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# Probability of Loss of Assured Safety in Systems with Multiple Time-Dependent Failure Modes: Incorporation of Delayed Link Failure in the Presence of Aleatory Uncertainty

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# Probability of Loss of Assured Safety in Systems with Multiple Time-Dependent Failure Modes: Incorporation of Delayed Link Failure in the Presence of Aleatory Uncertainty

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## Abstract

Probability of loss of assured safety (PLOAS) is modeled for weak link (WL)/strong link (SL) systems in which one or more WLs or SLs could potentially degrade into a precursor condition to link failure that will be followed by an actual failure after some amount of elapsed time. The following topics are considered: (i) Definition of precursor occurrence time cumulative distribution functions (CDFs) for individual WLs and SLs, (ii) Formal representation of PLOAS with constant delay times, (iii) Approximation and illustration of PLOAS with constant delay times, (iv) Formal representation of PLOAS with aleatory uncertainty in delay times, (v) Approximation and illustration of PLOAS with aleatory uncertainty in delay times, (vi) Formal representation of PLOAS with delay times defined by functions of link properties at occurrence times for failure precursors, (vii) Approximation and illustration of PLOAS with delay times defined by functions of link properties at occurrence times for failure precursors, and (viii) Procedures for the verification of PLOAS calculations for the three indicated definitions of delayed link failure.

## ACKNOWLEDGMENTS

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# NOMENCLATURE

| Abbreviation | Definition |
|---|---|
| CDF | cumulative distribution function |
| CI | confidence interval |
| DOE | Department of Energy |
| LOAS | loss of assured safety |
| PLOAS | probability of loss of assured safety |
| SB1 | first sampling-based procedure |
| SB2 | second sampling-based procedure |
| SL | strong link |
| SNL | Sandia National Laboratories |
| WL | weak link |

# 1. Introduction

As discussed in the Introduction to Ref. [1], weak link (WL)/strong link (SL) systems are important parts of the overall operational design of high-consequence systems [2-7]. In such designs, the SL system is very robust and is intended to permit operation of the entire system under, and only under, intended conditions (e.g., by transmitting a command to activate the system). In contrast, the WL system is intended to fail in a predictable and irreversible manner under accident conditions (e.g., in the event of a fire) and render the entire system inoperable before an accidental operation of the SL system. The likelihood that the WL system will fail to deactivate the entire system before the SL system fails (i.e., degrades into a configuration that could allow an accidental operation of the entire system) is referred to as probability of loss of assured safety (PLOAS). The descriptor loss of assured safety (LOAS) is used because failure of the WL system places the entire system in an inoperable configuration while failure of the SL system, although undesirable, does not necessarily result in an unintended operation of the entire system. Thus, safety is "assured" by failure of the WL system.

The present study investigates an accident context in which one or more WLs or SLs degrade into a precursor condition to link failure that will be followed by an actual failure after some amount of elapsed time. For example, the precursor condition might correspond to the beginning of a degradation process that will inevitably lead to link failure after a fixed or possibly randomly varying period of time (e.g., the precursor condition might correspond to a break in a boundary condition that allowed the initiation of a corrosion process that will ultimately fail the link). As another example, the precursor condition might correspond to a degraded condition of a link that will then result in link failure when the link experiences some form of random stress or perturbation (e.g., the precursor condition might correspond to a fracturing of a link that was later followed by link failure due to the random occurrence of an additional stress on the weakened link).

This study includes the effects of aleatory uncertainty on the failure of WLs and SLs. Specifically, aleatory uncertainty is assumed to be present in both (i) time-dependent property values for individual links and (ii) property values (either constant or time-dependent) at which individual links fail. As discussed in Refs. [8-20], aleatory uncertainty is used as a descriptor for random variability in the properties or behavior of a system. Specifically, aleatory uncertainty is distinct from epistemic uncertainty, which results from a lack of knowledge about the value of a quantity that has a fixed (i.e., unique) but poorly known value.

The study reported in Ref. [1] also considers the effects of aleatory uncertainty on the failure of both WLs and SLs but does not consider accident contexts in which one or more WLs or SLs degrade into a precursor condition to link failure that will be followed by an actual failure after some amount of elapsed time. Two earlier studies consider the effects of aleatory uncertainty in link failure values but do not consider (i) time-dependent link failure values and (ii) aleatory in link property values [21; 22]. Two other previous studies investigate verification procedures and tests for use in conjunction with the calculation of PLOAS for WL/SL systems [23; 24].

The following topics are considered in this presentation: (i) Definition of precursor occurrence time cumulative distribution functions (CDFs) for individual WLs or SLs (Sect. 2), (ii) Formal representation of PLOAS with constant delay times (Sect. 3), (iii) Approximation and illustration of PLOAS with constant delay times (Sect. 4), (iv) Formal representation of PLOAS with aleatory uncertainty in delay times (Sect. 5), (v) Approximation and illustration of PLOAS with aleatory uncertainty in delay times (Sect. 6), (vi) Formal representation of PLOAS with delay times defined by functions of link properties at occurrences times for failure precursors (Sect. 7), and (vii) Approximation and illustration of PLOAS with delay times defined by functions of link properties at occurrences times for failure precursors (Sect. 8). The presentation then ends with a summary discussion (Sect. 9).

An important motivation for this work is the importance of having verification procedures for delayed link failure results calculated with the CPLOAS program [25]. To this end, a variety of verification procedures are described and illustrated in Sects. 4, 6 and 8. Fortunately, these verification procedures showed that CPLOAS calculated all presented results correctly.

