Normal-mode-based analysis of electron plasma waves with second-order Hermitian formalism

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The classic problem of the dynamic evolution and Landau damping of linear Langmuir electron waves in a collisionless plasma with Maxwellian background is cast as a second-order, self-adjoint problem with a continuum spectrum of real and positive squared frequencies. The corresponding complete basis of singular normal modes is obtained, along with their orthogonality relation. This yields easily the general expression of the time-reversal-invariant solution for any initial-value problem. Examples are given for specific initial conditions that illustrate different behaviors of the Landau-damped macroscopic moments of the perturbations.

The linear Landau damping of collisionless plasma waves is one of the classic results in plasma physics. After Landau's original formulation[1], based on the Laplace transform of initial-value solutions for high-frequency electron Langmuir waves, an equivalent formulation based on the normal modes of such a system was developed by Van Kampen[2] and Case[3]. Despite its attractiveness, the Van Kampen-Case normal-mode approach is mathematically cumbersome because the considered normal modes are not eigenfunctions of a Hermitian operator. A recent work[4] has formulated the linear theory of low-frequency, collisionless sound waves in a quasineutral plasma, in terms of the complete basis of eigenfunctions of a Hermitian operator. The present paper develops the analogous Hermitian (self-adjoint) formalism for the normal modes of the classic, high-frequency electron plasma wave problem, which results in a transparent and mathematically straightforward analysis of the dynamics of such waves and their eventual damping. The normal-mode approach is especially well suited to take into account the most general initial conditions. This facilitates addressing the issue of perturbations different from Landau's standard solution whose asymptotic decay is exponential with an oscillation frequency and a damping rate given by the complex root of an "effective dispersion relation". Work by Belmont et al.[5], backed by detailed numerical simulations, has shown Langmuir wave solutions which decay exponentially at long times, with damping rates and oscillation frequencies that may differ from the standard Landau root. That study was based on wave packet superpositions of the original, non-Hermitian Van Kampen modes. The present Hermitian normal-mode formalism affords a simple proof of the stronger result that any time dependence of the macroscopic variables, such that its Fourier transform exists and is sufficiently well behaved, can be realized with a suitably chosen initial condition for the distribution function. Furthermore, it provides the explicit solution for the inverse problem of determining the initial distribution function that yields the prescribed functional dependence of its macroscopic moments versus time. The existence of solutions whose damping rate is not determined by the slope of the equilibrium distribution function at the resonant velocity has led to a closer examination of the wave-particle energy exchange paradigm[6,7].

Consider a small-amplitude, electrostatic perturbation ($\nabla \times \mathbf{E} = 0$) about a homogeneous, unmagnetized and Maxwellian plasma equilibrium with immobile ions. A linear analysis of such a perturbation can be based on the independent study of uncoupled spatial-plane-wave Fourier modes characterized by their wavevector $\mathbf{k}$. Then, for one such $\mathbf{k}$-mode, the curl-free condition on the electric field implies that $\mathbf{E}$ is in the direction of $\mathbf{k}$ ($\mathbf{E} = E\mathbf{k}/k$ with $k = |\mathbf{k}| > 0$) and the linearized electron Vlasov-Maxwell system yields

$$\frac{1}{c^2}\frac{\partial E(t)}{\partial t} = -j[f_1] \tag{1}$$

$$j[f_1] = -e\int_{-\infty}^{\infty} dv\ v f_1(v,t) \tag{2}$$

$$\frac{\partial f_1(v,t)}{\partial t} + ikv f_1(v,t) + \frac{eE(t)v}{T_0} f_{M0}(v^2) = 0. \tag{3}$$

Here, $j(t)$ is the magnitude of the electric current which is also parallel to $\mathbf{k}$ ($\mathbf{j} = j\mathbf{k}/k$), v is the phase-space velocity component in the direction of $\mathbf{k}$, and $f_{M0}(v^2)$ and $f_1(v,t)$ stand respectively for the electron equilibrium and perturbation distribution functions, integrated over the phase-space velocity components perpendicular to $\mathbf{k}$. Thus, the one-dimensional Maxwellian equilibrium distribution function is

$$f_{M0}(v^2) = n_0\left(\frac{m}{2\pi T_0}\right)^{1/2}\exp\left(-\frac{mv^2}{2T_0}\right) \tag{4}$$

where n_0 and T_0 are the electron equilibrium density and temperature, and m is the electron mass. The density moment of (3) yields the continuity equation

$$\frac{\partial \sigma(t)}{\partial t} + ikj(t) = 0 \tag{5}$$

where

$$\sigma(t) = -e\int_{-\infty}^{\infty} dv\ f_1(v,t) \tag{6}$$

is the charge density. Then, (1) and (5) guarantee that Gauss' law, $ikE(t) = c^2\sigma(t)$, is satisfied at all times provided it is satisfied by the initial condition at $t = 0$.

