, the two governing equations above are sufficient. In the case of the interaction of a punch with a loaded crack, where the loading is known and the crack face displacement discontinuities are not known, two additional equations need to be derived. The derivation of the first equation requires substitution of (22) in (11), while taking $z=c$. After proper simplification we get

$$
\begin{aligned}
\sigma_z(c) = \frac{1}{4\pi H}\Bigg\{ &\Delta\iint_S \frac{w_c}{R} dS + \frac{1}{(\gamma_1-\gamma_2)^2}\left[(\gamma_1+\gamma_2)(\gamma_1\Phi''(2c_1)+\gamma_2\Phi''(2c_2)) - 4\gamma_1\gamma_2\Phi''(c_1+c_2)\right] + \\
&+ \frac{\gamma_1+\gamma_2}{2(\gamma_1-\gamma_2)^2}\left[2\mathrm{X}''(c_1+c_2) - \mathrm{X}''(2c_1) - \mathrm{X}''(2c_2)\right]\Bigg\} - \\
&- \frac{1}{2\pi(\gamma_1-\gamma_2)}\Bigg\{ c\iint_S \left(\frac{1}{R_{1c}^3} - \frac{1}{R_{2c}^3}\right)\sigma(\rho_0,\phi_0)\rho_0 d\rho_0 d\phi_0 + \\
&+ \gamma_1\gamma_2 \iint_S \left(\frac{1}{R_{1c}^3} - \frac{1}{R_{2c}^3}\right)\mathrm{Re}\left[\bar{q}\,\tau(\rho_0,\phi_0)\right]\rho_0 d\rho_0 d\phi_0 \Bigg\}
\end{aligned}
\qquad (49)
$$

The notations Φ'' and X'' denote second derivative with respect to the argument in the parentheses. Derivation of the second equation requires substitution of (22) in (12) while taking $z = c$, which yields

$$
\begin{aligned}
\tau_z(c) = \frac{1}{2\pi^2(G_1^2 - G_2^2)}&\left[G_1\Delta\iint_{S_c}\frac{u_c}{R}dS_c + G_2\Lambda^2\iint_{S_c}\frac{\bar{u}_c}{R}dS_c\right] + \\
&+ \frac{1}{4\pi(\gamma_1-\gamma_2)}\Bigg[\iint_S\left(\frac{\gamma_1(2R_{2c}+c_2)}{R_{2c}^3(R_{2c}+c_2)^2} - \frac{\gamma_2(2R_{1c}+c_1)}{R_{1c}^3(R_{1c}+c_1)^2}\right)\bar{\tau}(\rho_0,\phi_0)q^2\rho_0 d\rho_0 d\phi_0 + \\
&+ \iint_S\left(\frac{\gamma_2 c_1}{R_{1c}^3} - \frac{\gamma_1 c_2}{R_{2c}^3}\right)\tau(\rho_0,\phi_0)\rho_0 d\rho_0 d\phi_0 + 2\iint_S\left(\frac{q}{R_{2c}^3} - \frac{q}{R_{1c}^3}\right)\sigma(\rho_0,\phi_0)\rho_0 d\rho_0 d\phi_0\Bigg] - \\
&- \frac{1}{4\pi}\left[c_3\iint_S\frac{\tau(\rho_0,\phi_0)}{R_{3c}^3}\rho_0 d\rho_0 d\phi_0 + \iint_S\frac{(2R_{3c}+c_3)}{R_{3c}^3(R_{3c}+c_3)^2}\bar{\tau}(\rho_0,\phi_0)q^2\rho_0 d\rho_0 d\phi_0 + \right] \\
&+ \frac{\gamma_1+\gamma_2}{4\pi^2 H(\gamma_1-\gamma_2)^2}\Lambda\left[\Phi'(2c_1) + \Phi'(2c_2) - 2\Phi'(c_1+c_2)\right] + \frac{\Lambda Y'(2c_3)}{4\pi^2(G_1+G_2)} + \\
&+ \frac{1}{4\pi^2(\gamma_1-\gamma_2)^2(G_1-G_2)}\Lambda\left\{4\gamma_1\gamma_2\mathrm{X}'(c_1+c_2) - (\gamma_1+\gamma_2)\left[\gamma_2\mathrm{X}'(2c_1) + \gamma_1\mathrm{X}'(2c_2)\right]\right\}
\end{aligned}
\qquad (50)
$$