## 2. Definition of Precursor Occurrence Time CDF for a Single WL or SL

The precursor occurrence time CDF for a single WL or SL is based on the following assumed properties of that link for the time interval $t_{mn} \le t \le t_{mx}$:

$$\bar{p}(t) = \text{nondecreasing positive function defining nominal link property for } t_{mn} \le t \le t_{mx}, \quad (2.1)$$

$$\bar{q}(t) = \text{nonincreasing positive function defining nominal precursor failure value for link property for } t_{mn} \le t \le t_{mx}, \quad (2.2)$$

$$d_A(\alpha) = \text{density function for positive variable } \alpha \text{ used to characterize aleatoryy uncertaintyin link property,} \quad (2.3)$$

$$d_B(\beta) = \text{density function for positive variable } \beta \text{ used to characterize aleatory uncertainty in link precursor failure value,} \quad (2.4)$$

$$p(t \mid \alpha) = \alpha \bar{p}(t) = \text{link property for } t_{mn} \le t \le t_{mx} \text{ given } \alpha, \quad (2.5)$$

and

$$q(t \mid \beta) = \beta \bar{q}(t) = \text{link precursor failure value for } t_{mn} \le t \le t_{mx} \text{ given } \beta. \quad (2.6)$$

Further, $d_A(\alpha)$ and $d_B(\beta)$ are assumed to be defined on intervals $[\alpha_{mn}, \alpha_{mx}]$ and $[\beta_{mn}, \beta_{mx}]$ and to equal zero outside these intervals.

The functions $p(t|\alpha)$ and $q(t|\beta)$ (i) define time-dependent values for a link property (e.g., temperature, pressure, …) and the precursor failure values for that property (e.g., failure temperature, failure pressure, …) and (ii) have distributions that derive from the distributions for $\alpha$ and $\beta$ characterized by the density functions $d_A(\alpha)$ and $d_B(\beta)$. For given values for $\alpha$ and $\beta$, the link enters a precursor condition to failure at the time $t$ for which the equality

$$\beta \bar{q}(t) = q(t \mid \beta) = p(t \mid \alpha) = \alpha \bar{p}(t) \quad (2.7)$$

holds. In turn, the distributions for $\alpha$ and $\beta$ result in a distribution of possible values for the precursor failure time $\tau$ that can be summarized by

$$CDF_P(t) = \text{probability that precursor conditions to link failure occur at or before time } t, \quad (2.8)$$

which is the CDF for the time at which precursor conditions to link failure occur.

The precursor time CDF in Eq. (2.8) can be represented as either a Riemann integral or a Stieltjes integral. When represented as a Riemann integral, the CDF in Eq. (2.8) has the form

$$CDF_P(t) = \int_{\alpha_{mn}}^{\alpha_{mx}} \left[ \int_{\beta_{mn}}^{F(\alpha,t)} d_B(\beta)\, \mathrm{d}\beta \right] d_A(\alpha)\, \mathrm{d}\alpha \tag{2.9}$$

with

$$F(\alpha,t) = \alpha\, \overline{p}(t) / \overline{q}(t) = \alpha / r(t) \text{ and } r(t) = \overline{q}(t) / \overline{p}(t). \tag{2.10}$$

Derivations and approximation procedures for $CDF_P(t)$ are given in Ref. [1].

## 3. PLOAS with Constant Delay Times: Formal Representation

In this case, a constant delay time $D > 0$ is assumed to exist between the time when the precursor conditions for link failure occur and the actual time at which link failure occurs. This results in the following CDF for delayed link failure time:

$$
\begin{aligned}
CDF_D(t) &= \text{probability that link failure occurs at or before time } t \text{ with} \\
&\quad\ \text{constant delay time } D > 0 \\
&= \int_0^t \delta\left[t - (\tau + D)\right] \mathrm{d}CDF_P(\tau) \\
&= \begin{cases} \int_0^{t-D} \mathrm{d}CDF_P(\tau) = CDF_P(t - D) & \text{for } 0 < t - D \\ 0 \quad \text{for} \quad t - D \le 0, \end{cases}
\end{aligned}
\tag{3.1}
$$

where

$$
\delta\left[t - (\tau + D)\right] = \begin{cases} 1 \text{ for } \ 0 \le t - (\tau + D) \\ 0 \text{ for } \ t - (\tau + D) < 0. \end{cases}
\tag{3.2}
$$

and $CDF_P(t)$ is defined in Eqs. (2.9).

In general, a WL/SL problem will involve $nWL$ WLs and $nSL$ SLs. In this case, delay times $D_{WL,j}$, $j = 1, 2, \ldots, nWL$, and $D_{SL,k}$, $k = 1, 2, \ldots, nSL$, can be defined for the individual links and corresponding failure time CDFs $CDF_{D,WL,j}(t)$, $j = 1, 2, \ldots, nWL$, and $CDF_{D,SL,k}(t)$, $k = 1, 2, \ldots, nSL$, for the individual links defined as in Eq. (3.1). In such a situation, it is likely that zero and nonzero delay times will be present. As extensively discussed in Ref. [1], LOAS can be determined with the relationships summarized in Table 1 once the indicated failure time CDFs are available.

The indicated verification tests in Table 1 are the known outcomes of assigning the same properties to all links [24]. This is not a realistic physical problem but serves as a useful verification test because it requires use of all the mathematics and programming underlying the calculation of PLOAS while, at the same time, having a known solution.

Table 1 Representation of time-dependent values $pF_i(t)$, $i$ = 1, 2, 3, 4, for PLOAS and associated verification tests for alternate definitions of LOAS for WL/SL systems with (i) $nWL$ WLs and $nSL$ SLs and (ii) independent distributions for link failure time defined by the CDFs $CDF_{WL,j}$, $j = 1, 2, ..., nWL,$ and $CDF_{SL,k}$, $k = 1, 2, ..., nSL$ ([22], Table 10).

