SANDIA REPORT
SAND2018-1849R
Unlimited Release
Printed July 2018

Property Values Associated with the Failure of Individual Links in a System with Multiple Weak and Strong Links

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Abstract

Representations are developed and illustrated for the distribution of link property values at the time of link failure in the presence of aleatory uncertainty in link properties. The following topics are considered: (i) defining properties for weak links and strong links, (ii) cumulative distribution functions (CDFs) for link failure time, (iii) integral-based derivation of CDFs for link property at time of link failure, (iv) sampling-based approximation of CDFs for link property at time of link failure, (v) verification of integral-based and sampling-based determinations of CDFs for link property at time of link failure, (vi) distributions of link properties conditional on time of link failure, and (vii) equivalence of two different integral-based derivations of CDFs for link property at time of link failure.

ACKNOWLEDGMENTS

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NOMENCLATURE

Abbreviation	Definition
CDF	cumulative distribution function
DOE	Department of Energy
LOAS	loss of assured safety
NNSA	National Nuclear Security Administration
PLOAS	probability of loss of assured safety
QMU	quantification of margins and uncertainty
SL	strong link
SNL	Sandia National Laboratories
WL	weak link

1. Introduction

Representations for the probability of loss of assured safety (PLOAS) for weak link (WL)/strong link (SL) systems [1-6] involving multiple time-dependent failure modes in the presence of both aleatory and epistemic uncertainty [7-17] are developed and illustrated in Ref. [18]. As described in Ref. [18], loss of assured safety (LOAS) occurs under accident conditions (e.g., a fire) when SL failures place the overall system in a potentially operational mode before deactivation of the overall system as a result of WL failures. In the following, representations are developed and illustrated for the distribution of link property values at the time of link failure in the presence of aleatory uncertainty in link properties. The presented work has been performed in support of the National Nuclear Security Administration's (NNSA's) mandate for the quantification of margins and uncertainties (QMU) in analyses of the United States' nuclear stockpile (see Refs. [19-22] for summary discussions of NNSA's mandate for QMU, Refs. [23-33] for additional background on the development of NNSA's mandate for QMU, and Refs. [34-45] for recent work on the implementation of NNSA's mandate for QMU).

The following topics are considered in this presentation: (i) defining properties for WLs and SLs (Sect. 2), (ii) cumulative distribution functions (CDFs) for link failure time (Sect. 3), (iii) integral-based derivation of CDFs for link property at time of link failure (Sects. 4, 5, 8 and 10), (iv) sampling-based approximation of CDFs for link property at time of link failure (Sect. 6), (v) verification of integral-based and sampling-based determinations of CDFs for link property at time of link failure (Sects. 7 and 11), (vi) distributions of link properties conditional on time of link failure (Sect. 9), and (vii) equivalence of two different integral-based derivations of CDFs for link property at time of link failure (Sect. 12). The presentation then ends with a concluding discussion (Sect. 13).

The results for link failure properties developed in this presentation will be extensively used in following reports on (i) time and failure property margins for systems involving multiple WLs and SLs [46] and (ii) delays in link failure time that are functions of link property value at the time of precursor link failure [47].

This report and the two associated reports [46; 47] are part of a sequence of results related to WL/SL systems [18; 48-51]. The earlier results deal primarily with the time at which WL/SL systems fail and the resultant probability that LOAS will occur. The probability that LOAS will occur is usually the outcome of greatest interest in the analysis of a WL/SL system. However, an over concentration on the final outcome of a complex analysis can lead to (i) loss insights and understanding with respect to the overall analysis and (ii) a possible failure to recognize errors that are present in the analysis. For these reasons, this report and the two indicated following reports deal with internal analysis results and additional summary results that can provide additional information in an analysis of a WL/SL system, including (i) times and property values at which individual links fail [18], (ii) times and property values at which systems of WLs or SLs fail [46], (iii) SL property values at which LOAS occurs [46], (iv) time and property value margins related to the occurrence of LOAS [46], and (v) a variety of verification procedures including comparisons

of LOAS related results obtained with quadrature-based procedures and sampling-based procedures [46; 47].

2. Link Properties

In a prior publication [18], representations for PLOAS are developed for systems in which the failure time CDF for a single WL or SL is based on the following assumed properties of that link for a time interval $t_{mn} \leq t \leq t_{mx}$, where t_{mn} and t_{mx} define the endpoints of the time interval considered for analysis:

$$\bar{p}(t) = \text{nondecreasing positive function defining nominal link property for } t_{mn} \leq t \leq t_{mx}, \quad (2.1)$$

$$\bar{q}(t) = \text{nonincreasing positive function defining nominal failure value for link property} \\ \text{for } t_{mn} \leq t \leq t_{mx}, \quad (2.2)$$

$$d_A(\alpha) = \text{density function for a positive variable } \alpha \text{ used to characterize aleatory} \\ \text{uncertainty in link property,} \quad (2.3)$$

$$d_B(\beta) = \text{density function for a positive variable } \beta \text{ used to characterize aleatory} \\ \text{uncertainty in link failure value,} \quad (2.4)$$

$$p(t \mid \alpha) = \alpha \bar{p}(t) = \text{ link property value for } t_{mn} \leq t \leq t_{mx} \text{ given } \alpha, \quad (2.5)$$

and

$$q(t \mid \beta) = \beta \bar{q}(t) = \text{ link failure value for } t_{mn} \leq t \leq t_{mx} \text{ given } \beta. \quad (2.6)$$

Further, $d_A(\alpha)$ and $d_B(\beta)$ are assumed (i) to be defined on intervals $[\alpha_{mn}, \alpha_{mx}]$ and $[\beta_{mn}, \beta_{mx}]$ and (ii) to equal zero outside these intervals. Although this does not have to be the case, it is anticipated that α and β will be assigned distributions with a mode of 1.0 in most analyses so that $\bar{p}(t)$ and $\bar{q}(t)$ will be the modes (i.e., most likely values) for $p(t|\alpha)$ and $q(t|\beta)$.

For given values for α and β, link failure occurs at the time t at which

$$\alpha \bar{p}(t) = \beta \bar{q}(t), \quad (2.7)$$

which corresponds to the time at which the property value curve $\alpha \bar{p}(t)$ and crosses the failure value curve $\beta \bar{q}(t)$.

As indicated in Eqs. (2.3) and (2.4), the distributions associated with the density functions $d_A(\alpha)$ and $d_B(\beta)$ are used to characterize aleatory uncertainty (i.e., random variability associated with the property value and failure value functions $\bar{p}(t)$ and $\bar{q}(t)$). However, if desired for a specific analysis, $d_A(\alpha)$ and $d_B(\beta)$ could, as an alternative, be defined and used to characterize epistemic uncertainty (i.e., lack of knowledge about the appropriate value for a quantity that has a fixed but poorly known value in the context of the analysis under consideration). In a previous

example analysis involving 5 SLs and 2 WLs, each link was assumed to have associated aleatory uncertainty characterized by independent density functions $d_A(\alpha)$ and $d_B(\beta)$. Further, each distribution for aleatory uncertainty was assumed to be epistemically uncertain due to epistemic uncertainty with respect to a parameter used in its definition (i.e., an epistemically uncertain standard deviation for normal distributions and an epistemically uncertain mode for triangular distributions; see Ref. [18], Sect. 10, for details). Thus, many possibilities exist for the possible use of $d_A(\alpha)$ and $d_B(\beta)$ in future analyses.

To avoid excessively complex notation, two important special cases of the definitions for $\overline{p}(t)$ and $\overline{q}(t)$ in Eqs. (2.1) and (2.2) are considered in this presentation for the derivation of closed-form integral representations for the distribution of property values at which an individual link could fail:

$$\text{Case 1} \sim \overline{p}(t) \text{ increasing and } \overline{q}(t) \text{ decreasing} \tag{2.8}$$

and

$$\text{Case 2} \sim \overline{p}(t) \text{ increasing and } \overline{q}(t) \text{ constant valued.} \tag{2.9}$$

Further, $d[\overline{q}(t)/\overline{p}(t)]/dt$ is assumed to exist on $[t_{mn}, t_{mx}]$ except for at most a finite number of values for t. In addition to closed-form integral representations for the distribution of property values at which an individual link could fail, sampling-based procedures for the estimation of the distribution of property values at which an individual link could fail are also presented. The sampling-based procedures are valid for the general definitions of $\overline{p}(t)$ and $\overline{q}(t)$ in Eqs. (2.1) and (2.2).

As examples, Case 1 in Eq. (2.8) corresponds to a situation in which a sealed region is undergoing time-dependent pressurization from heating while the strength of the region's boundary is degrading with increasing temperature. Case 2 in Eq. (2.9) corresponds to a situation in which a component is being heated until it reaches a constant but randomly variable failure temperature.

Three notional links that will be used for illustration are defined in Table 1 and shown in Fig. 1. As indicated for Link 1 in Fig. 1, the curves $\alpha_{mx}\overline{p}(\tau)$ and $\alpha_{mn}\overline{p}(\tau)$ are represented by dotted lines above and below $\overline{p}(\tau)$, and the curves $\beta_{mx}\overline{q}(\tau)$ and $\beta_{mn}\overline{q}(\tau)$ are represented by dotted lines above and below $\overline{q}(\tau)$.

Table 1 Defining properties of example Links 1, 2 and 3 used to illustrate the calculation of the probability that a link fails at a property value less than or equal to p at a time prior or equal to t.

General Properties for Links 1, 2 and 3

$$\bar{p}(\tau) = \frac{\bar{p}(\infty)\bar{p}(0)}{\bar{p}(0) + [\bar{p}(\infty) - \bar{p}(0)]\exp(-r_1\tau)}, \quad \bar{q}(\tau) = \frac{\bar{q}(0)}{1 + k\tau^{r_2}}$$

Properties of Link 1

$\bar{p}(\infty) = 875, \bar{p}(0) = 300, r_1 = 0.035$

$\bar{q}(0) = 725, k = 1.41 \times 10^{-4}, r_2 = 1.8$

$d_A(\alpha)$ uniform on $[\alpha_{mn}, \alpha_{mx}] = [0.6, 1.3]$

$d_B(\beta)$ triangular on $[\beta_{mn}, \beta_{mx}] = [0.7, 1.2]$ with mode 1.0

Corresponds to Case 1 (i.e., $\alpha_{mn} / \beta_{mn} < \alpha_{mx} / \beta_{mx}$) in Sect. 9.3

Properties of Link 2

$\bar{p}(\infty) = 900, \bar{p}(0) = 300, r_1 = 0.025$

$\bar{q}(0) = 775, k = 3.0 \times 10^{-5}, r_2 = 2.0$

$d_A(\alpha)$ triangular on $[\alpha_{mn}, \alpha_{mx}] = [0.85, 1.25]$ with mode 1.0

$d_B(\beta)$ triangular on $[\beta_{mn}, \beta_{mx}] = [0.65, 1.4]$ with mode 1.0

Corresponds to Case 2 (i.e., $\alpha_{mx} / \beta_{mx} < \alpha_{mn} / \beta_{mn}$) in Sect. 9.3

Properties of Link 3

$\bar{p}(\infty) = 850, \bar{p}(0) = 150, r_1 = 0.045$

$\bar{q}(0) = 900, k = 2.21 \times 10^{-4}, r_2 = 1.6$

$d_A(\alpha)$ uniform on $[\alpha_{mn}, \alpha_{mx}] = [0.76, 1.3]$

$d_B(\beta)$ uniform on $[\beta_{mn}, \beta_{mx}] = [0.76, 1.3]$

Corresponds to Case 3 (i.e., $\alpha_{mn} / \beta_{mn} = \alpha_{mx} / \beta_{mx}$) in Sect. 9.3
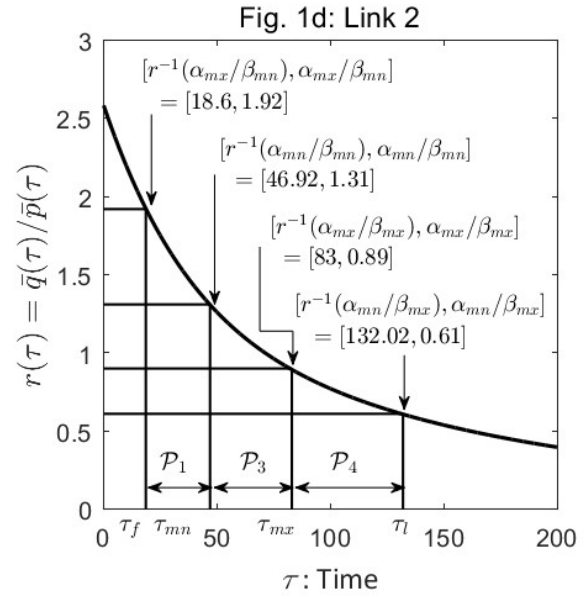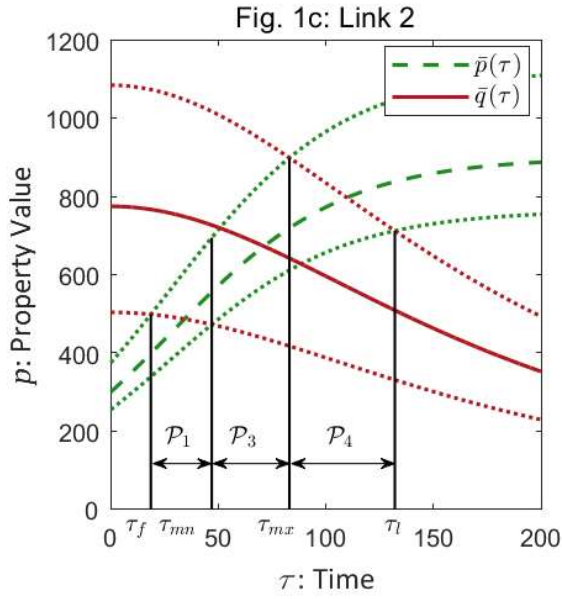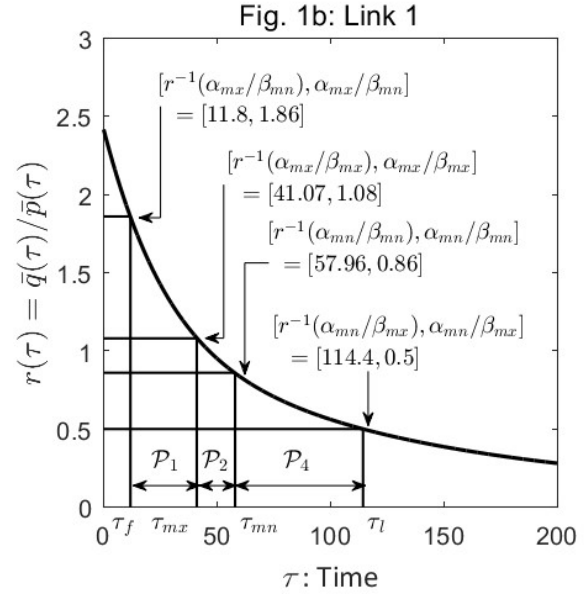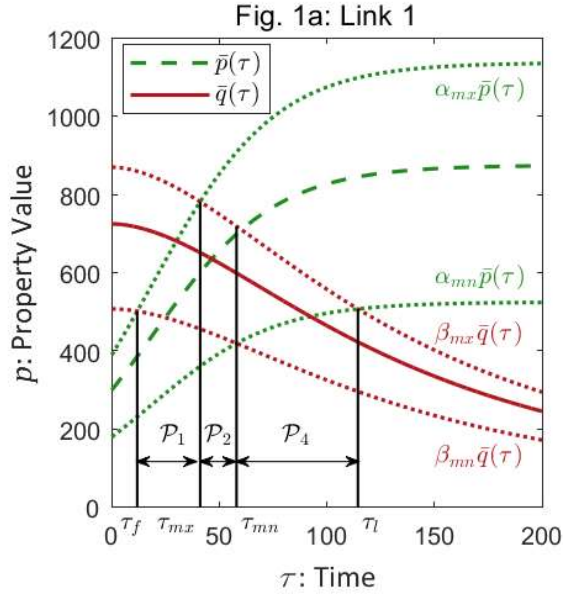
Fig. 1, Frames a, b, c and d. Figure caption and Frames e and f on next page.
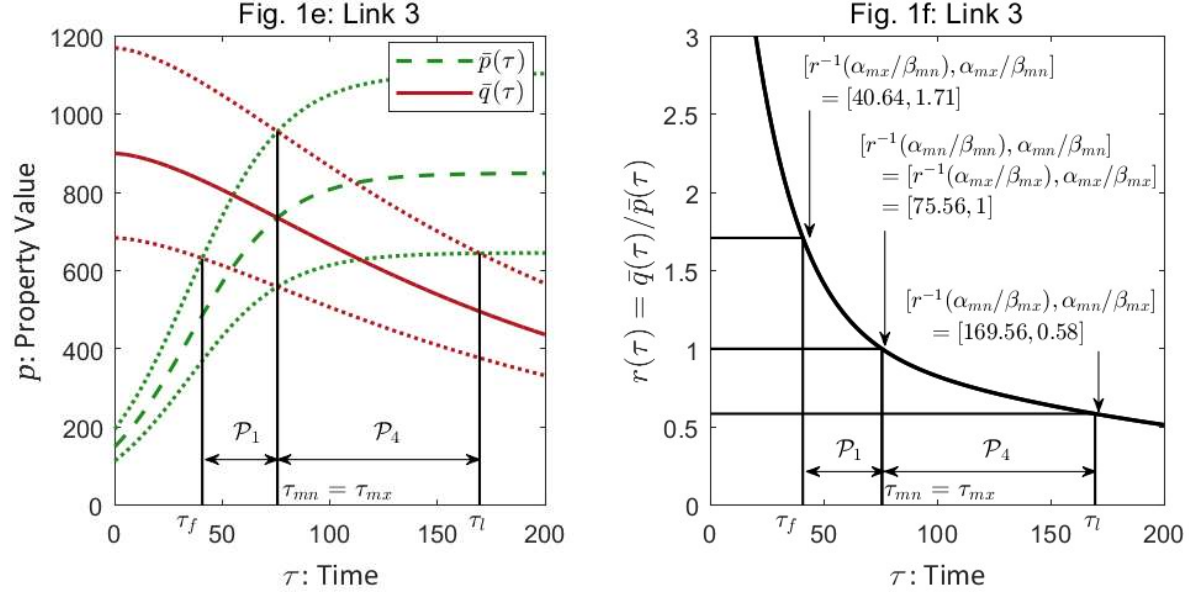
Fig. 1 Summary plots of the example Links 1, 2 and 3 defined in Table 1 used to illustrate the calculation of the probability that a link fails at a property value less than or equal to p at a time prior or equal to t with (i) $r(\tau)$, τ_f, τ_{mn}, τ_{mx} and τ_l defined in Eqs. (4.2)-(4.9) and (ii) $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ and $\mathcal{P}_4$ defined and discussed in Sects. 9.2 and 9.3.

The analytic representations for $\bar{p}(t)$ and $\bar{q}(t)$ defined in Table 1 and illustrated in Fig. 1 are used for representational convenience. In a real analysis, $\bar{p}(t)$ and $\bar{q}(t)$ would most likely be obtained as the discretized outcomes of complex numerical calculations rather than as simple continuous functions. In this situation, the options would be to (i) fit a continuous curve to the discretized results or (ii) deal directly with the discretized results. To maintain a level of realism, the numerical results obtained in this presentation with quadrature-based procedures using the MATLAB numerical package [52] and sampling-based procedures using the CPLOAS program [53] start with discretizations of the analytic representations for $\bar{p}(t)$ and $\bar{q}(t)$ in Table 1.

3. CDF for Link Failure Time

The definition of cumulative distribution functions (CDFs) for link failure time is now briefly summarized (see Ref. [18] for additional details). Suppose t is a potential link failure time and α_i is an element of the subdivision $\alpha_{mn} = \alpha_0 < \alpha_1 < \cdots < \alpha_n = \alpha_{mx}$ of $[\alpha_{mn}, \alpha_{mx}]$. For $p(t|\alpha_i)$ nondecreasing and $q(t|\beta)$ nonincreasing, link failure prior to time t conditional on α_i can occur at or before time t only for values of β satisfying

$$\beta \bar{q}(t) = q(t \mid \beta) \le p(t \mid \alpha_i) = \alpha_i \bar{p}(t), \tag{3.1}$$

which in turn implies the inequality

$$\beta \le \alpha_i \bar{p}(t) / \bar{q}(t) = F(\alpha_i, t). \tag{3.2}$$

As a consequence,

$$CDF_T(t) = \text{ probability that link fails in the time interval } [t_{mn}, t]$$
$$\cong \sum_{i=1}^{n} \left[\int_{\beta_{mn}}^{F(\alpha_i, t)} d_B(\beta) d\beta \right]_1 \left[d_A(\alpha_i) \Delta\alpha_i \right]_2, \tag{3.3}$$

where (i) $[\sim]_1$ is the probability that β is less than or equal to $F(\alpha_i, t)$, and (ii) $[\sim]_2$ is an approximation of the probability that α is in the interval $[\alpha_{i-1}, \alpha_i]$. In turn,

$$CDF_T(t) = \int_{\alpha_{mn}}^{\alpha_{mx}} \left[\int_{\beta_{mn}}^{F(\alpha, t)} d_B(\beta) d\beta \right] d_A(\alpha) d\alpha \tag{3.4}$$

in the limit as $\Delta\alpha_i \to 0$. On a technical note, the inequality $\beta_{mx} < F(\alpha, t)$ is possible in the inner integrals in Eqs. (3.3)-(3.4) but does not present a problem as $d_B(\beta) = 0$ for $\beta > \beta_{mx}$. In effect, the upper limit of integration for the inner integral in Eqs. (3.3)-(3.4) is $\min\{F(\alpha, t), \beta_{mx}\}$.

An alternative to the Riemann integral representation for $CDF_T(t)$ in Eq. (3.4) is the Stieltjes integral representation

$$CDF_T(t) = \int_{\alpha_{mn}}^{\alpha_{mx}} CDF_B[F(\alpha, t)] d CDF_A(\alpha), \tag{3.5}$$

where

$$CDF_A(\alpha) = \int_{\alpha_{mn}}^{\alpha} d_A(\tilde{\alpha}) d\tilde{\alpha} \text{ and } CDF_B(\beta) = \int_{\beta_{mn}}^{\beta} d_B(\tilde{\beta}) d\tilde{\beta} \tag{3.6}$$

are the CDFs for α and β, respectively. In computational practice, it may be easier to evaluate

$CDF_T(t)$ with the Stieltjes integral representation in Eq. (3.5) and precalculated values for the CDFs for α and β than to evaluate $CDF_T(t)$ with the Riemann integral representation in Eq. (3.4) and use of the density functions for α and β.

Another possibility is to use a sampling-based (i.e., Monte Carlo) procedure to estimate $CDF_T(t)$. With this approach, (i) a sample

$$\mathbf{s}_i = [\alpha_i, \beta_i], i = 1, 2, \cdots, n, \tag{3.7}$$

is generated from $[\alpha_{mn}, \alpha_{mx}] \times [\beta_{mn}, \beta_{mx}]$ is consistency with the distributions for α and β, and (ii) the associated results

$$\tau_i = \text{ time of link failure} = r^{-1}(\alpha_i / \beta_i) \tag{3.8}$$

obtained from

$$\alpha_i \overline{p}(\tau_i) = \beta_i \overline{q}(\tau_i) \quad \Rightarrow \quad \alpha_i / \beta_i = \overline{q}(\tau_i) / \overline{p}(\tau_i) = r(\tau_i) \quad \Rightarrow \quad \tau_i = r^{-1}(\alpha_i / \beta_i) \tag{3.9}$$

are determined for $i = 1, 2, \cdots, n$. Then, $CDF_T(t)$ is approximated by

$$CDF_T(t) \cong \sum_{i=1}^{n} \delta_t(\tau_i) / n \text{ with } \delta_t(\tau_i) = \begin{cases} 1 & \text{for } \tau_i \leq t \\ 0 & \text{otherwise.} \end{cases} \tag{3.10}$$

With respect to the definition of τ_i in Eq. (3.8), $r^{-1}(\alpha_i / \beta_i)$ will have a unique value if $\overline{p}(\tau)$ is increasing and $\overline{q}(\tau)$ is nonincreasing. However, $r^{-1}(\alpha_i / \beta_i)$ may not have a unique value if $\overline{p}(\tau)$ is assumed to be nondecreasing and $\overline{q}(\tau)$ is assumed to be nonincreasing. In this case, the curves $\alpha_i \overline{p}(\tau)$ and $\beta_i \overline{q}(\tau)$ must tracked to determine the time τ_i at which they initially intersect, which is equivalent to defining $r^{-1}(\alpha / \beta)$ by

$$r^{-1}(\alpha / \beta) = \min\{\tau : \alpha \overline{p}(\tau) = \beta \overline{q}(\tau)\} \tag{3.11}$$

when $\alpha \overline{p}(\tau) = \beta \overline{q}(\tau)$ holds for an interval of time rather than for a single point in time. This has the potential to occur only if $\overline{p}(\tau)$ and $\overline{q}(\tau)$ are constant valued for overlapping intervals of time.

Additional details on the definition and numerical evaluation of CDFs for link failure time is available in Ref. [18]. Further, the link failure time CDFs for the three links described and illustrated in Table 1 and Fig. 1 are shown in Fig. 2.
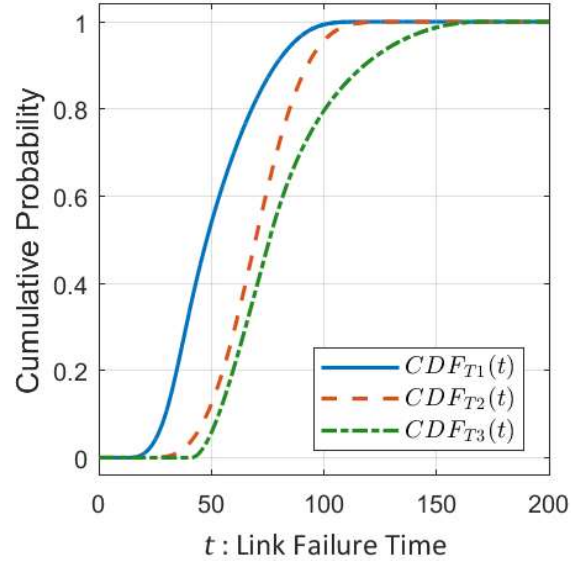
Fig. 2 Link failure time CDFs (i.e., $CDF_{Ti}(t)$ defined in Eqs. (3.4) and (3.5) for Links 1, 2 and 3 described and illustrated in Table 1 and Fig. 1.

4. CDFs for Link Property at Time of Link Failure for Case 1: $\bar{p}(t)$ Increasing and $\bar{q}(t)$ Decreasing

4.1 Preliminaries: CDFs for Link Property at Time of Link Failure

The following property for an individual link is now investigated:

$$CDF_P(p\,|\,[t_{mn},t]) = \text{ probability that link fails at a value less than or equal to } p$$
$$\text{in the time interval } [t_{mn},t]. \tag{4.1}$$

Possible link definitions and values for property value p at link failure are illustrated in Fig. 1 for Links 1, 2 and 3 defined in Table 1.

For use in this section and additional parts of this presentation, the following notation is introduced:

$$r(\tau) = \bar{q}(\tau)\,/\,\bar{p}(\tau) \tag{4.2}$$

$$\tau_f = \text{first possible time for link failure defined by } \alpha_{mx}\bar{p}(\tau) = \beta_{mn}\bar{q}(\tau)$$
$$= r^{-1}\left(\alpha_{mx}\,/\,\beta_{mn}\right) \tag{4.3}$$

$$p_f = \text{property value at which link failure occurs at time } \tau_f$$
$$= \alpha_{mx}\bar{p}(\tau_f) = \alpha_{mx}\bar{p}[r^{-1}(\alpha_{mx}\,/\,\beta_{mn})] \tag{4.4}$$

$$\tau_l = \text{last possible time for link failure defined by } \alpha_{mn}\bar{p}(\tau) = \beta_{mx}\bar{q}(\tau)$$
$$= r^{-1}\left(\alpha_{mn}\,/\,\beta_{mx}\right) \tag{4.5}$$

$$p_l = \text{property value at which link failure occurs at time } \tau_l$$
$$= \alpha_{mn}\bar{p}(\tau_l) = \alpha_{mn}\bar{p}[r^{-1}(\alpha_{mn}\,/\,\beta_{mx})] \tag{4.6}$$

$$\tau_{mn} = \text{time of minimum possible property value at link failure defined}$$
$$\text{by } \alpha_{mn}\bar{p}(\tau) = \beta_{mn}\bar{q}(\tau)$$
$$= r^{-1}\left(\alpha_{mn}\,/\,\beta_{mn}\right) \tag{4.7}$$

$$p_{mn} = \text{property value at which link failure occurs at time } \tau_{mn}$$
$$= \alpha_{mn}\bar{p}(\tau_{mn}) = \alpha_{mn}\bar{p}[r^{-1}(\alpha_{mn}\,/\,\beta_{mn})] \tag{4.8}$$

$$\tau_{mx} = \text{time of maximum possible property value at link failure defined}$$
$$\text{by } \alpha_{mx}\bar{p}(\tau) = \beta_{mx}\bar{q}(\tau)$$
$$= r^{-1}\left(\alpha_{mx}\,/\,\beta_{mx}\right) \tag{4.9}$$

$$p_{mx} = \text{property value at which link failure occurs at time } \tau_{mx}$$
$$= \alpha_{mx}\bar{p}(\tau_{mx}) = \alpha_{mx}\bar{p}[r^{-1}(\alpha_{mx}\,/\,\beta_{mx})] \tag{4.10}$$

$\tau_f(p) =$ first time that link failure could occur at property value p

$$= \begin{cases} \overline{q}^{-1}(p / \beta_{mn}) & \text{from } p = \beta_{mn} \overline{q}[\tau_f(p)] \text{ for } p_{mn} \le p < p_f \\ \overline{p}^{-1}(p / \alpha_{mx}) & \text{from } p = \alpha_{mx} \overline{p}[\tau_f(p)] \text{ for } p_f \le p \le p_{mx} \end{cases} \quad (4.11)$$

$\tau_l(p) =$ last time that link failure could occur at property value p

$$= \begin{cases} \overline{p}^{-1}(p / \alpha_{mn}) & \text{from } p = \alpha_{mx} \overline{p}[\tau_f(p)] \text{ for } p_{mn} \le p \le p_l \\ \overline{q}^{-1}(p / \beta_{mx}) & \text{from } p = \beta_{mx} \overline{q}[\tau_l(p)] \text{ for } p_l \le p \le p_{mx} \end{cases} \quad (4.12)$$

$\tau_{mn}(p) =$ first time that link failure could occur at a property value $\tilde{p} \le p$

$$= \begin{cases} \tau_f(p) = \overline{q}^{-1}(p / \beta_{mn}) & \text{for } p_{mn} \le p < p_f \\ \tau_f = r^{-1}(\alpha_{mx} / \beta_{mn}) & \text{for } p_f \le p \le p_{mx} \end{cases} \quad (4.13)$$

$\tau_{mx}(p) =$ last time that link failure could occur at a property value $\tilde{p} \le p$

$$= \begin{cases} \tau_l(p) = \overline{p}^{-1}(p / \alpha_{mn}) & \text{for } p_{mn} \le p \le p_l \\ \tau_l = r^{-1}(\alpha_{mn} / \beta_{mx}) & \text{for } p_l \le p \le p_{mx} \end{cases} \quad (4.14)$$

$\alpha_{mx}(p) =$ largest α value resulting in link failure at a property value $\tilde{p} \le p$

$$= p / \overline{p}[\tau_f(p)] \text{ from } p = \alpha_{mx}(p)\overline{p}[\tau_f(p)] \quad (4.15)$$

$$= \begin{cases} p / \overline{p}[\overline{q}^{-1}(p / \beta_{mn})] & \text{for } p_{mn} \le p < p_f \\ p / \overline{p}[\overline{p}^{-1}(p / \alpha_{mx})] = \alpha_{mx} & \text{for } p_f \le p \le p_{mx}. \end{cases}$$

Further, the results derived in Sect. 4 are for $\overline{p}(\tau)$ increasing and $\overline{q}(\tau)$ decreasing (i.e., for Case 1 as defined in Eq. (2.8)), which assures that $\overline{p}^{-1}(\tau)$, $\overline{q}^{-1}(\tau)$ and $r^{-1}(\tau)$ exist.