In simplifications of (49-50) the following identities were used.

$$
\frac{m_1 - 1}{m_1 + 1} = 2\pi H C_{44}(\gamma_1 - \gamma_2) = \frac{1 - m_2}{1 + m_2}, \qquad \gamma_3 = \pi C_{44}(G_1 + G_2) \qquad (51)
$$

Presuming that the normal tractions on the crack faces are prescribed as σ_c and the tangential tractions are equal τ_c, the two additional governing equations will have the form

$$
\sigma_z(c) = -\sigma_c, \qquad \tau_z(c) = -\tau_c \qquad (52)
$$

In the case of a circular crack and a circular punch, all the equations are solvable in quadratures and majority of integrals involved are computable in terms of elementary functions. We demonstrate the procedure in the next section.

4. Example: Interaction of a Circular Punch with a Circular Inclusion

As an illustration we consider the axisymmetric case of the interaction of a smooth flat circular punch of radius r_p with a rigid circular inclusion inside a crack of radius a located in the plane $z=c$. Both the punch and the circular inclusion are centered along the z-axis. The rigid inclusion produces the following displacement discontinuities on the crack faces

$$w_c(\rho)=w_{c0}\sqrt{1-(\rho/a)^2}\ , \qquad\qquad u_c=0 \tag{53}$$

Here w_{c0} is a known constant. We remind that w_c is equal to a half of the total displacement discontinuity. Since we presumed the punch to be smooth, this means that it exerts the normal pressure only, thus $\tau=0$. We can pose *two* types of problem for the flat punch: we may presume the normal force P to be given or we may presume that the z- coordinate of the punch in the position of equilibrium to be given as $w=w_0=const$. For simplicity sake, we take the second option. The problem now effectively reduces to just one integral equation (40), from which the normal tractions under the punch can be found; after that, the remaining equations will give us all other unknown quantities, like tangential displacements under the punch u and the normal σ_c and tangential τ_c tractions between the crack faces and the inclusion.

The equation (40) will now take the form

$$w+\frac{c}{\pi(\gamma_1-\gamma_2)}\iint_{S_c}\left(\frac{1}{R_{1c}^3}-\frac{1}{R_{2c}^3}\right)w_c dS_c=H\iint_S\frac{\sigma}{R}dS_0 \tag{54}$$

Utilization of all our presumptions and substitution of (53) into (54) yields

$$w_0+\frac{cw_{c0}}{\pi a(\gamma_1-\gamma_2)}\int_0^{2\pi}\int_0^a\left(\frac{1}{R_{1c}^3}-\frac{1}{R_{2c}^3}\right)\sqrt{a^2-\rho_0^2}\,\rho_0 d\rho_0 d\phi_0=H\int_0^{2\pi}\int_0^b\frac{\sigma(\rho_0)}{R}\rho_0 d\rho_0 d\phi_0 \tag{55}$$

For the sake of generality, we presumed the radius of contact to be yet unknown quantity $b\le r_p$; as inclusion would create a bump on the surface of the half-space, when the force P is relatively small, not all the surface of the punch will get into contact with the half-space, so that the radius of contact b will be defined from the condition that the normal traction under the punch would vanish at the boundary of the domain of contact.

The integral in the left-hand side of (55) is computable (Fabrikant, 2010) in terms of elementary functions, namely,

$$w_0 + \frac{cw_{c0}\left[f(c_1) - f(c_2)\right]}{\pi a(\gamma_1 - \gamma_2)} = H\int_0^{2\pi}\int_0^b \frac{\sigma(\rho_0)}{R}\rho_0 d\rho_0 d\phi_0 \tag{56}$$

Here

$$\begin{aligned} f(z) &= 2\pi\left[\frac{\sqrt{a^2 - l_1^2}}{z} - \sin^{-1}\left(\frac{a}{l_2}\right)\right], \\ l_1 = l_1(z) &= \frac{1}{2}\left[\sqrt{(a+\rho)^2 + z^2} - \sqrt{(a-\rho)^2 + z^2}\right], \\ l_2 = l_2(z) &= \frac{1}{2}\left[\sqrt{(a+\rho)^2 + z^2} + \sqrt{(a-\rho)^2 + z^2}\right] \end{aligned} \tag{57}$$