Writing f_1 as the sum of its even and odd parts with respect to v ($f_1 = f_1^{even} + f_1^{odd}$) and eliminating f_1^{even} and E, the linearized Vlasov-Maxwell system (1-3) reduces to the following second-order linear problem with respect to time for $f_1^{odd}(v,t)$:

$$-\frac{1}{k^2}\frac{\partial^2 f_1^{odd}}{\partial t^2} = L[f_1^{odd}] \tag{7}$$

where the linear operator L is

$$L[f_1^{odd}] = v^2 f_1^{odd} - \frac{ec^2}{k^2T_0} j[f_1^{odd}] v f_{M0}. \tag{8}$$

The operator L is self-adjoint in the Hilbert space of square-integrable distribution functions with the scalar product

$$\langle f|f'\rangle = \int_{-\infty}^{\infty} dv\ \frac{T_0}{f_{M0}(v^2)} f^*(v) f'(v), \tag{9}$$

because the scalar product $\langle f|L[f']\rangle$ can be cast in the Hermite-symmetric form

$$\langle f|L[f']\rangle = \frac{c^2}{k^2} j[f^*] j[f'] + \int_{-\infty}^{\infty} dv\ \frac{T_0}{f_{M0}(v^2)} v^2 f^*(v) f'(v) = \langle L[f]|f'\rangle \tag{10}$$

Besides,

$$\langle f|L[f]\rangle = \frac{c^2}{k^2}|j[f]|^2 + \int_{-\infty}^{\infty} dv\ \frac{T_0}{f_{M0}(v^2)} v^2 |f(v)|^2 > 0 \tag{11}$$

so L is a positive operator.

The normal modes of the second-order problem (7) are separable solutions of the form

$$f_1^{odd}(v,t)\ = v\ h^\lambda(v^2)\ \exp(-i\omega t) \tag{12}$$

where the label λ is the squared phase velocity ($\lambda \equiv \omega^2/k^2$) so that vh^λ is an eigenfunction of the operator L with eigenvalue λ. Since L is self-adjoint and positive, the λ spectrum is real and positive, therefore the normal-mode frequencies ω are real. Then, calling $\zeta \equiv v^2$ and normalizing h^λ to

$$-\frac{1}{e} j[vh^\lambda] = \int_0^\infty d\zeta\ \zeta^{1/2} h^\lambda(\zeta) = 1, \tag{13}$$

the normal-mode eigenvalue equation can be expressed as

$$(\zeta - \lambda) h^\lambda(\zeta) = -\frac{m\omega_p^2}{k^2 T_0 n_0} f_{M0}(\zeta) \tag{14}$$

where $\omega_p^2 \equiv c^2e^2n_0/m$ is the square of the plasma frequency. For any $\lambda > 0$, this has the singular solution

$$h^\lambda(\zeta) = -\frac{m\omega_p^2}{k^2T_0n_0}\mathcal{P}\frac{f_{M0}(\zeta)}{(\zeta-\lambda)} + \Lambda(\lambda)\lambda^{-1/2}\delta(\zeta-\lambda) \tag{15}$$

where $\mathcal{P}$ stands for the Cauchy principal value and δ is the Dirac distribution. The coefficient $\Lambda(\lambda)$ is specified by the condition that h^λ satisfy the normalization condition (13). This yields

$$\Lambda(\lambda) = 1 + \frac{m\omega_p^2}{k^2T_0}\ W(\hat{\lambda}) \tag{16}$$

where $\hat{\lambda} \equiv m\lambda(2T_0)^{-1}$ is the ratio of the squared phase velocity to the squared electron thermal velocity, and

$$W(\hat{\lambda})\ \equiv\ \pi^{-1/2}\int_0^\infty d\hat{\zeta}\ \hat{\zeta}^{1/2}\ \mathcal{P}\frac{\exp(-\hat{\zeta})}{\hat{\zeta}-\hat{\lambda}} \tag{17}$$

which is related to the real part of the plasma dispersion function Z by $W(\hat{\lambda}) = 1 + \hat{\lambda}^{1/2} ReZ(\hat{\lambda}^{1/2})$. It satisfies $W(0) = 1$ and has the asymptotic behavior $W(\hat{\lambda} \to \infty) = -(2\hat{\lambda})^{-1}$.