Case 1: Failure of all SLs before failure of any WL (Eqs. (2.1) and (2.5), Ref. [24])

$$pF_1(t) = \sum_{k=1}^{nSL}\left(\int_{t_{mn}}^{t}\left\{\prod_{l=1,l\neq k}^{nSL} CDF_{SL,l}(\tau)\right\}\left\{\prod_{j=1}^{nWL}\left[1 - CDF_{WL,j}(\tau)\right]\right\}\mathrm{d}CDF_{SL,k}(\tau)\right)$$

Verification test: $pF_1(\infty) = nWL!\,nSL!/(nWL + nSL)!$

Case 2: Failure of any SL before failure of any WL (Eqs. (3.1) and (3.4), Ref. [24])

$$pF_2(t) = \sum_{k=1}^{nSL}\left(\int_{t_{mn}}^{t}\left\{\prod_{l=1,l\neq k}^{nSL}\left[1 - CDF_{SL,l}(\tau)\right]\right\}\left\{\prod_{j=1}^{nWL}\left[1 - CDF_{WL,j}(\tau)\right]\right\}\mathrm{d}CDF_{SL,k}(\tau)\right)$$

Verification test: $pF_2(\infty) = nSL/(nWL + nSL)$

Case 3: Failure of all SLs before failure of all WLs (Eqs. (4.1) and (4.4), Ref. [24])

$$pF_3(t) = \sum_{k=1}^{nSL}\left(\int_{t_{mn}}^{t}\left\{\prod_{l=1,l\neq k}^{nSL} CDF_{SL,l}(\tau)\right\}\left\{1 - \prod_{j=1}^{nWL} CDF_{WL,j}(\tau)\right\}\mathrm{d}CDF_{SL,k}(\tau)\right)$$

Verification test: $pF_3(\infty) = nWL/(nWL + nSL)$

Case 4: Failure of any SL before failure of all WLs (Eqs. (5.1) and (5.4), Ref. [24])

$$pF_4(t) = \sum_{k=1}^{nSL}\left(\int_{t_{mn}}^{t}\left\{\prod_{l=1,l\neq k}^{nSL}\left[1 - CDF_{SL,l}(\tau)\right]\right\}\left\{1 - \prod_{j=1}^{nWL} CDF_{WL,j}(\tau)\right\}\mathrm{d}CDF_{SL,k}(\tau)\right)$$

Verification test: $pF_4(\infty) = 1 - \left[nWL!\,nSL!/(nWL + nSL)!\right]$

## 4. PLOAS with Constant Delay Times: Approximation and Illustration

As discussed in Sect. 4 of Ref. [1], a quadrature procedure provides one possibility for evaluating the integrals in Table 1 that define the probabilities $pF_1(t)$, $pF_2(t)$, $pF_3(t)$ and $pF_4(t)$ for LOAS. Specifically, the preceding probabilities can be approximated by

$$pF_1(t) \cong \sum_{k=1}^{nSL}\left(\sum_{i=1}^{nSD}\left\{\prod_{l=1,l\neq k}^{nSL} CDF_{SL,l}\left(t_{i-1}\right)\right\}\left\{\prod_{j=1}^{nWL}\left[1-CDF_{WL,j}\left(t_i\right)\right]\right\}\Delta CDF_{SL,k}\left(t_i\right)\right), \tag{4.1}$$

$$pF_2(t) \cong \sum_{k=1}^{nSL}\left(\sum_{i=1}^{nSD}\left\{\prod_{l=1,l\neq k}^{nSL}\left[1-CDF_{SL,l}\left(t_{i-1}\right)\right]\right\}\left\{\prod_{j=1}^{nWL}\left[1-CDF_{WL,j}\left(t_i\right)\right]\right\}\Delta CDF_{SL,k}\left(t_i\right)\right), \tag{4.2}$$

$$pF_3(t) \cong \sum_{k=1}^{nSL}\left(\sum_{i=1}^{nSD}\left\{\prod_{l=1,l\neq k}^{nSL} CDF_{SL,l}\left(t_{i-1}\right)\right\}\left\{1-\prod_{j=1}^{nWL} CDF_{WL,j}\left(t_i\right)\right\}\Delta CDF_{SL,k}\left(t_i\right)\right), \tag{4.3}$$

and

$$pF_4(t) \cong \sum_{k=1}^{nSL}\left(\sum_{i=1}^{nSD}\left\{\prod_{l=1,l\neq k}^{nSL}\left[1-CDF_{SL,l}\left(t_{i-1}\right)\right]\right\}\left\{1-\prod_{j=1}^{nWL} CDF_{WL,j}\left(t_i\right)\right\}\Delta CDF_{SL,k}\left(t_i\right)\right), \tag{4.4}$$

where $t_{mn} = t_0 < t_1 < t_2 < \cdots < t_{nSD} = t$ is a subdivision of $[t_{mn}, t]$.

As discussed in Sect. 5 of Ref. [1], sampling-based procedures can also be used to approximate $pF_1(t)$, $pF_2(t)$, $pF_3(t)$ and $pF_4(t)$. Specifically, $pF_i(t)$, $i$ = 1, 2, 3 or 4, can be approximated by

$$\begin{aligned} pF_i(t) &\cong \sum\nolimits_{l=1}^{nR} \delta_i\left[t \mid \mathbf{t}_l\right]/nR \\ &= \sum\nolimits_{l=1}^{nR} \delta_i\left\{t \mid \left[tWL_{1l}, tWL_{2l}, ..., tWL_{nWL,l}, tSL_{1l}, tSL_{2l}, ..., tSL_{nSL,l}\right]\right\}/nR, \end{aligned} \tag{4.5}$$

where

$$t = \text{ time at which PLOAS (i.e., } pF_i(t) \text{ in Table 1) is to be determined,} \tag{4.6}$$

$$tWL_j = \text{ time at which WL } j \text{ fails, } j = 1, 2, ..., nWL, \tag{4.7}$$

$$tSL_j = \text{ time at which SL } j \text{ fails, } j = 1, 2, ..., nSL, \tag{4.8}$$