Now we need to solve the integral equation (56) with respect to σ. Its solution is well known (Fabrikant, 1989).

$$\sigma(r) = -\frac{1}{\pi^2 Hr}\frac{\partial}{\partial r}\int_r^b \frac{tdt}{\sqrt{t^2 - r^2}}\frac{\partial}{\partial t}\int_0^t \frac{w(\rho)\rho d\rho}{\sqrt{t^2 - \rho^2}} \tag{58}$$

where

$$w(\rho) = w_0 + \frac{cw_{c0}\left[f(c_1) - f(c_2)\right]}{\pi a(\gamma_1 - \gamma_2)} \tag{59}$$

The integrals in (58) are computable and the final result is

$$\sigma(r) = -\frac{1}{\pi^2 H}\left\{\frac{1}{\sqrt{b^2 - r^2}}\left[w_0 + w_{c0}\frac{\gamma_1\theta(c_1) - \gamma_2\theta(c_2)}{a(\gamma_1 - \gamma_2)}\right] + w_{c0}\frac{\gamma_1\Psi(r,c_1) - \gamma_2\Psi(r,c_2)}{a(\gamma_1 - \gamma_2)}\right\} \tag{60}$$

where

$$\begin{aligned} \theta(z) = a &- \frac{b}{2}\ln\left(\frac{l_{2b}(z) + l_{1b}(z)}{l_{2b}(z) - l_{1b}(z)}\right) + \\ &+ z\left\{\frac{1}{2}\cos^{-1}\left[\frac{(a^2 + z^2)^2 + b^2(z^2 - a^2)}{(a^2 + z^2)(l_{2b}^2(z) - l_{1b}^2(z))}\right] - \sin^{-1}\left(\frac{a}{\sqrt{a^2 + z^2}}\right)\right\} \end{aligned} \tag{61}$$

$$\Psi(r,z) = \frac{1}{2}\int_r^b \ln\left(\frac{l_{2t}(z) + l_{1t}(z)}{l_{2t}(z) - l_{1t}(z)}\right)\frac{dt}{\sqrt{t^2 - r^2}} - \mathrm{Re}\left[\frac{ai}{\sqrt{(a+iz)^2 - r^2}}\tan^{-1}\left(\frac{\sqrt{b^2 - r^2}}{\sqrt{r^2 - (a+iz)^2}}\right)\right] \tag{62}$$

with

$$l_{1b}(z)=\frac{1}{2}\left[\sqrt{(a+b)^2+z^2}-\sqrt{(a-b)^2+z^2}\right],\quad l_{2b}(z)=\frac{1}{2}\left[\sqrt{(a+b)^2+z^2}+\sqrt{(a-b)^2+z^2}\right] \quad (63)$$

$$l_{1t}(z)=\frac{1}{2}\left[\sqrt{(a+t)^2+z^2}-\sqrt{(a-t)^2+z^2}\right],\quad l_{2t}(z)=\frac{1}{2}\left[\sqrt{(a+t)^2+z^2}+\sqrt{(a-t)^2+z^2}\right] \quad (64)$$

The details of the derivation are given in the Appendix. One can see that the expression (60) for $\sigma(r)$ consists of two parts: the first one has a singularity at the edge $r=b$ of the domain of contact, while the second term vanishes at $r=b$; thus, we can find the radius b from the condition

$$w_0+w_{c0}\frac{\gamma_1\theta(c_1)-\gamma_2\theta(c_2)}{a(\gamma_1-\gamma_2)}=0 \quad (65)$$

The normal force, applied to the punch, can be found by integration of $\sigma(r)$ over the circle $r\le b$. The result is

$$P=\frac{2}{\pi H}\left\{b\left[w_0+w_{c0}\frac{\gamma_1\theta(c_1)-\gamma_2\theta(c_2)}{a(\gamma_1-\gamma_2)}\right]+w_{c0}\frac{\gamma_1\chi(c_1)-\gamma_2\chi(c_2)}{a(\gamma_1-\gamma_2)}\right\} \quad (66)$$

where

$$\chi(z)=\frac{1}{4}\left(b^2-a^2+z^2\right)\ln\left(\frac{l_{2b}(z)+l_{1b}(z)}{l_{2b}(z)-l_{1b}(z)}\right)+\frac{1}{2}az\left[\tan^{-1}\left(\frac{a-b}{z}\right)-\tan^{-1}\left(\frac{a+b}{z}\right)\right]+$$
$$+a\,\mathrm{Re}\left[i(a+iz)\tan^{-1}\left(\frac{b}{i(a+iz)}\right)-\frac{b}{2}\right] \quad (67)$$