The scalar products among these normal modes,

$$\langle vh^\lambda|vh^{\lambda'}\rangle = \int_0^\infty d\zeta\ \zeta^{1/2}\frac{T_0}{f_{M0}(\zeta)} h^\lambda(\zeta) h^{\lambda'}(\zeta), \tag{18}$$

are evaluated with the help of the identity

$$\mathcal{P}\frac{1}{\zeta-\lambda}\mathcal{P}\frac{1}{\zeta-\lambda'} = \mathcal{P}\frac{1}{\lambda-\lambda'}\left(\mathcal{P}\frac{1}{\zeta-\lambda} - \mathcal{P}\frac{1}{\zeta-\lambda'}\right) + \pi^2\delta(\lambda-\lambda')\delta(\zeta-\lambda). \tag{19}$$

Recalling the definition (17) of the function W and carrying the integration of Dirac deltas, one obtains

$$\langle vh^\lambda|vh^{\lambda'}\rangle = \frac{T_0}{\lambda^{1/2} f_{M0}(\lambda)} D(\lambda)\delta(\lambda-\lambda') \tag{20}$$

where

$$D(\lambda) = \left[1 + \frac{m\omega_p^2}{k^2T_0}W(\hat{\lambda})\right]^2 + \pi^2\left(\frac{m\omega_p^2}{k^2T_0}\right)^2 \frac{\lambda f_{M0}^2(\lambda)}{n_0^2}, \tag{21}$$

so normal modes with different λ eigenvalues are orthogonal as expected from the self-adjointness of the operator L.

Once the normal modes (12,15-17) of the electron plasma wave system have been obtained, one can readily solve for any initial-value problem. The normal modes $vh^\lambda(v^2)$ constitute a complete continuum basis in the space of odd, square-integrable functions with the scalar product (9), because they are singular eigenfunctions of a self-adjoint operator. Therefore, any initial conditions for f_1^{odd} belonging to such Hilbert space can be expanded as

$$f_1^{odd}(v,0) = v \int_0^\infty d\lambda\ \lambda^{-1/2} C(\lambda)\ h^\lambda(v^2) \tag{22}$$

$$\frac{\partial f_1^{odd}(v,0)}{\partial t} = kv \int_0^\infty d\lambda\ S(\lambda)\ h^\lambda(v^2). \tag{23}$$

Then, the solution of the corresponding initial-value problem is

$$f_1^{odd}(v,t) = v \int_0^\infty d\lambda\ \lambda^{-1/2} h^\lambda(v^2) \Big[C(\lambda)\cos(\lambda^{1/2}kt) + S(\lambda)\sin(\lambda^{1/2}kt) \Big]. \tag{24}$$

Recalling the normalization condition (13) and changing the integration variable back to ω, one obtains the expression for the current perturbation

$$j(t) = -\frac{2e}{k} \int_0^\infty d\omega \left[C\Big(\frac{\omega^2}{k^2}\Big)\cos\omega t + S\Big(\frac{\omega^2}{k^2}\Big)\sin\omega t \right] \tag{25}$$

which means that, up to the multiplicative constant specified in Eq.(25), $C(\omega^2/k^2)$ is the cosine Fourier transform of $j(t)$ and $S(\omega^2/k^2)$ is its sine Fourier transform. From Maxwell's equation (1) and Gauss's law or the continuity equation (5), the electric field and the electron density perturbation are

$$E(t) = \frac{2ec^2}{k} \int_0^\infty \frac{d\omega}{\omega} \left[C\Big(\frac{\omega^2}{k^2}\Big)\sin\omega t - S\Big(\frac{\omega^2}{k^2}\Big)\cos\omega t \right] \tag{26}$$

$$n_1(t) = -\frac{\sigma(t)}{e} = -2i \int_0^\infty \frac{d\omega}{\omega} \left[C\Big(\frac{\omega^2}{k^2}\Big)\sin\omega t - S\Big(\frac{\omega^2}{k^2}\Big)\cos\omega t \right] \tag{27}$$