The CDF $CDF_P(p \,|\, [t_{mn}, t])$ for link property at time of link failure defined in Eq. (4.1) can be formally represented by integrals involving the link parameters α and β. Consistent with the examples in Fig. 1, the configurations

$$\text{Configuration 1: } p_f \le p \le p_{mx} \text{ with } \tau_f < t \le \tau_f(p), \quad (4.16)$$

$$\text{Configuration 2: } p_{mn} \le p \le p_{mx} \text{ with } \tau_f(p) < t \le \tau_l(p), \quad (4.17)$$

$$\text{Configuration 3: } p_l \le p \le p_{mx} \text{ with } \tau_l(p) < t \le \tau_l \quad (4.18)$$

involving t and p result in different integral representations for $CDF_P(p \,|\, [t_{mn}, t])$ in terms of α and β. The indicated representations for $CDF_P(p \,|\, [t_{mn}, t])$ are derived in the following three subsections. Further, the derivations are illustrated with the two links (i.e., Links 4 and 5) defined in Fig. 3.
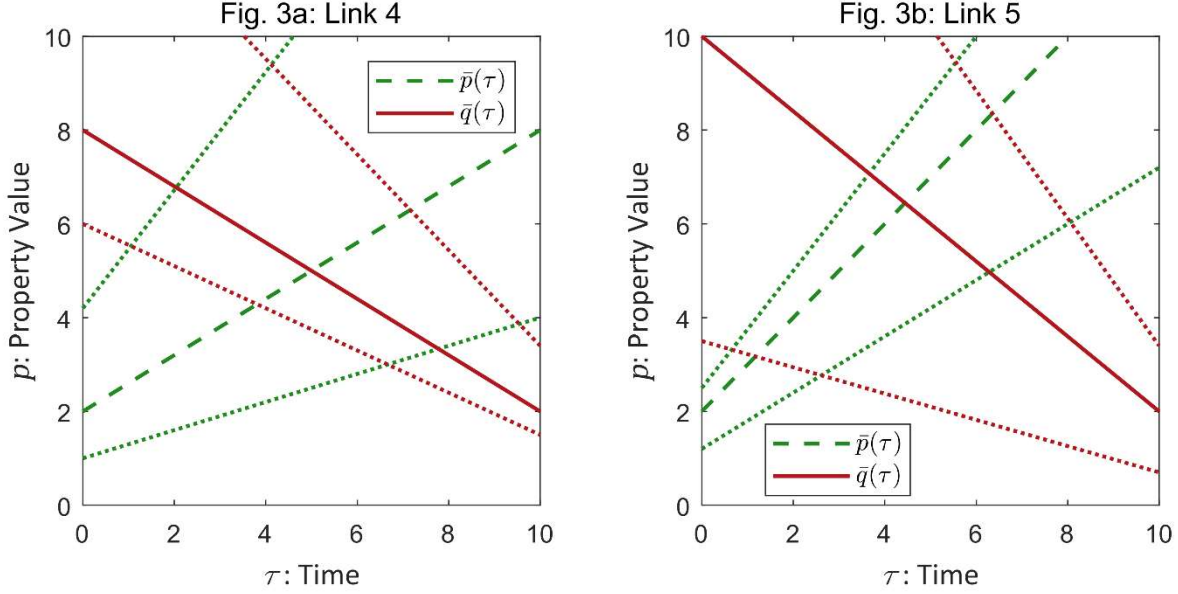
Fig. 3 Example Links 4 and 5 used to illustrate the derivation of integral representations of $CDF_P(p, [t_{mn}, t])$ for Configurations 1, 2 and 3 defined in Eqs. (4.16)-(4.18): (a) Link 4 with $\overline{p}(\tau) = 2.0 + 0.6\tau$, $\overline{q}(\tau) = 8.0 - 0.6\tau$, $[\alpha_{mn}, \alpha_{mx}] = [0.5, 2.1]$, and $[\beta_{mn}, \beta_{mx}] = [0.75, 1.7]$, and (b) Link 5 with $\overline{p}(\tau) = 2.0 + \tau$, $\overline{q}(\tau) = 10.0 - 0.8\tau$, $[\alpha_{mn}, \alpha_{mx}] = [0.6, 1.25]$, and $[\beta_{mn}, \beta_{mx}] = [0.35, 1.7]$.

4.2 Integral Representation of $CDF_P(p \,|\, [t_{mn}, t])$ for Configuration 1

Given the conditions imposed on t and p for Configuration 1 (i.e., $p_f \le p \le p_{mx}$ with $\tau_f < t \le \tau_f(p)$), a failure value $\tilde{p} \le p$ can only occur for curves $\alpha \overline{p}(\tau)$ that cross the vertical line $\mathcal{L}$ illustrated in Fig. 4 connecting the points

$$[t, \alpha_{mx}(p)\overline{p}(t)] = [t, \alpha_{mx}\overline{p}(t)] \text{ and } [t, \alpha_{mn}(t)\overline{p}(t)] = \begin{cases} [t, \beta_{mn}\overline{q}(t)] & \text{for } t < \tau_{mn} \\ [t, \alpha_{mn}\overline{p}(t)] & \text{for } \tau_{mn} \le t, \end{cases} \quad (4.19)$$

where (i) $\alpha_{mx}(p) = \alpha_{mx}$ as indicated in Eq. (4.15) and (ii) $\alpha_{mn}(t)$ defined by

$$\alpha_{mn}(t) = \begin{cases} \beta_{mn}\overline{q}(t) / \overline{p}(t) = \beta_{mn}r(t) & \text{for } t < \tau_{mn} \text{ from } \alpha_{mn}(t)\overline{p}(t) = \beta_{mn}\overline{q}(t) \\ \alpha_{mn} & \text{for } \tau_{mn} \le t \text{ from } \alpha_{mn}(t)\overline{p}(t) = \alpha_{mn}\overline{p}(t) \end{cases} \quad (4.20)$$

is the α value such that the curve $\alpha_{mn}(t)\overline{p}(\tau)$ passes through the point $[t, \beta_{mn}\overline{q}(t)]$ for $t < \tau_{mn}$ as illustrated in Fig. 4a and the point $[t, \alpha_{mn}\overline{p}(t)]$ for $\tau_{mn} \le t$ as illustrated in Fig. 4b. In turn, the set

$$\mathcal{A} = \{\alpha : \alpha_{mn}(t) \le \alpha \le \alpha_{mx}\} \quad (4.21)$$

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contains the α values for the curves crossing the line $\mathcal{L}$.

Fig. 4 Illustration of regions (i.e., colored areas) integrated over to obtain $CDF_P(p,[t_{mn},t])$ for Configuration 1 defined in Eq. (4.16): (a) Link 4 with $t_{mn}=0$, $p=9$, and $t=3$, and (b) Link 5 with $t_{mn}=0$, $p=8.5$, and $t=4$.

For the following, a subdivision

$$\alpha_{mn}(t) = \alpha_0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n = \alpha_{mx}(p) \tag{4.22}$$

of $[\alpha_{mn}(t),\alpha_{mx}(p)]$ is assumed. For $\alpha_i \in \mathcal{A}$, (i) the value β_i for which $\beta_i \bar{q}(t) = \alpha_i \bar{p}(t)$ is given by

$$\beta_i = \alpha_i \bar{p}(t) / \bar{q}(t) = \alpha_i / r(t) = F(\alpha_i,t), \tag{4.23}$$

and (ii) as a consequence of the monotonicity of $\bar{p}(\tau)$ and $\bar{q}(\tau)$,

$$\text{prob}(\beta \le \beta_i | \alpha_i \in \mathcal{A}) = \text{ probability that link fails at } \tilde{p} \le p \text{ conditional on } \alpha_i \in \mathcal{A}$$
$$= \int_{\beta_{mn}}^{F(\alpha_i,t)} d_B(\beta)d\beta. \tag{4.24}$$

In turn, given the results in Eqs. (4.23) and (4.24),

$$CDF_P(p \,|\, [t_{mn}, t]) = CDF_P(p \,|\, [\tau_f, t]) \text{ for } \tau_f < t \leq \tau_f(p)$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} \left[\int_{\beta_{mn}}^{F(\alpha_i, t)} d_B(\beta) \mathrm{d}\beta \right] d_A(\alpha_i) \Delta\alpha_i$$

$$= \int_{\alpha_{mn}(t)}^{\alpha_{mx}} \left[\int_{\beta_{mn}}^{F(\alpha, t)} d_B(\beta) \mathrm{d}\beta \right] d_A(\alpha) \mathrm{d}\alpha$$

$$= \int_{\alpha_{mn}(t)}^{\alpha_{mx}} \left[\int_{\beta_{mn}}^{\alpha/r(t)} d_B(\beta) \mathrm{d}\beta \right] d_A(\alpha) \mathrm{d}\alpha$$

(4.25)

with $F(\alpha, t) = \alpha \,/\, r(t)$ as defined in Eq. (4.23).

4.3 Integral Representation of $CDF_P(p \,|\, [t_{mn}, t])$ for Configuration 2

Given the conditions imposed on t and p for Configuration 2 (i.e., $p_{mn} \leq p \leq p_{mx}$ with $\tau_f(p) < t \leq \tau_l(p)$), a failure value $\tilde{p} \leq p$ can only occur for curves $\alpha \overline{p}(\tau)$ that cross the horizontal line $\mathcal{L}_1$ illustrated in Fig. 5 connecting the points

$$\left[\tau_f(p), p \right] = \left[\tau_f(p), \alpha_{mx}(p) \overline{p}[\tau_f(p)] \right] \text{ and } [t, p] = \left[t, \alpha(t, p) \overline{p}(t) \right]$$

(4.26)

or the vertical line $\mathcal{L}_2$ also illustrated in Fig. 5 connecting the points

$$[t, p] \text{ and } [t, \alpha_{mn}(t) \overline{p}(t)] = \begin{cases} [t, \beta_{mn} \overline{q}(t)] & \text{for } t < \tau_{mn} \text{ as in Fig. 5a} \\ [t, \alpha_{mn} \overline{p}(t)] & \text{for } \tau_{mn} \leq t \text{ as in Fig. 5b,} \end{cases}$$

(4.27)

where (i) $\alpha_{mx}(p)$ defined by

$$\alpha_{mx}(p) = p \,/\, \overline{p}[\tau_f(p)]$$

$$= \begin{cases} p \,/\, \overline{p}[\overline{q}^{-1}(p \,/\, \beta_{mn})] & \text{for } p_{mn} \leq p \leq p_f \text{ as in Fig. 5a} \\ p \,/\, \overline{p}[\overline{p}^{-1}(p \,/\, \alpha_{mx})] = \alpha_{mx} & \text{for } p_f \leq p \leq p_{mx} \text{ as in Fig. 5b} \end{cases}$$

(4.28)

in Eq. (4.15) is the α value such that the curve $\alpha_{mx}(p) \overline{p}(\tau)$ passes through the point $[\tau_f(p), p]$, (ii) $\alpha(t, p)$ defined by

$$p = \alpha(t, p) \overline{p}(t) \quad \Rightarrow \quad \alpha(t, p) = p \,/\, \overline{p}(t)$$

(4.29)

is the α value such that the curve $\alpha(t, p) \overline{p}(\tau)$ passes through the point $[t, p]$, and (iii) $\alpha_{mn}(t)$ is defined the same as in Eq. (4.20). In turn, the sets

$$\mathcal{A}_2 = \{\alpha : \alpha_{mn}(t) \leq \alpha \leq \alpha(t, p)\} \text{ and } \mathcal{A}_1 = \{\alpha : \alpha(t, p) \leq \alpha \leq \alpha_{mx}(p)\}$$

(4.30)

contain the α values for the curves $\alpha \bar{p}(\tau)$ crossing lines $\mathcal{L}_2$ and $\mathcal{L}_1$, respectively.
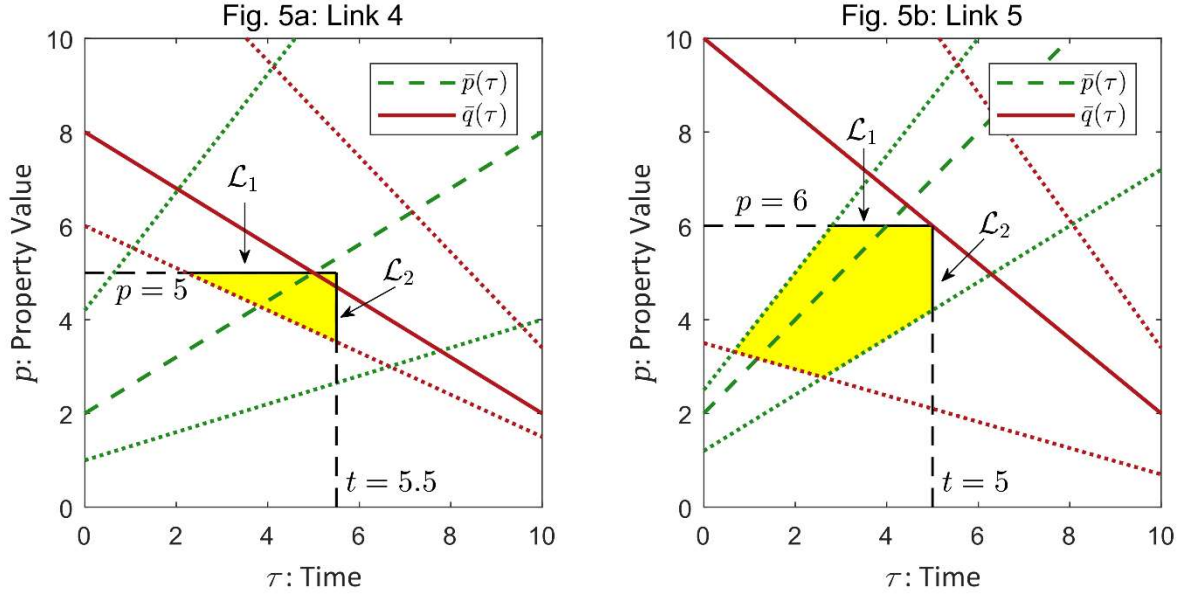
Fig. 5 Illustration of regions (i.e., colored areas) integrated over to obtain $CDF_P(p\,|\,[t_{mn},t])$ for Configuration 2 defined in Eq. (4.17): (a) Link 4 with $t_{mn}=0$, $p=5$, and $t=5.5$, and (b) Link 5 with $t_{mn}=0$, $p=6$, and $t=5$.

For the following, a subdivision

$$\alpha_{mn}(t)=\alpha_0<\alpha_1<\alpha_2<\cdots<\alpha_n=\alpha_{mx}(p) \tag{4.31}$$

of

$$[\alpha_{mn}(t),\alpha_{mx}(p)]=[\alpha_{mn}(t),\alpha(t,p)]\cup[\alpha(t,p),\alpha_{mx}(p)]=\mathcal{A}_2\cup\mathcal{A}_1 \tag{4.32}$$

is assumed with $\alpha_m=\alpha(t,p)$.

For $\alpha_i\in\mathcal{A}_2$ and similarly to Eqs. (4.23) and (4.24), (i) the value β_i for which $\beta_i\bar{q}(t)=\alpha_i\bar{p}(t)$ is given by

$$\beta_i=\alpha_i\bar{p}(t)\,/\,\bar{q}(t)=\alpha_i\,/\,r(t)=F(\alpha_i,t), \tag{4.33}$$

and (ii) as a consequence of the monotonicity of $\bar{p}(\tau)$ and $\bar{q}(\tau)$,

$$\begin{aligned}\mathrm{prob}(\beta\leq\beta_i|\alpha_i\in\mathcal{A}_2)=&\ \text{probability that link fails at }\tilde{p}\leq p\text{ conditional on }\alpha_i\in\mathcal{A}_2\\ =&\int_{\beta_{mn}}^{F(\alpha_i,t)}d_B(\beta)\mathrm{d}\beta.\end{aligned} \tag{4.34}$$

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For $\alpha_i \in \mathcal{A}_1$, (i) the time τ_i at which $\alpha_i \overline{p}(\tau_i) = p$ is given by $\tau_i = \overline{p}^{-1}(p/\alpha_i)$, (ii) the value β_i for which $\beta_i \overline{q}(\tau_i) = p$ is given by

$$\beta_i = p / \overline{q}(\tau_i) = p / \overline{q}[\overline{p}^{-1}(p/\alpha_i)] = G(\alpha_i, p), \tag{4.35}$$

and (iii) as a consequence of the monotonicity of $\overline{p}(\tau)$ and $\overline{q}(\tau)$,

$$\text{prob}(\beta \le \beta_i | \alpha_i \in \mathcal{A}_1) = \text{ probability that link fails at value } \tilde{p} \le p \text{ conditional}$$
$$\text{on the occurrence of } \alpha_i \in \mathcal{A}_1 \tag{4.36}$$
$$= \int_{\beta_{mn}}^{G(\alpha_i, p)} d_\beta(\beta) d\beta.$$

In turn, given the assumption $\tau_f(p) < t \le \tau_l(p)$ and the results in Eqs. (4.34) and (4.36),

$$CDF_P(p \,|\, [t_{mn}, t]) = CDF_P(p \,|\, [\tau_{mn}(p), t]) \text{ for } \tau_f(p) < t \le \tau_l(p)$$
$$= \lim_{n \to \infty} \left\{ \sum_{i=1}^{m-1} \left[\int_{\beta_{mn}}^{F(\alpha_i, t)} d_B(\beta) d\beta \right] d_A(\alpha_i) \Delta\alpha_i + \sum_{i=m}^{n} \left[\int_{\beta_{mn}}^{G(\alpha_i, p)} d_B(\beta) d\beta \right] d_A(\alpha_i) \Delta\alpha_i \right\}$$
$$= \int_{\alpha_{mn}(t)}^{\alpha(t,p)} \left[\int_{\beta_{mn}}^{F(\alpha, t)} d_B(\beta) d\beta \right] d_A(\alpha) d\alpha + \int_{\alpha(t,p)}^{\alpha_{mx}(p)} \left[\int_{\beta_{mn}}^{G(\alpha, p)} d_B(\beta) d\beta \right] d_A(\alpha) d\alpha \tag{4.37}$$
$$= \int_{\alpha_{mn}(t)}^{p/\overline{p}(t)} \left[\int_{\beta_{mn}}^{\alpha/r(t)} d_B(\beta) d\beta \right] d_A(\alpha) d\alpha + \int_{p/\overline{p}(t)}^{\alpha_{mx}(p)} \left[\int_{\beta_{mn}}^{p/\overline{q}[\overline{p}^{-1}(p/\alpha)]} d_B(\beta) d\beta \right] d_A(\alpha) d\alpha$$

with $\alpha_{mn}(t)$, $\alpha_{mx}(p)$, $\alpha(t, p) = p / \overline{p}(t)$, $F(\alpha, t) = \alpha / r(t)$ and $G(\alpha, p) = p / \overline{q}[\overline{p}^{-1}(p/\alpha)]$ defined in Eqs. (4.20), (4.28), (4.29), (4.33) and (4.35), respectively.

Further, the preceding representation for $CDF_P(p \,|\, [t_{mn}, t])$ simplifies to

$$CDF_P(p \,|\, [t_{mn}, t]) = \int_{\alpha_{mn}}^{\alpha_{mx}(p)} \left[\int_{\beta_{mn}}^{p/\overline{q}[\overline{p}^{-1}(p/\alpha)]} d_B(\beta) d\beta \right] d_A(\alpha) d\alpha \tag{4.38}$$

for $t = \tau_l(p)$ and $p_{mn} < p \le p_l$ as a result of (i) the relationships

$$\tau_l(p) = \overline{p}^{-1}(p/\alpha_{mn}), \ \alpha_{mn}[\tau_l(p)] = \alpha_{mn} \text{ and } p / \overline{p}[\tau_l(p)] = p / \overline{p}[\overline{p}^{-1}(p/\alpha_{mn})] = \alpha_{mn} \tag{4.39}$$

that exist when $[\tau_l(p), p] = [\overline{p}^{-1}(p/\alpha_{mn}), p]$ is a point on the curve $\alpha_{mn}\overline{p}(\tau)$ and (ii) the consequent equality

$$\int_{\alpha_{mn}(t)}^{p/\overline{p}(t)}\left[\int_{\beta_{mn}}^{\alpha/r(t)}d_B(\beta)\mathrm{d}\beta\right]d_A(\alpha)\mathrm{d}\alpha = \int_{\alpha_{mn}}^{\alpha_{mn}}\left[\int_{\beta_{mn}}^{\alpha/r(t)}d_B(\beta)\mathrm{d}\beta\right]d_A(\alpha)\mathrm{d}\alpha = 0. \qquad (4.40)$$

4.4 Integral Representation of $CDF_P(p\,|\,[t_{mn},t])$ for Configuration 3

Given the conditions imposed on t and p for Configuration 3 (i.e., $p_l \le p \le p_{mx}$ with $\tau_l(p) < t \le \tau_l$), a failure value $\tilde{p} \le p$ can only occur for curves $\alpha\overline{p}(\tau)$ that cross the horizontal line $\mathcal{L}_1$ illustrated in Fig. 6 connecting the points

$$\left[\tau_f(p),p\right]=\left[\tau_f(p),\alpha_{mx}(p)\overline{p}[\tau_f(p)]\right] \text{ and } \left[\tau_l(p),p\right]=\left[\tau_l(p),\alpha[\tau_l(p),p]\overline{p}(t)\right], \quad (4.41)$$

the curve $\mathcal{L}_2$ illustrated in Fig. 6 corresponding to $\beta_{mx}\overline{q}(\tau)$ connecting the points

$$\left[\tau_l(p),p\right]=\left[\tau_l(p),\beta_{mx}\overline{q}[\tau_l(p)]\right] \text{ and } \left[t,\alpha[t,\beta_{mx}\overline{q}(t)]\overline{p}(t)\right]=\left[t,\beta_{mx}\overline{q}(t)\right], \qquad (4.42)$$

or the vertical line $\mathcal{L}_3$ also illustrated in Fig. 6 connecting the points

$$\left[t,\alpha[t,\beta_{mx}\overline{q}(t)]\overline{p}(t)\right] \text{ and } \left[t,\alpha_{mn}(t)\overline{p}(t)\right] = \begin{cases} [t,\beta_{mn}\overline{q}(t)] & \text{for } t < \tau_{mn} \text{ as in Fig. 6a} \\ [t,\alpha_{mn}\overline{p}(t)] & \text{for } \tau_{mn} \le t \text{ as in Fig. 6b,} \end{cases} \qquad (4.43)$$

where (i) $\alpha_{mx}(p)$ defined in Eqs. (4.15) and (4.28) is the α value such that the curve $\alpha_{mx}(p)\overline{p}(\tau)$ passes through the point $[\tau_f(p),p]$, (ii) $\alpha[\tau_l(p),p]$ and $\alpha[t,\beta_{mx}\overline{q}(t)]$ defined by

$$\alpha[\tau_l(p),p]\overline{p}[\tau_l(p)]=p \quad \Rightarrow \quad \alpha[\tau_l(p),p]=p\,/\,\overline{p}[\tau_l(p)]=p\,/\,\overline{p}[\overline{q}^{-1}(p\,/\,\beta_{mx})] \qquad (4.44)$$

and

$$\alpha[t,\beta_{mx}\overline{q}(t)]\overline{p}(t)=\beta_{mx}\overline{q}(t) \quad \Rightarrow \quad \alpha[t,\beta_{mx}\overline{q}(t)]=\beta_{mx}\overline{q}(t)\,/\,\overline{p}(t)=\beta_{mx}r(t) \qquad (4.45)$$

are the α values such that the curve $\alpha[\tau_l(p),p]\overline{p}(\tau)$ passes through the point $[\tau_l(p),p]$ and the curve $\alpha[t,\beta_{mx}\overline{q}(t)]\overline{p}(\tau)$ passes through the point $[t,\beta_{mx}\overline{q}(t)]$, and (iii) $\alpha_{mn}(t)$ is defined the same as in Eq. (4.20). In turn, the sets

$$\mathcal{A}_3 = \left\{\alpha : \alpha_{mn}(t) \le \alpha \le \alpha[t,\beta_{mx}\overline{q}(t)]\right\}, \qquad (4.46)$$

$$\mathcal{A}_2 = \left\{\alpha : \alpha[t,\beta_{mx}\overline{q}(t)] \le \alpha \le \alpha[\tau_l(p),p]\right\} \qquad (4.47)$$

and

$$\mathcal{A}_1 = \left\{\alpha : \alpha[\tau_l(p),p] \le \alpha \le \alpha_{mx}(p)\right\} \qquad (4.48)$$

contain the α values for the curves $\alpha\bar{p}(\tau)$ crossing lines $\mathcal{L}_3$, $\mathcal{L}_2$ and $\mathcal{L}_1$, respectively.
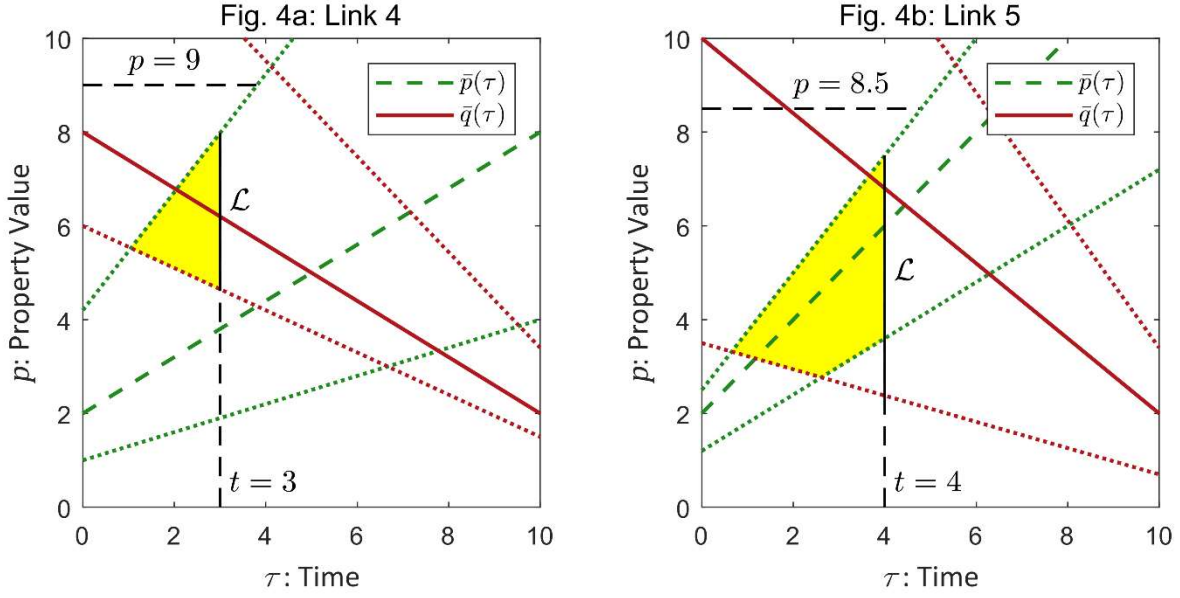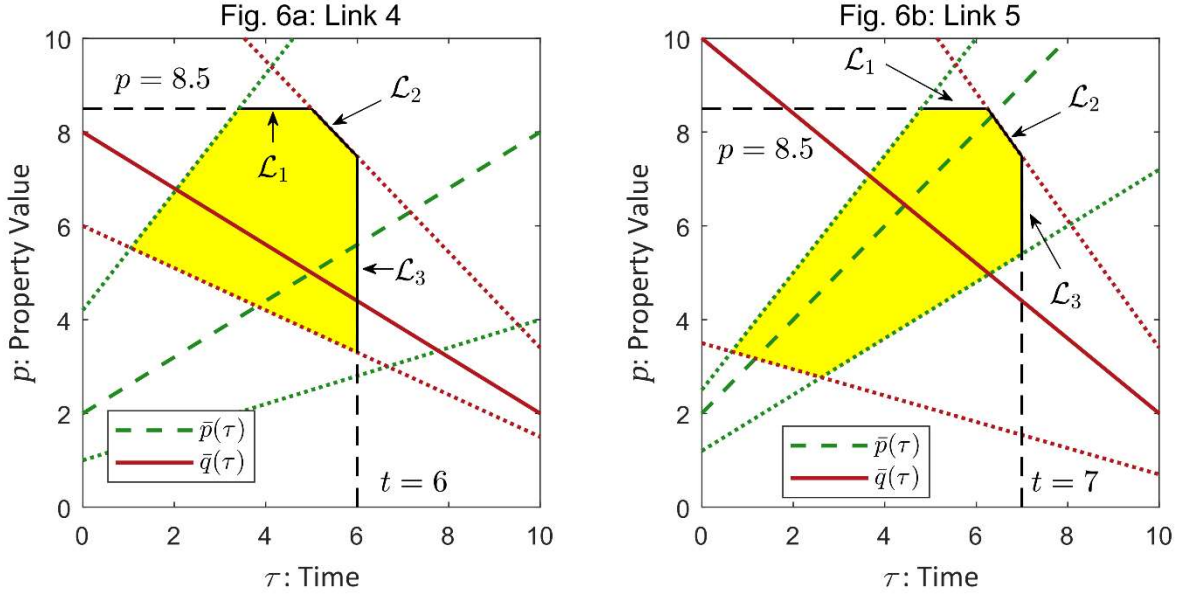

Fig. 6 Illustration of regions (i.e., colored areas) integrated over to obtain $CDF_P(p,[t_{mn},t])$ for Configuration 3 defined in Eq. (4.18): (a) Link 4 with $t_{mn} = 0$, $p = 8.5$, and $t = 6$, and (b) Link 5 with $t_{mn} = 0$, $p = 8.5$, and $t = 7$.