The details of the derivation are given in the Appendix. We note that all the results of this article are valid for the case of isotropy, if we find the limit in each formula for $\gamma_1\to\gamma_2\to 1$ and

$$H=\frac{1-\nu^2}{\pi E},\quad \alpha=\frac{1-2\nu}{2(1-\nu)},\quad \beta=\frac{1+\nu}{\pi E},\quad G_1=\frac{(2-\nu)(1+\nu)}{\pi E},\quad G_2=\frac{\nu(1+\nu)}{\pi E} \quad (68)$$

where E is the modulus of elasticity and ν is the Poisson's coefficient.

Conclusion

For the first time in the literature, we successfully considered here the most general case of interaction of an arbitrary punch with a general crack or inclusion, located in the plane parallel to the boundary of a transversely isotropic elastic half-space. When the crack is subjected to both normal and tangential tractions and the punch is not smooth, the problem is reduced to four simultaneous integral equations with elementary kernels, two of them are real and the remaining two in a complex form. The case of a general punch interacting with an inclusion reduces to just two integral equations, one real and another complex. Only one equation needs to be solved in the case of smooth punch. The method developed in this article can be expanded and applied to more complicated cases of a piezo-electric or even magneto-electro-elastic half-spaces.

Acknowledgement

This research was sponsored by the Division of Materials Sciences and Engineering, Basic Energy Sciences, U.S. Department of Energy. A portion of this research was conducted at the Center for Nanophase Materials Sciences which is a DOE Office of Science User Facility.

Appendix

We give here the details of the derivation of the formulae (60-67). The following integral is to be computed in order to get (60)

$$I = -\frac{1}{\pi^2 Hr}\frac{\partial}{\partial r}\int_r^b \frac{tdt}{\sqrt{t^2 - r^2}}\frac{\partial}{\partial t}\int_0^t \frac{f(z)\rho d\rho}{\sqrt{t^2-\rho^2}} \tag{69}$$

where $f(z)$ is defined in (56). The first integral to compute is

$$I_1 = \int_0^t \sin^{-1}\left(\frac{a}{l_2}\right)\frac{\rho d\rho}{\sqrt{t^2-\rho^2}} \tag{70}$$

Integration by parts in (70) yields

$$I_1 = t\sin^{-1}\left(\frac{a}{\sqrt{a^2+z^2}}\right) - \int_0^t \sqrt{t^2-\rho^2}\,\frac{l_1\sqrt{l_2^2-a^2}}{l_2\left(l_2^2-l_1^2\right)}d\rho \tag{71}$$

Now we introduce new variable

$$y = l_2 \tag{72}$$

From which it follows

$$\rho = y\sqrt{1+\frac{z^2}{a^2-y^2}}, \qquad \frac{\partial l_2}{\partial \rho} = \frac{\rho\left(l_2^2-a^2\right)}{l_2\left(l_2^2-l_1^2\right)}, \qquad d\rho = \frac{l_2\left(l_2^2-l_1^2\right)}{\rho\left(l_2^2-a^2\right)}dy \tag{73}$$

Substitution of (72-73) into (71) and utilization of the identity $l_1 l_2 = a\rho$ gives us

$$I_1 = t\sin^{-1}\left(\frac{a}{\sqrt{a^2+z^2}}\right) - a\int_{l_{20}(z)}^{l_{2t}(z)} \frac{\sqrt{\left[y^2-l_{1t}^2(z)\right]\left[l_{2t}^2(z)-y^2\right]}dy}{y\left(y^2-a^2\right)} \tag{74}$$

Here l_{1t} and l_{2t} are defined in (64) and

$$l_{20}(z) = \sqrt{a^2+z^2} \tag{75}$$

The integration in (74) is elementary and the final result is

$$I_1 = t\sin^{-1}\left(\frac{a}{\sqrt{a^2+z^2}}\right) - \frac{1}{2}\left\{a\left[\tan^{-1}\left(\frac{a^2+z^2-t^2}{2zt}\right) - \frac{\pi}{2}\right] + \right.$$
$$\left. + t\cos^{-1}\left[\frac{\left(a^2+z^2\right)^2 + t^2\left(z^2-a^2\right)}{\left(a^2+z^2\right)\left(l_{2t}^2(z)-l_{1t}^2(z)\right)}\right] + z\ln\left[\frac{l_{2t}(z)+l_{1t}(z)}{l_{2t}(z)-l_{1t}(z)}\right]\right\} \tag{76}$$