This solution exhibits the invariance under the time reversal,

$$t \to -t, \qquad f_1^{odd}(v,0) \to -f_1^{odd}(v,0), \qquad \partial f_1^{odd}(v,0)/\partial t \to \partial f_1^{odd}(v,0)/\partial t, \tag{28}$$

as should be the case for the considered dissipation-free, collisionless Vlasov-Maxwell model (1-3). It also exhibits the Landau damping of the macroscopic variables for $t \to \pm\infty$, as the consequence of the superposition of a continuum of spectral components with rapidly varying phases. To be precise, the Riemann-Lebesgue lemma guarantees such long-time decay of the macroscopic variables (25-27), for initial conditions for which $C(\omega^2/k^2)$ and $S(\omega^2/k^2)$ are regular and integrable functions of ω, and $S(\omega^2/k^2 \to 0) \le O(\omega/k)$. On the contrary, the time dependence of the macroscopic moments of the normal modes (12) is undamped and purely oscillatory because their associated $C(\omega^2/k^2)$ and $S(\omega^2/k^2)$ are singular Dirac deltas. This analysis provides also a straightforward linear proof that any $j(t)$, $E(t)$ or $n_1(t)$, such that its Fourier transform exists and is sufficiently well behaved, can be realized with the appropriately chosen initial condition defined explicitly by Eqs.(22,23,25-27). The Fourier transform has to be sufficiently well behaved for the integrals (22,23) to converge.

Explicit applications of the above formalism are given next, by considering two examples of specific initial conditions. For the first one, take

$$f_1(v,0) = \frac{n_1(0)}{n_0} f_{M0}(v^2), \quad E(0) = \frac{iec^2}{k} n_1(0). \tag{29}$$

This implies

$$f_1^{odd}(v,0) = 0 \tag{30}$$

and, from Eq.(3),

$$\frac{\partial f_1^{odd}(v,0)}{\partial t} = -ikv \left(1 + \frac{m\omega_p^2}{k^2 T_0} \right) \frac{n_1(0)}{n_0} f_{M0}(v^2). \tag{31}$$

Making use of the orthogonality relation (20,21), the projection of this initial condition onto the normal-mode basis yields $C(\lambda) = 0$ and

$$S(\lambda) = -i \left(1 + \frac{m\omega_p^2}{k^2 T_0} \right) \frac{n_1(0)\lambda^{1/2} f_{M0}(\lambda)}{n_0 D(\lambda)}. \tag{32}$$

Accordingly, the time evolution of the density perturbation is given by

$$n_1(t) = \Big(\frac{2}{\pi}\Big)^{1/2} \int_0^\infty d\omega\ \tilde{n}(\omega)\ \cos\omega t, \tag{33}$$

where the Fourier transform of $n_1(t)$ is

$$\tilde{n}(\omega) = (2\pi)^{1/2} \left(1 + \frac{m\omega_p^2}{k^2 T_0} \right) \frac{n_1(0) f_{M0}(\omega^2/k^2)}{n_0 k D(\omega^2/k^2)} \tag{34}$$

which can have a sharp resonant peak if $D(\omega^2/k^2)$ becomes close to zero for a narrow frequency interval. This happens if and only if the wave phase velocity is much greater than the electron thermal velocity, i.e. $\omega^2/k^2 \gg 2T_0/m = v_{th}^2$, so that $[1 + m\omega_p^2 k^{-2} T_0^{-1} W(\hat\lambda)]^2$ can have a zero with $\hat\lambda \gg 1$, for which the other positive term in the expression of $D(\lambda)$ (21) is small. Then, using the large-argument asymptotic form of $W(\hat\lambda)$, one can approximate

$$1 + \frac{m\omega_p^2}{k^2 T_0} W(\hat\lambda) \simeq 1 - \frac{\omega_p^2}{\omega^2} \tag{35}$$

which has a zero at $\omega = \omega_p$. Substituting the approximation (35) and setting $\omega = \omega_p$ in the remaining terms of (34), in the limit $\omega_p \gg kv_{th}$, one obtains

$$\frac{\tilde{n}(\omega)}{n_1(0)} \simeq \Big(\frac{8}{\pi}\Big)^{1/2} \frac{\eta_0\ \omega_p^3}{(\omega^2 - \omega_p^2)^2 + 4\eta_0^2\ \omega_p^4} \tag{36}$$

where

$$\eta_0 = \pi^{1/2}\Big(\frac{\omega_p}{kv_{th}}\Big)^3 \exp\Big(-\frac{\omega_p^2}{k^2v_{th}^2}\Big) \ll 1. \quad (37)$$

Finally, after substituting (36) in (33) and carrying out the integration over ω, the corresponding approximation for $n_1(t)$ is

$$\frac{n_1(t)}{n_1(0)} \simeq \exp(-\eta_0\omega_p|t|)\Big[\cos(\omega_p t) + \eta_0 \sin(\omega_p|t|)\Big] \quad (38)$$

in agreement with the classic result[1,2,3] for the weakly Landau-damped electron plasma wave. Consistent with the time-reversal invariance, $n_1(t)$ is an even function of time with the same decaying behavior as $t \to \pm\infty$. This example is representative of the standard initial conditions characterized by $C(\lambda)$ and $S(\lambda)$ functions that are inversely proportional to $D(\lambda)$, thus resulting in Landau's standard rate of exponential decay for a Langmuir oscillation at the plasma frequency.