For the following, a subdivision

$$\alpha_{mn}(t) = \alpha_0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n = \alpha_{mx}(p) \tag{4.49}$$

of

$$[\alpha_{mn}(t), \alpha_{mx}(p)] = \left[\alpha_{mn}(t), \alpha[t, \beta_{mx}\bar{q}(t)]\right] \cup \left[\alpha[t, \beta_{mx}\bar{q}(t)], \alpha[\tau_l(p), p]\right]$$
$$\cup \left[\alpha[\tau_l(p), p], \alpha_{mx}(p)\right] \tag{4.50}$$
$$= \mathcal{A}_3 \cup \mathcal{A}_2 \cup \mathcal{A}_1$$

is assumed with

$$\alpha_r = \alpha[t, \beta_{mx}\bar{q}(t)] = \beta_{mx}\bar{q}(t) / \bar{p}(t) \text{ and } \alpha_s = \alpha[\tau_l(p), p] = p / \bar{p}[\tau_l(p)]. \tag{4.51}$$

For $\alpha_i \in \mathcal{A}_3$ and similarly to Eqs. (4.23) and (4.24), (i) the value β_i for which $\beta_i\bar{q}(t) = \alpha_i\bar{p}(t)$ is given by

$$\beta_i = \alpha_i\bar{p}(t) / \bar{q}(t) = \alpha_i / r(t) = F(\alpha_i, t), \tag{4.52}$$

and (ii) as a consequence of the monotonicity of $\bar{p}(\tau)$ and $\bar{q}(\tau)$,

$$\text{prob}(\beta \le \beta_i | \alpha_i \in \mathcal{A}_3) = \text{ probability that link fails at } \tilde{p} \le p \text{ conditional on } \alpha_i \in \mathcal{A}_3$$
$$= \int_{\beta_{mn}}^{F(\alpha_i, t)} d_B(\beta) d\beta. \tag{4.53}$$

For $\alpha_i \in \mathcal{A}_2$, (i) $\beta_{mx}\bar{q}(\tau_i) = \alpha_i\bar{p}(\tau_i)$, where τ_i is the time at which $\alpha_i\bar{p}(\tau)$ crosses the curve $\mathcal{L}_2$ and (ii) as a consequence of the monotonicity of $\bar{p}(\tau)$ and $\bar{q}(\tau)$,

$$\text{prob}(\beta \le \beta_{mx} | \alpha_i \in \mathcal{A}_2) = \int_{\beta_{mn}}^{\beta_{mx}} d_B(\beta) d\beta$$
$$= \text{ probability that link fails at } \tilde{p} \le p \text{ conditional on } \alpha_i \in \mathcal{A}_2 \tag{4.54}$$
$$= 1.0.$$

For $\alpha_i \in \mathcal{A}_1$ and similarly to Eqs. (4.35) and (4.36), (i) the time τ_i at which $\alpha_i\bar{p}(\tau_i) = p$ is given by $\tau_i = \bar{p}^{-1}(p/\alpha_i)$, (ii) the value β_i for which $\beta_i\bar{q}(\tau_i) = p$ is given by

$$\beta_i = p / \bar{q}(\tau_i) = p / \bar{q}[\bar{p}^{-1}(p/\alpha_i)] = G(\alpha_i, p), \tag{4.55}$$

and (iii) as a consequence of the monotonicity of $\bar{p}(\tau)$ and $\bar{q}(\tau)$,

$$\text{prob}(\beta \le \beta_i | \alpha_i \in \mathcal{A}_1) = \text{ probability that link fails at } \tilde{p} \le p \text{ conditional on } \alpha_i \in \mathcal{A}_1$$
$$= \int_{\beta_{mn}}^{G(\alpha_i, p)} d_B(\beta) d\beta. \tag{4.56}$$

In turn, given the assumption $\tau_l(p) < t \le \tau_{mx}(p)$ and the results in Eqs. (4.53), (4.54) and (4.56),

$$CDF_P(p \mid [t_{mn}, t]) = CDF_P(p \mid [\tau_{mn}(p), t]) \text{ for } \tau_l(p) < t \leq \tau_l$$

$$= \lim_{n \to \infty} \left\{ \sum_{i=1}^{r} \left[\int_{\beta_{mn}}^{F(\alpha_i, t)} d_B(\beta) d\beta \right] d_A(\alpha_i) \Delta \alpha_i + \sum_{i=r+1}^{s} \left[\int_{\beta_{mn}}^{\beta_{mn}} d_B(\beta) d\beta \right] d_A(\alpha_i) \Delta \alpha_i \right.$$

$$\left. + \sum_{i=s+1}^{n} \left[\int_{\beta_{mn}}^{G(\alpha_i, p)} d_B(\beta) d\beta \right] d_A(\alpha_i) \Delta \alpha_i \right\}$$

$$= \int_{\alpha_{mn}(t)}^{\alpha[t, \beta_{mx}\bar{q}(t)]} \left[\int_{\beta_{mn}}^{F(\alpha, t)} d_B(\beta) d\beta \right] d_A(\alpha) d\alpha + \int_{\alpha[t, \beta_{mx}\bar{q}(t)]}^{\alpha[\tau_l(p), p]} d_A(\alpha) d\alpha \qquad (4.57)$$

$$+ \int_{\alpha[\tau_l(p), p]}^{\alpha_{mx}(p)} \left[\int_{B_{mn}}^{G(\alpha, p)} d_B(\beta) d\beta \right] d_A(\alpha) d\alpha$$

$$= \int_{\alpha_{mn}(t)}^{\beta_{mx}r(t)} \left[\int_{\beta_{mn}}^{\alpha/r(t)} d_B(\beta) d\beta \right] d_A(\alpha) d\alpha + \int_{\beta_{mx}r(t)}^{p/\bar{p}[\tau_l(p)]} d_A(\alpha) d\alpha$$

$$+ \int_{p/\bar{p}[\tau_l(p)]}^{\alpha_{mx}(p)} \left[\int_{\beta_{mn}}^{p/\bar{q}[\bar{p}^{-1}(p/\alpha)]} d_B(\beta) d\beta \right] d_A(\alpha) d\alpha$$

with $\alpha_{mn}(t)$, $\alpha_{mx}(p)$, $\alpha[\tau_l(p), p] = p / \bar{p}[\tau_l(p)]$, $\tau_l(p) = \bar{q}^{-1}(p / \beta_{mx})$, $\alpha[t, \beta_{mx}\bar{q}(t)] = \beta_{mx}r(t)$, $F(\alpha, t) = \alpha / r(t)$ and $G(\alpha, p) = p / \bar{q}[\bar{p}^{-1}(p / \alpha)]$ defined in Eqs. (4.20), (4.28), (4.44), (4.12), (4.45), (4.52) and (4.55), respectively.

Further, the preceding representation for $CDF_P(p, [t_{mn}, t])$ simplifies to

$$CDF_P(p \mid [t_{mn}, t]) = \int_{\alpha_{mn}}^{p/\bar{p}[\tau_l(p)]} d_A(\alpha) d\alpha + \int_{p/\bar{p}[\tau_l(p)]}^{\alpha_{mx}(p)} \left[\int_{\beta_{mn}}^{p/\bar{q}[\bar{p}^{-1}(p/\alpha)]} d_B(\beta) d\beta \right] d_A(\alpha) d\alpha \quad (4.58)$$

for $t = \tau_l$ as a result of (i) the equalities

$$\alpha_{mn}(t) = \alpha_{mn} = \beta_{mx}r(t) \text{ for } t = \tau_l = r^{-1}\left(\alpha_{mn} / \beta_{mx}\right) \qquad (4.59)$$

as discussed in conjunction with Eqs. (4.37)-(4.40) and (ii) the consequent equality

$$\int_{\alpha_{mn}(t)}^{\beta_{mx}r(t)} \left[\int_{\beta_{mn}}^{\alpha/r(t)} d_B(\beta) d\beta \right] d_A(\alpha) d\alpha = \int_{\alpha_{mn}}^{\alpha_{mn}} \left[\int_{\beta_{mn}}^{\alpha/r(t)} d_B(\beta) d\beta \right] d_A(\alpha) d\alpha = 0. \qquad (4.60)$$

4.5 Summary and Illustration of CDFs for Link Property at Time of Link Failure for Case 1

The integral representations for $CDF_P(p, [t_{mn}, t])$ derived in Sects. 4.2-4.4 are summarized in Table 2. Further, the CDFs $CDF_P(p, [t_{mn}, \tau_l])$ for the three links defined in Table 1 and illustrated in Fig. 1 are shown in Fig. 7 . The CDFs in Fig. 7 are obtained by numerically evaluating the integrals in the following representations for $CDF_P(p, [t_{mn}, \tau_l])$:

$$CDF_P(p \mid [t_{mn}, \tau_l]) = CDF_P(p \mid [t_{mn}, \tau_l(p)])$$

$$= \int_{\alpha_{mn}}^{\alpha_{mx}(p)} \left[\int_{\beta_{mn}}^{p/\overline{q}[\overline{p}^{-1}(p/\alpha)]} d_B(\beta) \mathrm{d}\beta \right] d_A(\alpha) \mathrm{d}\alpha \qquad (4.61)$$

for Configuration 2 in Table 2 and $p_{mn} < p \le p_l$, and

$$CDF_P(p \mid [t_{mn}, \tau_l]) = \int_{\alpha_{mn}}^{p/\overline{p}[\tau_l(p)]} d_A(\alpha) \mathrm{d}\alpha + \int_{p/\overline{p}[\tau_l(p)]}^{\alpha_{mx}(p)} \left[\int_{\beta_{mn}}^{p/\overline{q}[\overline{p}^{-1}(p/\alpha)]} d_B(\beta) \mathrm{d}\beta \right] d_A(\alpha) \mathrm{d}\alpha \qquad (4.62)$$

for Configuration 3 in Table 2 and $p_l \le p \le p_{mx}$. Numerical evaluation of the preceding integrals was performed with the MATLAB software package [52; 54] with (i) the **fit** function with the spline option used to fit property value functions and inverse functions and (ii) the **TwoD** function [55] used for integration.

Table 2 Summary of integral representations for $CDF_P(p,[t_{mn},t])$ derived in Sects. 4.2-4.4 for Configurations 1-3 defined in Eqs. (4.16)-(4.18).

Selected Notation

$$\alpha_{mn}(t) = \begin{cases} \beta_{mn}\overline{q}(t)/\overline{p}(t) = \beta_{mn}r(t) & \text{for } t < \tau_{mn} \\ \alpha_{mn} & \text{for } \tau_{mn} \le t \end{cases}$$

$$\alpha_{mx}(p) = \begin{cases} p/\overline{p}[\overline{p}^{-1}(p/\alpha_{mx})] = \alpha_{mx} & \text{for } p_f \le p \le p_{mx} \\ p/\overline{p}[\overline{q}^{-1}(p/\beta_{mn})] & \text{for } p_{mn} \le p \le p_f \end{cases}$$

Configuration 1: $p_f \le p \le p_{mx}$ with $\tau_f < t \le \tau_f(p)$

$$CDF_P(p\,|\,[t_{mn},t]) = CDF_P(p\,|\,[\tau_f,t]) \text{ for } \tau_f < t \le \tau_f(p)$$

$$= \int_{\alpha_{mn}(t)}^{\alpha_{mx}} \left[\int_{\beta_{mn}}^{\alpha/r(t)} d_B(\beta)\mathrm{d}\beta \right] d_A(\alpha)\mathrm{d}\alpha$$

Configuration 2: $p_{mn} \le p \le p_{mx}$ with $\tau_f(p) < t \le \tau_l(p)$

$$CDF_P(p\,|\,[t_{mn},t]) = CDF_P(p\,|\,[\tau_{mn}(p),t]) \text{ for } \tau_f(p) < t \le \tau_l(p)$$

$$= \int_{\alpha_{mn}(t)}^{p/\overline{p}(t)} \left[\int_{\beta_{mn}}^{\alpha/r(t)} d_B(\beta)\mathrm{d}\beta \right] d_A(\alpha)\mathrm{d}\alpha + \int_{p/\overline{p}(t)}^{\alpha_{mx}(p)} \left[\int_{\beta_{mn}}^{p/\overline{q}[\overline{p}^{-1}(p/\alpha)]} d_B(\beta)\mathrm{d}\beta \right] d_A(\alpha)\mathrm{d}\alpha$$

$$= \int_{\alpha_{mn}}^{\alpha_{mx}(p)} \left[\int_{\beta_{mn}}^{p/\overline{q}[\overline{p}^{-1}(p/\alpha)]} d_B(\beta)\mathrm{d}\beta \right] d_A(\alpha)\mathrm{d}\alpha \text{ for } t = \tau_l(p) \text{ and } p_{mn} < p \le p_l$$

Configuration 3: $p_l \le p \le p_{mx}$ with $\tau_l(p) < t \le \tau_l$

$$CDF_P(p\,|\,[t_{mn},t]) = CDF_P(p\,|\,[\tau_{mn}(p),t]) \text{ for } \tau_l(p) < t \le \tau_l$$

$$= \int_{\alpha_{mn}(t)}^{\beta_{mx}r(t)} \left[\int_{\beta_{mn}}^{\alpha/r(t)} d_B(\beta)\mathrm{d}\beta \right] d_A(\alpha)\mathrm{d}\alpha + \int_{\beta_{mx}r(t)}^{p/\overline{p}[\tau_l(p)]} d_A(\alpha)\mathrm{d}\alpha$$

$$+ \int_{p/\overline{p}[\tau_l(p)]}^{\alpha_{mx}(p)} \left[\int_{\beta_{mn}}^{p/\overline{q}[\overline{p}^{-1}(p/\alpha)]} d_B(\beta)\mathrm{d}\beta \right] d_A(\alpha)\mathrm{d}\alpha$$

$$= \int_{\alpha_{mn}}^{p/\overline{p}[\tau_l(p)]} d_A(\alpha)\mathrm{d}\alpha + \int_{p/\overline{p}[\tau_l(p)]}^{\alpha_{mx}(p)} \left[\int_{\beta_{mn}}^{p/\overline{q}[\overline{p}^{-1}(p/\alpha)]} d_B(\beta)\mathrm{d}\beta \right] d_A(\alpha)\mathrm{d}\alpha \text{ for } t = \tau_l$$
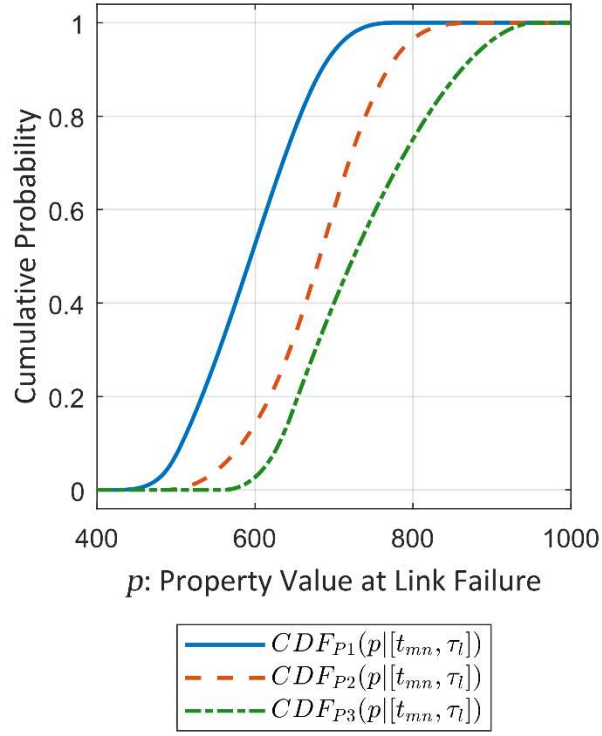
Fig. 7 Property value at link failure CDFs (i.e., $CDF_{Pi}(p \,|\, [t_{mn}, t]) = CDF_{Pi}(p \,|\, [t_{mn}, \tau_l])$ for Link i, $i = 1, 2, 3$) determined over all possible times of link failure (i.e., for time $t = \tau_l$ defined by $\alpha_{mn} \overline{p}(t) = \beta_{mx} \overline{q}(t)$ in Eq. (4.5)) obtained as indicated in Table 2 for Links 1, 2 and 3 described and illustrated in Table 1 and Fig. 1.

5. CDFs for Link Property at Time of Link Failure for Case 2: $\bar{p}(t)$ Increasing and $\bar{q}(t)$ Constant-Valued

The special, but important, case with $\bar{q}(\tau)$ equal a constant k (i.e., $\bar{q}(\tau) = k$) is now considered. Examples of links with this property are defined in Table 3 and shown in Fig. 8.

Table 3 Defining properties of Links 6, 7 and 8 used to illustrate the calculation of the probability that a link fails at a property value less than or equal to p at a time prior or equal to t for $\bar{q}(\tau)$ constant-valued.

General Properties for Links 6, 7 and 8

$$\bar{p}(\tau) = \frac{\bar{p}(\infty)\bar{p}(0)}{\bar{p}(0) + [\bar{p}(\infty) - \bar{p}(0)]\exp(-r_1\tau)}, \quad \bar{q}(\tau) \text{ constant valued}$$

Properties of Link 6

$\bar{p}(\infty) = 1400, \bar{p}(0) = 225, r_1 = 0.06, \bar{q}(\tau) = 725$
$d_A(\alpha)$ triangular on $[\alpha_{mn}, \alpha_{mx}] = [0.77, 1.15]$ with mode 1.0
$d_B(\beta)$ unifrom on $[\beta_{mn}, \beta_{mx}] = [0.8, 1.35]$

Properties of Link 7

$\bar{p}(\infty) = 1600, \bar{p}(0) = 200, r_1 = 0.09, \bar{q}(\tau) = 700$
$d_A(\alpha)$ triangular on $[\alpha_{mn}, \alpha_{mx}] = [0.6, 1.7]$ with mode 1.0
$d_B(\beta)$ triangular on $[\beta_{mn}, \beta_{mx}] = [0.65, 1.25]$ with mode 1.0

Properties of Link 8

$\bar{p}(\infty) = 2000, \bar{p}(0) = 200, r_1 = 0.05, \bar{q}(\tau) = 550$
$d_A(\alpha)$ uniform on $[\alpha_{mn}, \alpha_{mx}] = [0.65, 1.4]$
$d_B(\beta)$ triangular on $[\beta_{mn}, \beta_{mx}] = [0.65, 1.4]$ with mode 1.0
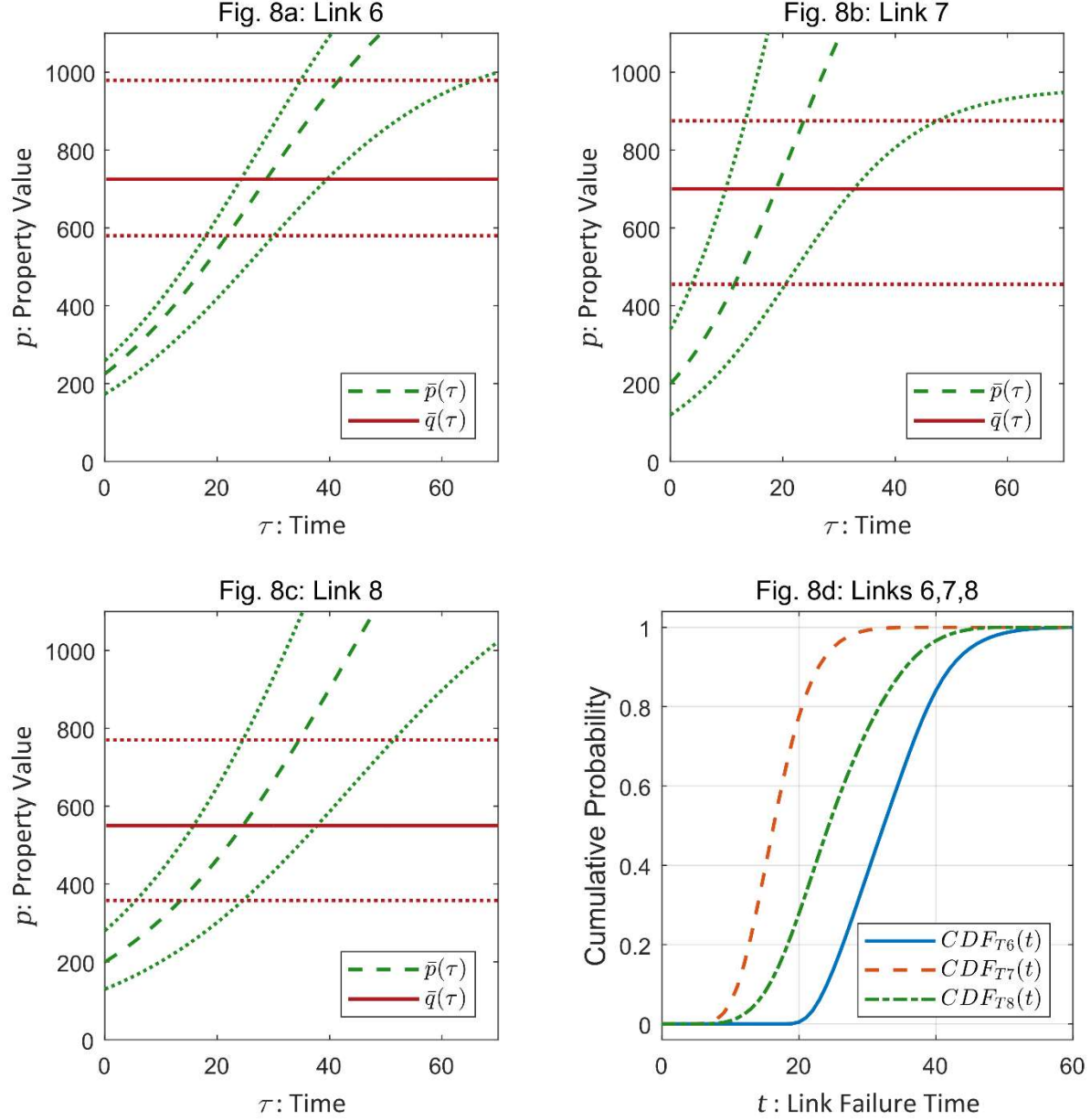
Fig. 8 Summary plots for Links 6, 7 and 8 defined in Table 3 and used to illustrate the calculation of the probability that a link fails at a property value less than or equal to p at a time prior or equal to t with $\overline{q}(\tau)$ constant-valued (i.e., $\overline{q}(\tau) = k$): (a) Link 6 properties, (b) Link 7 properties, (c) Link 8 properties, and (d) link failure time CDFs (i.e., $CDF_{Ti}(t)$ defined in Eqs. (3.4) and (3.5) for Link i) for Links 6, 7 and 8.

As before, the function

$$r(\tau) = \overline{q}(\tau) / \overline{p}(\tau) = k / \overline{p}(\tau) \tag{5.1}$$

is a decreasing function and thus has an inverse r^{-1}. As indicated in Fig. 8, (i) the lower boundary

of possible link failure times has constant value of $\beta_{mn}\overline{q}(\tau)=k\beta_{mn}$ over the time interval

$$\left[r^{-1}\left(\alpha_{mx}/\beta_{mn}\right),r^{-1}\left(\alpha_{mn}/\beta_{mn}\right)\right]=\left[\tau_f,\tau_{mn}\right]=\left[\underline{\tau}_{mn},\overline{\tau}_{mn}\right] \tag{5.2}$$

and (i) the upper boundary of possible link failure times has constant value of $\beta_{mx}\overline{q}(\tau)=k\beta_{mx}$ over the time interval

$$\left[r^{-1}\left(\alpha_{mx}/\beta_{mx}\right),r^{-1}\left(\alpha_{mn}/\beta_{mx}\right)\right]=\left[\tau_{mx},\tau_l\right]=\left[\underline{\tau}_{mx},\overline{\tau}_{mx}\right]. \tag{5.3}$$

The times $\tau_f,\tau_{mn},\tau_{mx},\tau_l$ as defined above have the same definitions with r^{-1} as in Sect. 4.1. However, as examination of Fig. 8 shows, they correspond to slightly different properties of a link when $\overline{q}(t)=k$. Specifically with $\overline{q}(t)=k$, (i) $\tau_f=\underline{\tau}_{mn}$ corresponds to both the first possible time for link failure and the first time at which the smallest possible value for link failure could occur, (ii) $\tau_{mn}=\overline{\tau}_{mn}$ corresponds to the last time at which the smallest possible value for link failure could occur, (iii) $\tau_{mx}=\underline{\tau}_{mx}$ corresponds to the first possible time at which the largest value for link failure could occur, and (iv) $\tau_l=\overline{\tau}_{mx}$ corresponds to both the last possible time at which the largest value for link failure could occur and the last possible time at which any link failure could occur. The changed notations

$$\tau_f=\underline{\tau}_{mn},\ \tau_{mn}=\overline{\tau}_{mn},\ \tau_{mx}=\underline{\tau}_{mx},\ \tau_l=\overline{\tau}_{mx} \tag{5.4}$$

in Eqs. (5.2) and (5.3) are introduced to provide an indication of what the times $\tau_f,\tau_{mn},\tau_{mx},\tau_l$ correspond to for a constant-valued $\overline{q}(t)$.

As summarized below, the derivation of $CDF_P(p\,|\,[t_{mn},t])$ for $\overline{q}(t)=k$ is similar to the derivations of $CDF_P(p\,|\,[t_{mn},t])$ in Sects. 4.2 and 4.3 for link configurations 1 and 2 defined in Eqs. (4.16) and (4.17). Specifically, the following two cases with

$$\tau_f(p)=\overline{p}^{-1}(p/\alpha_{mx})\ \text{ and }\ \tau_l(p)=\overline{p}^{-1}(p/\alpha_{mn}) \tag{5.5}$$

as defined in Eqs. (4.11) and (4.12) need to be considered: (i) $\underline{\tau}_{mn}<t\le\tau_f(p)$ and $\tau_f(p)<t\le\tau_l(p)$.

For $\underline{\tau}_{mn}<t\le\tau_f(p)$, a derivation similar to the one in Sect. 4.2 results in the following form for $CDF_P(p\,|\,[t_{mn},t])$:

$$CDF_P(p \,|\, [t_{mn}, t]) = CDF_P(p \,|\, [\underline{\tau}_{mn}, t]) \quad \text{for } \underline{\tau}_{mn} < t \leq \underline{\tau}_{mx}$$

$$= \int_{\alpha_{mn}(t)}^{\alpha_{mx}} \left[\int_{\beta_{mn}}^{F(\alpha,t)} d_B(\beta) d\beta \right] d_A(\alpha) d\alpha \tag{5.6}$$

$$= \int_{\alpha_{mn}(t)}^{\alpha_{mx}} \left[\int_{\beta_{mn}}^{\alpha \overline{p}(t)/k} d_B(\beta) d\beta \right] d_A(\alpha) d\alpha,$$

with

$$F(\alpha, t) = \alpha / r(t) = \alpha \overline{p}(t) / k, \tag{5.7}$$

and

$$\alpha_{mn}(t) = \begin{cases} k\beta_{mn} / \overline{p}(t) \text{ from } \alpha \overline{p}(t) = \beta_{mn} \overline{q}(t) = \beta_{mn} k \text{ for } \underline{\tau}_{mn} < t \leq \overline{\tau}_{mn} \\ \alpha_{mn} \text{ for } \overline{\tau}_{mn} < t \leq \overline{\tau}_{mx}. \end{cases} \tag{5.8}$$

With respect to Eqs. (5.7) and (5.8), (i) the role of $F(\alpha, t)$ is described in Eqs. (4.23)-(4.25) and (ii) $\alpha_{mn}(t)$ is defined to incorporate the effects of the curves

$$[\tau, \beta_{mn} \overline{q}(\tau) = \beta_{mn} k], \underline{\tau}_{mn} < t \leq \overline{\tau}_{mn}, \text{ and } [\tau, \alpha_{mn} \overline{p}(\tau)], \overline{\tau}_{mn} < t \leq \overline{\tau}_{mx}, \tag{5.9}$$

on lower limits of integration for α.

For $\tau_f(p) < t \leq \tau_l(p)$, a derivation similar to the one in Sect. 4.3 results in the following form for $CDF_P(p \,|\, [t_{mn}, t])$:

$$CDF_P(p \,|\, [t_{mn}, t]) = CDF_P(p \,|\, [\tau_{mn}(p), t]) \quad \text{for } \tau_f(p) < t \leq \tau_l(p)$$

$$= \int_{\alpha_{mn}(t)}^{\alpha(t,p)} \left[\int_{\beta_{mn}}^{F(\alpha,t)} d_B(\beta) d\beta \right] d_A(\alpha) d\alpha + \int_{\alpha(t,p)}^{\alpha_{mx}} \left[\int_{\beta_{mn}}^{G(\alpha,p)} d_B(\beta) d\beta \right] d_A(\alpha) d\alpha \tag{5.10}$$

$$= \int_{\alpha_{mn}(t)}^{p/\overline{p}(t)} \left[\int_{\beta_{mn}}^{\alpha \overline{p}(t)/k} d_B(\beta) d\beta \right] d_A(\alpha) d\alpha + \int_{p/\overline{p}(t)}^{\alpha_{mx}} \left[\int_{\beta_{mn}}^{p/k} d_B(\beta) d\beta \right] d_A(\alpha) d\alpha$$

with (i) $F(\alpha, t)$ and $\alpha_{mn}(t)$ defined in Eqs. (5.7) and (5.8), (ii) $\alpha(t, p) = p / \overline{p}(t)$ defined in Eq. (4.29), and (iii) $G(\alpha, p) = p / \overline{q}[\overline{p}^{-1}(p/\alpha)]$ defined in Eq. (4.35)-(4.36). Further, the preceding representation for $CDF_P(p \,|\, [t_{mn}, t])$ simplifies to

$$CDF_P(p \,|\, [t_{mn}, t]) = \int_{\alpha_{mn}}^{\alpha_{mx}} \left[\int_{\beta_{mn}}^{p/k} d_B(\beta) d\beta \right] d_A(\alpha) d\alpha = \int_{\beta_{mn}}^{p/k} d_B(\beta) d\beta \tag{5.11}$$

for $t = \tau_l(p)$ as indicated in conjunction with Eqs. (4.38) and (4.39). A further simplification to

$$CDF_P(p \,|\, [t_{mn}, t]) = [p / k - \beta_{mn}] / [\beta_{mx} - \beta_{mn}] \tag{5.12}$$

results for $t = \tau_l(p)$ and β uniform on $[\beta_{mn}, \beta_{mx}]$.

As an example, the results of evaluating $CDF_P(p \,|\, [t_{mn}, t])$ for Links 4, 5 and 6 defined in Table 3 with the integral representation in Eq. (5.11) are shown in Fig. 9. The integrals defining the CDFs in Fig. 9 were evaluated with the same numerical procedures used to evaluate the integrals defining the CDFs in Fig. 7.
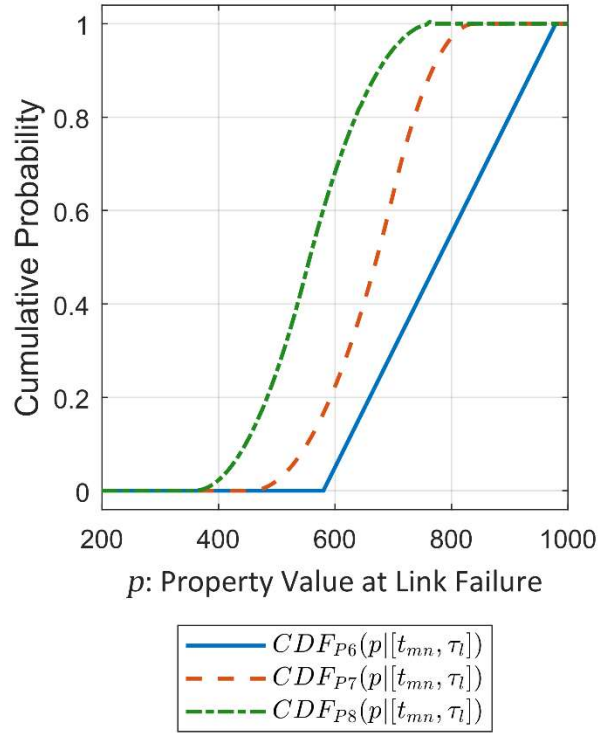
Fig. 9 Property value at link failure CDFs (i.e., $CDF_{Pi}(p \,|\, [t_{mn}, t]) = CDF_{Pi}(p \,|\, [t_{mn}, \tau_l])$ for Link i, i = 6, 7, 8) determined over all possible times of link failure (i.e., for time $t = \tau_l$ defined by $\alpha_{mn}\overline{p}(t) = \beta_{mx}\overline{q}(t)$ in Eq. (4.5)) obtained as indicated in Eq. (5.11) for Links 6, 7 and 8 described and illustrated in Table 3 and Fig. 8.