For the sake of future reference, we present below two more integrals

$$\int_0^t \frac{l_1\sqrt{l_2^2-a^2}\,\rho^2 d\rho}{l_2\sqrt{t^2-\rho^2}\left(l_2^2-l_1^2\right)} = \frac{1}{2}\left\{a\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{a^2+z^2-t^2}{2zt}\right)\right] - z\ln\left[\frac{l_{2t}(z)+l_{1t}(z)}{l_{2t}(z)-l_{1t}(z)}\right]\right\} \tag{77}$$

$$\int_0^t \frac{l_1\sqrt{l_2^2-a^2}\,d\rho}{l_2\sqrt{t^2-\rho^2}\left(l_2^2-l_1^2\right)} = \frac{1}{2t}\cos^{-1}\left[\frac{\left(a^2+z^2\right)^2 + t^2\left(z^2-a^2\right)}{\left(a^2+z^2\right)\left(l_{2t}^2(z)-l_{1t}^2(z)\right)}\right] \tag{78}$$

The next step is computation of the derivative

$$\frac{\partial}{\partial t}\int_0^t \sin^{-1}\left(\frac{a}{l_2}\right)\frac{\rho d\rho}{\sqrt{t^2-\rho^2}} = \sin^{-1}\left(\frac{a}{\sqrt{a^2+z^2}}\right) - \frac{1}{2}\cos^{-1}\left[\frac{\left(a^2+z^2\right)^2 + t^2\left(z^2-a^2\right)}{\left(a^2+z^2\right)\left(l_{2t}^2(z)-l_{1t}^2(z)\right)}\right] \tag{79}$$

Now we need to compute the integral

$$I_2 = \int_r^b \cos^{-1}\left[\frac{\left(a^2+z^2\right)^2 + t^2\left(z^2-a^2\right)}{\left(a^2+z^2\right)\left(l_{2t}^2(z)-l_{1t}^2(z)\right)}\right]\frac{tdt}{\sqrt{t^2-r^2}} \tag{80}$$

Integration by part in (80) yields

$$I_2 = \sqrt{b^2-r^2}\cos^{-1}\left[\frac{\left(a^2+z^2\right)^2 + b^2\left(z^2-a^2\right)}{\left(a^2+z^2\right)\left(l_{2b}^2(z)-l_{1b}^2(z)\right)}\right] - 4az\int_r^b \frac{\sqrt{t^2-r^2}\,tdt}{\left[(t+a)^2+z^2\right]\left[(t-a)^2+z^2\right]} \tag{81}$$

We remind that $l_{1b}(z)$ and $l_{2b}(z)$ are defined in (63). According to (69), we need to compute the derivative

$$-\frac{1}{r}\frac{\partial I_2}{\partial r} = \frac{1}{\sqrt{b^2-r^2}}\cos^{-1}\left[\frac{\left(a^2+z^2\right)^2 + b^2\left(z^2-a^2\right)}{\left(a^2+z^2\right)\left(l_{2b}^2(z)-l_{1b}^2(z)\right)}\right]$$
$$-4az\int_r^b \frac{tdt}{\sqrt{t^2-r^2}\left[(t+a)^2+z^2\right]\left[(t-a)^2+z^2\right]} \tag{82}$$

The integration in (82) is done by the following transformation of the integrand

$$\frac{t}{\left[(t+a)^2+z^2\right]\left[(t-a)^2+z^2\right]}=\frac{1}{8iaz}\left(\frac{1}{t+a+iz}-\frac{1}{t+a-iz}-\frac{1}{t-a+iz}+\frac{1}{t-a-iz}\right) \tag{83}$$

We use the following basic integral

$$\int\frac{dt}{\left[t+(a+iz)\right]\sqrt{t^2-r^2}}=\frac{1}{\sqrt{(a+iz)^2-r^2}}\ln\frac{\sqrt{r-(a+iz)}\sqrt{r-t}+\sqrt{r+(a+iz)}\sqrt{r+t}}{\sqrt{r-(a+iz)}\sqrt{r-t}-\sqrt{r+(a+iz)}\sqrt{r+t}} \tag{84}$$