$$\mathbf{t} = \left[tWL_1, tWL_2, ..., tWL_{nWL}, tSL_1, tSL_2, ..., tSL_{nSL}\right], \tag{4.9}$$

$$\delta_1(t \mid \mathbf{t}) = \begin{cases} 1 \text{ if } \max\{tSL_1, tSL_2, ..., tSL_{nSL}\} \le \min\{t, tWL_1, tWL_2, ..., tWL_{nSL}\} \\ 0 \text{ otherwise,} \end{cases} \quad (4.10)$$

$$\delta_2(t \mid \mathbf{t}) = \begin{cases} 1 \text{ if } \min\{tSL_1, tSL_2, ..., tSL_{nSL}\} \le \min\{t, tWL_1, tWL_2, ..., tWL_{nSL}\} \\ 0 \text{ otherwise,} \end{cases} \quad (4.11)$$

$$\delta_3(t \mid \mathbf{t}) = \begin{cases} 1 \text{ if } \max\{tSL_1, tSL_2, ..., tSL_{nSL}\} \le \min\{t, \max\{tWL_1, tWL_2, ..., tWL_{nSL}\}\} \\ 0 \text{ otherwise,} \end{cases} \quad (4.12)$$

$$\delta_4(t \mid \mathbf{t}) = \begin{cases} 1 \text{ if } \min\{tSL_1, tSL_2, ..., tSL_{nSL}\} \le \min\{t, \max\{tWL_1, tWL_2, ..., tWL_{nSL}\}\} \\ 0 \text{ otherwise,} \end{cases} \quad (4.13)$$

and

$$\mathbf{t}_l = \left[tWL_{1l}, tWL_{2l}, ..., tWL_{nWL,l}, tSL_{1l}, tSL_{2l}, ..., tSL_{nSL,l}\right], l = 1, 2, ..., nR, \quad (4.14)$$

is a random sample from the possible values for **t** generated in consistency with the distributions for the failure times $tWL_j$, $j$ = 1, 2, …, $nWL$, and $tSL_j$, $j$ = 1, 2, …, $nSL$.

The example WL/SL system defined in Table 2 and shown in Fig. 1 is used for illustration. Specifically, this example involves a WL/SL system with 2 WLs and 2 SLs. For consistency, this is the same example used in an earlier article on margins related to LOAS for WL/SL systems ([26], Table 1).

Table 2 Defining properties of two WLs and two SLs used in the illustration of the definition and calculation of delayed failure for WL/SL systems ([26], Table 1).

| General Properties for Links |
| --- |

$$\bar{p}(\tau) = \frac{\bar{p}(\infty)\bar{p}(0)}{\bar{p}(0) + [\bar{p}(\infty) - \bar{p}(0)]\exp(-r_1\tau)} \quad \text{for all links}$$

$\bar{q}(\tau)$ constant-valued for WL 1 and SL 1

$$\bar{q}(\tau) = \frac{\bar{q}(0)}{1 + k\tau^{r_2}} \quad \text{for WL 2 and SL 2}$$

| Additional Properties of WL 1 |
| --- |

$\bar{p}(\infty) = 950, \bar{p}(0) = 300, r_1 = 0.02, \bar{q}(\tau) = 650$

$d_A(\alpha)$ triangular on $[\alpha_{mn}, \alpha_{mx}] = [0.88, 1.15]$ with mode 1.0

$d_B(\beta)$ triangular on $[\beta_{mn}, \beta_{mx}] = [0.8, 1.15]$ with mode 1.0

| Additional Properties of WL 2 |
| --- |

$\bar{p}(\infty) = 850, \bar{p}(0) = 300, r_1 = 0.02$

$\bar{q}(0) = 650, k = 2.21\times10^{-4}, r_2 = 1.5$

$d_A(\alpha)$ triangular on $[\alpha_{mn}, \alpha_{mx}] = [0.85, 1.2]$ with mode 1.0

$d_B(\beta)$ triangular on $[\beta_{mn}, \beta_{mx}] = [0.75, 1.2]$ with mode 1.0

| Additional Properties of SL 1 |
| --- |

$\bar{p}(\infty) = 1025, \bar{p}(0) = 300, r_1 = 0.025, \bar{q}(\tau) = 775$

$d_A(\alpha)$ triangular on $[\alpha_{mn}, \alpha_{mx}] = [0.9, 1.15]$ with mode 1.0

$d_B(\beta)$ uniform on $[\beta_{mn}, \beta_{mx}] = [0.8, 1.15]$

| Additional Properties of SL 2 |
| --- |

$\bar{p}(\infty) = 950, \bar{p}(0) = 300, r_1 = 0.025$

$\bar{q}(0) = 750, k = 1.41\times10^{-4}, r_2 = 1.5$

$d_A(\alpha)$ triangular on $[\alpha_{mn}, \alpha_{mx}] = [0.8, 1.1]$ with mode 1.0