6. Sampling-based Procedure to Estimate $CDF_P(p\,|\,[t_{mn},t])$

Another possibility is to use a sampling-based (i.e., Monte Carlo) procedure to estimate $CDF_P(p\,|\,[t_{mn},t])$. This approach uses (i) a sample of the form $\mathbf{s}_i = [\alpha_i, \beta_i]$, $i = 1, 2, \cdots, n$, of the form indicated in Eq. (3.7), (ii) the corresponding link failure times τ_i defined in Eqs. (3.8)-(3.11), and (iii) the link failure values $p_i = \alpha_i \bar{p}(\tau_i)$. Then,

$$CDF_P(p\,|\,[t_{mn},t]) \cong \sum_{i=1}^{n} \delta_p(p_i\,|\,t)/n \ \text{ with } \ \delta_p(p_i\,|\,t) = \begin{cases} 1 & \text{for } p_i \leq p \ \text{ and } \ \tau_i \leq t \\ 0 & \text{otherwise.} \end{cases} \tag{6.1}$$

As an example, CDFs for link property value at link failure obtained with use of samples of size $nS = 10^6$ as indicated in Eq. (6.1) are illustrated in Fig. 10.
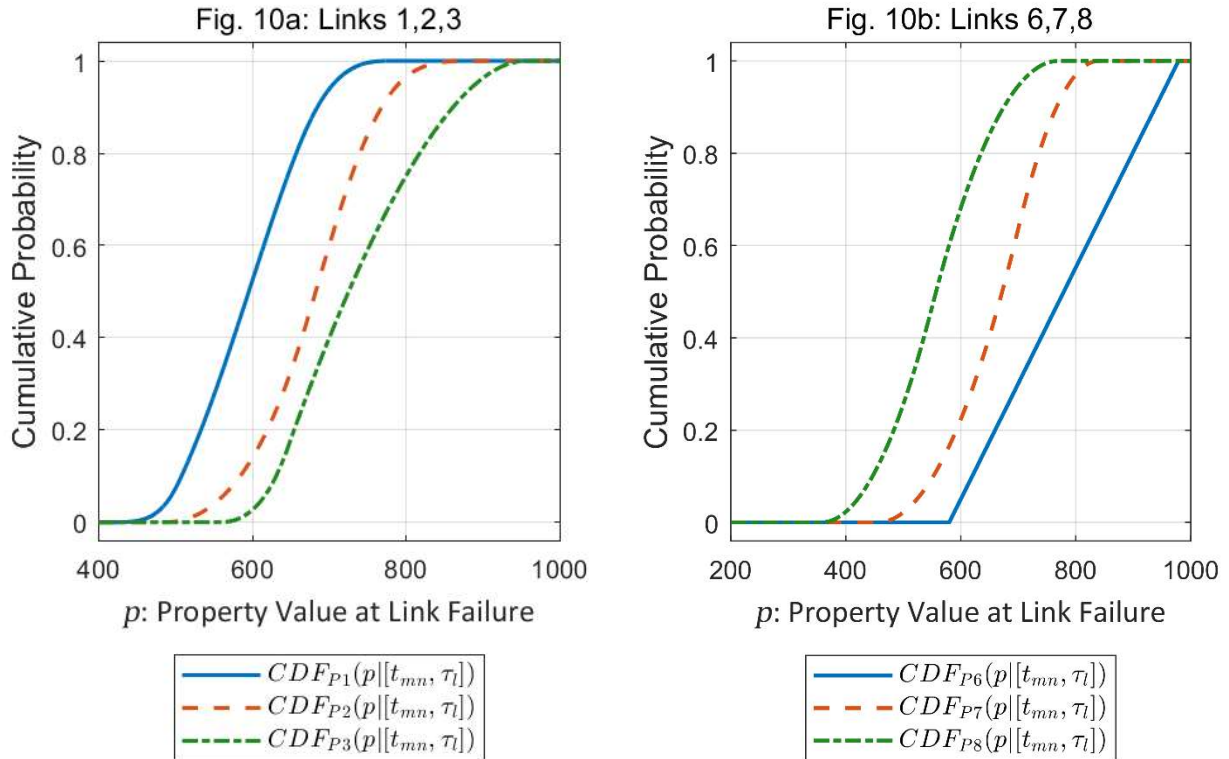
Fig. 10 Property value at link failure CDFs (i.e., $CDF_{Pi}(p\,|\,[t_{mn},t]) = CDF_{Pi}(p\,|\,[t_{mn},\tau_l])$) for Link i, $i = 1, 2, 3, 6, 7, 8$) determined over all possible times of link failure (i.e., for time $t = \tau_l$ defined by $\alpha_{mn}\bar{p}(t) = \beta_{mx}\bar{q}(t)$ in Eq. (4.5)) obtained with use of samples of size $nS = 10^6$ as indicated in Eq. (6.1) for (a) Links 1, 2 and 3 described and illustrated in Table 1 and Fig. 1 and (b) Links 6, 7 and 8 described and illustrated in Table 3 and Fig. 8.

7. Verification and the Estimation of $CDF_P(p\,|\,[t_{mn},t])$

As discussed and illustrated in Refs. [49; 50] for WL/SL systems and in Refs. [56-65] for many additional contexts, model/analysis verification based on the comparison of results obtained in two independent analyses is an important part of the assessment of models and software used in the analysis of high consequence systems. Model verification and model validation are two related, but different and often confused, concepts. Two widely used definitions are (Ref. [65], p. 3):

Verification: The process of determining that a model implementation accurately represents the developers' conceptual description of the model and the solution of the model.

Validation: The process of determining the degree to which a model is an accurate representation of the real world from the perspective of the intended uses of the model.

Thus, verification relates to assessing the correctness of the mathematical development and implementation of a model. It is in this sense that verification is used in this presentation. Further, verification is interpreted broadly enough to include a checking of the correctness of the formal mathematical derivation of a model. In contrast, validation relates to assessing the degree to which a model represents the actual behavior of the processes under consideration. In general, validation involves the comparison of model predictions with experimental results. Such comparisons are not part of this presentation.

The explicit integral-based representations for $CDF_P(p\,|\,[t_{mn},t])$ derived in Sects. 4.2.2-4.2.3 are summarized in Table 2. These integrals can be estimated with quadrature procedures. Due to the changing forms of the integrals for different values of p and t, this can be a complex and inefficient way to estimate the CDF for link failure values that occur prior to time t (i.e., the CDF defined by $(p, CDF_P(p\,|\,[t_{mn},t]))$ for all link failure values p that occur prior to time t).

However, the integral representations for $CDF_P(p\,|\,[t_{mn},t])$ in Table 2 do have a useful role to play in analysis verification. Specifically, for selected values of p and t, the corresponding integral from Table 2 can be evaluated and used to verify that the sampling-based procedure in Eq. (6.1) is providing a correct and reasonably accurate approximation to $CDF_P(p\,|\,[t_{mn},t])$. As is the case here, the existence of two independent procedures to calculate a specific analysis result is a significant verification capability.

As an example, estimates for $CDF_P(p\,|\,[t_{mn},t])$ are presented in Fig. 11 for the links in Fig. 1 and Fig. 8 obtained by (i) numerical evaluation of the integrals in Eqs. (4.61), (4.62) and (5.11) and (ii) sampling-based evaluation as indicated in Eq. (6.1). The overlay of the CDFs obtained with the two procedures provides a strong indication that both procedures are correctly derived and implemented. This is a particularly strong verification result as the two procedures differ in both conceptual (i.e., mathematical) basis and computational implementation.
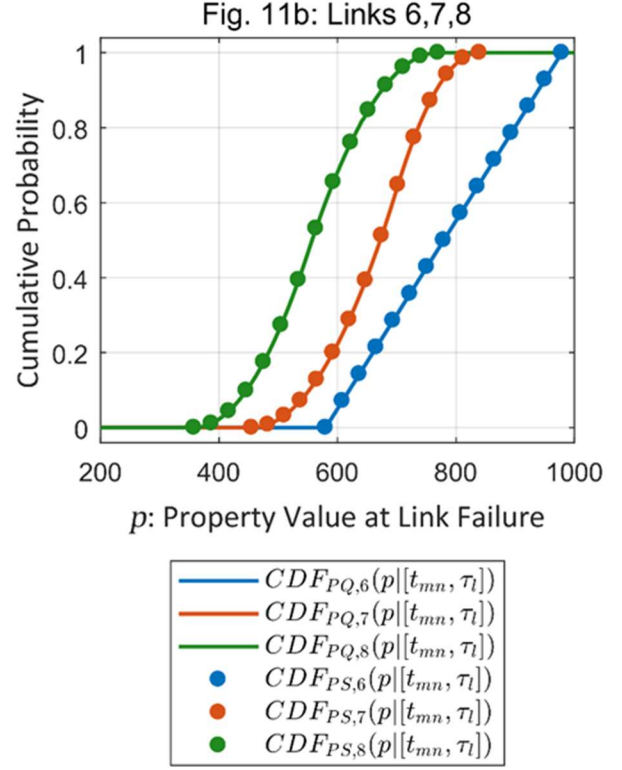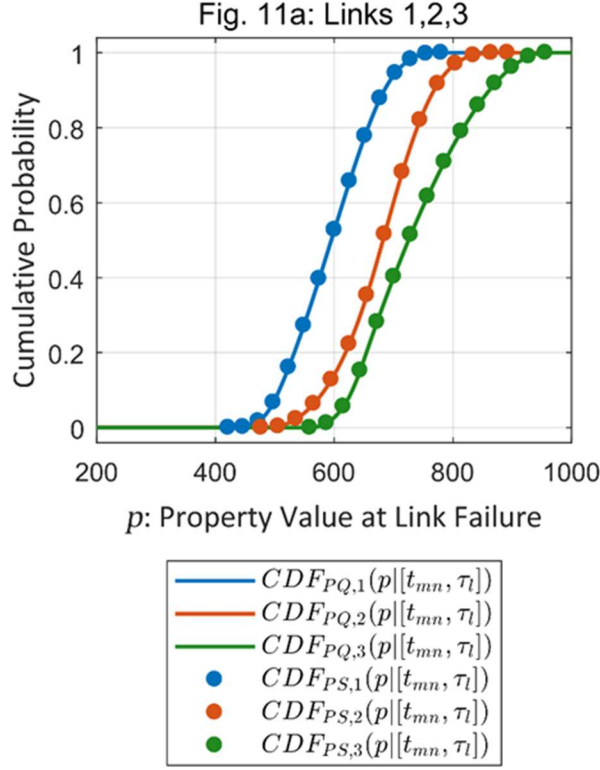
Fig. 11 Comparisons of quadrature-based evaluations $CDF_{PQ,i}(p\,|\,[t_{mn}, \tau_l])$ and sampling-based evaluations $CDF_{PS,i}(p\,|\,[t_{mn}, \tau_l])$ for $CDF_{Pi}(p\,|\,[t_{mn}, t]) = CDF_{Pi}(p\,|\,[t_{mn}, \tau_l])$ for Link i, i = 1, 2, 3, 6, 7, 8, in Figs. 7, 9 and 10: (a) Comparisons for Links 1, 2 and 3, and (b) Comparisons for Links 6, 7 and 8.

8. Integral Representation of $CDF_P(p\,|\,[t_{mn},t])$ Based on Time τ and Failure Value p

An alternate form for the definition of $CDF_P(p\,|\,[t_{mn},t])$ is

$$
\begin{aligned}
CDF_P(p\,|\,[t_{mn},t]) &= \int_{t_{mn}}^{t} CDF_P(p\,|\,\tau)\mathrm{d}CDF_T(\tau) \\
&= \int_{t_{mn}}^{t} CDF_P(p\,|\,\tau)d_T(\tau)\mathrm{d}\tau,
\end{aligned}
\tag{8.1}
$$

where (i) $CDF_P(p\,|\,\tau)$ is the CDF for link property value p at link failure conditional on link failure occurring at time τ, (ii) $CDF_T(\tau)$ is the CDF defined in Eq. (3.4) for link failure time, and (iii) $d_T(\tau)$ is the density function for link failure time defined by

$$
\begin{aligned}
d_T(\tau) &= \mathrm{d}CDF_T(\tau)/\mathrm{d}\tau \\
&= \frac{\mathrm{d}}{\mathrm{d}\tau}\left\{\int_{\alpha_{mn}}^{\alpha_{mx}}\left[\int_{\beta_{mn}}^{F(\alpha,\tau)} d_B(\beta)\mathrm{d}\beta\right]d_A(\alpha)\mathrm{d}\alpha\right\} \\
&\qquad \text{with } F(\alpha,\tau) = \alpha\overline{p}(\tau)/\overline{q}(\tau) = \alpha/r(\tau) \text{ and } r(\tau) = \overline{q}(\tau)/\overline{p}(\tau) \\
&= \int_{\alpha_{mn}}^{\alpha_{mx}}\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\int_{\beta_{mn}}^{F(\alpha,\tau)} d_B(\beta)\mathrm{d}\beta\right]d_A(\alpha)\mathrm{d}\alpha \\
&= \int_{\alpha_{mn}}^{\alpha_{mx}} d_B[F(\alpha,\tau)][\mathrm{d}F(\alpha,\tau)/\mathrm{d}\tau]d_A(\alpha)\mathrm{d}\alpha \\
&= \int_{\alpha_{mn}}^{\alpha_{mx}} d_B[\alpha/r(\tau)]\left[\alpha\frac{\mathrm{d}}{\mathrm{d}\tau}[1/r(\tau)]\right]d_A(\alpha)\mathrm{d}\alpha \\
&= \left[\mathrm{d}[1/r(\tau)]/\mathrm{d}\tau\right]\int_{\alpha_{mn}}^{\alpha_{mx}} d_B[\alpha/r(\tau)]\alpha d_A(\alpha)\mathrm{d}\alpha \\
&= \left[\mathrm{d}[\overline{p}(\tau)/\overline{q}(\tau)]/\mathrm{d}\tau\right]\int_{\alpha_{mn}}^{\alpha_{mx}} d_B[\alpha/r(\tau)]\alpha d_A(\alpha)\mathrm{d}\alpha.
\end{aligned}
\tag{8.2}
$$

The derivation of $CDF_P(p\,|\,\tau)$ is rather complex and is presented in Sect. 9.5. Then, in Sect. 11, it is shown that the representation for $CDF_P(p\,|\,[t_{mn},t])$ in Eq. (8.1) is mathematically equivalent to the representations for $CDF_P(p\,|\,[t_{mn},t])$ summarized in Table 2. The equivalent outcomes of two different derivations for $CDF_P(p\,|\,[t_{mn},t])$ provide an additional verification result indicating that $CDF_P(p\,|\,[t_{mn},t])$ has been derived correctly.

9. Distribution for Link Property Conditional on Time of Link Failure

9.1 Preliminaries: Distribution for Link Property Conditional on Time of Link Failure

For systems of WLs and SLs, the distributions for link property values conditional on time of link failure play an important role in the formal representation of system properties such as (i) distributions of link property values at time of link system failure for WL systems and SL systems, (ii) distributions of WL link property values and SL property values at the time that LOAS occurs for a WL/SL system, (iii) distributions of margins for a WL/SL system defined by the difference of SL property value at time of SL system failure and WL property value at the time of WL system failure, (iv) distributions of margins for a WL/SL system defined by the difference of SL property value at time of SL system failure and SL property value at the time of WL system failure, and (v) delays in link failure time that are functions of link property value at the time of precursor link failure. The indicated system properties are developed in two following reports [46; 47]. \

This section considers a single link (i.e., a WL or a SL) with properties as described in Sect. 2. Derivations follow for

$$d_P(p \mid \tau) = \text{density function for link property } p \text{ conditional on link failure}$$
$$\text{occurring at time } \tau \tag{9.1}$$

and the interval of definition $[p_{mn}(\tau), p_{mx}(\tau)]$ for $d_P(p \mid \tau)$. Further, values for a number of additional closely related quantities are also obtained.

9.2 Sample Space $[p_{mn}(\tau), p_{mx}(\tau)]$ Associated with Density Function $d_P(p \mid \tau)$

Values for $d_P(p \mid \tau)$ and its interval of definition $[p_{mn}(\tau), p_{mx}(\tau)]$ are determined by (i) the positive-valued density functions $d_A(\alpha)$ and $d_B(\beta)$ defined on intervals $[\alpha_{mn}, \alpha_{mx}]$ and $[\beta_{mn}, \beta_{mx}]$ indicated in Sect. 2 and (ii) the ratio $r(\tau) = \overline{q}(\tau) / \overline{p}(\tau)$ for the functions $\overline{p}(\tau)$ and $\overline{q}(\tau)$ defined in Sect. 2 (see Eqs. (2.1)-(2.2)). In set notation, the interval $[p_{mn}(\tau), p_{mx}(\tau)]$ is defined by

$$[p_{mn}(\tau), p_{mx}(\tau)] = \mathcal{S}(p \mid \tau)$$
$$= \{p : p = \alpha \overline{p}(\tau) = \beta \overline{q}(\tau), \alpha \in [\alpha_{mn}, \alpha_{mx}], \beta \in [\beta_{mn}, \beta_{mx}]\}. \tag{9.2}$$

In turn, membership in the set $\mathcal{S}(p \mid \tau)$, which is the sample space for p, is determined by the values for α and β that satisfy the equalities

$$\alpha = [\overline{q}(\tau) / \overline{p}(\tau)]\beta = r(\tau)\beta. \tag{9.3}$$

As illustrated in Fig. 12, the preceding linear relationship between α and β for a fixed value for τ leads to four possibilities for the definition of $\mathcal{S}(p \mid \tau)$. In correspondence with Line 1 in Fig. 12, the first possibility is

$$
\begin{aligned}
\mathcal{S}_1(p \mid \tau \in \mathcal{P}_1) &= \left\{ p : p = \alpha \overline{p}(\tau) \text{ for } \alpha \in \mathcal{A}_1(\alpha \mid \tau \in \mathcal{P}_1) = \left\{ \alpha : r(\tau)\beta_{mn} \le \alpha \le \alpha_{mx} \right\} \right\} \\
&= \left[r(\tau)\beta_{mn}\overline{p}(\tau), \alpha_{mx}\overline{p}(\tau) \right] \\
&= \left[\beta_{mn}\overline{q}(\tau), \alpha_{mx}\overline{p}(\tau) \right]
\end{aligned}
\tag{9.4}
$$

with

$$
\mathcal{P}_1 = \left\{ \tau : \alpha_{mn} \le r(\tau)\beta_{mn} \le \alpha_{mx} \le r(\tau)\beta_{mx} \right\}.
\tag{9.5}
$$

Specifically, if $\tau \in \mathcal{P}_1$, then (i) the possible values for α fall in the interval $[r(\tau)\beta_{mn}, \alpha_{mx}]$, (ii) the corresponding interval of values for β consistent with the equality $\alpha = r(\tau)\beta$ in Eq. (9.3) is $[\beta_{mn}, \alpha_{mx} / r(\tau)]$, and (iii) the resultant interval of values for p is $[\beta_{mn}\overline{q}(\tau), \alpha_{mx}\overline{p}(\tau)]$.

Structure: $\alpha\overline{p}(\tau) = \beta\overline{q}(\tau) \Rightarrow \alpha/\beta = \overline{q}(\tau)/\overline{p}(\tau) = r \Rightarrow \alpha = r\beta$
Line 1: $\beta_{mn} \le \beta \le \alpha_{mx}/r_1$ for $\alpha_{mn} \le r_1\beta_{mn} \le \alpha_{mx} \le r_1\beta_{mx}$
Line 2: $\beta_{mn} \le \beta \le \beta_{mx}$ for $\alpha_{mn} \le r_2\beta_{mn} \le r_2\beta_{mx} \le \alpha_{mx}$
Line 3: $\alpha_{mn}/r_3 \le \beta \le \alpha_{mx}/r_3$ for $r_3\beta_{mn} \le \alpha_{mn} \le \alpha_{mx} \le r_3\beta_{mx}$
Line 4: $\alpha_{mn}/r_4 \le \beta \le \beta_{mx}$ for $r_4\beta_{mn} \le \alpha_{mn} \le r_4\beta_{mx} \le \alpha_{mx}$
No line in box for $\alpha = r\beta$ if $\alpha_{mx} \le r\beta_{mn}$ or $r\beta_{mx} \le \alpha_{mn}$
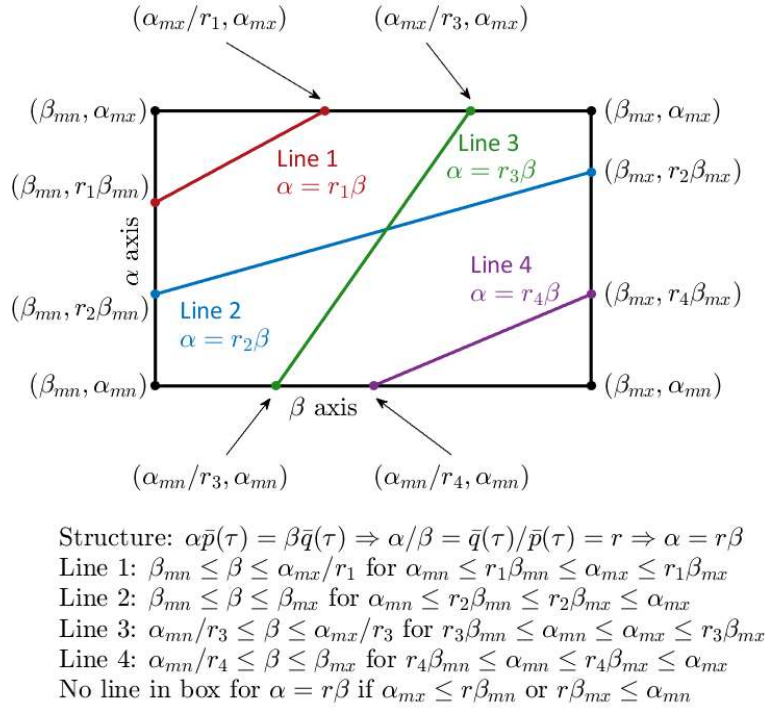
Fig. 12 Possible relations between α and β for $\alpha_{mn} \le \alpha \le \alpha_{mx}$, $\beta_{mn} \le \beta \le \beta_{mx}$ and $\alpha = [\overline{q}(\tau) / \overline{p}(\tau)]\beta = r(\tau)\beta$.

Similarly, $\mathcal{S}_2(p \mid \tau \in \mathcal{P}_2)$, $\mathcal{S}_3(p \mid \tau \in \mathcal{P}_3)$ and $\mathcal{S}_4(p \mid \tau \in \mathcal{P}_4)$ are defined by

$$S_2(p \mid \tau \in \mathcal{P}_2) = \left\{ p : p = \alpha \bar{p}(\tau) \text{ for } \alpha \in \mathcal{A}_2(\alpha \mid \tau \in \mathcal{P}_2) = \left\{ \alpha : r(\tau)\beta_{mn} \leq \alpha \leq r(\tau)\beta_{mx} \right\} \right\}$$
$$= \left[\beta_{mn}\bar{q}(\tau), \beta_{mx}\bar{q}(\tau) \right] \tag{9.6}$$

with $\mathcal{P}_2 = \{\tau : \alpha_{mn} \leq r(\tau)\beta_{mn} \leq r(\tau)\beta_{mx} \leq \alpha_{mx}\}$ in correspondence with Line 2 in Fig. 12;

$$S_3(p \mid \tau \in \mathcal{P}_3) = \left\{ p : p = \alpha \bar{p}(\tau) \text{ for } \alpha \in \mathcal{A}_3(\alpha \mid \tau \in \mathcal{P}_3) = \left\{ \alpha : \alpha_{mn} \leq \alpha \leq \alpha_{mx} \right\} \right\}$$
$$= \left[\alpha_{mn}\bar{p}(\tau), \alpha_{mx}\bar{p}(\tau) \right] \tag{9.7}$$

with $\mathcal{P}_3 = \{\tau : r(\tau)\beta_{mn} \leq \alpha_{mn} \leq \alpha_{mx} \leq r(\tau)\beta_{mx}\}$ in correspondence with Line 3 in Fig. 12; and

$$S_4(p \mid \tau \in \mathcal{P}_4) = \left\{ p : p = \alpha \bar{p}(\tau) \text{ for } \alpha \in \mathcal{A}_4(\alpha \mid \tau \in \mathcal{P}_4) = \left\{ \alpha : \alpha_{mn} \leq \alpha \leq r(\tau)\beta_{mx} \right\} \right\}$$
$$= \left[\alpha_{mn}\bar{p}(\tau), \beta_{mx}\bar{q}(\tau) \right] \tag{9.8}$$

with $\mathcal{P}_4 = \{\tau : r(\tau)\beta_{mn} \leq \alpha_{mn} \leq r(\tau)\beta_{mx} \leq \alpha_{mx}\}$ in correspondence with Line 4 in Fig. 12. If either of the inequalities $\alpha_{mx} < r(\tau)\beta_{mn}$ or $r(\tau)\beta_{mx} < \alpha_{mn}$ holds, then $S(p \mid \tau)$ is the null set.

9.3 Exact Nature of and Relationships between Sets $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ and $\mathcal{P}_4$

The exact nature of and relationships between the sets

$$\mathcal{P}_1 = \left\{ \tau : \alpha_{mn} \leq r(\tau)\beta_{mn} \leq \alpha_{mx} \leq r(\tau)\beta_{mx} \right\} \tag{9.9}$$

$$\mathcal{P}_2 = \left\{ \tau : \alpha_{mn} \leq r(\tau)\beta_{mn} \leq r(\tau)\beta_{mx} \leq \alpha_{mx} \right\} \tag{9.10}$$

$$\mathcal{P}_3 = \left\{ \tau : r(\tau)\beta_{mn} \leq \alpha_{mn} \leq \alpha_{mx} \leq r(\tau)\beta_{mx} \right\} \tag{9.11}$$

and

$$\mathcal{P}_4 = \left\{ \tau : r(\tau)\beta_{mn} \leq \alpha_{mn} \leq r(\tau)\beta_{mx} \leq \alpha_{mx} \right\} \tag{9.12}$$

is not immediately apparent from the preceding definitions. However, consideration of the cases

$$\text{Case 1: } \alpha_{mn} / \beta_{mn} < \alpha_{mx} / \beta_{mx}, \tag{9.13}$$

$$\text{Case 2: } \alpha_{mx} / \beta_{mx} < \alpha_{mn} / \beta_{mn} \tag{9.14}$$

and

$$\text{Case 3: } \alpha_{mn} / \beta_{mn} = \alpha_{mx} / \beta_{mx} \tag{9.15}$$

provides both (i) simple definitions for the sets $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ and $\mathcal{P}_4$ and (ii) a clear description of the relationships between these sets.

For Case 1 (i.e., $\alpha_{mn} / \beta_{mn} < \alpha_{mx} / \beta_{mx}$), the definitions for $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ and $\mathcal{P}_4$ become

$$
\begin{aligned}
\mathcal{P}_1 &= \left\{ \tau : \alpha_{mn} \leq r(\tau)\beta_{mn} \leq \alpha_{mx} \leq r(\tau)\beta_{mx} \right\} \\
&= \left\{ \tau : \alpha_{mn} / \beta_{mn} \leq r(\tau), r(\tau) \leq \alpha_{mx} / \beta_{mn}, \alpha_{mx} / \beta_{mx} \leq r(\tau) \right\} \\
&= \left\{ \tau : \max \left\{ \alpha_{mn} / \beta_{mn}, \alpha_{mx} / \beta_{mx} \right\} \leq r(\tau), r(\tau) \leq \alpha_{mx} / \beta_{mn} \right\} \\
&= \left\{ \tau : \alpha_{mx} / \beta_{mx} \leq r(\tau), r(\tau) \leq \alpha_{mx} / \beta_{mn} \right\} \\
&= \left\{ \tau : \alpha_{mx} / \beta_{mx} \leq r(\tau) \leq \alpha_{mx} / \beta_{mn} \right\} \\
&= \left\{ \tau : r^{-1} \left(\alpha_{mx} / \beta_{mn} \right) \leq \tau \leq r^{-1} \left(\alpha_{mx} / \beta_{mx} \right) \right\},
\end{aligned}
\tag{9.16}
$$

$$
\begin{aligned}
\mathcal{P}_2 &= \left\{ \tau : \alpha_{mn} \leq r(\tau)\beta_{mn} \leq r(\tau)\beta_{mx} \leq \alpha_{mx} \right\} \\
&= \left\{ \tau : \alpha_{mn} / \beta_{mn} \leq r(\tau), r(\tau) \leq \alpha_{mx} / \beta_{mx} \right\} \\
&= \left\{ \tau : \alpha_{mn} / \beta_{mn} \leq r(\tau) \leq \alpha_{mx} / \beta_{mx} \right\} \\
&= \left\{ \tau : r^{-1} \left(\alpha_{mx} / \beta_{mx} \right) \leq \tau \leq r^{-1} \left(\alpha_{mn} / \beta_{mn} \right) \right\},
\end{aligned}
\tag{9.17}
$$

$$
\begin{aligned}
\mathcal{P}_3 &= \left\{ \tau : r(\tau)\beta_{mn} \leq \alpha_{mn} \leq \alpha_{mx} \leq r(\tau)\beta_{mx} \right\} \\
&= \left\{ \tau : r(\tau) \leq \alpha_{mn} / \beta_{mn}, \alpha_{mx} / \beta_{mx} \leq r(\tau) \right\} \\
&= \left\{ \tau : \alpha_{mx} / \beta_{mx} \leq r(\tau) \leq \alpha_{mn} / \beta_{mn} \right\} \\
&= \varnothing,
\end{aligned}
\tag{9.18}
$$

and

$$
\begin{aligned}
\mathcal{P}_4 &= \left\{ \tau : r(\tau)\beta_{mn} \leq \alpha_{mn} \leq r(\tau)\beta_{mx} \leq \alpha_{mx} \right\} \\
&= \left\{ \tau : r(\tau) \leq \alpha_{mn} / \beta_{mn}, \alpha_{mn} / \beta_{mx} \leq r(\tau), r(\tau) \leq \alpha_{mx} / \beta_{mx} \right\} \\
&= \left\{ \tau : \alpha_{mn} / \beta_{mx} \leq r(\tau), r(\tau) \leq \min \left\{ \alpha_{mn} / \beta_{mn}, \alpha_{mx} / \beta_{mx} \right\} \right\} \\
&= \left\{ \tau : \alpha_{mn} / \beta_{mx} \leq r(\tau), r(\tau) \leq \alpha_{mn} / \beta_{mn} \right\} \\
&= \left\{ \tau : \alpha_{mn} / \beta_{mx} \leq r(\tau) \leq \alpha_{mn} / \beta_{mn} \right\} \\
&= \left\{ \tau : r^{-1} \left[\alpha_{mn} / \beta_{mn} \right] \leq r(\tau) \leq r^{-1} \left[\alpha_{mn} / \beta_{mx} \right] \right\}.
\end{aligned}
\tag{9.19}
$$

In turn, the inequalities

$$
\alpha_{mn} / \beta_{mx} \leq r(\tau_4) \leq \alpha_{mn} / \beta_{mn} \leq r(\tau_2) \leq \alpha_{mx} / \beta_{mx} \leq r(\tau_1) \leq \alpha_{mx} / \beta_{mn}
\tag{9.20}
$$

and

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$$\underbrace{r^{-1}\left(\alpha_{mx}/\beta_{mn}\right)}_{\tau_f} \leq \overset{\in \mathcal{P}_1}{\tau_1} \leq \underbrace{r^{-1}\left(\alpha_{mx}/\beta_{mx}\right)}_{\tau_{mx}} \leq \overset{\in \mathcal{P}_2}{\tau_2} \leq \underbrace{r^{-1}\left(\alpha_{mn}/\beta_{mn}\right)}_{\tau_{mn}} \leq \overset{\in \mathcal{P}_4}{\tau_4} \leq \underbrace{r^{-1}\left(\alpha_{mn}/\beta_{mx}\right)}_{\tau_l} \quad (9.21)$$

with τ_i indicating membership in $\mathcal{P}_i$ for $i = 1, 2$ and 4 provide summaries for Case 1 (i.e., $\alpha_{mn}/\beta_{mn} < \alpha_{mx}/\beta_{mx}$) of (i) the definitions of the nonnull sets $\mathcal{P}_1, \mathcal{P}_2$ and $\mathcal{P}_4$ and (ii) the relationships between these sets.

The relationships formally summarized in Eqs. (9.20) and (9.21) are illustrated in Fig. 1a,b for a notional link defined in Table 1 with properties consistent with Case 1 (i.e., for $\alpha_{mn}/\beta_{mn} < \alpha_{mx}/\beta_{mx}$). Specifically, Fig. 1b shows the relationships between the function $r(\tau) = \overline{q}(\tau)/\overline{p}(\tau)$ and the times τ_f, τ_{mx}, τ_{mn} and τ_l defined in Eqs. (4.3)-(4.9). Further, the indicated times and their relationships to link failure properties are illustrated in Fig. 1a,b. The sets $\mathcal{P}_1, \mathcal{P}_2$ and $\mathcal{P}_4$ correspond to the intervals $[\tau_f, \tau_{mx}]$, $[\tau_{mx}, \tau_{mn}]$ and $[\tau_{mn}, \tau_l]$ on the abscissas in Fig. 1a,b.