Substitution of (83) into integral (82) will lead to four integrals of the type (84). Further simplification will give us

$$\int\frac{tdt}{\left[(t+a)^2+z^2\right]\left[(t-a)^2+z^2\right]\sqrt{t^2-r^2}}=\frac{1}{2az}\mathrm{Re}\left[\frac{1}{\sqrt{(a+iz)^2-r^2}}\tan^{-1}\left(\frac{\sqrt{t^2-r^2}}{\sqrt{r^2-(a+iz)^2}}\right)\right] \tag{85}$$

Using the above results, we can finally write

$$\begin{aligned}
&-\frac{1}{r}\frac{\partial}{\partial r}\int_r^b\left[\sin^{-1}\left(\frac{a}{\sqrt{a^2+z^2}}\right)-\frac{1}{2}\cos^{-1}\left(\frac{\left(a^2+z^2\right)^2+t^2\left(z^2-a^2\right)}{\left(a^2+z^2\right)\left(l_{2t}^2(z)-l_{1t}^2(z)\right)}\right)\right]\frac{tdt}{\sqrt{t^2-r^2}}=\\
&=\frac{1}{\sqrt{b^2-r^2}}\left[\sin^{-1}\left(\frac{a}{\sqrt{a^2+z^2}}\right)-\frac{1}{2}\cos^{-1}\left(\frac{\left(a^2+z^2\right)^2+b^2\left(z^2-a^2\right)}{\left(a^2+z^2\right)\left(l_{2b}^2(z)-l_{1b}^2(z)\right)}\right)\right]+\\
&+\mathrm{Re}\left[\frac{1}{\sqrt{(a+iz)^2-r^2}}\tan^{-1}\left(\frac{\sqrt{b^2-r^2}}{\sqrt{r^2-(a+iz)^2}}\right)\right]
\end{aligned} \tag{86}$$

Now we can move to the computation of

$$I_3=\int_0^t\sqrt{a^2-l_1^2}\,\frac{\rho d\rho}{\sqrt{t^2-\rho^2}} \tag{87}$$

Integration by parts gives us

$$I_3=at-\int_0^t\sqrt{t^2-\rho^2}\,\frac{\rho\sqrt{a^2-l_1^2}}{\left(l_2^2-l_1^2\right)}d\rho \tag{88}$$

As before, we introduce new variable

$$l_1 = y, \qquad \rho = y\sqrt{1+\frac{z^2}{a^2-y^2}}, \quad \frac{\partial l_1}{\partial \rho} = \frac{\rho\left(a^2-l_1^2\right)}{l_1\left(l_2^2-l_1^2\right)}, \qquad d\rho = \frac{l_1\left(l_2^2-l_1^2\right)}{\rho\left(a^2-l_1^2\right)}dy \tag{89}$$

Substitution of (89) in (88) gives us after simplification

$$I_3 = at - \int_0^{l_{1t}(z)} \frac{\sqrt{\left[l_{1t}^2(z)-y^2\right]\left[l_{2t}^2(z)-y^2\right]}\,ydy}{a^2-y^2} \tag{90}$$

The integral in (90) is elementary and the final result is

$$I_3 = \int_0^t \sqrt{a^2-l_1^2}\,\frac{\rho d\rho}{\sqrt{t^2-\rho^2}} = \frac{at}{2} + \frac{az}{2}\cos^{-1}\left(\frac{a^2+z^2-t^2}{l_{2t}(z)-l_{1t}(z)}\right) - \frac{1}{4}\left(t^2+z^2-a^2\right)\ln\left[\frac{l_{2t}(z)+l_{1t}(z)}{l_{2t}(z)-l_{1t}(z)}\right] \tag{91}$$

The differentiation with respect to t simplifies the result as follows

$$\frac{\partial}{\partial t}\int_0^t \sqrt{a^2-l_1^2}\,\frac{\rho d\rho}{\sqrt{t^2-\rho^2}} = a - \frac{t}{2}\ln\left[\frac{l_{2t}(z)+l_{1t}(z)}{l_{2t}(z)-l_{1t}(z)}\right] \tag{92}$$