$d_B(\beta)$ uniform on $[\beta_{mn}, \beta_{mx}] = [0.85, 1.3]$

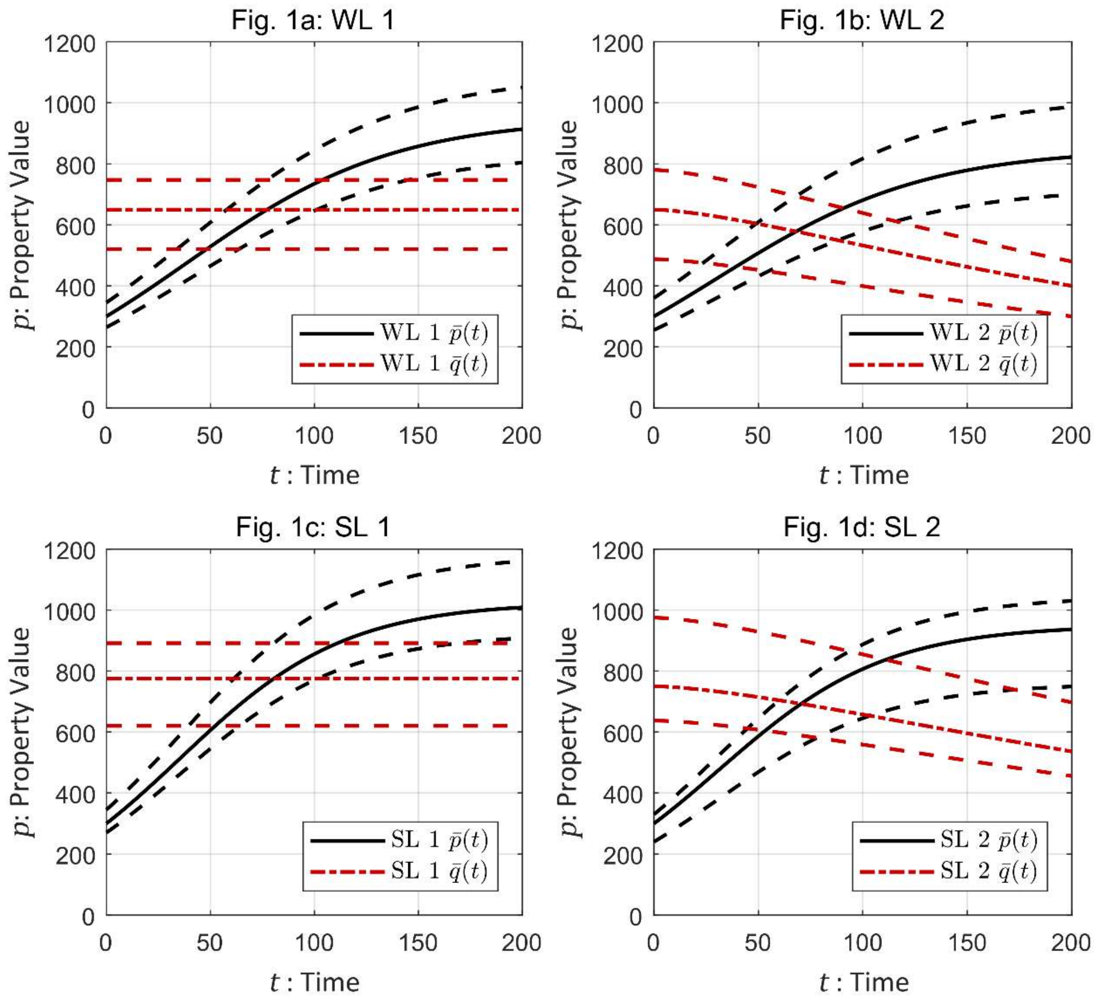


Fig. 1 Summary plots of the properties of two WLs and two SLs used in the illustration of the definition and calculation of delayed failure for WL/SL systems: (a) WL 1, (b) WL 2, (c) SL 1, and (d) SL 2 ([26], Fig. 1).

For purposes of illustration, the following delay times are assumed for the WLs and SLs indicated in Table 2:

$$D_{WL1} = 5.0 \text{ and } D_{WL2} = 8.0 \tag{4.15}$$

for WLs 1 and 2, and

$$D_{SL1} = 12.0 \text{ and } D_{SL2} = 14.0 \tag{4.16}$$

for SLs 1 and 2.

For comparison, PLOAS values obtained with and without the inclusion of the indicated delay times for the individual links are presented in Fig. 2 for each of the four failure patterns in Table

1 (i.e., for $pF_i(t), i = 1,2,3,4$). If the delays in Eqs. (4.15) and (4.16) were all equal to zero, then what is indicated as precursor results in Fig. 2 with subscript "*P*'s" would be the actual PLOAS results.

Verification of the correctness of numerical calculations is an important part of any analysis [27-34]. For the present analysis, one approach to verification is to determine PLOAS with both the quadrature procedures indicated in Eqs. (4.1)-(4.4) and the sampling-based procedures indicated in Eqs. (4.5)-(4.14). Additional verification can be obtained by using several different sampling-based procedures. For illustration, results obtained with two different sampling-based procedures for the estimation of PLOAS are compared with the quadrature-based results in Fig. 2d. The two sampling-based procedures differ only in how the vector $\mathbf{t}_l$ in Eq. (4.14) is sampled.

For the first sampling-based procedure (SB1), the link failure times are sampled directly from the failure time CDFs in Fig. 2c (i.e., from the final CDFs for link failure time that have incorporated the delay times in Eqs. (4.15) and (4.16)). For the second sampling-based procedure (SB2), values for $\alpha$ and $\beta$ are sampled for each link and used to determine precursor failure times; then, the delay time for each link is added to the link's sampled precursor failure times to obtain the vectors $\mathbf{t}_l$ in Eq. (4.14). Of the two sampling-based procedures, SB2 is the more effective verification procedure as it does not use the failure time CDFs that are used in the quadrature procedures indicated in Eqs. (4.1)-(4.4).

As shown in Table 3 for $t$ = 200, the quadrature procedure and the two sampling procedures produce what are effectively the same values for PLOAS. This is a strong verification result indicating that all three procedures are correctly defined and implemented.