For Case 2 (i.e., $\alpha_{mx}/\beta_{mx} < \alpha_{mn}/\beta_{mn}$), derivations analogous to those shown in Eqs. (9.16)-(9.19) for Case 1 establish the following forms for $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ and $\mathcal{P}_4$:

$$\begin{aligned}
\mathcal{P}_1 &= \{\tau : \alpha_{mn}/\beta_{mn} \leq r(\tau) \leq \alpha_{mx}/\beta_{mn}\} \\
&= \{\tau : r^{-1}\left(\alpha_{mx}/\beta_{mn}\right) \leq r(\tau) \leq r^{-1}\left(\alpha_{mn}/\beta_{mn}\right)\},
\end{aligned} \quad (9.22)$$

$$\mathcal{P}_2 = \{\tau : \alpha_{mn}/\beta_{mn} \leq r(\tau) \leq \alpha_{mx}/\beta_{mx}\} = \varnothing, \quad (9.23)$$

$$\begin{aligned}
\mathcal{P}_3 &= \{\tau : \alpha_{mx}/\beta_{mx} \leq r(\tau) \leq \alpha_{mn}/\beta_{mn}\} \\
&= \{\tau : r^{-1}\left(\alpha_{mn}/\beta_{mn}\right) \leq \tau \leq r^{-1}\left(\alpha_{mx}/\beta_{mx}\right)\},
\end{aligned} \quad (9.24)$$

and

$$\begin{aligned}
\mathcal{P}_4 &= \{\tau : \alpha_{mn}/\beta_{mx} \leq r(\tau) \leq \alpha_{mx}/\beta_{mx}\} \\
&= \{\tau : r^{-1}\left[\alpha_{mx}/\beta_{mx}\right] \leq \tau \leq r^{-1}\left[\alpha_{mn}/\beta_{mx}\right]\}.
\end{aligned} \quad (9.25)$$

In turn, the inequalities

$$\alpha_{mn}/\beta_{mx} \leq r(\tau_4) \leq \alpha_{mx}/\beta_{mx} \leq r(\tau_3) \leq \alpha_{mn}/\beta_{mn} \leq r(\tau_1) \leq \alpha_{mx}/\beta_{mn} \quad (9.26)$$

and

$$\underbrace{r^{-1}\left(\alpha_{mx}/\beta_{mn}\right)}_{\tau_f}\leq\overset{\in\mathcal{P}_1}{\overset{\frown}{\tau_1}}\leq\underbrace{r^{-1}\left(\alpha_{mn}/\beta_{mn}\right)}_{\tau_{mn}}\leq\overset{\in\mathcal{P}_3}{\overset{\frown}{\tau_3}}\leq\underbrace{r^{-1}\left(\alpha_{mx}/\beta_{mx}\right)}_{\tau_{mx}}\leq\overset{\in\mathcal{P}_4}{\overset{\frown}{\tau_4}}\leq\underbrace{r^{-1}\left(\alpha_{mn}/\beta_{mx}\right)}_{\tau_l} \quad (9.27)$$

with τ_i indicating membership in $\mathcal{P}_i$ for i = 1, 3 and 4 provide summaries for Case 2 (i.e., $\alpha_{mx}/\beta_{mx} < \alpha_{mn}/\beta_{mn}$) of (i) the definitions of the nonnull sets $\mathcal{P}_1, \mathcal{P}_3$ and $\mathcal{P}_4$ and (ii) the relationships between these sets.

The relationships formally summarized in Eqs. (9.26) and (9.27) are illustrated in Fig. 1c,d for a notional link defined in Table 1 with properties consistent with Case 2 (i.e., for $\alpha_{mx}/\beta_{mx} < \alpha_{mn}/\beta_{mn}$). Specifically, Fig. 1d shows the relationships between the function $r(\tau) = \overline{q}(\tau)/\overline{p}(\tau)$ and the times $\tau_f, \tau_{mx}, \tau_{mn}$ and τ_l. Further, the indicated times and their relationships to link failure properties are illustrated in Fig. 1c,d. The sets $\mathcal{P}_1, \mathcal{P}_3$ and $\mathcal{P}_4$ correspond to the intervals $[\tau_f, \tau_{mn}]$, $[\tau_{mn}, \tau_{mx}]$ and $[\tau_{mx}, \tau_l]$ on the abscissas in Fig. 1c,d. Although τ_f, τ_{mx}, τ_{mn} and τ_l are defined the same for Case 1 and Case 2, their ordering in time is different (i.e., $\tau_f < \tau_{mx} < \tau_{mn} < \tau_f$ for Case 1 and $\tau_f < \tau_{mn} < \tau_{mx} < \tau_f$ for Case 2).

For Case 3 (i.e., $\alpha_{mn}/\beta_{mn} = \alpha_{mx}/\beta_{mx}$), derivations analogous to those shown in Eqs. (9.16)-(9.19) for Case 1 establish the following forms for $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ and $\mathcal{P}_4$:

$$\begin{aligned} \mathcal{P}_1 &= \{\tau : \alpha_{mn}/\beta_{mn} = \alpha_{mx}/\beta_{mx} \leq r(\tau) \leq \alpha_{mx}/\beta_{mn}\} \\ &= \{\tau : r^{-1}\left(\alpha_{mx}/\beta_{mn}\right) \leq \tau \leq r^{-1}\left(\alpha_{mx}/\beta_{mx}\right)\}, \end{aligned} \quad (9.28)$$

$$\begin{aligned} \mathcal{P}_2 = \mathcal{P}_3 &= \{\tau : \alpha_{mx}/\beta_{mx} \leq r(\tau) \leq \alpha_{mx}/\beta_{mx}\} \\ &= \{\tau : \tau = r^{-1}\left(\alpha_{mx}/\beta_{mx}\right)\}, \end{aligned} \quad (9.29)$$

and

$$\begin{aligned} \mathcal{P}_4 &= \{\tau : \alpha_{mn}/\beta_{mx} \leq r(\tau) \leq \alpha_{mx}/\beta_{mx}\} \\ &= \{\tau : r^{-1}\left[\alpha_{mx}/\beta_{mx}\right] \leq \tau \leq r^{-1}\left[\alpha_{mn}/\beta_{mx}\right]\}. \end{aligned} \quad (9.30)$$

In turn, the inequalities

$$\alpha_{mn}/\beta_{mx} \leq r(\tau_4) \leq \alpha_{mx}/\beta_{mx} \leq r(\tau_1) \leq \alpha_{mx}/\beta_{mn} \quad (9.31)$$

and

$$\underbrace{r^{-1}\left(\alpha_{mx}/\beta_{mn}\right)}_{\tau_f} \le \overset{\in \mathcal{P}_1}{\tau_1} \le \underbrace{r^{-1}\left(\alpha_{mx}/\beta_{mx}\right)}_{\tau_{mx}} \le \overset{\in \mathcal{P}_4}{\tau_4} \le \underbrace{r^{-1}\left(\alpha_{mn}/\beta_{mx}\right)}_{\tau_l} \tag{9.32}$$

with τ_i indicating membership in $\mathcal{P}_i$ for $i = 1$ and 4 provide summaries for Case 3 (i.e., $\alpha_{mn}/\beta_{mn} = \alpha_{mx}/\beta_{mx}$) of (i) the definitions of the nondegenerate sets $\mathcal{P}_1$ and $\mathcal{P}_4$ and (ii) the relationships between these sets. Similarly to the examples for Cases 1 and 2 in Fig. 1a-d, the relationships formally summarized in Eqs. (9.31) and (9.32) for Case 3 are illustrated in Fig. 1e,f with $\mathcal{P}_1$ and $\mathcal{P}_4$ corresponding to the intervals $[\tau_f, \tau_{mn} = \tau_{mx}]$ and $[\tau_{mn} = \tau_{mx}, \tau_l]$ on the abscissas in Fig. 1e, f.

Definitions of the sets $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ and $\mathcal{P}_4$ initially defined in Eqs. (9.9)-(9.12) for Cases 1, 2, 3 defined in Eqs. (9.13)-(9.15) are summarized in Table 4. The role of $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ and $\mathcal{P}_4$ is to identify the intervals of definition for $[p_{mn}(\tau), p_{mx}(\tau)]$ with

$$[p_{mn}(\tau), p_{mx}(\tau)] = \begin{cases} \left[\beta_{mn}\overline{q}(\tau), \alpha_{mx}\overline{p}(\tau)\right] & \text{for } \tau \in \mathcal{P}_1 \\ \left[\beta_{mn}\overline{q}(\tau), \beta_{mx}\overline{q}(\tau)\right] & \text{for } \tau \in \mathcal{P}_2 \\ \left[\alpha_{mn}\overline{p}(\tau), \alpha_{mx}\overline{p}(\tau)\right] & \text{for } \tau \in \mathcal{P}_3 \\ \left[\alpha_{mn}\overline{p}(\tau), \beta_{mx}\overline{q}(\tau)\right] & \text{for } \tau \in \mathcal{P}_4 \end{cases} \tag{9.33}$$

as summarized in Eqs. (9.4)-(9.8) and illustrated in Fig. 1 and Fig. 8.

9.4 Density Function for α Conditional on Link Failure at Time τ

The density function $d_{Pi}(p \mid \tau \in \mathcal{P}_i)$ for p defined on $\mathcal{S}_i(p \mid \tau \in \mathcal{P}_i)$ for $i = 1, 2, 3, 4$ can be obtained from the corresponding density function $d_{Ai}(\alpha \mid \tau \in \mathcal{P}_i)$ for α, where

$$\begin{aligned} \mathcal{A}_i(\alpha \mid \tau \in \mathcal{P}_i) &= \left[\alpha_{mn}(\tau), \alpha_{mx}(\tau)\right] \text{ for } \tau \in \mathcal{P}_i \\ &= \begin{cases} \left[r(\tau)\beta_{mn}, \alpha_{mx}\right] & \text{for } \tau \in \mathcal{P}_1 \text{ (see Eq. (9.4))} \\ \left[r(\tau)\beta_{mn}, r(\tau)\beta_{mx}\right] & \text{for } \tau \in \mathcal{P}_2 \text{ (see Eq. (9.6))} \\ \left[\alpha_{mn}, \alpha_{mx}\right] & \text{for } \tau \in \mathcal{P}_3 \text{ (see Eq. (9.7))} \\ \left[\alpha_{mn}, r(\tau)\beta_{mx}\right] & \text{for } \tau \in \mathcal{P}_4 \text{ (see Eq. (9.8))} \end{cases} \end{aligned} \tag{9.34}$$

is the corresponding sample space for α. Specifically, given that p is defined by $p(\alpha) = \alpha\overline{p}(\tau)$, the density function $d_{Pi}(p \mid \tau \in \mathcal{P}_i)$ for p defined on $\mathcal{S}_i(p \mid \tau \in \mathcal{P}_i)$ for $i = 1, 2, 3, 4$ is given by

$$d_{Pi}(p \mid \tau \in \mathcal{P}_i) = \left(1/\overline{p}(\tau)\right) d_{Ai}\left(p/\overline{p}(\tau) \mid \tau \in \mathcal{P}_i\right) \tag{9.35}$$

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through an application of the relationship

$$d_U(u = cx) = (1/c)d_X(u/c) \tag{9.36}$$

for (i) a constant c > 0, (ii) $d_X(x)$ the density function for x on $[x_{mn}, x_{mx}]$, and (iii) $d_U(u)$ the density function for u on $[cx_{mn}, cx_{mx}]$ ([66], Table 7.1, p. 381).

Table 4 Definition of sets $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ and $\mathcal{P}_4$ initially defined in Eqs. (9.9)-(9.12) for Cases 1, 2, 3 defined in Eqs. (9.13)-(9.15) with (i) τ_f = first possible time for link failure, (ii) τ_{mx} = time of maximum possible property value at link failure, (iii) τ_{mn} = time of minimum possible property value at link failure, and (iv) τ_l = last possible time for link failure.

Case 1: $\alpha_{mn} / \beta_{mn} < \alpha_{mx} / \beta_{mx}$

$$\mathcal{P}_1 = [\tau_f, \tau_{mx}] = \left\{ \tau : r^{-1}\left(\alpha_{mx} / \beta_{mn}\right) \leq \tau \leq r^{-1}\left(\alpha_{mx} / \beta_{mx}\right) \right\}$$

$$\mathcal{P}_2 = [\tau_{mx}, \tau_{mn}] = \left\{ \tau : r^{-1}\left(\alpha_{mx} / \beta_{mx}\right) \leq \tau \leq r^{-1}\left(\alpha_{mn} / \beta_{mn}\right) \right\}$$

$$\mathcal{P}_3 = \varnothing$$

$$\mathcal{P}_4 = [\tau_{mn}, \tau_l] = \left\{ \tau : r^{-1}\left(\alpha_{mn} / \beta_{mn}\right) \leq r(\tau) \leq r^{-1}\left(\alpha_{mn} / \beta_{mx}\right) \right\}$$

Case 2: $\alpha_{mx} / \beta_{mx} < \alpha_{mn} / \beta_{mn}$

$$\mathcal{P}_1 = [\tau_f, \tau_{mn}] = \left\{ \tau : r^{-1}\left(\alpha_{mx} / \beta_{mn}\right) \leq r(\tau) \leq r^{-1}\left(\alpha_{mn} / \beta_{mn}\right) \right\}$$

$$\mathcal{P}_2 = \varnothing$$

$$\mathcal{P}_3 = [\tau_{mn}, \tau_{mx}] = \left\{ \tau : r^{-1}\left(\alpha_{mn} / \beta_{mn}\right) \leq \tau \leq r^{-1}\left(\alpha_{mx} / \beta_{mx}\right) \right\}$$

$$\mathcal{P}_4 = [\tau_{mx}, \tau_l] = \left\{ \tau : r^{-1}\left[\alpha_{mx} / \beta_{mx}\right] \leq \tau \leq r^{-1}\left[\alpha_{mn} / \beta_{mx}\right] \right\}$$

Case 3: $\alpha_{mn} / \beta_{mn} = \alpha_{mx} / \beta_{mx}$

$$\mathcal{P}_1 = [\tau_f, \tau_{mn} = \tau_{mx}] = \left\{ \tau : r^{-1}\left(\alpha_{mx} / \beta_{mn}\right) \leq \tau \leq r^{-1}\left(\alpha_{mx} / \beta_{mx}\right) \right\}$$

$$\mathcal{P}_2 = \mathcal{P}_3 = r^{-1}\left(\alpha_{mx} / \beta_{mx}\right) = \left\{ \tau : \tau = r^{-1}\left(\alpha_{mx} / \beta_{mx}\right) \right\}$$

$$\mathcal{P}_4 = [\tau_{mn} = \tau_{mx}, \tau_l] = \left\{ \tau : r^{-1}\left(\alpha_{mx} / \beta_{mx}\right) \leq \tau \leq r^{-1}\left(\alpha_{mn} / \beta_{mx}\right) \right\}$$

As indicated in Eq. (9.35), the determination of $d_{Pi}(p \mid \tau \in \mathcal{P}_i)$ is straight forward provided the density function $d_{Ai}(\alpha \mid \tau \in \mathcal{P}_i)$ for α defined on $\mathcal{A}_i(\alpha \mid \tau \in \mathcal{P}_i)$ can be determined. The determination of $d_{Ai}(\alpha \mid \tau \in \mathcal{P}_i)$ is now addressed. There is a certain level of complexity to this determination because the conditionality on the ratio $r = r(\tau) = \overline{q}(\tau) / \overline{p}(\tau)$ that derives from the required equality $\alpha \overline{p}(\tau) = \beta \overline{q}(\tau)$ results in $d_{Ai}(\alpha \mid \tau \in \mathcal{P}_i)$ being dependent on the density functions $d_A(\alpha)$ and $d_B(\beta)$ and their associated intervals of definition $[\alpha_{mn}, \alpha_{mx}]$ and $[\beta_{mn}, \beta_{mx}]$.

The derivation for $d_{Ai}(\alpha \mid \tau \in \mathcal{P}_i)$ starts with a determination of the joint density function $d(u = \alpha / \beta, \alpha)$ for $u = \alpha/\beta$ and α. In general, the joint density function $d(x, y)$ for variables x and y with densities $d_X(x)$ and $d_Y(y)$ is given by ([67], p. 88)

$$d(x, y) = d_X(x \mid y)d_Y(y) = d_X(x)d_Y(y \mid x). \tag{9.37}$$

In consistency with Eq. (9.37), $d(u = \alpha / \beta, \alpha)$ can be represented in two forms:

$$d(u = \alpha / \beta, \alpha) = \begin{cases} d_A(\alpha \mid u = \alpha / \beta)d_U(u = \alpha / \beta) \\ d_A(\alpha)d_U(u = \alpha / \beta \mid \alpha). \end{cases} \tag{9.38}$$

Further, the density function $d_U(u = \alpha / \beta)$ is defined by

$$d_U(u = \alpha / \beta) = \int_{\alpha_{mn}}^{\alpha_{mx}} \left(\alpha / u^2\right) d_A(\alpha) d_B(\alpha / u) \mathrm{d}\alpha \tag{9.39}$$

as stated in ([66], Table 7.2, p. 385), and the density function $d_U(u = \alpha / \beta \mid \alpha)$ is defined by

$$d_U(u = \alpha / \beta \mid \alpha) = (\alpha / u^2)d_B(\alpha / u) \text{ with } \beta(u) = \alpha / u \tag{9.40}$$

through an application of the relationship

$$\begin{aligned} d_U(u) &= \left|\mathrm{d}x(u) / \mathrm{d}u\right| d_X[x(u)] \text{ for } u = u(x) \\ &= \left(c/u^2\right) d_X[c/u] \text{ for } u(x) = c / x, x(u) = c / u, c > 0, \end{aligned} \tag{9.41}$$

where $d_X(x)$ and $d_U(u)$ are the density functions for x and u, respectively ([66], Eq. (2.93a, p. 377). Eq. (9.36) is a special case of the first equality in Eq. (9.41).

Representations for the density function $d_A(\alpha \mid u = \alpha / \beta)$ in Eq. (9.38) are given by

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$$d_A(\alpha \mid u = \alpha/\beta) = d(u = \alpha/\beta, \alpha)/d_U(u = \alpha/\beta)$$
$$= d_A(\alpha)d_U(u = \alpha/\beta \mid \alpha)/d_U(u = \alpha/\beta)$$
$$= \frac{d_A(\alpha)(\alpha/u^2)d_B(\alpha/u)}{\int_{\alpha_{mn}}^{\alpha_{mx}}(\alpha/u^2)d_A(\alpha)d_B(\alpha/u)\,d\alpha} \qquad (9.42)$$
$$= \frac{\alpha d_A(\alpha)d_B(\alpha/u)}{\int_{\alpha_{mn}}^{\alpha_{mx}}\alpha d_A(\alpha)d_B(\alpha/u)\,d\alpha},$$

where (i) the first two equalities follow from Eq. (9.38), (ii) the third equality follows from Eqs. (9.39) and (9.40), and (iii) the fourth equality follows from the cancellation of u^2. In turn, the desired density function

$$d_{Ai}(\alpha \mid \tau \in \mathcal{P}_i) = d_{Ai}(\alpha \mid u = \alpha/\beta = \bar{q}(\tau)/\bar{p}(\tau) = r(\tau) \text{ for } \tau \in \mathcal{P}_i)$$
$$= \frac{\alpha d_A(\alpha)d_B[\alpha/r(\tau)]}{\int_{\alpha_{mn}}^{\alpha_{mx}}\alpha d_A(\alpha)d_B[\alpha/r(\tau)]\,d\alpha} \qquad (9.43)$$

is obtained for $\alpha\bar{p}(\tau) = \beta\bar{q}(\tau)$ from the final equality in Eq. (9.42) by replacing u with $r(\tau)$.

The final forms of the integral

$$I(r(\tau)) = \int_{\alpha_{mn}}^{\alpha_{mx}}\alpha d_A(\alpha)d_B(\alpha/r(\tau))\,d\alpha \qquad (9.44)$$

in the denominator of Eq. (9.43), and hence the final forms for $d_{Ai}(\alpha \mid \tau \in \mathcal{P}_i)$, depend on which of the sets $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4$ defined in conjunction with Eqs. (9.4)-(9.8) τ belongs to. This membership also determines the sample space associated with $d_{Ai}(\alpha \mid \tau \in \mathcal{P}_i)$; specifically, the sample space for $d_{Ai}(\alpha \mid \tau \in \mathcal{P}_i)$ is the set $\mathcal{A}_i(\alpha \mid \tau \in \mathcal{P}_i)$ as indicated in Eqs. (9.4)-(9.8). The final forms of the integral $I(r(\tau))$ are now obtained and will depend on the subsets of $[\alpha_{mn}, \alpha_{mx}]$ on which either $\alpha/r(\tau) < \beta_{mn}$ or $\beta_{mx} < \alpha/r(\tau)$ holds and, consequently, $d_B(\alpha/r(\tau)) = 0$.

Membership of τ in $\mathcal{P}_1$ requires that $r(\tau)$ satisfies the inequalities $\alpha_{mn} \leq r(\tau)\beta_{mn} \leq \alpha_{mx} \leq r(\tau)\beta_{mx}$, with the results that (i) $\alpha/r(\tau) < \beta_{mn}$ for $\alpha_{mn} \leq \alpha < r(\tau)\beta_{mn}$ and (ii) $\beta_{mn} \leq \alpha/r(\tau) \leq \beta_{mx}$ for $r(\tau)\beta_{mn} \leq \alpha \leq \alpha_{mx}$. As a consequence of the preceding inequalities, the final form of the integral in Eq. (9.44) for $\tau \in \mathcal{P}_1$ is

$$I_1\left(r(\tau)\,|\,\tau \in \mathcal{P}_1\right) = \int_{\alpha_{mn}}^{\alpha_{mx}} \alpha\, d_A(\alpha)\, d_B\left(\alpha\,/\,r(\tau)\right) \mathrm{d}\alpha$$

$$= \int_{\alpha_{mn}}^{r(\tau)\beta_{mn}} \alpha\, d_A(\alpha)\, \overbrace{d_B\left(\alpha\,/\,r(\tau)\right)}^{=0} \mathrm{d}\alpha + \int_{r(\tau)\beta_{mn}}^{\alpha_{mx}} \alpha\, d_A(\alpha)\, \overbrace{d_B\left(\alpha\,/\,r(\tau)\right)}^{>0} \mathrm{d}\alpha \qquad (9.45)$$

$$= \int_{r(\tau)\beta_{mn}}^{\alpha_{mx}} \alpha\, d_A(\alpha)\, d_B\left(\alpha\,/\,r(\tau)\right) \mathrm{d}\alpha.$$

Membership of τ in $\mathcal{P}_2$ requires that $r(\tau)$ satisfies the inequalities $\alpha_{mn} \le r(\tau)\beta_{mn} \le r(\tau)\beta_{mx} \le \alpha_{mx}$, with the results that (i) $\alpha\,/\,r(\tau) < \beta_{mn}$ for $\alpha_{mn} \le \alpha < r(\tau)\beta_{mn}$, (ii) $\beta_{mn} \le \alpha\,/\,r(\tau) \le \beta_{mx}$ for $r\beta_{mn} \le \alpha \le r(\tau)\beta_{mn}$, and (iii) $\beta_{mx} < \alpha\,/\,r(\tau)$ for $r(\tau)\beta_{mx} < \alpha \le \alpha_{mx}$. As a consequence of the preceding inequalities, the final form of the integral in Eq. (9.44) for $\tau \in \mathcal{P}_2$ is

$$I_2\left(r(\tau)\,|\,\tau \in \mathcal{P}_2\right) = \int_{\alpha_{mn}}^{\alpha_{mx}} \alpha\, d_A(\alpha)\, d_B\left(\alpha\,/\,r(\tau)\right) \mathrm{d}\alpha$$

$$= \int_{\alpha_{mn}}^{r(\tau)\beta_{mn}} \alpha\, d_A(\alpha)\, \overbrace{d_B\left(\alpha\,/\,r(\tau)\right)}^{=0} \mathrm{d}\alpha + \int_{r(\tau)\beta_{mn}}^{r(\tau)\beta_{mx}} \alpha\, d_A(\alpha)\, \overbrace{d_B\left(\alpha\,/\,r(\tau)\right)}^{>0} \mathrm{d}\alpha$$

$$+ \int_{r(\tau)\beta_{mx}}^{\alpha_{mx}} \alpha\, d_A(\alpha)\, \overbrace{d_B\left(\alpha\,/\,r(\tau)\right)}^{=0} \mathrm{d}\alpha \qquad (9.46)$$

$$= \int_{r(\tau)\beta_{mn}}^{r(\tau)\beta_{mx}} \alpha\, d_A(\alpha)\, d_B\left(\alpha\,/\,r\right)(\tau) \mathrm{d}\alpha.$$

Membership of τ in $\mathcal{P}_3$ requires that $r(\tau)$ satisfies the inequalities $r(\tau)\beta_{mn} \le \alpha_{mn} \le \alpha_{mx} \le r(\tau)\beta_{mx}$, with the result that $\beta_{mn} \le \alpha\,/\,r(\tau) \le \beta_{mx}$ for $\alpha_{mn} \le \alpha \le \alpha_{mx}$. As a consequence of the preceding inequalities,

$$I_3\left(r(\tau)\,|\,\tau \in \mathcal{P}_3\right) = \int_{\alpha_{mn}}^{\alpha_{mx}} \alpha\, d_A(\alpha)\, \overbrace{d_B\left(\alpha\,/\,r(\tau)\right)}^{>0} \mathrm{d}\alpha, \qquad (9.47)$$

for $\tau \in \mathcal{P}_3$, which is the same as integral in Eq. (9.44). Membership of τ in $\mathcal{P}_4$ requires that $r(\tau)$ satisfies the inequalities $r(\tau)\beta_{mn} \le \alpha_{mn} \le r(\tau)\beta_{mx} \le \alpha_{mx}$, with the results that (i) $\beta_{mn} \le \alpha\,/\,r(\tau) \le \beta_{mx}$ for $\alpha_{mn} \le \alpha \le r(\tau)\beta_{mx}$ and (ii) $\beta_{mx} \le \alpha\,/\,r(\tau)$ for $r(\tau)\beta_{mx} \le \alpha \le \alpha_{mx}$. As a consequence of the preceding inequalities, the final form of the integral in Eq. (9.44) for $\tau \in \mathcal{P}_4$ is

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$$I_4\left(r(\tau)\,|\,\tau \in \mathcal{P}_4\right) = \int_{\alpha_{mn}}^{\alpha_{mx}} \alpha\, d_A\left(\alpha\right) d_B\left(\alpha\,/\,r(\tau)\right) \mathrm{d}\alpha$$

$$= \int_{\alpha_{mn}}^{r(\tau)\beta_{mx}} \alpha\, d_A\left(\alpha\right) \overbrace{d_B\left(\alpha\,/\,r(\tau)\right)}^{>0} \mathrm{d}\alpha + \int_{r(\tau)\beta_{mx}}^{\alpha_{mx}} \alpha\, d_A\left(\alpha\right) \overbrace{d_B\left(\alpha\,/\,r(\tau)\right)}^{=0} \mathrm{d}\alpha \qquad (9.48)$$

$$= \int_{\alpha_{mn}}^{r(\tau)\beta_{mx}} \alpha\, d_A\left(\alpha\right) d_B\left(\alpha\,/\,r(\tau)\right) \mathrm{d}\alpha.$$

The results in Eqs. (9.45)-(9.48) and their role in the definition of $d_{Ai}(\alpha\,|\,\tau \in \mathcal{P}_i)$ for $i = 1, 2, 3, 4$ are summarized in Table 5.

The core relationships established in Eqs. (9.45)-(9.48) can be summarized as

$$I\left(r(\tau)\right) = \int_{\alpha_{mn}}^{\alpha_{mx}} \alpha\, d_A\left(\alpha\right) d_B\left(\alpha\,/\,r(\tau)\right) \mathrm{d}\alpha$$

$$= \begin{cases} \int_{r(\tau)\beta_{mn}}^{\alpha_{mx}} \alpha\, d_A\left(\alpha\right) d_B\left(\alpha\,/\,r(\tau)\right) \mathrm{d}\alpha & \text{for } \tau \in \mathcal{P}_1 \\[2mm] \int_{r(\tau)\beta_{mn}}^{r(\tau)\beta_{mx}} \alpha\, d_A\left(\alpha\right) d_B\left(\alpha\,/\,r(\tau)\right) \mathrm{d}\alpha & \text{for } \tau \in \mathcal{P}_2 \\[2mm] \int_{\alpha_{mn}}^{\alpha_{mx}} \alpha\, d_A\left(\alpha\right) d_B\left(\alpha\,/\,r(\tau)\right) \mathrm{d}\alpha & \text{for } \tau \in \mathcal{P}_3 \\[2mm] \int_{\alpha_{mn}}^{r(\tau)\beta_{mx}} \alpha\, d_A\left(\alpha\right) d_B\left(\alpha\,/\,r(\tau)\right) \mathrm{d}\alpha & \text{for } \tau \in \mathcal{P}_4 \end{cases}$$

$$= \begin{cases} I_1\left(r(\tau)\,|\,\tau \in \mathcal{P}_1\right) \\ I_2\left(r(\tau)\,|\,\tau \in \mathcal{P}_2\right) \\ I_3\left(r(\tau)\,|\,\tau \in \mathcal{P}_3\right) \\ I_4\left(r(\tau)\,|\,\tau \in \mathcal{P}_4\right) \end{cases} \qquad (9.49)$$

and will be useful in later derivations.