For the future reference, we can quote the following integral

$$\int_0^t \frac{\sqrt{a^2-l_1^2}\,\rho d\rho}{\sqrt{t^2-\rho^2}\left(l_2^2-l_1^2\right)} = \frac{1}{2}\ln\left[\frac{l_{2t}(z)+l_{1t}(z)}{l_{2t}(z)-l_{1t}(z)}\right] \tag{93}$$

We can also quote the following useful derivatives

$$\frac{\partial}{\partial t}\ln\left[\frac{l_{2t}(z)+l_{1t}(z)}{l_{2t}(z)-l_{1t}(z)}\right] = \frac{2a\left(a^2+z^2-t^2\right)}{\left[l_{2t}^2(z)-l_{1t}^2(z)\right]^2} \tag{94}$$

$$\frac{\partial}{\partial t}\tan^{-1}\left(\frac{a^2+z^2-t^2}{2zt}\right) = -2z\frac{l_{2t}^2(z)+l_{1t}^2(z)}{\left[l_{2t}^2(z)-l_{1t}^2(z)\right]^2} \tag{95}$$

$$\frac{\partial}{\partial t}\tan^{-1}\left[\frac{2azt^2}{\left(a^2+z^2\right)^2+t^2\left(z^2-a^2\right)}\right] = \frac{4azt}{\left[l_{2t}^2(z)-l_{1t}^2(z)\right]^2} \tag{96}$$

$$\frac{\partial}{\partial t}\cos^{-1}\left[\frac{a^2+z^2-t^2}{l_{2t}(z)-l_{1t}(z)}\right] = 2z\frac{l_{2t}^2(z)+l_{1t}^2(z)}{\left[l_{2t}^2(z)-l_{1t}^2(z)\right]^2} \tag{97}$$

The next integral to compute is

$$I_4 = -\frac{1}{r}\frac{\partial}{\partial r}\int_r^b \left\{ a - \frac{t}{2}\ln\left[\frac{l_{2t}(z)+l_{1t}(z)}{l_{2t}(z)-l_{1t}(z)}\right]\right\}\frac{tdt}{\sqrt{t^2-r^2}} \tag{98}$$

Integration by parts in (98) and differentiation of the result with respect to r gives us

$$I_4 = \frac{1}{\sqrt{b^2-r^2}}\left\{ a - \frac{b}{2}\ln\left[\frac{l_{2b}(z)+l_{1b}(z)}{l_{2b}(z)-l_{1b}(z)}\right]\right\} + \text{Re}\left[\frac{z-ai}{\sqrt{(a+iz)^2-r^2}}\tan^{-1}\left(\frac{\sqrt{b^2-r^2}}{\sqrt{b^2-(a+iz)^2}}\right)\right] + \\ +\frac{1}{2}\int_r^b \ln\left[\frac{l_{2t}(z)+l_{1t}(z)}{l_{2t}(z)-l_{1t}(z)}\right]\frac{dt}{\sqrt{t^2-r^2}} \tag{99}$$

The last integral in (99) does not seem to be computable in terms of elementary functions. In derivation of (99) we used the following integrals.

$$\int_r^b \frac{t^3dt}{\left[(t+a)^2+z^2\right]\left[(t-a)^2+z^2\right]\sqrt{t^2-r^2}} = \frac{1}{2az}\text{Re}\left[\frac{(a+iz)^2}{\sqrt{(a+iz)^2-r^2}}\tan^{-1}\left(\frac{\sqrt{b^2-r^2}}{\sqrt{r^2-(a+iz)^2}}\right)\right] \tag{100}$$

$$\int_r^b \frac{at\left(a^2+z^2-t^2\right)dt}{\left[l_{2t}^2(z)-l_{1t}^2(z)\right]^2\sqrt{t^2-r^2}} = \text{Re}\left[\frac{z-ia}{\sqrt{(a+iz)^2-r^2}}\tan^{-1}\left(\frac{\sqrt{b^2-r^2}}{\sqrt{r^2-(a+iz)^2}}\right)\right] \tag{101}$$

Combination of (86) and (99) gives us (60-62)

Finding of the total force P requires computation of the following integrals.

$$I_5 = 2\pi\int_0^b rdr\int_r^b \ln\left[\frac{l_{2t}(z)+l_{1t}(z)}{l_{2t}(z)-l_{1t}(z)}\right]\frac{dt}{\sqrt{t^2-r^2}} \tag{102}$$