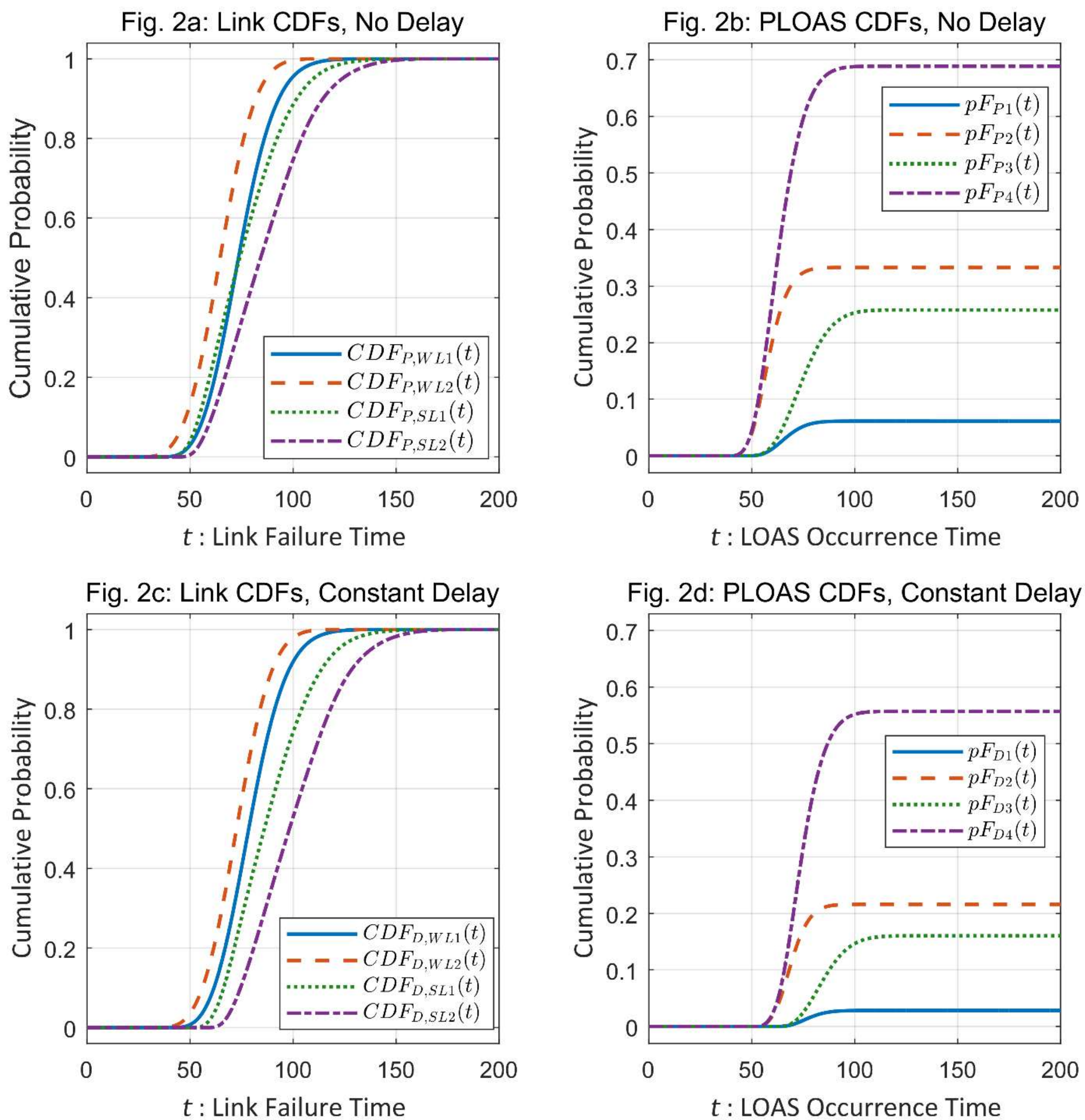


Fig. 2 Time-dependent PLOAS results obtained (i) with the quadrature procedures indicated in Eqs. (4.1)-(4.4) and a subdivision of [0, 200] of size $nSD = 10^4$ and (ii) with and without inclusion of the constant delay times in Eqs. (4.15) and (4.16) for the individual links: (a) CDFs $CDF_{P,WL1}(t)$, $CDF_{P,WL2}(t)$, $CDF_{P,SL1}(t)$ and $CDF_{P,SL2}(t)$ for occurrence time for precursor to link failure time (i.e., for link failure times obtained with delay times of zero), (b) CDFs $pF_{Pi}(t), i = 1,2,3,4,$ for PLOAS obtained without inclusion of delay times, (c) CDFs $CDF_{D,WL1}(t)$, $CDF_{D,WL2}(t)$, $CDF_{D,SL1}(t)$ and $CDF_{D,SL2}(t)$ for link failure time obtained with inclusion of delay times, and (d) CDFs $pF_{Di}(t), i = 1,2,3,4,$ for PLOAS obtained with inclusion of delay times.

Table 3 Comparison of PLOAS results at $t = 200$ with inclusion of the constant delay times in Eqs. (4.15) and (4.16) obtained with (i) quadrature procedures in Eqs. (4.1)-(4.4) with $nSD = 10^4$, and (ii) sampling-based procedures SB1 and SB2 with $nR = 10^6$.

Case 1: Failure of all SLs before failure of any WL

$$pF_1(200) = \begin{cases} 0.0283 & \text{Quadrature} \\ 0.0284 & \text{Sampling SB1, 95\% CI=}[0.0280, 0.0287] \\ 0.0283 & \text{Sampling SB2, 95\% CI=}[0.0280, 0.0286] \end{cases}$$

Case 2: Failure of any SL before failure of any WL

$$pF_2(200) = \begin{cases} 0.2159 & \text{Quadrature} \\ 0.2161 & \text{Sampling SB1, 95\% CI=}[0.2153, 0.2169] \\ 0.2153 & \text{Sampling SB2, 95\% CI=}[0.2145, 0.2161] \end{cases}$$

Case 3: Failure of all SLs before failure of all WLs

$$pF_3(200) = \begin{cases} 0.1605 & \text{Quadrature} \\ 0.1614 & \text{Sampling SB1, 95\% CI=}[0.1607, 0.1621] \\ 0.1603 & \text{Sampling SB2, 95\% CI=}[0.1596, 0.1611] \end{cases}$$