Table 5 Summary of density functions $d_{Ai}(\alpha \mid \tau \in \mathcal{P}_i)$ and corresponding CDFs $CDF_{Ai}(\alpha \mid \tau \in \mathcal{P}_i)$ for $i = 1, 2, 3, 4$ for variable α conditional on link failure at time τ and resultant ratio $r(\tau) = \bar{q}(\tau) / \bar{p}(\tau)$.

$$d_{A1}(\alpha \mid \tau \in \mathcal{P}_1) = \frac{\alpha d_A(\alpha) d_B\left(\alpha / r(\tau)\right)}{\int_{r(\tau)\beta_{mn}}^{\alpha_{mx}} \alpha d_A(\alpha) d_B\left(\alpha / r(\tau)\right) d\alpha}, \; CDF_{A1}(\alpha \mid \tau \in \mathcal{P}_1) = \frac{\int_{r(\tau)\beta_{mn}}^{\alpha} \tilde{\alpha} d_A(\tilde{\alpha}) d_B\left(\tilde{\alpha} / r(\tau)\right) d\tilde{\alpha}}{\int_{r(\tau)\beta_{mn}}^{\alpha_{mx}} \alpha d_A(\alpha) d_B\left(\alpha / r(\tau)\right) d\alpha}$$

for $\mathcal{P}_1 = \{\tau : \alpha_{mn} \le r(\tau)\beta_{mn} \le \alpha_{mx} \le r(\tau)\beta_{mx}\}$, $r(\tau) = \bar{q}(\tau) / \bar{p}(\tau)$, and $\alpha \in \mathcal{A}_1(\alpha \mid \tau \in \mathcal{P}_1) = [r(\tau)\beta_{mn}, \alpha_{mx}]$

$$d_{A2}(\alpha \mid \tau \in \mathcal{P}_2) = \frac{\alpha d_A(\alpha) d_B\left(\alpha / r(\tau)\right)}{\int_{r(\tau)\beta_{mn}}^{r(\tau)\beta_{mx}} \alpha d_A(\alpha) d_B\left(\alpha / r(\tau)\right) d\alpha}, CDF_{A2}(\alpha \mid \tau \in \mathcal{P}_2) = \frac{\int_{r(\tau)\beta_{mn}}^{\alpha} \tilde{\alpha} d_A(\tilde{\alpha}) d_B\left(\tilde{\alpha} / r(\tau)\right) d\tilde{\alpha}}{\int_{r(\tau)\beta_{mn}}^{r(\tau)\beta_{mx}} \alpha d_A(\alpha) d_B\left(\alpha / r(\tau)\right) d\alpha}$$

for $\mathcal{P}_2 = \{\tau : \alpha_{mn} \le r(\tau)\beta_{mn} \le r(\tau)\beta_{mx} \le \alpha_{mx}\}$, $r(\tau) = \bar{q}(\tau) / \bar{p}(\tau)$, and
$\alpha \in \mathcal{A}_2(\alpha \mid \tau \in \mathcal{P}_2) = [r(\tau)\beta_{mn}, r(\tau)\beta_{mx}]$

$$d_{A3}(\alpha \mid \tau \in \mathcal{P}_3) = \frac{\alpha d_A(\alpha) d_B\left(\alpha / r(\tau)\right)}{\int_{\alpha_{mn}}^{\alpha_{mx}} \alpha d_A(\alpha) d_B\left(\alpha / r(\tau)\right) d\alpha}, CDF_{A3}(\alpha \mid \tau \in \mathcal{P}_3) = \frac{\int_{\alpha_{mn}}^{\alpha} \tilde{\alpha} d_A(\tilde{\alpha}) d_B\left(\tilde{\alpha} / r(\tau)\right) d\tilde{\alpha}}{\int_{\alpha_{mn}}^{\alpha_{mx}} \alpha d_A(\alpha) d_B\left(\alpha / r(\tau)\right) d\alpha}$$

for $\mathcal{P}_3 = \{\tau : r(\tau)\beta_{mn} \le \alpha_{mn} \le \alpha_{mx} \le r(\tau)\beta_{mx}\}$, $r(\tau) = \bar{q}(\tau) / \bar{p}(\tau)$, and $\alpha \in \mathcal{A}_3(\alpha \mid \tau \in \mathcal{P}_3) = [\alpha_{mn}, \alpha_{mx}]$

$$d_{A4}(\alpha \mid \tau \in \mathcal{P}_4) = \frac{\alpha d_A(\alpha) d_B\left(\alpha / r(\tau)\right)}{\int_{\alpha_{mn}}^{r(\tau)\beta_{mx}} \alpha d_A(\alpha) d_B\left(\alpha / r(\tau)\right) d\alpha}, CDF_{A4}(\alpha \mid \tau \in \mathcal{P}_4) = \frac{\int_{\alpha_{mn}}^{\alpha} \tilde{\alpha} d_A(\tilde{\alpha}) d_B\left(\tilde{\alpha} / r(\tau)\right) d\tilde{\alpha}}{\int_{\alpha_{mn}}^{r(\tau)\beta_{mx}} \alpha d_A(\alpha) d_B\left(\alpha / r(\tau)\right) d\alpha}$$

for $\mathcal{P}_4 = \{\tau : r(\tau)\beta_{mn} \le \alpha_{mn} \le r(\tau)\beta_{mx} \le \alpha_{mx}\}$, $r(\tau) = \bar{q}(\tau) / \bar{p}(\tau)$, and
$\alpha \in \mathcal{A}_4(\alpha \mid \tau \in \mathcal{P}_4) = [\alpha_{mn}, r(\tau)\beta_{mx}]$

Once the density functions $d_{Ai}(\alpha \mid \tau \in \mathcal{P}_i)$ and associated intervals of definition $[\alpha_{mn}(\tau), \alpha_{mx}(\tau)]$ are determined as indicated in Eqs. (9.34) and (9.45)-(9.48), the corresponding CDFs $CDF_{Ai}(\alpha \mid \tau \in \mathcal{P}_i)$ are defined by integration of $d_{Ai}(\alpha \mid \tau \in \mathcal{P}_i)$ from $\alpha_{mn}(\tau)$ to α for $\alpha \in [\alpha_{mn}(\tau), \alpha_{mx}(\tau)]$. Specifically,

$$
\begin{aligned}
CDF_{Ai}\left(\alpha \mid \tau \in \mathcal{P}_i\right) &= \int_{\alpha_{mn}(\tau)}^{\alpha} d_{Ai}\left(\tilde{\alpha} \mid \tau \in \mathcal{P}_i\right) \mathrm{d}\tilde{\alpha} \\
&= \int_{\alpha_{mn}(\tau)}^{\alpha} \left\{ \frac{\tilde{\alpha} d_A(\tilde{\alpha}) d_B\left(\tilde{\alpha}/r(\tau)\right)}{I_i\left(r(\tau) \mid \tau \in \mathcal{P}_i\right)} \right\} \mathrm{d}\tilde{\alpha} \\
&= \int_{\alpha_{mn}(\tau)}^{\alpha} \tilde{\alpha} d_A(\tilde{\alpha}) d_B\left(\tilde{\alpha}/r(\tau)\right) \mathrm{d}\tilde{\alpha} \Big/ I_i\left(r(\tau) \mid \tau \in \mathcal{P}_i\right),
\end{aligned}
\tag{9.50}
$$

where, for $\tau \in \mathcal{P}_i$, the corresponding sample space $\mathcal{A}_i(\alpha \mid \tau \in \mathcal{P}_i)$ for α and the integral $I_i(\tau \mid \tau \in \mathcal{P}_i)$ are defined in Eqs. (9.34) and (9.49), respectively. Together with the density functions $d_{Ai}(\alpha \mid \tau \in \mathcal{P}_i)$, the CDFs $CDF_{Ai}(\alpha \mid \tau \in \mathcal{P}_i)$ are summarized in Table 5.

A partial check on the correctness of the density functions $d_{Ai}(\alpha \mid \tau \in \mathcal{P}_i)$ can be obtained by verifying that the integrals of these functions over their domains of definition $\mathcal{A}_i(\alpha \mid \tau \in \mathcal{P}_i)$ are equal to 1.0. This partial check follows immediately from the CDFs defined in Eq. (9.50). Specifically, evaluation of $CDF_{Ai}(\alpha \mid \tau \in \mathcal{P}_i)$ for the maximum value $\alpha_{mx}(\tau)$ of α in the set $\mathcal{A}_i(\alpha \mid \tau \in \mathcal{P}_i)$ results in $CDF_{Ai}(\alpha_{mx}(\tau) \mid \tau \in \mathcal{P}_i) = 1.0$ as a consequence of the numerator and denominator in defining expression for $CDF_{Ai}(\alpha_{mx}(\tau) \mid \tau \in \mathcal{P}_i)$ being equal. Thus, the integral of $d_{Ai}(\alpha \mid \tau \in \mathcal{P}_i)$ over the corresponding sample space $\mathcal{A}_i(\alpha \mid \tau \in \mathcal{P}_i)$ for α is equal to 1.0.

9.5 Density Function $d_P(p \mid \tau)$ for Link Property Conditional on Link Failure at Time τ

Now that the density functions $d_{Ai}(\alpha \mid \tau \in \mathcal{P}_i)$, $i = 1, 2, 3, 4$, for α are defined as summarized in Table 5, the density functions $d_{Pi}(p \mid \tau \in \mathcal{P}_i)$ for p on

$$
S_i(p \mid \tau \in \mathcal{P}_i) = [p_{mn}(\tau), p_{mx}(\tau)] = \begin{cases} \left[\beta_{mn}\bar{q}(\tau), \alpha_{mx}\bar{p}(\tau)\right] & \text{for } i = 1 \text{ (see Eq. (9.4))} \\ \left[\beta_{mn}\bar{q}(\tau), \beta_{mx}\bar{q}(\tau)\right] & \text{for } i = 2 \text{ (see Eq. (9.6))} \\ \left[\alpha_{mn}\bar{p}(\tau), \alpha_{mx}\bar{p}(\tau)\right] & \text{for } i = 3 \text{ (see Eq. (9.7))} \\ \left[\alpha_{mn}\bar{p}(\tau), \beta_{mx}\bar{q}(\tau)\right] & \text{for } i = 4 \text{ (see Eq. (9.8))} \end{cases}
\tag{9.51}
$$

can be obtained as indicated in Eq. (9.35). Specifically,

$$d_{Pi}(p \mid \tau \in \mathcal{P}_i) = \left(1 / \overline{p}(\tau)\right) d_{Ai}\left(p / \overline{p}(\tau) \mid \tau \in \mathcal{P}_i\right)$$

$$= \left(1 / \overline{p}(\tau)\right) \left(\frac{\left(p / \overline{p}(\tau)\right) d_A\left[p / \overline{p}(\tau)\right] d_B\left[p / \overline{p}(\tau) \times 1 / r(\tau)\right]}{I_i\left(r(\tau) \mid \tau \in \mathcal{P}_i\right)} \right) \quad (9.52)$$

$$= \frac{\left(p / \overline{p}^2(\tau)\right) d_A\left[p / \overline{p}(\tau)\right] d_B\left[p / \overline{q}(\tau)\right]}{I_i\left(r(\tau) \mid \tau \in \mathcal{P}_i\right)},$$

where (i) the first equality follows from Eq. (9.35), (ii) the second equality follows from the definitions of $d_{Ai}(\alpha \mid \tau \in \mathcal{P}_i)$ as summarized in Table 5 with the associated integrals represented by $I_i[r(\tau) \mid \tau \in \mathcal{P}_i]$ summarized in Eq. (9.49), and (iii) the third equality follows from an algebraic rearrangement of terms. The density functions $d_{Pi}(p \mid \tau \in \mathcal{P}_i)$ defined in Eq. (9.52) are summarized in Table 6.

As done in Eq. (9.50) to obtain the CDFs $CDF_{Ai}(\alpha_{mx}(\tau) \mid \tau \in \mathcal{P}_i)$ for α, the CDFs $CDF_{Pi}(p \mid \tau \in \mathcal{P}_i)$, $i = 1, 2, 3, 4$, for p can be obtained by integrations of $d_{Pi}(p \mid \tau \in \mathcal{P}_i)$ over the corresponding sample spaces $\mathcal{S}_i(p \mid \tau \in \mathcal{P}_i)$ for p. Specifically,

$$CDF_{Pi}\left(p \mid \tau \in \mathcal{P}_i\right) = \int_{p_{mn}(\tau)}^{p} d_{Pi}\left(\tilde{p} \mid \tau \in \mathcal{P}_i\right) d\tilde{p}$$

$$= \int_{p_{mn}(\tau)}^{p} \left\{ \frac{\left(\tilde{p} / \overline{p}^2(\tau)\right) d_A\left[\tilde{p} / \overline{p}(\tau)\right] d_B\left[\tilde{p} / \overline{q}(\tau)\right]}{I_i\left(r(\tau) \mid \tau \in \mathcal{P}_i\right)} \right\} d\tilde{p} \quad (9.53)$$

$$= \int_{p_{mn}(\tau)}^{p} \left(\tilde{p} / \overline{p}^2(\tau)\right) d_A\left[\tilde{p} / \overline{p}(\tau)\right] d_B\left[\tilde{p} / \overline{q}(\tau)\right] d\tilde{p} \Big/ I_i\left(\tau \mid \tau \in \mathcal{P}_i\right)$$

for $p \in \mathcal{S}_i(p \mid \tau \in \mathcal{P}_i) = [p_{mn}(\tau), p_{mx}(\tau)]$. The integral in the numerator of the final term in the preceding equality can be rewritten through a change of variables as

$$\int_{p_{mn}(\tau)}^{p} \left(\tilde{p} / \overline{p}^2(\tau)\right) d_A\left[\tilde{p} / \overline{p}(\tau)\right] d_B\left[\tilde{p} / \overline{q}(\tau)\right] d\tilde{p}$$

$$= \int_{p_{mn}(\tau)}^{p} \alpha(\tilde{p}) d_A\left[\alpha(\tilde{p})\right] d_B\left[\alpha(\tilde{p}) / r(\tau)\right] \left[d\alpha(\tilde{p}) / d\tilde{p}\right] d\tilde{p} \quad \text{for } \alpha(\tilde{p}) = \tilde{p} / \overline{p}(\tau)$$

$$= \int_{\alpha\left(p_{mn}(\tau)\right)}^{\alpha(p)} \alpha d_A(\alpha) d_B\left(\alpha / r(\tau)\right) d\alpha \quad \text{by change of variables} \quad (9.54)$$

$$= \int_{p_{mn}(\tau) / \overline{p}(\tau)}^{p / \overline{p}(\tau)} \alpha d_A(\alpha) d_B\left(\alpha / r(\tau)\right) d\alpha.$$

In turn, the representation

$$CDF_{Pi}\left(p \mid \tau \in \mathcal{P}_i\right) = \int_{p_{mn}(\tau) / \overline{p}(\tau)}^{p / \overline{p}(\tau)} \alpha d_A(\alpha) d_B\left(\alpha / r(\tau)\right) d\alpha \Big/ I_i\left(\tau \mid \tau \in \mathcal{P}_i\right) \quad (9.55)$$

results by combining the final expressions in Eqs. (9.53) and (9.54).

Table 6 Summary of density functions $d_{Pi}(p \mid \tau \in \mathcal{P}_i)$ and corresponding CDFs $CDF_{Pi}(p \mid \tau \in \mathcal{P}_i)$ for system property p conditional on link failure at time τ and resultant ratio $r(\tau) = \overline{q}(\tau)/\overline{p}(\tau)$.

$$d_{P1}(p \mid \tau \in \mathcal{P}_1) = \frac{\left(p/\overline{p}^2(\tau)\right)d_A\left[p/\overline{p}(\tau)\right]d_B\left[p/\overline{q}(\tau)\right]}{\int_{r(\tau)\beta_{mn}}^{\alpha_{mx}} \alpha d_A(\alpha) d_B(\alpha/r(\tau))\mathrm{d}\alpha}, CDF_{P1}(p \mid \tau \in \mathcal{P}_1) = \frac{\int_{r(\tau)\beta_{mn}}^{p/\overline{p}(\tau)} \alpha d_A(\alpha) d_B(\alpha/r(\tau))\mathrm{d}\alpha}{\int_{r(\tau)\beta_{mn}}^{\alpha_{mx}} \alpha d_A(\alpha) d_B(\alpha/r(\tau))\mathrm{d}\alpha}$$

for $\mathcal{P}_1 = \left\{\tau : \alpha_{mn} \leq r(\tau)\beta_{mn} \leq \alpha_{mx} \leq r(\tau)\beta_{mx}\right\}$, $r(\tau) = \overline{q}(\tau)/\overline{p}(\tau)$, and

$p \in \mathcal{S}_1(p \mid \tau \in \mathcal{P}_1) = \left[\beta_{mn}\overline{q}(\tau), \alpha_{mx}\overline{p}(\tau)\right]$

$$d_{P2}(p \mid \tau \in \mathcal{P}_2) = \frac{\left(p/\overline{p}^2(\tau)\right)d_A\left[p/\overline{p}(\tau)\right]d_B\left[p/\overline{q}(\tau)\right]}{\int_{r(\tau)\beta_{mn}}^{r(\tau)\beta_{mx}} \alpha d_A(\alpha) d_B(\alpha/r(\tau))\mathrm{d}\alpha}, CDF_{P2}(p \mid \tau \in \mathcal{P}_2) = \frac{\int_{r(\tau)\beta_{mn}}^{p/\overline{p}(\tau)} \alpha d_A(\alpha) d_B(\alpha/r(\tau))\mathrm{d}\alpha}{\int_{r(\tau)\beta_{mn}}^{r(\tau)\beta_{mx}} \alpha d_A(\alpha) d_B(\alpha/r(\tau))\mathrm{d}\alpha}$$

for $\mathcal{P}_2 = \left\{\tau : \alpha_{mn} \leq r(\tau)\beta_{mn} \leq r(\tau)\beta_{mx} \leq \alpha_{mx}\right\}$, $r(\tau) = \overline{q}(\tau)/\overline{p}(\tau)$, and

$p \in \mathcal{S}_2(p \mid \tau \in \mathcal{P}_2) = \left[\beta_{mn}\overline{q}(\tau), \beta_{mx}\overline{q}(\tau)\right]$

$$d_{P3}(p \mid \tau \in \mathcal{P}_3) = \frac{\left(p/\overline{p}^2(\tau)\right)d_A\left[p/\overline{p}(\tau)\right]d_B\left[p/\overline{q}(\tau)\right]}{\int_{\alpha_{mn}}^{\alpha_{mx}} \alpha d_A(\alpha) d_B(\alpha/r(\tau))\mathrm{d}\alpha}, CDF_{P3}(p \mid \tau \in \mathcal{P}_3) = \frac{\int_{\alpha_{mn}}^{p/\overline{p}(\tau)} \alpha d_A(\alpha) d_B(\alpha/r(\tau))\mathrm{d}\alpha}{\int_{\alpha_{mn}}^{\alpha_{mx}} \alpha d_A(\alpha) d_B(\alpha/r(\tau))\mathrm{d}\alpha}$$

for $\mathcal{P}_3 = \left\{\tau : r(\tau)\beta_{mn} \leq \alpha_{mn} \leq \alpha_{mx} \leq r(\tau)\beta_{mx}\right\}$, $r(\tau) = \overline{q}(\tau)/\overline{p}(\tau)$, and

$p \in \mathcal{S}_3(p \mid \tau \in \mathcal{P}_3) = \left[\alpha_{mn}\overline{p}(\tau), \alpha_{mx}\overline{p}(\tau)\right]$

$$d_{P4}(p \mid \tau \in \mathcal{P}_4) = \frac{\left(p/\overline{p}^2(\tau)\right)d_A\left[p/\overline{p}(\tau)\right]d_B\left[p/\overline{q}(\tau)\right]}{\int_{\alpha_{mn}}^{r(\tau)\beta_{mx}} \alpha d_A(\alpha) d_B(\alpha/r(\tau))\mathrm{d}\alpha}, CDF_{P4}(p \mid \tau \in \mathcal{P}_4) = \frac{\int_{\alpha_{mn}}^{p/\overline{p}(\tau)} \alpha d_A(\alpha) d_B(\alpha/r(\tau))\mathrm{d}\alpha}{\int_{\alpha_{mn}}^{r(\tau)\beta_{mx}} \alpha d_A(\alpha) d_B(\alpha/r(\tau))\mathrm{d}\alpha}$$

for $\mathcal{P}_4 = \left\{\tau : r(\tau)\beta_{mn} \leq \alpha_{mn} \leq r(\tau)\beta_{mx} \leq \alpha_{mx}\right\}$, $r(\tau) = \overline{q}(\tau)/\overline{p}(\tau)$, and

$p \in \mathcal{S}_4(p \mid \tau \in \mathcal{P}_4) = \left[\alpha_{mn}\overline{p}(\tau), \beta_{mx}\overline{q}(\tau)\right]$

Next, substitutions in Eq. (9.55) for $p_{mn}(\tau)$ as indicated for the sets $\mathcal{S}_i(p \mid \tau \in \mathcal{P}_i)$ summarized in Eq. (9.51) and the integrals $I_i(\tau \mid \tau \in \mathcal{P}_i)$ defined in Eqs. (9.45)-(9.48) produce the following representations for $CDF_{Pi}(p \mid \tau \in \mathcal{P}_i)$:

$$
\begin{aligned}
CDF_{P1}(p \mid \tau \in \mathcal{P}_1) &= \frac{\displaystyle\int_{p_{mn}(\tau)/\overline{p}(\tau)}^{p/\overline{p}(\tau)} \alpha d_A(\alpha) d_B(\alpha / r(\tau)) \mathrm{d}\alpha}{\displaystyle\int_{r(\tau)\beta_{mn}}^{\alpha_{mx}} \alpha d_A(\alpha) d_B(\alpha / r(\tau)) \mathrm{d}\alpha} \\[2mm]
&= \frac{\displaystyle\int_{r(\tau)\beta_{mn}}^{p/\overline{p}(\tau)} \alpha d_A(\alpha) d_B(\alpha / r(\tau)) \mathrm{d}\alpha}{\displaystyle\int_{r(\tau)\beta_{mn}}^{\alpha_{mx}} \alpha d_A(\alpha) d_B(\alpha / r(\tau)) \mathrm{d}\alpha}
\end{aligned}
\tag{9.56}
$$

with $\mathcal{S}_1(p \mid \tau \in \mathcal{P}_1) = \left[\beta_{mn}\overline{q}(\tau), \alpha_{mx}\overline{p}(\tau)\right]$;

$$
\begin{aligned}
CDF_{P2}(p \mid \tau \in \mathcal{P}_2) &= \frac{\displaystyle\int_{p_{mn}(\tau)/\overline{p}(\tau)}^{p/\overline{p}(\tau)} \alpha d_A(\alpha) d_B(\alpha / r(\tau)) \mathrm{d}\alpha}{\displaystyle\int_{r(\tau)\beta_{mn}}^{r(\tau)\beta_{mx}} \alpha d_A(\alpha) d_B(\alpha / r(\tau)) \mathrm{d}\alpha} \\[2mm]
&= \frac{\displaystyle\int_{r(\tau)\beta_{mn}}^{p/\overline{p}(\tau)} \alpha d_A(\alpha) d_B(\alpha / r(\tau)) \mathrm{d}\alpha}{\displaystyle\int_{r(\tau)\beta_{mn}}^{r(\tau)\beta_{mx}} \alpha d_A(\alpha) d_B(\alpha / r(\tau)) \mathrm{d}\alpha}
\end{aligned}
\tag{9.57}
$$

with $\mathcal{S}_2(p \mid \tau \in \mathcal{P}_2) = \left[\beta_{mn}\overline{q}(\tau), \beta_{mx}\overline{q}(\tau)\right]$;

$$
\begin{aligned}
CDF_{P3}(p \mid \tau \in \mathcal{P}_3) &= \frac{\displaystyle\int_{p_{mn}(\tau)/\overline{p}(\tau)}^{p/\overline{p}(\tau)} \alpha d_A(\alpha) d_B(\alpha / r(\tau)) \mathrm{d}\alpha}{\displaystyle\int_{\alpha_{mn}}^{\alpha_{mx}} \alpha d_A(\alpha) d_B(\alpha / r(\tau)) \mathrm{d}\alpha} \\[2mm]
&= \frac{\displaystyle\int_{\alpha_{mn}}^{p/\overline{p}(\tau)} \alpha d_A(\alpha) d_B(\alpha / r(\tau)) \mathrm{d}\alpha}{\displaystyle\int_{\alpha_{mn}}^{\alpha_{mx}} \alpha d_A(\alpha) d_B(\alpha / r(\tau)) \mathrm{d}\alpha}
\end{aligned}
\tag{9.58}
$$

with $\mathcal{S}_3(p \mid \tau \in \mathcal{P}_3) = \left[\alpha_{mn}\overline{p}(\tau), \alpha_{mx}\overline{p}(\tau)\right]$; and

$$CDF_{P4}(p \mid \tau \in \mathcal{P}_4) = \frac{\int_{p_{mn}(\tau)/\bar{p}(\tau)}^{p/\bar{p}(\tau)} \alpha d_A(\alpha) d_B(\alpha / r(\tau)) d\alpha}{\int_{\alpha_{mn}}^{r(\tau)\beta_{mx}} \alpha d_A(\alpha) d_B(\alpha / r(\tau)) d\alpha}$$

$$= \frac{\int_{\alpha_{mn}}^{p/\bar{p}(\tau)} \alpha d_A(\alpha) d_B(\alpha / r(\tau)) d\alpha}{\int_{\alpha_{mn}}^{r(\tau)\beta_{mx}} \alpha d_A(\alpha) d_B(\alpha / r(\tau)) d\alpha}$$

(9.59)

with $\mathcal{S}_4(p \mid \tau \in \mathcal{P}_4) = \left[\alpha_{mn}\bar{p}(\tau), \beta_{mx}\bar{q}(\tau)\right]$. Together with the density functions $d_{Pi}(p \mid \tau \in \mathcal{P}_i)$, the CDFs $CDF_{Pi}(p \mid \tau \in \mathcal{P}_i)$ are summarized in Table 6.

A partial check on the correctness of the density functions $CDF_{Pi}(p \mid \tau \in \mathcal{P}_i)$ can be obtained by verifying that the integrals of these functions over their domains of definition $\mathcal{S}_i(p \mid \tau \in \mathcal{P}_i)$ are equal to 1.0. This partial check follows immediately from the CDFs in Eqs. (9.56)-(9.59). Specifically, evaluation of $CDF_{Pi}(p \mid \tau \in \mathcal{P}_i)$ for the maximum value $p_{mx}(\tau)$ of β in the set $\mathcal{S}_i(p \mid \tau \in \mathcal{P}_i)$ results in $CDF_{Pi}(p_{mx}(\tau) \mid \tau \in \mathcal{P}_i) = 1.0$ as a consequence of the numerator and denominator in defining expression for $d_{Pi}(p \mid \tau \in \mathcal{P}_i)$ being equal. Thus, the integral of $d_{Pi}(p \mid \tau \in \mathcal{P}_i)$ over the corresponding sample space $\mathcal{S}_i(p \mid \tau \in \mathcal{P}_i)$ for p is equal to 1.0.

9.6 Representation of Joint Density Functions $d_A(\alpha \mid \tau) d_T(\tau)$ and $d_P(p \mid \tau) d_T(\tau)$

The joint density functions

$$d_{AT}(\alpha, \tau) = d_A(\alpha \mid \tau) d_T(\tau) \text{ and } d_{PT}(p, \tau) = d_P(p \mid \tau) d_T(\tau) \qquad (9.60)$$

play a role in the derivation of several quantities of interest (e.g., cumulative distribution for link property at time of link failure in Sect. 10 and margins involving SL properties in Ref. [46]). As summarized below in Eqs. (9.61)-(9.63), the individual density functions $d_A(\alpha \mid \tau)$, $d_P(p \mid \tau)$ and $d_T(\tau)$ have complicated forms that depend on the membership of τ in one of the sets $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4$ defined and discussed in Sect. 9.2 and also listed in Table 4. Then, as shown in Eqs. (9.64) and (9.65), the effect of membership of τ in one of the sets $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4$ is reduced for the joint density functions $d_A(\alpha \mid \tau) d_T(\tau)$ and $d_A(\alpha \mid \tau) d_T(\tau)$ defined in Eq. (9.60).

With the indicated restriction of τ to $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ or $\mathcal{P}_4$, the representations for $d_A(\alpha \mid \tau)$, $d_P(p \mid \tau)$ and $d_T(\tau)$ are

$$d_A(\alpha \mid \tau) = \begin{cases} d_{A1}(\alpha \mid \tau \in \mathcal{P}_1) = \dfrac{\alpha d_A(\alpha)d_B\left(\alpha / r(\tau)\right)}{\displaystyle\int_{r(\tau)\beta_{mn}}^{\alpha_{mx}} \alpha d_A(\alpha)d_B\left(\alpha / r(\tau)\right)\mathrm{d}\alpha} \\[4ex] d_{A2}(\alpha \mid \tau \in \mathcal{P}_2) = \dfrac{\alpha d_A(\alpha)d_B\left(\alpha / r(\tau)\right)}{\displaystyle\int_{r(\tau)\beta_{mn}}^{r(\tau)\beta_{mx}} \alpha d_A(\alpha)d_B\left(\alpha / r(\tau)\right)\mathrm{d}\alpha} \\[4ex] d_{A3}(\alpha \mid \tau \in \mathcal{P}_3) = \dfrac{\alpha d_A(\alpha)d_B\left(\alpha / r(\tau)\right)}{\displaystyle\int_{\alpha_{mn}}^{\alpha_{mx}} \alpha d_A(\alpha)d_B\left(\alpha / r(\tau)\right)\mathrm{d}\alpha} \\[4ex] dA_{A4}(\alpha \mid \tau \in \mathcal{P}_4) = \dfrac{\alpha d_A(\alpha)d_B\left(\alpha / r(\tau)\right)}{\displaystyle\int_{\alpha_{mn}}^{r(\tau)\beta_{mx}} \alpha d_A(\alpha)d_B\left(\alpha / r(\tau)\right)\mathrm{d}\alpha} \end{cases} \tag{9.61}$$

as indicated in Table 5,

$$d_P(p \mid \tau) = \begin{cases} d_{P1}(p \mid \tau \in \mathcal{P}_1) = \dfrac{\left(p / \bar{p}^2(\tau)\right)d_A\left[p / \bar{p}(\tau)\right]d_B\left[p / \bar{q}(\tau)\right]}{\displaystyle\int_{r(\tau)\beta_{mn}}^{\alpha_{mx}} \alpha d_A(\alpha)d_B\left(\alpha / r(\tau)\right)\mathrm{d}\alpha} \\[4ex] d_{P2}(p \mid \tau \in \mathcal{P}_2) = \dfrac{\left(p / \bar{p}^2(\tau)\right)d_A\left[p / \bar{p}(\tau)\right]d_B\left[p / \bar{q}(\tau)\right]}{\displaystyle\int_{r(\tau)\beta_{mn}}^{r(\tau)\beta_{mx}} \alpha d_A(\alpha)d_B\left(\alpha / r(\tau)\right)\mathrm{d}\alpha} \\[4ex] d_{P3}(p \mid \tau \in \mathcal{P}_3) = \dfrac{\left(p / \bar{p}^2(\tau)\right)d_A\left[p / \bar{p}(\tau)\right]d_B\left[p / \bar{q}(\tau)\right]}{\displaystyle\int_{\alpha_{mn}}^{\alpha_{mx}} \alpha d_A(\alpha)d_B\left(\alpha / r(\tau)\right)\mathrm{d}\alpha} \\[4ex] d_{P4}(p \mid \tau \in \mathcal{P}_4) = \dfrac{\left(p / \bar{p}^2(\tau)\right)d_\alpha\left[p / \bar{p}(\tau)\right]d_\beta\left[p / \bar{q}(\tau)\right]}{\displaystyle\int_{\alpha_{mn}}^{r(\tau)\beta_{mx}} \alpha d_A(\alpha)d_B\left(\alpha / r(\tau)\right)\mathrm{d}\alpha} \end{cases} \tag{9.62}$$

as indicated in Table 6, and

$$d_T(\tau) = \begin{cases} d_{T1}(\tau \mid \tau \in \mathcal{P}_1) = \left[\mathrm{d}\left[\bar{p}(\tau) / \bar{q}(\tau)\right] / \mathrm{d}\tau\right]\displaystyle\int_{r(\tau)\beta_{mn}}^{\alpha_{mx}} \alpha d_A(\alpha)d_B\left(\alpha / r(\tau)\right)\mathrm{d}\alpha \\[3ex] d_{T2}(\tau \mid \tau \in \mathcal{P}_2) = \left[\mathrm{d}\left[\bar{p}(\tau) / \bar{q}(\tau)\right] / \mathrm{d}\tau\right]\displaystyle\int_{r(\tau)\beta_{mn}}^{r(\tau)\beta_{mx}} \alpha d_A(\alpha)d_B\left(\alpha / r(\tau)\right)\mathrm{d}\alpha \\[3ex] d_{T3}(\tau \mid \tau \in \mathcal{P}_3) = \left[\mathrm{d}\left[\bar{p}(\tau) / \bar{q}(\tau)\right] / \mathrm{d}\tau\right]\displaystyle\int_{\alpha_{mn}}^{\alpha_{mx}} \alpha d_A(\alpha)d_B\left(\alpha / r(\tau)\right)\mathrm{d}\alpha_3 \\[3ex] d_{T4}(\tau \mid \tau \in \mathcal{P}_4) = \left[\mathrm{d}\left[\bar{p}(\tau) / \bar{q}(\tau)\right] / \mathrm{d}\tau\right]\displaystyle\int_{\alpha_{mn}}^{r(\tau)\beta_{mx}} \alpha d_A(\alpha)d_B\left(\alpha / r(\tau)\right)\mathrm{d}\alpha \end{cases} \tag{9.63}$$

as a consequence of the definition of $d_T(\tau)$ in Eq. (8.2) and the equalities summarized in Eq. (9.49).