The second example will illustrate the solution to the inverse problem of finding the distribution function initial condition for a specified, non-standard time variation of the perturbed density. To this effect, take

$$\frac{n_1(t)}{n_1(0)} = \exp\Big(-\frac{t^2}{2\tau^2}\Big) \quad (39)$$

where τ is an arbitrary time constant. This non-oscillatory, Gaussian decay of the density is neither Landau's standard one nor any of the non-standard forms of exponential decay shown in Ref. 5. Equation (39) has a regular Fourier transform,

$$\frac{n_1(t)}{n_1(0)} = \Big(\frac{2}{\pi}\Big)^{1/2}\int_0^\infty d\omega \;\cos\omega t \;\tau \exp\Big(-\frac{\omega^2\tau^2}{2}\Big), \quad (40)$$

hence the coefficients of the normal-mode expansion of the initial condition are determined by Eq.(27) to be $C(\lambda) = 0$ and

$$S(\lambda) = -i\frac{n_1(0)k\tau}{(2\pi)^{1/2}}\lambda^{1/2}\exp\Big(-\frac{k^2\tau^2\lambda}{2}\Big) \quad (41)$$

which is not inversely proportional to $D(\lambda)$, as expected from the non-standard nature of $n_1(t)$. Then, Eqs.(3,22,23) yield the expressions for the initial distribution function

$$f_1^{odd}(v,0) = 0, \quad (42)$$

$$f_1^{even}(v,0) = \frac{i}{kv}\frac{\partial f_1^{odd}(v,0)}{\partial t} - \frac{m\omega_p^2 n_1(0)}{k^2T_0n_0}f_{M0}(v^2) \quad (43)$$

and

$$\frac{i}{kv}\frac{\partial f_1^{odd}(v,0)}{\partial t} = \frac{n_1(0)k\tau}{(2\pi)^{1/2}}\int_0^\infty d\lambda\;\lambda^{1/2}\exp\Big(-\frac{k^2\tau^2\lambda}{2}\Big)h^\lambda(v^2). \quad (44)$$

After substituting Eqs.(15,16) for the normal-mode eigenfunctions $h^\lambda(v^2)$ and carrying out the integration over λ, the final result is

$$f_1(v,0) = \frac{n_1(0)}{(2\pi)^{1/2}}\Bigg\{k\tau\exp\Big(-\frac{k^2\tau^2v^2}{2}\Big)\Bigg[1 + \frac{m\omega_p^2}{k^2T_0}W\Big(\frac{mv^2}{2T_0}\Big)\Bigg] + \frac{m^{3/2}\omega_p^2}{k^2T_0^{3/2}}\exp\Big(-\frac{mv^2}{2T_0}\Big)\Bigg[W\Big(\frac{k^2\tau^2v^2}{2}\Big) - 1\Bigg]\Bigg\} \quad (45)$$

which is a smooth, regular function of the phase-space velocity. For large values of the time constant τ, it approaches the singular limit

$$\lim_{\tau\to\infty} f_1(v,0) = n_1(0)\Bigg[\Bigg(1 + \frac{m\omega_p^2}{k^2T_0}\Bigg)\delta(v) - \frac{m\omega_p^2}{k^2T_0n_0}f_{M0}(v^2)\Bigg]. \quad (46)$$

It is generally accepted that, although there are infinitely many standard as well as non-standard possible initial conditions, the standard ones are statistically far more likely. A precise characterization and proof of this proposition might be difficult but would be desirable.

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[1] L. Landau, J. Phys. (U.S.S.R.) **10**, 25 (1946).
[2] N.G. Van Kampen, Physica **21**, 949 (1955).
[3] K.M. Case, Ann. Phys. **7**, 349 (1959).
[4] J.J. Ramos, J. Plasma Phys. **83**, 725830601 (2017).
[5] G. Belmont, F. Mottez, T. Chust and S. Hess, Phys. Plasmas **15**, 052310 (2008).
[6] T. Chust, G. Belmont, F. Mottez and S. Hess, Phys. Plasmas **16**, 092104 (2009).
[7] D.F. Escande and Y. Elskens, Plasma Phys. Control. Fusion **45**, A115 (2003).