Case 4: Failure of any SL before failure of all WLs

$$pF_4(200) = \begin{cases} 0.5572 & \text{Quadrature} \\ 0.5569 & \text{Sampling SB1, 95\% CI=}[0.5559, 0.5578] \\ 0.5575 & \text{Sampling SB2, 95\% CI=}[0.5566, 0.5585] \end{cases}$$

As indicated in Table 1, an additional verification procedure is provided by evaluating the probabilities $pF_i(t), i = 1,2,3,4,$ with the same properties assigned to all links. Although this is not a physically realistic problem, it has value as a verification test because (i) it requires use of all the programmed calculations employed in the determination of $pF_i(t)$ and (ii) the correct asymptotic value $pF_i(\infty)$ for $pF_i(t)$ is known as stated in Table 1. For the example considered in this section with $nWL = 3$, $nSL = 2$ and the same properties assigned to all links, the resultant possibilities are

$$pF_i(\infty) = \begin{cases} pF_1(\infty) = nWL!nSL!/(nWL + nSL)! = 1/10 = 0.1 \text{ for } i = 1 \\ pF_2(\infty) = nSL/(nWL + nSL) = 2/5 = 0.4 \text{ for } i = 2 \\ pF_3(\infty) = nWL/(nWL + nSL) = 3/5 = 0.6 \text{ for } i = 3 \\ pF_4(\infty) = 1 - nWL!nSL!/(nWL + nSL)! = 9/10 = 0.9 \text{ for } i = 4. \end{cases} \quad (4.17)$$

As shown in Fig. 3, the verification condition

$$\lim_{t \to \infty} pF_i(t) = pF_i(\infty) \quad (4.18)$$

is satisfied for $pF_{Di}(t)$ determined with (i) the constant link failure delays defined in Eqs. (4.15) and (4.16), (ii) $nWL = 3$ and $nSL = 2$, and (iii) the properties of WL 1 in Table 2 used for all links.

The results presented in Fig. 2, Table 3 and Fig. 3 were obtained with use of the CPLOAS program [25; 35] and thus provide verification results that indicate that CPLOAS is correctly implementing constant delays in link failure in the calculation of the PLOAS values $pF_{Di}(t), i = 1,2,3,4$.

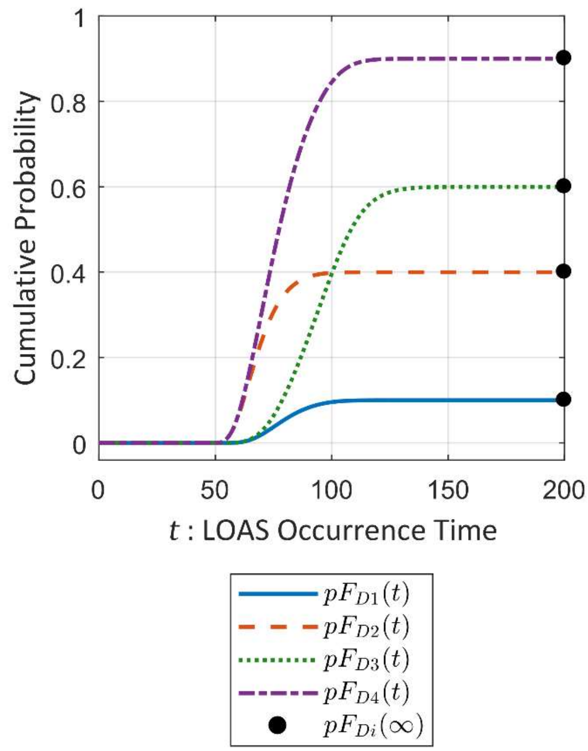


Fig. 3 Verification test for $pF_{Di}(t), i = 1,2,3,4,$ with (i) the constant link failure delays defined in Eqs. (4.15) and (4.16), (ii) $nWL = 3$ and $nSL = 2$, and (iii) the properties of WL 1 in Table 2 used for all links.

## 5. PLOAS with Aleatory Uncertainty in Delay Times: Formal Representation

Similarly to link properties and link precursor occurrence times, aleatory uncertainty can also be incorporated into the definition of failure delay time. Specifically, this involves specifying the following quantities for individual links:

$$\bar{D} = \text{nominal delay time between the occurence of precursor conditions to link failure and the time of link failure,} \tag{5.1}$$

$$d_G(\gamma) = \text{density function defined on } [\gamma_{mn}, \gamma_{mx}] \text{ with } \gamma_{mn} > 0 \text{ for variable } \gamma \text{ used to characterize aleatory uncertainty in delay time } \bar{D}, \tag{5.2}$$

$$\gamma\bar{D} = \text{ delay in link failure time given } \gamma. \tag{5.3}$$

This results in the following CDF for link failure time:

$$\begin{aligned} CDF_D(t) &= \text{ probability that link failure occurs at or before time for a variable delay } D \\ &= \int_0^t \left( \int_{\gamma_{mn}}^{\gamma_{mx}} \delta\left[t - (\tau + \gamma\bar{D})\right] d_G(\gamma) \mathrm{d}\gamma \right) \mathrm{d}CDF_{Pt}(\tau), \end{aligned} \tag{5.4}$$

with $\delta(\sim)$ defined as in Eq. (3.2). The preceding representation for $CDF_D(t)$ reduces to the representation for $CDF_D(t)$ in Eq. (3.1) when $d_G(\gamma)$ is a suitably defined Dirac delta function (i.e., if there is a single value for $\gamma$ with a probability of 1.0).