Given the relationships in Eqs. (9.61)-(9.63), the joint density functions $d_A(\alpha\,|\,\tau)d_T(\tau)$ and $d_P(p\,|\,\tau)d_T(\tau)$ can be expressed as

$$
\begin{aligned}
d_{AT}(\alpha,\tau) &= d_A(\alpha\,|\,\tau)d_T(\tau) \\
&= \begin{cases}
d_{A1}(\alpha\,|\,\tau\in\mathcal{P}_1)d_{T1}(\tau\,|\,\tau\in\mathcal{P}_1) & \text{for } \tau\in\mathcal{P}_1 \\
d_{A2}(\alpha\,|\,\tau\in\mathcal{P}_1)d_{T2}(\tau\,|\,\tau\in\mathcal{P}_2) & \text{for } \tau\in\mathcal{P}_2 \\
d_{A3}(\alpha\,|\,\tau\in\mathcal{P}_1)d_{T3}(\tau\,|\,\tau\in\mathcal{P}_3) & \text{for } \tau\in\mathcal{P}_3 \\
d_{A4}(\alpha\,|\,\tau\in\mathcal{P}_1)d_{T4}(\tau\,|\,\tau\in\mathcal{P}_4) & \text{for } \tau\in\mathcal{P}_4
\end{cases} \\
&= \left\{ d\!\left[\frac{\overline{p}(\tau)}{\overline{q}(\tau)}\right] \Big/ d\tau \right\} \alpha\, d_A(\alpha) d_B\big(\alpha\,/\,r(\tau)\big)
\end{aligned}
\tag{9.64}
$$

and

$$
\begin{aligned}
d_{PT}(p,\tau) &= d_P(p\,|\,\tau)d_T(\tau) \\
&= \begin{cases}
d_{P1}(p\,|\,\tau\in\mathcal{P}_1)d_{T1}(\tau\,|\,\tau\in\mathcal{P}_1) & \text{for } \tau\in\mathcal{P}_1 \\
d_{P2}(p\,|\,\tau\in\mathcal{P}_1)d_{T2}(\tau\,|\,\tau\in\mathcal{P}_2) & \text{for } \tau\in\mathcal{P}_2 \\
d_{P3}(p\,|\,\tau\in\mathcal{P}_1)d_{T3}(\tau\,|\,\tau\in\mathcal{P}_3) & \text{for } \tau\in\mathcal{P}_3 \\
d_{P4}(p\,|\,\tau\in\mathcal{P}_1)d_{T4}(\tau\,|\,\tau\in\mathcal{P}_4) & \text{for } \tau\in\mathcal{P}_4
\end{cases} \\
&= \left\{ d\!\left[\frac{\overline{p}(\tau)}{\overline{q}(\tau)}\right] \Big/ d\tau \right\}\!\left\{\frac{p}{\overline{p}^2(\tau)}\right\} d_A\big[p\,/\,\overline{p}(\tau)\big] d_B\big[p\,/\,\overline{q}(\tau)\big],
\end{aligned}
\tag{9.65}
$$

with the final equalities in Eqs. (9.64) and (9.65) resulting from the cancellation of terms in the numerator and denominator of each of the products

$$
d_{Ai}(\alpha\,|\,\tau\in\mathcal{P}_i)d_{Ti}(\tau\,|\,\tau\in\mathcal{P}_i) \quad \text{and} \quad d_{Pi}(p\,|\,\tau\in\mathcal{P}_i)d_{Ti}(\tau\,|\,\tau\in\mathcal{P}_i).
\tag{9.66}
$$

10. Double Integral Representation of $CDF_P(p\,|\,[t_{mn},t])$ Based on Time τ and Failure Value p

The results in Table 6 for link properties at specific link failure times provide a basis for determining the cumulative probability $CDF_P(p\,|\,[t_{mn},t])$ for link property p at time τ of link failure for link failure times in the time interval $[t_{mn},t]$. Specifically, the representation for $CDF_P(p\,|\,[t_{mn},t])$ in Eq. (8.1) for $\bar{p}(\tau)$ increasing and $\bar{q}(\tau)$ either decreasing or constant-valued can be restated as

$$
\begin{aligned}
CDF_P(p\,|\,[t_{mn},t]) =_1 \ & CDF_P(p\,|\,[\tau_{mn}(p),\tau_{mx}(t,p)]) \quad \text{for } CDF_P(p\,|\,[t_{mn},t]) > 0 \\
=_2 \ & \int_{\tau_{mn}(p)}^{\tau_{mx}(t,p)} \left[\int_{p_{mn}(\tau)}^{p_{mx}(p,\tau)} d_P(\tilde{p}\,|\,\tau)\mathrm{d}\tilde{p} \right] d_T(\tau)\mathrm{d}\tau \\
=_3 \ & \int_{\tau_{mn}(p)}^{\tau_{mx}(t,p)} \left[\int_{p_{mn}(\tau)}^{p_{mx}(p,\tau)} d_P(\tilde{p}\,|\,\tau) d_T(\tau)\mathrm{d}\tilde{p} \right] \mathrm{d}\tau \quad (10.1) \\
=_4 \ & \int_{\tau_{mn}(p)}^{\tau_{mx}(t,p)} \left\{ \int_{p_{mn}(\tau)}^{p_{mx}(p,\tau)} \left\{ \mathrm{d}\left[\frac{\bar{p}(\tau)}{\bar{q}(\tau)} \right] / \mathrm{d}\tau \right\} \left\{ \frac{\tilde{p}}{\bar{p}^2(\tau)} \right\} d_A[\tilde{p}\,/\,\bar{p}(\tau)] d_B[\tilde{p}\,/\,\bar{q}(\tau)]\mathrm{d}\tilde{p} \right\} \mathrm{d}\tau \\
=_5 \ & \int_{\tau_{mn}(p)}^{\tau_{mx}(t,p)} \left\{ \int_{p_{mn}(\tau)}^{p_{mx}(p,\tau)} \left\{ \mathrm{d}\left[\frac{1}{r(\tau)} \right] / \mathrm{d}\tau \right\} \left\{ \frac{\tilde{p}}{\bar{p}^2(\tau)} \right\} d_A[\tilde{p}\,/\,\bar{p}(\tau)] d_B[\tilde{p}\,/\,\bar{q}(\tau)]\mathrm{d}\tilde{p} \right\} \mathrm{d}\tau,
\end{aligned}
$$

with (i) $\tau_{mn}(p)$ = first time with a link failure value $\le p$, (ii) $\tau_{mx}(t,p) = \min\{t, \tau_{mx}(p) = $ last time with a link failure value $\le p\}$, (iii) $p_{mx}(p,\tau) = \min\{p, p_{mx}(\tau)\}$ for $\tau_{mn}(p) \le \tau \le \tau_{mx}(t,p)$, (iv) $d_P(\tilde{p}\,|\,\tau)$ is the density function for link failure value conditional on link failure at time τ, (v) $d_T(\tau)$ defined in Eqs. (8.2) and (9.63), and (vi) the substitution producing Equality 4 following from Eq. (9.65). As shown in Eq. (12.9), the quotient $\tilde{p}\,/\,\bar{p}^2(\tau)$ in Eq. (10.1) can be removed by the change of variables $\alpha(\tilde{p}) = \tilde{p}\,/\,\bar{p}(\tau)$. For convenience, the limits of integration in Eq. (10.1) are summarized in Table 7.

Table 7 Integration limits in Eqs. (10.1) and (10.10) for $\overline{p}(\tau)$ increasing and $\overline{q}(\tau)$ either decreasing or constant-valued.

$\tau_{mn}(p) = $ first time that link failure could occur at a property value $\tilde{p} \leq p$

$$= \begin{cases} \tau_f = r^{-1}(\alpha_{mx} / \beta_{mn}) & \text{for } p_f \leq p \leq p_{mx} \\ \tau_f(p) = \overline{q}^{-1}(p / \beta_{mn}) & \text{for } p_{mn} \leq p < p_f \text{ (not relevant for } \overline{q}(\tau) = c \text{ because } p_{mn} = p_f) \end{cases}$$

$\tau_{mx}(p) = $ last time that link failure could occur at a property value $\tilde{p} \leq p$

$$= \begin{cases} \tau_l = r^{-1}(\alpha_{mn} / \beta_{mx}) & \text{for } p_l < p \leq p_{mx} \text{ (not relevant for } \overline{q}(\tau) = c \text{ because } p_l = p_{mx}) \\ \tau_l(p) = \overline{p}^{-1}(p / \alpha_{mn}) & \text{for } p_{mn} \leq p \leq p_l \end{cases}$$

$\tau_{mx}(t, p) = \min\{t, \tau_{mx}(p)\}$

$[p_{mn}(\tau), p_{mx}(\tau)] = $ interval of link failure values p at time τ for $\alpha_{mn} / \beta_{mn} < \alpha_{mx} / \beta_{mx}$

$$= \begin{cases} [\beta_{mn}\overline{q}(\tau), \alpha_{mx}\overline{p}(\tau)] & \text{for } \tau \in P_1 = \{\tau : \tau_f = r^{-1}(\alpha_{mx} / \beta_{mn}) \leq \tau \leq r^{-1}(\alpha_{mx} / \beta_{mx}) = \tau_{mx}\} \\ [\beta_{mn}\overline{q}(\tau), \beta_{mx}\overline{q}(\tau)] & \text{for } \tau \in P_2 = \{\tau : \tau_{mx} = r^{-1}(\alpha_{mx} / \beta_{mx}) \leq \tau \leq r^{-1}(\alpha_{mn} / \beta_{mn}) = \tau_{mn}\} \\ [\alpha_{mn}\overline{p}(\tau), \beta_{mx}\overline{q}(\tau)] & \text{for } \tau \in P_4 = \{\tau : \tau_{mn} = r^{-1}(\alpha_{mn} / \beta_{mn}) \leq \tau \leq r^{-1}(\alpha_{mn} / \beta_{mx}) = \tau_l\} \end{cases}$$

$[p_{mn}(\tau), p_{mx}(\tau)] = $ interval of link failure values p at time τ for $\alpha_{mx} / \beta_{mx} < \alpha_{mn} / \beta_{mn}$

$$= \begin{cases} [\beta_{mn}\overline{q}(\tau), \alpha_{mx}\overline{p}(\tau)] & \text{for } \tau \in P_1 = \{\tau : \tau_f = r^{-1}(\alpha_{mx} / \beta_{mn}) \leq \tau \leq r^{-1}(\alpha_{mn} / \beta_{mn}) = \tau_{mn}\} \\ [\alpha_{mn}\overline{p}(\tau), \alpha_{mx}\overline{p}(\tau)] & \text{for } \tau \in P_3 = \{\tau : \tau_{mn} = r^{-1}(\alpha_{mn} / \beta_{mn}) \leq \tau \leq r^{-1}(\alpha_{mx} / \beta_{mx}) = \tau_{mx}\} \\ [\alpha_{mn}\overline{p}(\tau), \beta_{mx}\overline{q}(\tau)] & \text{for } \tau \in P_4 = \{\tau : \tau_{mx} = r^{-1}(\alpha_{mx} / \beta_{mx}) \leq \tau \leq r^{-1}(\alpha_{mn} / \beta_{mx}) = \tau_l\} \end{cases}$$

$[p_{mn}(\tau), p_{mx}(\tau)] = $ interval of link failure values p at time τ for $\alpha_{mn} / \beta_{mn} = \alpha_{mx} / \beta_{mx}$

$$= \begin{cases} [\beta_{mn}\overline{q}(\tau), \alpha_{mx}\overline{p}(\tau)] & \text{for } \tau \in P_1 = \{\tau : \tau_f = r^{-1}(\alpha_{mx} / \beta_{mn}) \leq \tau \leq r^{-1}(\alpha_{mx} / \beta_{mx}) = \tau_{mx}\} \\ [\alpha_{mn}\overline{p}(\tau), \beta_{mx}\overline{q}(\tau)] & \text{for } \tau \in P_4 = \{\tau : \tau_{mx} = r^{-1}(\alpha_{mx} / \beta_{mx}) \leq \tau \leq r^{-1}(\alpha_{mn} / \beta_{mx}) = \tau_l\} \end{cases}$$

$p_{mx}(p, \tau) = \min\{p, p_{mx}(\tau)\}$

Examples of the regions integrated over in Eq. (10.1) are presented in Fig. 13. In Fig. 13a, $p = 5.75$ and the highlighted region corresponds to the region integrated over to determine

$$CDF_P(p = 5.75 \,|\, [\tau_{mn}(p), \tau_{mx}(t, p)]) = CDF_P(5.75 \,|\, [\tau_f, \tau_l]) \quad \text{for} \ \ t = \tau_l \tag{10.2}$$

with the time-dependent property bounds defined by

$$p_{mn}(\tau) = \begin{cases} \beta_{mn} \bar{q}(\tau) & \text{for} \ \ \tau_f < \tau \le \tau_{mn} \\ \alpha_{mn} \bar{p}(\tau) & \text{for} \ \ \tau_{mn} < \tau \le \tau_l \end{cases} \tag{10.3}$$

and

$$p_{mx}(p = 5.75, \tau) = \begin{cases} \alpha_{mx} \bar{p}(\tau) & \text{for} \ \ \tau_f < \tau \le \tau_f(5.75) \\ p & \text{for} \ \ \tau_f(p) < \tau \le \tau_l(5.75) \\ \beta_{mx} \bar{q}(\tau) & \text{for} \ \ \tau_l(5.75) < \tau \le \tau_l. \end{cases} \tag{10.4}$$

The region integrated over to determine

$$CDF_P(p = 5.75 \,|\, [\tau_{mn}(p), \tau_{mx}(t, p)]) = CDF_P(5.75 \,|\, [\tau_f, t]) \quad \text{for} \ \ \tau_f < t < \tau_l \tag{10.5}$$

is the subset of the highlighted region in Fig. 13a bounded on the right by a vertical line originating at t on the time axis (e.g., as illustrated by t_1, t_2, t_3, t_4 in Fig. 13a). In Fig. 13b, $p = 4.5$ and the highlighted region corresponds to the region integrated over to determine

$$CDF_P(p = 4.5 \,|\, [\tau_{mn}(p), \tau_{mx}(t, p)]) = CDF_P(4.5 \,|\, [\tau_f(4.5), \tau_l]) \quad \text{for} \ \ t = \tau_l \tag{10.6}$$

with $p_{mn}(\tau)$ defined the same as in Eq. (10.3) and

$$p_{mx}(p = 4.5, \tau) = \begin{cases} p & \text{for} \ \ \tau_f(4.5) < \tau \le \tau_l(4.5) \\ \beta_{mx} \bar{q}(\tau) & \text{for} \ \ \tau_l(4.5) < \tau \le \tau_l. \end{cases} \tag{10.7}$$

The region integrated over to determine

$$CDF_P(p = 4.5 \,|\, [\tau_{mn}(p), \tau_{mx}(t, p)]) = CDF_P(4.5 \,|\, [\tau_f(4.5), t]) \quad \text{for} \ \ \tau_f(4.5) < t < \tau_l \tag{10.8}$$

is the subset of the highlighted region in Fig. 13b bounded on the right by a vertical line originating at t on the time axis (e.g., as illustrated by t_1, t_2, t_3 in Fig. 13b).
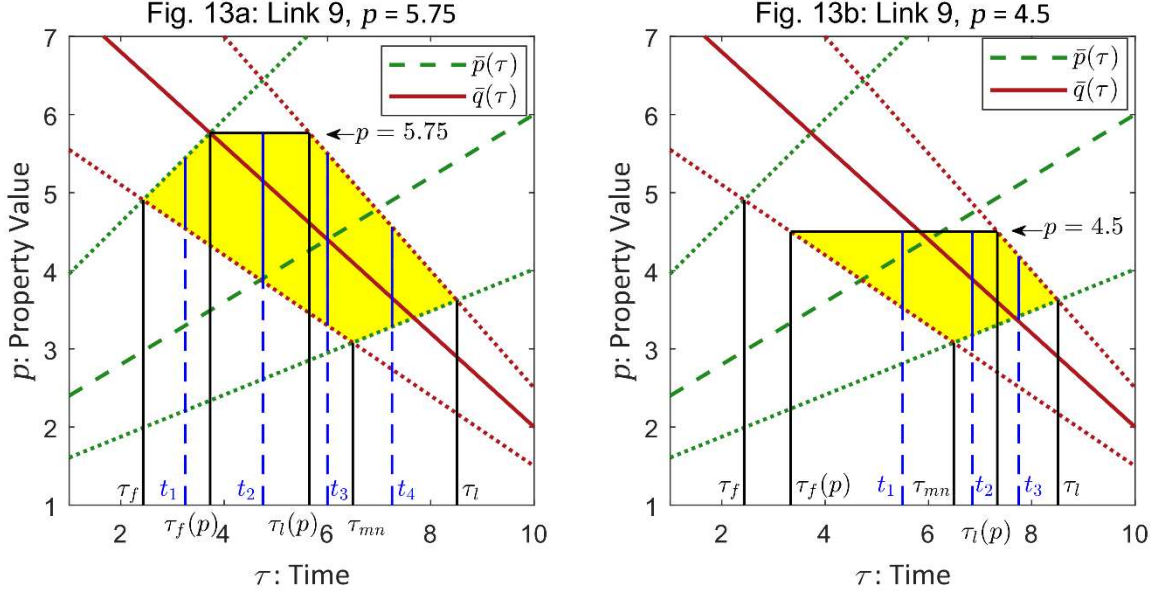
Fig. 13 Illustration of regions integrated over to obtain $CDF_P(p\,|\,[t_{mn},t])$ in Eq. (10.1) for Link 9 defined by $\bar{p}(\tau) = 2 + 0.4\tau$, $\bar{q}(\tau) = 8 - 0.6\tau$, $[\alpha_{mn},\alpha_{mx}] = [0.67,1.65]$, and $[\beta_{mn},\beta_{mx}] = [0.75,1.25]$: (a) Link 9 with $p = 5.75$, and (b) Link 9 with $p = 4.5$.

The representation for $CDF_P(p\,|\,[t_{mn},t])$ in Eq. (10.1) and the representations for $CDF_P(p\,|\,[t_{mn},t])$ developed in Sect. 4 and summarized in Table 2 do not look very similar. However, they are equivalent as shown in Sect. 12.

Most use of the representation for $CDF_P(p\,|\,[t_{mn},t])$ in Eq. (10.1) will probably be for

$$t = \tau_{mx}(p) = \text{ last time with a link failure value } \le p. \tag{10.9}$$

In this case, the representation for $CDF_P(p\,|\,[t_{mn},t])$ in Eq. (10.1) becomes

$$CDF_P(p\,|\,[t_{mn},\tau_{mx}(p)]) = CDF_P(p\,|\,[\tau_{mn}(p),\tau_{mx}(p)]) \text{ for } CDF_P(p\,|\,[t_{mn},\tau_{mx}(p)]) > 0$$

$$= \int_{\tau_{mn}(p)}^{\tau_{mx}(p)} \left\{ \int_{P_{mn}(\tau)}^{P_{mx}(p,\tau)} \left\{ d\left[\frac{1}{r(\tau)}\right] / d\tau \right\} \left\{ \frac{\tilde{p}}{\bar{p}^2(\tau)} \right\} d_A\left[\tilde{p}\,/\,\bar{p}(\tau)\right] d_B\left[\tilde{p}\,/\,\bar{q}(\tau)\right] d\tilde{p} \right\} d\tau. \tag{10.10}$$

Although the representation for $CDF_P(p\,|\,[t_{mn},t])$ in Eq. (10.1) looks very complicated, it probably provides a more efficient structure for a quadrature procedure to evaluate the CDF defined by

$$\left[p, CDF_P(p\,|\,[\tau_{mn}(p),\tau_{mx}(t,p)])\right] \tag{10.11}$$

for

$$\alpha_{mn} \overline{p}[r^{-1}(\alpha_{mn} / \beta_{mn})] = p_{mn} \le p \le p_{mx} = \alpha_{mx} \overline{p}[r^{-1}(\alpha_{mx} / \beta_{mx})] \tag{10.12}$$

than the integral representations for $CDF_P(p \,|\, [t_{mn}, t])$ developed in Sect. 4 and summarized in Table 2. This statement is made because evaluation of the integral in Eq. (10.1) involves integrating $d_{PT}(\tilde{p}, \tau)$ over a subregion $\mathcal{R}(p)$ of the region $\mathcal{R}$ defined by the curves $\alpha_{mn} \overline{p}(\tau)$, $\alpha_{mx} \overline{p}(\tau)$, $\beta_{mn} \overline{q}(\tau)$ and $\beta_{mx} \overline{q}(\tau)$ as illustrated in Fig. 1 with time τ on the abscissa and property value p on the ordinate. Specifically, (i) the upper boundary of $\mathcal{R}(p)$ is defined by a horizontal line originating from a value of p on the ordinate and (ii) the right boundary is defined by a vertical line originating from a value t on the abscissa. As a result, the numerical evaluation of $CDF_P(p \,|\, [\tau_{mn}(p), \tau_{mx}(t, p)])$ can make full use of the calculations performed to obtain $CDF_P(\tilde{p} \,|\, [\tau_{mn}(\tilde{p}), \tau_{mx}(t, \tilde{p})])$ for $p_{mn} \le \tilde{p} < p$ as this evaluation for increasing values of p simply involves systematically increasing the upper limit of integration for p. A similar relationship holds for increasing values for t. However, the need to include the derivatives $d[1/r(\tau)] / d\tau$ in the integrand could pose a numerical challenge.

If desired, $CDF_P(p \,|\, [t_{mn}, t])$ can also be defined with the order of integration in Eq. (10.1) reversed so that the outer integral is on property value p and the inner integral is on time τ. The result of this reversal is

$$CDF_P(p \,|\, [t_{mn}, t]) = CDF_P(p \,|\, [\tau_{mn}(p), \tau_{mx}(t, p)]) \text{ for } CDF_P(p \,|\, [t_{mn}, t]) > 0$$

$$= \int_{p_{mn}(p,t)}^{p_{mx}(p,t)} \left\{ \int_{\tau_{mn}(p,t)}^{\tau_{mx}(p,t)} \left\{ d\left[\frac{1}{r(\tau)}\right] / d\tau \right\} \left\{ \frac{p}{\overline{p}^2(\tau)} \right\} d_A\left[p / \overline{p}(\tau)\right] d_B\left[p / \overline{q}(\tau)\right] d\tau \right\} dp \tag{10.13}$$

with the limits of integration defined in Table 8. As the outer variable of integration in Eq. (10.13) is property value, the double integral in Eq. (10.13) may be more convenient for approximating the CDF indicated in Eq. (10.11) than the double integral in Eq. (10.1).

Table 8 Integration limits in Eq. (10.13) for $\bar{p}(\tau)$ increasing and $\bar{q}(\tau)$ either decreasing or constant-valued.

$$p_{mn}(p,t) = \begin{cases} \text{undefined} & \text{for } p_{mn} \leq p \leq p_f \text{ and } t < t_f(p) \\ \beta_{mn}\bar{q}(t) \text{ for (i) } p_{mn} < p < p_f \text{ and } \tau_f(p) < t < \tau_{mn} \text{ (not relevant for } \bar{q}(\tau) = c \\ \qquad \text{because } p_{mn} = p_f) \text{ or (ii) } p_f \leq p \leq p_{mx} \text{ and } \tau_f \leq t < \tau_{mn} \\ p_{mn} \text{ for } \tau_{mn} \leq \min\{\tau_f(p), t\} \end{cases}$$

$$p_{mx}(p,t) = \begin{cases} \text{undefined} & \text{for } p_{mn} \leq p \leq p_f \text{ and } t < t_f(p) \\ \alpha_{mx}\bar{p}(t) & \text{for } p_f \leq p \leq p_{mx} \text{ and } t_f \leq t < t_f(p) \\ p & \text{for } t_f(p) \leq t \end{cases}$$

$$\tau_{mn}(p,t) = \begin{cases} \text{undefined} & \text{for } p_{mn} \leq p \leq p_f \text{ and } t < t_f(p) \\ \bar{q}^{-1}(p/\beta_{mn}) \text{ for } p_{mn} < p < p_f \text{ and } \tau_f(p) < t \text{ (not relevant for } \bar{q}(\tau) = c \\ \qquad \text{because } p_{mn} = p_f) \\ \bar{p}^{-1}(p/\alpha_{mx}) \text{ for } p_f \leq p \leq p_{mx} \text{ and } \tau_f \leq t \end{cases}$$

$$\tau_{mx}(p,t) = \begin{cases} \text{undefined} & \text{for } p_{mn} \leq p \leq p_f \text{ and } t < t_f(p) \\ \min\{t, \bar{p}^{-1}(p/\alpha_{mn})\} & \text{for } p_{mn} \leq p < p_l \\ \min\{t, \bar{q}^{-1}(p/\beta_{mx})\} & \text{for } p_l \leq p \leq p_{mx} \text{ (not relevant for } \bar{q}(\tau) = c \\ \qquad \text{because } p_l = p_{mx}) \end{cases}$$

11. Illustration and Verification of Double Integral Representation of $CDF_P(p\,|\,[t_{mn},t])$ **Based on Time** τ **and Failure Value** p

As an example, two evaluations of the CDFs defined by $CDF_P(p\,|\,[\tau_{mn},\tau_{mx}(p)])$ for $p_{mn} \leq p < p_{mx}$ and the links defined in Table 1 and illustrated in Fig. 1 are shown in Fig. 14, with (i) one evaluation obtained with sampling as indicated in Eq. (6.1) with samples of size $nS = 10^6$ and (ii) the other evaluation obtained by numerical approximation of the double integral in Eq. (10.10) with the MATLAB program **TwoD** [55]. The similarity of the CDFs obtained with the two evaluation procedures provides a strong verification result that the representations for $CDF_P(p\,|\,[t_{mn},t])$ and $CDF_P(p\,|\,[\tau_{mn},\tau_{mx}(p)])$ in Eqs. (10.1) and (10.10) have been correctly derived. Further, as shown by the essentially identical match of the results for the sampling-based procedure and the quadrature-based procedure in Fig. 11a, the numerical approximations of the double integral in Eq. (10.10) also matches the numerical approximations of the integrals derived in Sect. 4 and summarized in Table 2.
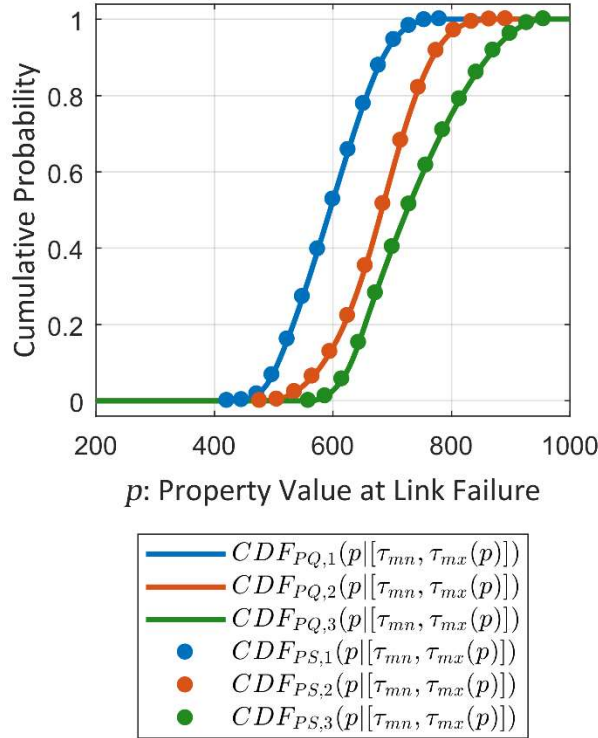
Fig. 14 Two evaluations of the CDFs defined by $CDF_P(p\,|\,[\tau_{mn},\tau_{mx}(p)])$ for $p_{mn} \leq p < p_{mx}$ and Links 1, 2, 3 defined in Table 1 and illustrated in Fig. 1, with (i) $CDF_{PQ,i}(p\,|\,[\tau_{mn},\tau_{mx}(p)])$ for Link i obtained by numerical approximation of the double integral in Eq. (10.10) and (ii) $CDF_{PS,i}(p\,|\,[\tau_{mn},\tau_{mx}(p)])$ for Link i obtained with sampling as indicated in Eq. (6.1).

An important aspect of the positive verification results for Eqs. (10.1) and (10.10) is that a number of results obtained as parts of their derivation will be important components of results

obtained in later analyses [46; 47]. The verification of Eqs. (10.1) and (10.10) provides a strong indication that results underlying their derivation have also been derived correctly.

12. Equivalence of Different Representations for $CDF_P(p\,|\,[t_{mn},t])$

12.1 Preliminaries: Equivalence of Different Representations for $CDF_P(p\,|\,[t_{mn},t])$

The equivalence of the representation for $CDF_P(p\,|\,[t_{mn},t])$ in Eq. (10.1) and the representations for $CDF_P(p\,|\,[t_{mn},t])$ developed in Sect. 4 and summarized in Table 2 (i.e., for the three configurations defined in Eqs. (4.16)-(4.18) is now established. As in sect. 4, the results are derived for $\bar{p}(\tau)$ increasing and $\bar{q}(\tau)$ decreasing (i.e., for Case 1 as defined in Eq. (2.8)), which assures that $\bar{p}^{-1}(\tau)$, $\bar{q}^{-1}(\tau)$ and $r^{-1}(\tau)$ exist. Establishing this equivalence provides an additional verification of the correctness of both (i) the representations for $CDF_P(p\,|\,[t_{mn},t])$ derived in Sect. 4 and (ii) the representation for $CDF_P(p\,|\,[t_{mn},t])$ in Eq. (10.1).

In concept, the desired equivalence can be obtained with use of the change of variables theorem for double integrals (see Ref. [68], Sect. 14.4, for technical details). Specifically, this theorem states that

$$\iint_{\mathcal{N}} f[x(u,v),y(u,v)]\,|\,D(u,v)\,|\,\mathrm{d}u\mathrm{d}v = \iint_{\mathcal{M}} f(x,y)\mathrm{d}x\mathrm{d}y \tag{12.1}$$

for (i) the mapping

$$x = x(u,v),\, y = y(u,v) \tag{12.2}$$

from the space $\mathcal{N} \sim \{(u,v)\}$ to the space $\mathcal{M} \sim \{(x,y)\}$ and (ii)

$$D(u,v) = \begin{vmatrix} \partial x(u,v)/\partial u & \partial x(u,v)/\partial v \\ \partial y(u,v)/\partial u & \partial y(u,v)/\partial v \end{vmatrix}. \tag{12.3}$$

Use of the indicated change of variables theorem with the integral in Eq. (10.1) defining $CDF_P(p\,|\,[t_{mn},t])$ produces

$$
\begin{aligned}
CDF_P(p\,|\,[t_{mn},t]) &= CDF_P(p\,|\,[\tau_{mn}(p),\tau_{mx}(t,p)])\;\;\text{for } CDF_P(p\,|\,[t_{mn},t]) > 0 \\
&= \int_{\tau_{mn}(p)}^{\tau_{mx}(t,p)} \left\{ \int_{p_{mn}(\tau)}^{p_{mx}(p,\tau)} \{\mathrm{d}[1/r(\tau)]/\mathrm{d}\tau\}\{\tilde{p}/\bar{p}^2(\tau)\} d_A[\tilde{p}/\bar{p}(\tau)] d_B[\tilde{p}/\bar{q}(\tau)]\mathrm{d}\tilde{p} \right\}\mathrm{d}\tau \\
&= \int_{\tau_{mn}(p)}^{\tau_{mx}(t,p)} \left\{ \int_{p_{mn}(\tau)}^{p_{mx}(p,\tau)} |D(\alpha,\beta)|\,f[\alpha(\tau,\tilde{p}),\beta(\tau,\tilde{p})]\mathrm{d}\tilde{p} \right\}\mathrm{d}\tau \\
&= \iint_{\mathcal{M}} d_A(\alpha)d_B(\beta)\mathrm{d}\beta\mathrm{d}\alpha
\end{aligned}
\tag{12.4}
$$

with

$$\alpha = \alpha(\tau,\tilde{p}) = \tilde{p}/\bar{p}(\tau),\, \beta = \beta(\tau,\tilde{p}) = \tilde{p}/\bar{q}(\tau), \tag{12.5}$$

$$f[\alpha(\tau,\tilde{p}),\beta(\tau,\tilde{p})] = f[\tilde{p}/\overline{p}(\tau),\tilde{p}/\overline{q}(\tau)] = d_A[\tilde{p}/\overline{p}(\tau)]d_B[\tilde{p}/\overline{q}(\tau)], \qquad (12.6)$$

$$\begin{aligned}
D(\alpha,\beta) &= \begin{vmatrix} \partial\alpha(\tau,\tilde{p})/\partial\tau & \partial\alpha(\tau,\tilde{p})/\partial\tilde{p} \\ \partial\beta(\tau,\tilde{p})/\partial\tau & \partial\beta(\tau,\tilde{p})/\partial\tilde{p} \end{vmatrix} \\[2mm]
&= \begin{vmatrix} \partial[\tilde{p}/\overline{p}(\tau)]/\partial\tau & \partial[\tilde{p}/\overline{p}(\tau)]/\partial\tilde{p} \\ \partial[\tilde{p}/\overline{q}(\tau)]/\partial\tau & \partial[\tilde{p}/\overline{q}(\tau)]/\partial\tilde{p} \end{vmatrix} \\[2mm]
&= \begin{vmatrix} -\tilde{p}\overline{p}\,'(\tau)/\overline{p}^2(\tau) & 1/\overline{p}(\tau) \\ -\tilde{p}\overline{q}\,'(\tau)/\overline{q}^2(\tau) & 1/\overline{q}(\tau) \end{vmatrix} \text{ with } \begin{cases} \overline{p}\,'(\tau) = \mathrm{d}\overline{p}(\tau)/\mathrm{d}\tau \\ \overline{q}\,'(\tau) = \mathrm{d}\overline{q}(\tau)/\mathrm{d}\tau \end{cases} \\[2mm]
&= [-\tilde{p}\overline{q}\,'(\tau)/\overline{q}^2(\tau)][1/\overline{p}(\tau)] - [-\tilde{p}\overline{p}\,'(\tau)/\overline{p}^2(\tau)][1/\overline{q}(\tau)] \qquad (12.7) \\[2mm]
&= \left[\frac{-\overline{p}(\tau)\overline{q}\,'(\tau) + \overline{q}(\tau)\overline{p}\,'(\tau)}{\overline{q}^2(\tau)}\right]\left[\frac{\tilde{p}}{\overline{p}^2(\tau)}\right] \\[2mm]
&= \left\{\mathrm{d}[1/r(\tau)]/\mathrm{d}\tau\right\}\left\{\tilde{p}/\overline{p}^2(\tau)\right\} > 0 \text{ from } \begin{cases} 1/r(\tau) = \overline{p}(\tau)/\overline{q}(\tau) \\ 1/r(\tau) \text{ increasing} \end{cases} \\[2mm]
&= |D(\alpha,\beta)|,
\end{aligned}$$

and $\mathcal{M}$ defined by the transformation of

$$\mathcal{N} = [\tau_{mn}(p),\tau_{mx}(t,p)] \times [p_{mn}(\tau),p_{mx}(p,\tau)] \qquad (12.8)$$

as indicated in Eq. (12.5). The integrand in the final integral in Eq. (12.4) matches the integrands in the representations for $CDF_P(p\,|\,[t_{mn},t])$ developed in Sect. 4 and summarized in Table 2. However, deriving the corresponding limits of integration that in effect define the set $\mathcal{M}$ is difficult. Fortunately, the change of variables leading to the final integral in Eq. (12.4) can be derived using specific properties of the first integral in Eq. (12.4) in a manner that leads to definitions for $\mathcal{M}$ that correspond to specific definitions for $\mathcal{N}$.