We interchange the order of integration in (102), integrate with respect to r and then use the integration by parts to integrate with respect to t. The final result is

$$I_5 = \pi\left\{\left(b^2-a^2+z^2\right)\ln\left[\frac{l_{2b}(z)+l_{1b}(z)}{l_{2b}(z)-l_{1b}(z)}\right] + 2ab + 2az\left[\tan^{-1}\left(\frac{a+b}{z}\right) - \tan^{-1}\left(\frac{a-b}{z}\right)\right]\right\} \tag{103}$$

The next integral is computable by parts:

$$2\pi\int_0^b \frac{-ia}{\sqrt{(a+iz)^2-r^2}}\tan^{-1}\left(\frac{\sqrt{b^2-r^2}}{\sqrt{r^2-(a+iz)^2}}\right) r dr = 2\pi ia\left[(a+iz)\tan^{-1}\left(\frac{b}{i(a+iz)}\right)+ib\right] \qquad (104)$$

The following integrals were used

$$\int_0^b \frac{dt}{\left[(t+a)^2+z^2\right]\left[(t-a)^2+z^2\right]} = \\ = \frac{1}{4(a^2+z^2)}\left\{\frac{1}{a}\ln\left[\frac{l_{2b}(z)+l_{1b}(z)}{l_{2b}(z)-l_{1b}(z)}\right]+\frac{1}{z}\left[\tan^{-1}\left(\frac{a+b}{z}\right)-\tan^{-1}\left(\frac{a-b}{z}\right)\right]\right\} \qquad (105)$$

$$\int_0^b \frac{t^2 dt}{\left[(t+a)^2+z^2\right]\left[(t-a)^2+z^2\right]} = \frac{1}{4az}\left\{z\ln\left[\frac{l_{2b}(z)-l_{1b}(z)}{l_{2b}(z)+l_{1b}(z)}\right]+a\left[\tan^{-1}\left(\frac{a+b}{z}\right)-\tan^{-1}\left(\frac{a-b}{z}\right)\right]\right\}$$
(106)

$$\int_0^b \frac{t^4 dt}{\left[(t+a)^2+z^2\right]\left[(t-a)^2+z^2\right]} = \\ = b+\frac{1}{4az}\left\{z(z^2-3a^2)\ln\left[\frac{l_{2b}(z)+l_{1b}(z)}{l_{2b}(z)-l_{1b}(z)}\right]+a(a^2-3z^2)\left[\tan^{-1}\left(\frac{a+b}{z}\right)-\tan^{-1}\left(\frac{a-b}{z}\right)\right]\right\} \qquad (107)$$

$$\int_0^b \frac{(a^2+z^2-t^2)t^2 dt}{\left[(t+a)^2+z^2\right]\left[(t-a)^2+z^2\right]} = \\ = \frac{a^2-z^2}{2a}\ln\left[\frac{l_{2b}(z)+l_{1b}(z)}{l_{2b}(z)-l_{1b}(z)}\right]+z\left[\tan^{-1}\left(\frac{a+b}{z}\right)-\tan^{-1}\left(\frac{a-b}{z}\right)\right]-b \qquad (108)$$

$$\int_r^b \frac{t^2 dt}{\left[(t+a)^2+z^2\right]\left[(t-a)^2+z^2\right]\sqrt{t^2-r^2}} = \frac{1}{2az}\mathrm{Re}\left[\frac{(a+iz)}{\sqrt{(a+iz)^2-r^2}}\tan^{-1}\left(\frac{(a+iz)\sqrt{b^2-r^2}}{b\sqrt{r^2-(a+iz)^2}}\right)\right] \qquad (109)$$

$$\int_r^b \frac{t^4 dt}{\left[(t+a)^2+z^2\right]\left[(t-a)^2+z^2\right]\sqrt{t^2-r^2}} = \\ = \cosh^{-1}\left(\frac{b}{r}\right)+\frac{1}{2az}\mathrm{Re}\left[\frac{(a+iz)^3}{\sqrt{(a+iz)^2-r^2}}\tan^{-1}\left(\frac{(a+iz)\sqrt{b^2-r^2}}{b\sqrt{r^2-(a+iz)^2}}\right)\right] \qquad (110)$$

The above integrals are sufficient for the computation of (66-67).

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