The indicated change of variables is developed in the following manner starting from the representation for $CDF_P(p\,|\,[t_{mn},t])$ in Eq. (10.1):

$$CDF_P(p\,|\,[t_{mn},t]) =_1 CDF_P(p\,|\,[\tau_{mn}(p),t]) \quad \text{for } \tau_{mn}(p) < t \le \tau_{mx}(p)$$

$$=_2 \int_{\tau_{mn}(p)}^{t} \left\{ \int_{p_{mn}(\tau)}^{p_{mx}(p,\tau)} \left\{ d[1/r(\tau)]/d\tau \right\} \left\{ \tilde{p}/\overline{p}^2(\tau) \right\} d_A\left[\tilde{p}/\overline{p}(\tau)\right] d_B\left[\tilde{p}/\overline{q}(\tau)\right] d\tilde{p} \right\} d\tau$$

$$=_3 \int_{\tau_{mn}(p)}^{t} \left\{ \int_{\alpha[p_{mn}(\tau)]}^{\alpha[p_{mx}(p,\tau)]} \left\{ d[1/r(\tau)]/d\tau \right\} \alpha d_A(\alpha) d_B\left(\alpha/r(\tau)\right) d\alpha \right\} d\tau \quad \text{for } \alpha(\tilde{p}) = \tilde{p}/\overline{p}(\tau)$$

$$=_4 \int_{\tau_{mn}(p)}^{t} \left\{ \int_{p_{mn}(\tau)/\overline{p}(\tau)}^{p_{mx}(p,\tau)/\overline{p}(\tau)} \left\{ d[1/r(\tau)]/d\tau \right\} \alpha d_A(\alpha) d_B\left(\alpha/r(\tau)\right) d\alpha \right\} d\tau \tag{12.9}$$

$$=_5 \int_{\alpha_{mn}(t)}^{\alpha_{mx}(p)} \left\{ \int_{\tau_{mn}(\alpha)}^{\tau_{mx}(t,\alpha)} \left\{ d[1/r(\tau)]/d\tau \right\} \alpha d_B\left(\alpha/r(\tau)\right) d\tau \right\} d_A(\alpha) d\alpha$$

$$=_6 \int_{\alpha_{mn}(t)}^{\alpha_{mx}(p)} \left\{ \int_{\beta[\tau_{mn}(\alpha)]}^{\beta[\tau_{mx}(t,\alpha)]} d_B(\beta) d\beta \right\} d_A(\alpha) d\alpha \quad \text{for } \beta(\tau) = \alpha/r(\tau)$$

$$=_7 \int_{\alpha_{mn}(t)}^{\alpha_{mx}(p)} \left\{ \int_{\alpha/r[\tau_{mn}(\alpha)]}^{\alpha/r[\tau_{mx}(t,\alpha)]} d_B(\beta) d\beta \right\} d_A(\alpha) d\alpha,$$

where:

(i) Equalities 1 and 2 follow from Eq. (10.1) with

$$t = \tau_{mx}(t,p) = \min\{t, \tau_{mx}(p)\} \quad \text{for } \tau_{mn}(p) < t \le \tau_{mx}(p) \tag{12.10}$$

as indicated in the definition of $\tau_{mx}(t,p)$ following Eq. (10.1). As examples, regions potentially being integrated over are illustrated by the high-lighted areas in Fig. 13a and Fig. 13b with the limits of integration for τ defined by the intervals

$$[\tau_{mn}(p),t] = \begin{cases} [\tau_f, t] & \text{for } p_f < p \le p_{mx} \text{ in Fig. 13a} \\ [\tau_f(p), t] & \text{for } p_{mn} < p \le p_f \text{ in Fig. 13b} \end{cases} \tag{12.11}$$

on the abscissa and the limits of integration for $\tilde{p}$ on the ordinate defined by the intervals $[p_{mn}(\tau), p_{mx}(p,\tau)]$ with

$$p_{mn}(\tau) = \begin{cases} \beta_{mn}\overline{q}(\tau) & \text{for } \tau_f < \tau \le \tau_{mn} \\ \alpha_{mn}\overline{p}(\tau) & \text{for } \tau_{mn} < \tau \le \tau_l \end{cases} \tag{12.12}$$

and

$$p_{mx}(p,\tau) = \begin{cases} \alpha_{mx}\overline{p}(\tau) & \text{for } \tau_f < \tau \le \tau_f(p) \\ p & \text{for } \tau_f(p) < \tau \le \tau_l(p) \\ \beta_{mx}\overline{q}(\tau) & \text{for } \tau_l(p) < \tau \le \tau_l \end{cases} \tag{12.13}$$

for $\tau_{mn}(p) \le \tau \le t$.

(ii) Equalities 3 and 4 result from a change of variables with $\alpha(\tilde{p}) = \tilde{p} / \overline{p}(\tau)$. In continuation of the example in Fig. 13a, the region potentially being integrated over is transformed into the region shown in Fig. 15a. The limit of integration for τ defined by the interval $[\tau_{mn}(p), t] = [\tau_f, t]$ on the abscissa is unchanged, but the limits of integration for α on the ordinate are now defined by the intervals

$$[\alpha_{mn}(\tau), \alpha_{mx}(p, \tau)] = [p_{mn}(\tau) / \overline{p}(\tau), p_{mx}(p, \tau) / \overline{p}(\tau)] \tag{12.14}$$

with

$$\alpha_{mn}(\tau) = p_{mn}(\tau) / \overline{p}(\tau) = \begin{cases} \beta_{mn}\overline{q}(\tau) / \overline{p}(\tau) = \beta_{mn}r(\tau) & \text{for } \tau_f < \tau \leq \tau_{mn} \\ \alpha_{mn}\overline{p}(\tau) / \overline{p}(\tau) = \alpha_{mn} & \text{for } \tau_{mn} < \tau \leq \tau_l \end{cases} \tag{12.15}$$

and

$$\alpha_{mx}(p, \tau) = p_{mx}(p, \tau) / \overline{p}(\tau) = \begin{cases} \alpha_{mx}\overline{p}(\tau) / \overline{p}(\tau) = \alpha_{mx} & \text{for } \tau_{mn} < \tau \leq \tau_f(p) \\ p / \overline{p}(\tau) = p / \overline{p}(\tau) & \text{for } \tau_f(p) < \tau \leq \tau_l(p) \\ \beta_{mx}\overline{q}(\tau) / \overline{p}(\tau) = \beta_{mx}r(\tau) & \text{for } \tau_l(p) < \tau \leq \tau_l. \end{cases} \tag{12.16}$$

The examples in Fig. 13a and Fig. 15a are for Link 9 with $p_f \leq p \leq p_{mx}$. Additional examples for Link 9 with $p_{mn} < p < p_f$ are given in Fig. 13b and Fig. 15b. For this example, $\alpha_{mn}(\tau)$ is defined the same as in Eq. (12.15), and $\alpha_{mx}(p, \tau)$ is defined by

$$\alpha_{mx}(p, \tau) = p_{mx}(p, \tau) / \overline{p}(\tau) = \begin{cases} p / \overline{p}(\tau) = p / \overline{p}(\tau) & \text{for } \tau_f(p) < \tau \leq \tau_l(p) \\ \beta_{mx}\overline{q}(\tau) / \overline{p}(\tau) = \beta_{mx}r(\tau) & \text{for } \tau_l(p) < \tau \leq \tau_l. \end{cases} \tag{12.17}$$

This distinction is important because it affects the range of α values that can result in link failure as illustrated in Fig. 15a and Fig. 15b.
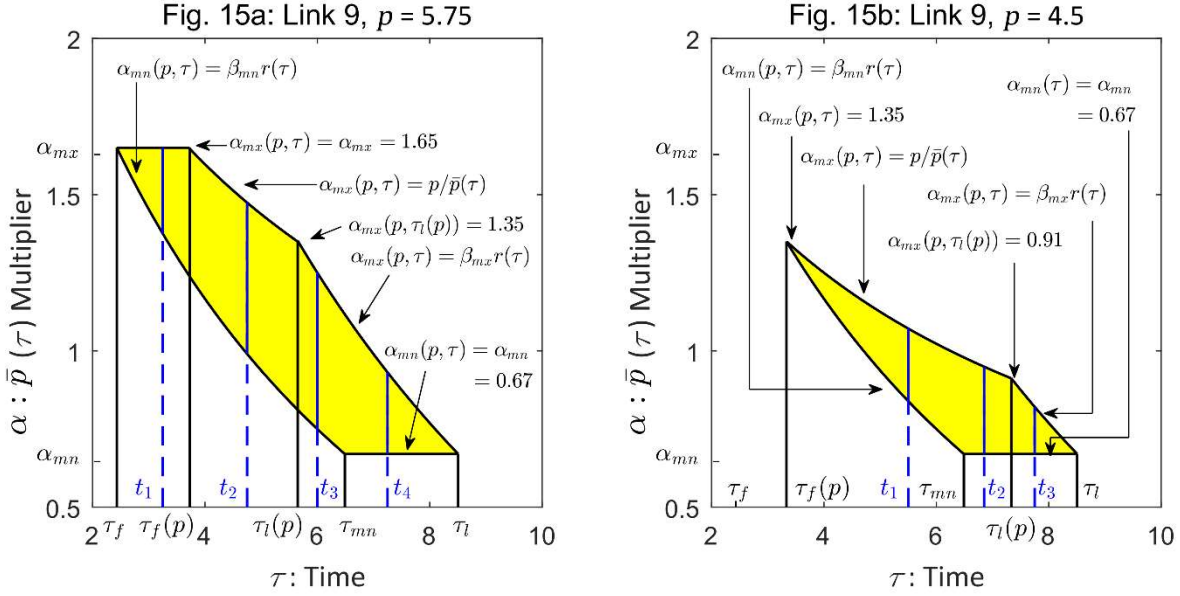
Fig. 15 Illustration of change of variables $\alpha(\tilde{p}) = \tilde{p} / \overline{p}(\tau)$ in Equalities 3 and 4 of Eq. (12.9): (a) Link 9 with $p = 5.75$, and (b) Link 9 with $p = 4.5$.

(iii) Equality 5 results from a reversal in the order of integration. After this reversal, the outer integral is over an interval $[\alpha_{mn}(t), \alpha_{mx}(p)]$ of values for α and the inner integral is over intervals $[\tau_{mn}(\alpha), \tau_{mx}(t,\alpha)]$ of values for τ. As illustrated by the examples in Fig. 15, $[\alpha_{mn}(t), \alpha_{mx}(p)]$ is an interval of values for α on the ordinates of Fig. 15a and Fig. 15b, and $[\tau_{mn}(\alpha), \tau_{mx}(t,\alpha)]$ is an interval of values for τ on the abscissas of Fig. 15a and Fig. 15b. The definitions of $\alpha_{mn}(\tau)$ and $\alpha_{mx}(p,\tau)$ in Eqs. (12.15)-(12.17) lead to the following definitions $\alpha_{mn}(\tau)$ and $\alpha_{mx}(p)$:

$$\alpha_{mn}(t) = p_{mn}(t) / \overline{p}(t) = \begin{cases} \beta_{mn}r(t) & \text{for } \tau_f < t \le \tau_{mn} \\ \alpha_{mn} & \text{for } \tau_{mn} < t \le \tau_l \end{cases} \tag{12.18}$$

from Eq. (12.15), and

$$\begin{aligned} \alpha_{mx}(p) &= \max\{\alpha_{mx}(p,\tau) : \tau_{mn}(p) < \tau \le t\} \\ &= \begin{cases} \alpha_{mx} & \text{for } p_f < p \le p_{mx} \\ p / \overline{p}[\tau_f(p)] = p / \overline{p}[\overline{q}^{-1}(p / \beta_{mn})] & \text{for } p_{mn} < p \le p_f \end{cases} \end{aligned} \tag{12.19}$$

consistent with Eqs. (12.16) and (12.17) as previously indicated in Eq. (4.15). Further, the minimum and maximum possible values $\tau_{mn}(\alpha)$ and $\tau_{mx}(\alpha)$ for τ conditional on a specific value for α are

$$\tau_{mn}(\alpha) = r^{-1}(\alpha / \beta_{mn}) \text{ from } \alpha = \alpha_{mn}(\tau) = \beta_{mn}r(\tau) \text{ for } \alpha_{mn} \le \alpha \le \alpha_{mx}(p) \tag{12.20}$$

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and

$$\tau_{mx}(\alpha) = \begin{cases} \overline{p}^{-1}(p/\alpha) \text{ from } \alpha = \alpha_{mx}(p,\tau) = p/\overline{p}(\tau) \text{ for } \alpha_{mx}[p,\tau_l(p)] \le \alpha \le \alpha_{mx}(p) \\ r^{-1}(\alpha/\beta_{mx}) \text{ from } \alpha = \alpha_{mx}(p,\tau) = \beta_{mx}r_{mx}(\tau) \text{ for } \alpha_{mn} \le \alpha \le \alpha_{mx}[p,\tau_l(p)]. \end{cases}$$ (12.21)

In turn, the value of $\tau_{mx}(t,\alpha)$ depends on the values for t, α and $\tau_{mx}(\alpha)$. Specifically,

$$\tau_{mx}(t,\alpha) = t \text{ for } \alpha_{mn}(t) \le \alpha \le \alpha_{mx}(p) = \alpha_{mx}$$ (12.22)

for $p_f \le p \le p_{mx}$ and $\tau_f \le t \le \tau_f(p)$ as illustrated by the region in Fig. 15a bounded on the right by t_1;

$$\tau_{mx}(t,\alpha) = \begin{cases} t \text{ for } \alpha_{mn}(t) \le \alpha \le \alpha_{mx}(p,t) = p/\overline{p}(t) \\ \tau_{mx}(\alpha) = \overline{p}^{-1}(p/\alpha) \text{ for } \alpha_{mx}(p,t) \le \alpha \le \alpha_{mx}(p) \end{cases}$$ (12.23)

for $p_{mn} \le p \le p_{mx}$ and $\tau_f(p) < t \le \tau_l(p)$ as illustrated by (i) the region in Fig. 15a bounded on the right by t_2, and (ii) the regions in Fig. 15b bounded on the right by t_1 and t_2; and

$$\tau_{mx}(t,\alpha) = \begin{cases} t \text{ for } \alpha_{mn}(t) \le \alpha \le \alpha_{mx}(p,t) = \beta_{mx}r(t) \\ \tau_{mx}(\alpha) \text{ for } \alpha_{mx}(p,t) \le \alpha \le \alpha_{mx}(p) \end{cases}$$

$$= \begin{cases} t \text{ for } \alpha_{mn}(t) \le \alpha \le \alpha_{mx}(p,t) = \beta_{mx}r(t) \\ \tau_{mx}(\alpha) = r^{-1}(\alpha/\beta_{mx}) \text{ for } \alpha_{mx}(p.t) \le \alpha \le \alpha_{mx}[p,\tau_l(p)] = p/\overline{p}[\tau_l(p)] \\ \tau_{mx}(\alpha) = \overline{p}^{-1}(p/\alpha) \text{ for } \alpha_{mx}[p,\tau_l(p)] \le \alpha \le \alpha_{mx}(p) \end{cases}$$ (12.24)

for $p_l \le p \le p_{mx}$ and $\tau_l(p) < t \le \tau_l$ as illustrated by (i) the regions in Fig. 15a bounded on the right by t_3 and t_4, and (ii) the region in Fig. 15b bounded on the right by t_3.

(iv) Equalities 6 and 7 result from a change of variables with $\beta(\tau) = \alpha/r(\tau)$. After the change of variables, the outer integral is still over an interval $[\alpha_{mn}(t),\alpha_{mx}(p)]$ of values for α and the inner integral is over intervals $[\alpha/r[\tau_{mn}(\alpha)],\alpha/r[\tau_{mx}(t,\alpha)]]$ of values for β. Specifically, $\alpha/r[\tau_{mn}(\alpha)]$ is defined by

$$\alpha/r[\tau_{mn}(\alpha)] = \alpha/r[r^{-1}(\alpha/\beta_{mn})] = \beta_{mn} \text{ for } \alpha_{mn} \le \alpha \le \alpha_{mx}(p)$$ (12.25)

with $\tau_{mn}(\alpha)$ defined in Eq. (12.20), and $\alpha/r[\tau_{mx}(t,\alpha)]$ is defined by (a)

$$\alpha/r[\tau_{mx}(t,\alpha)] = \alpha/r(t) \text{ for } \alpha_{mn}(t) \le \alpha \le \alpha_{mx}(p,t) = \alpha_{mx}$$ (12.26)

with $\tau_{mx}(t,\alpha)$ defined in Eq. (12.22) for $p_f \le p \le p_{mx}$ and $\tau_f \le t \le \tau_f(p)$, (b)

$$\alpha / r[\tau_{mx}(t,\alpha)] = \begin{cases} \alpha / r(t) \ \ \text{for} \ \ \alpha_{mn}(t) \le \alpha \le \alpha_{mx}(p,t) = p / \overline{p}(t) \\ \alpha / r[\overline{p}^{-1}(p / \alpha)] = p / \overline{q}[\overline{p}^{-1}(p / \alpha)] \ \ \text{for} \ \ \alpha_{mx}(p,t) \le \alpha \le \alpha_{mx}(p) \end{cases} \tag{12.27}$$

with $\tau_{mx}(t,\alpha)$ defined in Eq. (12.23) for $p_{mn} \le p \le p_{mx}$ and $\tau_f(p) < t \le \tau_l(p)$, and (c)

$$\begin{aligned} &\alpha / r[\tau_{mx}(t,\alpha)] \\ &= \begin{cases} \alpha / r(t) \ \ \text{for} \ \ \alpha_{mn}(t) \le \alpha \le \alpha_{mx}(p,t) = \beta_{mx} r(t) \\ \alpha / r[r^{-1}(\alpha / \beta_{mx})\}] = \beta_{mx} \ \ \text{for} \ \ \alpha_{mx}(p,t) \le \alpha \le \alpha_{mx}[p,\tau_l(p)] = p / \overline{p}[\tau_l(p)] \\ \alpha / r[\overline{p}^{-1}(p / \alpha)] = p / \overline{q}[\overline{p}^{-1}(p / \alpha)] \ \ \text{for} \ \ \alpha_{mx}[p,\tau_l(p)] \le \alpha \le \alpha_{mx}(p) \end{cases} \end{aligned} \tag{12.28}$$

with $\tau_{mx}(t,\alpha)$ defined in Eq. (12.24) for $p_l \le p \le p_{mx}$ and $\tau_l(p) < t \le \tau_l$.

12.2 Representation of $CDF_P(p \,|\, [t_{mn}, t])$ for Configuration 1 in Eq. (4.16)

The equivalence of the representations for $CDF_P(p \,|\, [t_{mn}, t])$ in Eqs. (4.25) and (12.9) for the conditions imposed on t and p for Configuration 1 (i.e., $p_f \le p \le p_{mx}$ with $\tau_f \le t \le \tau_f(p)$) is now established. The indicated conditions on t and p for Configuration 1 exactly match the conditions in the example in Fig. 13a and Fig. 15a used to illustrate Eq. (12.9) when t is assumed to satisfy the equality $\tau_f \le t \le \tau_f(p)$. In this case, the final representation for $CDF_P(p \,|\, [t_{mn}, t])$ in Eq. (12.9) is

$$\begin{aligned} CDF_P(p \,|\, [t_{mn}, t]) &= CDF_P(p \,|\, [\tau_f, t]) \ \ \text{for} \ \tau_f < t \le \tau_f(p) \\ &= \int_{\alpha_{mn}(t)}^{\alpha_{mx}(t,p)} \left\{ \int_{\alpha / r[\tau_{mn}(\alpha)]}^{\alpha / r[\tau_{mx}(t,\alpha)]} d_B(\beta) \mathrm{d}\beta \right\} d_A(\alpha) \mathrm{d}\alpha \\ &= \int_{\alpha_{mn}(t)}^{\alpha_{mx}} \left\{ \int_{\beta_{mn}}^{\alpha / r(t)} d_B(\beta) \mathrm{d}\beta \right\} d_A(\alpha) \mathrm{d}\alpha \end{aligned} \tag{12.29}$$

with (i) $\alpha_{mn}(t)$ defined the same in Eqs. (4.20) and (12.18), (ii) $\alpha_{mx}(t,p) = \alpha_{mx}$ defined in Eq. (12.19), (iii) $\alpha / r[\tau_{mn}(\alpha)] = \beta_{mn}$ defined in Eq. (12.25), and (iv) $\alpha / r[\tau_{mx}(t,\alpha)] = \alpha / r(t)$ defined in Eq. (12.26) and the notation $F(\alpha,t) = \alpha / r(t)$ used in Eq. (4.25). As comparison of the results in Eqs. (4.25) and (12.29) shows, the derivations for Eqs. (4.25) and (12.9) produce equivalent representations for $CDF_P(p \,|\, [t_{mn}, t])$ for Configuration 1 (i.e., $p_f \le p \le p_{mx}$ with $\tau_f \le t \le \tau_f(p)$).

12.3 Representation of $CDF_P(p\,|[t_{mn},t])$ for Configuration 2 in Eq. (4.17)

The equivalence of the representations for $CDF_P(p\,|[t_{mn},t])$ in Eqs. (4.37) and (12.9) for the conditions imposed on t and p for Configuration 2 (i.e., $p_{mn}\leq p\leq p_{mx}$ with $\tau_f(p)<t\leq\tau_l(p)$) is now established. The integral in Equation 7 of Eq. (12.9) can be divided into two integrals with the result that $CDF_P(p\,|[t_{mn},t])$ then has the form

$$CDF_P(p\,|[t_{mn},t]) = CDF_P(p\,|[\tau_{mn}(p),t]) \text{ for } \tau_f(p)<t\leq\tau_l(p)$$
$$= \int_{\alpha_{mn}(t)}^{\alpha_{mx}(p,t)}\left\{\int_{\alpha/r[\tau_{mn}(\alpha)]}^{\alpha/r[\tau_{mx}(t,\alpha)]}d_B(\beta)\mathrm{d}\beta\right\}d_A(\alpha)\mathrm{d}\alpha + \int_{\alpha_{mx}(p,t)}^{\alpha_{mx}(p)}\left\{\int_{\alpha/r[\tau_{mn}(\alpha)]}^{\alpha/r[\tau_{mx}(t,\alpha)]}d_B(\beta)\mathrm{d}\beta\right\}d_A(\alpha)\mathrm{d}\alpha \quad (12.30)$$

with (i) $\alpha_{mn}(t)$ defined in Eq. (12.18), (ii) $\alpha_{mx}(p,t)=p/\bar{p}(t)$ defined in Eq. (12.16), (iii) $\alpha/r[\tau_{mn}(\alpha)]=\beta_{mn}$ defined in Eq. (12.25), (iv) $\alpha/r[\tau_{mx}(t,\alpha)]$ defined in Eq. (12.27), and (v) $\alpha_{mx}(p)$ defined the same in Eqs. (4.15) and (12.19). Substituting the values for $\alpha_{mx}(p,t)=p/\bar{p}(t)$, $\alpha/r[\tau_{mn}(\alpha)]=\beta_{mn}$ and $\alpha/r[\tau_{mx}(t,\alpha)]$ into Eq. (12.30) produces

$$CDF_P(p\,|[t_{mn},t]) = CDF_P(p\,|[\tau_{mn}(p),t]) \text{ for } \tau_f(p)<t\leq\tau_l(p)$$
$$= \int_{\alpha_{mn}(t)}^{p/\bar{p}(t)}\left\{\int_{\beta_{mn}}^{\alpha/r(t)}d_B(\beta)\mathrm{d}\beta\right\}d_A(\alpha)\mathrm{d}\alpha + \int_{p/\bar{p}(t)}^{\alpha_{mx}(p)}\left\{\int_{\beta_{mn}}^{p/\bar{q}[\bar{p}^{-1}(p/\alpha)]}d_B(\beta)\mathrm{d}\beta\right\}d_A(\alpha)\mathrm{d}\alpha. \quad (12.31)$$

As comparison of the results in Eqs. (4.37) and (12.31) shows, the derivations for Eqs. (4.25) and (12.9) produce equivalent representations for $CDF_P(p\,|[t_{mn},t])$ for Configuration 2 (i.e., $p_{mn}\leq p\leq p_{mx}$ with $\tau_f(p)<t\leq\tau_l(p)$).

12.4 Representation of $CDF_P(p\,|[t_{mn},t])$ for Configuration 3 in Eq. (4.18)

The equivalence of the representations for $CDF_P(p\,|[t_{mn},t])$ in Eqs. (4.57) and (12.9) for the conditions imposed on t and p for Configuration 3 (i.e., $p_l\leq p\leq p_{mx}$ with $\tau_l(p)<t\leq\tau_l$) is now established. The integral in Equation 7 of Eq. (12.9) can be divided into three integrals with the result that $CDF_P(p\,|[t_{mn},t])$ then has the form

$$CDF_P(p\,|[t_{mn},t]) = CDF_P(p\,|[\tau_{mn}(p),t]) \text{ for } \tau_l(p)<t\leq\tau_l$$
$$= \int_{\alpha_{mn}(t)}^{\alpha_{mx}(p,t)}\left\{\int_{\alpha/r[\tau_{mn}(\alpha)]}^{\alpha/r[\tau_{mx}(t,\alpha)]}d_B(\beta)\mathrm{d}\beta\right\}d_A(\alpha)\mathrm{d}\alpha$$
$$+ \int_{\alpha_{mx}(p,t)}^{\alpha_{mx}[p,\tau_l(p)]}\left\{\int_{\alpha/r[\tau_{mn}(\alpha)]}^{\alpha/r[\tau_{mx}(t,\alpha)]}d_B(\beta)\mathrm{d}\beta\right\}d_A(\alpha)\mathrm{d}\alpha \quad (12.32)$$
$$+ \int_{\alpha_{mx}[p,\tau_l(p)]}^{\alpha_{mx}(p)}\left\{\int_{\alpha/r[\tau_{mn}(\alpha)]}^{\alpha/r[\tau_{mx}(t,\alpha)]}d_B(\beta)\mathrm{d}\beta\right\}d_A(\alpha)\mathrm{d}\alpha,$$

with (i) $\alpha_{mn}(t)$ defined in Eq. (12.18), (ii) $\alpha/r[\tau_{mn}(\alpha)] = \beta_{mn}$ defined in Eq. (12.25), (iii) $\alpha_{mx}(p,t) = p/\overline{p}(t)$, $\alpha_{mx}[p,\tau_l(p)] = p/\overline{p}[\tau_l(p)]$ and $\alpha/r[\tau_{mx}(t,\alpha)]$ defined in Eq. (12.28) and $\tau_l(p) = \overline{q}^{-1}(p/\beta_{mx})$ defined in Eq. (4.12), and (iv) $\alpha_{mx}(p)$ defined in Eq. (12.19). Substituting the indicated values for $\alpha/r[\tau_{mn}(\alpha)]$, $\alpha_{mx}(p,t)$, $\alpha_{mx}[p,\tau_l(p)]$ and $\alpha/r[\tau_{mx}(t,\alpha)]$ into Eq. (12.32) produces

$$CDF_P(p\,|\,[t_{mn},t]) = CDF_P(p\,|\,[\tau_{mn}(p),t]) \text{ for } \tau_l(p) < t \le \tau_l$$

$$= \int_{\alpha_{mn}(t)}^{\beta_{mx}r(t)} \left\{ \int_{\beta_{mn}}^{\alpha/r(t)} d_B(\beta)\mathrm{d}\beta \right\} d_A(\alpha)\mathrm{d}\alpha + \int_{\beta_{mx}r(t)}^{p/\overline{p}[\tau_l(p)]} \left\{ \int_{\beta_{mn}}^{\beta_{mx}} d_B(\beta)\mathrm{d}\beta \right\} d_A(\alpha)\mathrm{d}\alpha$$

$$+ \int_{p/\overline{p}[\tau_l(p)]}^{\alpha_{mx}(p)} \left\{ \int_{\beta_{mn}}^{p/\overline{q}[\overline{p}^{-1}(p/\alpha)]} d_B(\beta)\mathrm{d}\beta \right\} d_A(\alpha)\mathrm{d}\alpha \qquad (12.33)$$

$$= \int_{\alpha_{mn}(t)}^{\beta_{mx}r(t)} \left\{ \int_{\beta_{mn}}^{\alpha/r(t)} d_B(\beta)\mathrm{d}\beta \right\} d_A(\alpha)\mathrm{d}\alpha + \int_{\beta_{mx}r(t)}^{p/\overline{p}[\tau_l(p)]} d_A(\alpha)\mathrm{d}\alpha$$

$$+ \int_{p/\overline{p}[\tau_l(p)]}^{\alpha_{mx}(p)} \left\{ \int_{\beta_{mn}}^{p/\overline{q}[\overline{p}^{-1}(p/\alpha)]} d_B(\beta)\mathrm{d}\beta \right\} d_A(\alpha)\mathrm{d}\alpha,$$

which is the same as the representation for $CDF_P(p\,|\,[t_{mn},t])$ in Eq. (4.57). As comparison of the results in Eqs. (4.57) and (12.31) shows, the derivations for Eqs. (4.57) and (12.9) produce equivalent representations for $CDF_P(p\,|\,[t_{mn},t])$ for Configuration 3 (i.e., $p_l \le p \le p_{mx}$ with $\tau_l(p) < t \le \tau_l$).

13. Summary Discussion

Weak link/strong link (WL/SL) systems are important components in the overall design of high consequence systems. In such systems, loss of assured safety (LOAS) occurs under accident conditions (e.g., a fire) when SL failures place the overall system in a potentially operational mode before deactivation of the overall system as a result of WL failures. In this presentation, multiple representations are developed and illustrated for the distribution of link property values at the time of link failure in the presence of aleatory uncertainty in link properties. Specifically, two integral-based representations and one sampling-based representation for the distribution of link property values at the time of link failure are developed.

The derivation and numerical implementation of the three representations are independent of each other even though they are intended to produce the same distribution of link property values at the time of link failure. As demonstrated, all three derivations and their associated numerical implementations result in the same distributions of link failure. This agreement provides a strong verification result that all three derivations are correct.

Of the three derivations, the sampling-based (i.e., Monte Carlo) procedure is the easiest to understand and implement. However, verification of sampling-based procedures can be difficult. Thus, even though the integral-based representations may not be the preferred representations from an explanatory and implementation perspective, their existence provides a way to provide independently obtained results for use in verifying the correctness of the sampling-based procedure.

In addition to the distributions for link property values at the time of link failure, a number of intermediate results are also obtained that will be extensively used in two following reports on (i) time and failure property margins for systems involving multiple WLs and SLs [46] and (ii) delays in link failure time that are functions of link property value at the time of precursor link failure [47] .

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