As a consequence of the relationships

$$\delta\left[t - (\tau + \gamma\bar{D})\right] = 0 \text{ for } t - (\tau + \gamma\bar{D}) < 0 \quad \Rightarrow \quad (t - \tau) / \bar{D} < \gamma \tag{5.5}$$

and

$$\delta\left[t - (\tau + \gamma\bar{D})\right] = 1.0 \text{ for } 0 \le t - (\tau + \gamma\bar{D}) \quad \Rightarrow \quad \gamma \le (t - \tau) / \bar{D}, \tag{5.6}$$

the inner integral in Eq. (5.4) can be expressed as

$$\int_{\gamma_{mn}}^{\gamma_{mx}} \delta\left[t - (\tau + \gamma\bar{D})\right] d_G(\gamma) \mathrm{d}\gamma = \begin{cases} 1.0 \text{ for } \gamma_{mx} \le (t - \tau) / \bar{D} \\ \int_{\gamma_{mn}}^{(t-\tau)/\bar{D}} d_G(\gamma) \mathrm{d}\gamma \text{ for } \gamma_{mn} < (t - \tau) / \bar{D} < \gamma_{mx} \\ 0 \text{ for } (t - \tau) / \bar{D} \le \gamma_{mn}. \end{cases} \tag{5.7}$$

As discussed in Sect. 3, nominal delay times and associated density functions characterizing aleatory uncertainty can be defined for the individual links in a problem involving multiple WLs and SLs. Then, CDFs for the individual links defined as indicated in Eq. (5.4) can be used with the representations in Table 1 in the calculation of LOAS (e.g., as indicated in Eqs. (4.1)-(4.4)).

## 6. PLOAS with Aleatory Uncertainty in Delay Times: Approximation and Illustration

As in Sec. 4, the WL/SL system defined in Table 2 and Fig. 1 is used for illustration. For this illustration, the nominal delay time $\bar{D}$ for each link is taken to be the same as the corresponding constant delay time defined in Eqs. (4.15) and (4.16) (i.e., $\bar{D}_{WL1}$ = 5.0, $\bar{D}_{WL2}$ = 8.0, $\bar{D}_{SL1}$ = 12.0, $\bar{D}_{SL2}$ = 14.0). Further, the density function $d_G(\gamma)$ for each link characterizing the aleatory uncertainty associated with the delay in link failure is defined as follows: uniform on [0.5, 1.5] for WL 1, uniform on [0.5, 1.5] for WL 2, triangular on [0.6, 1.4] with mode = 1.0 for SL 1, and triangular on [1.0, 3.0] with mode = 1.0 for SL 2.

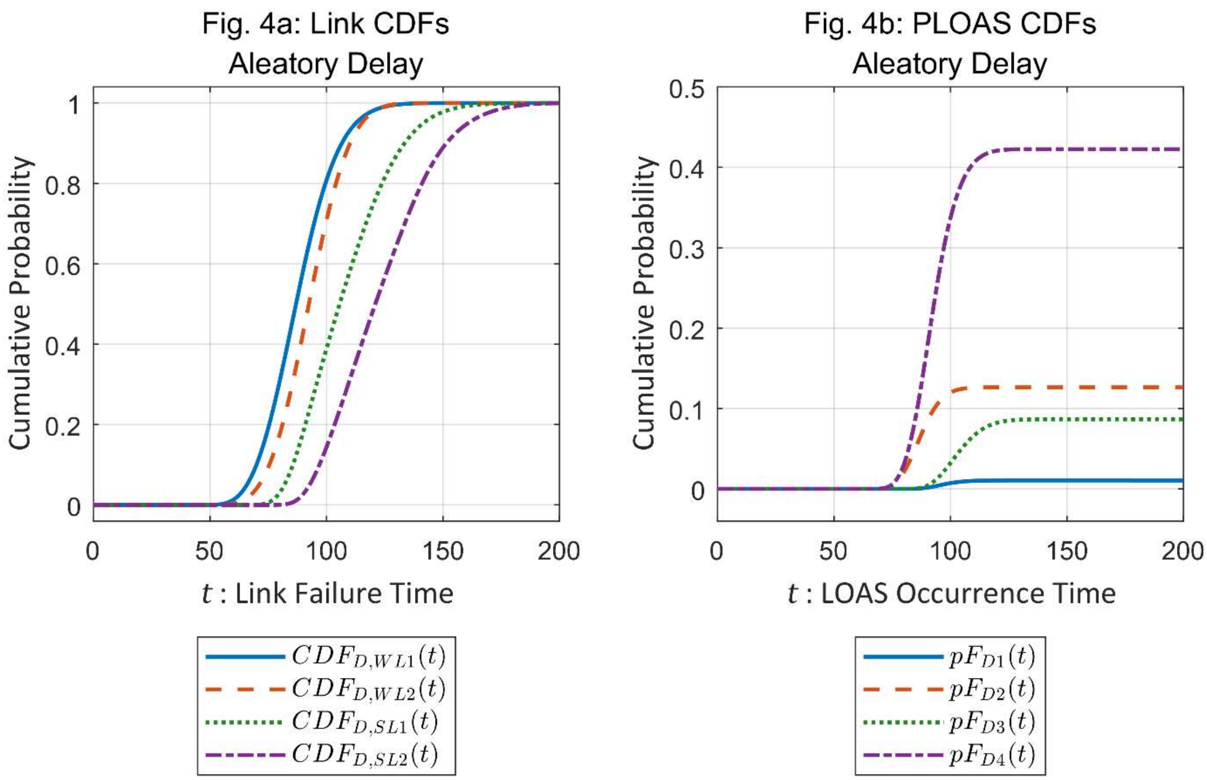


Fig. 4 Time-dependent PLOAS results for aleatory variation in link failure time obtained with (i) the quadrature procedures indicated in Eqs. (4.1)-(4.4) and a subdivision of [0, 200] of size $nSD = 10^4$ and (ii) $\bar{D}_{WL1}$ = 5.0 and $d_G(\gamma)$ uniform on [0.5, 1.5] for WL 1, $\bar{D}_{WL2}$ = 8.0 and $d_G(\gamma)$ uniform on [0.5, 1.5] for WL 2, $\bar{D}_{SL1}$ = 12.0 and $d_G(\gamma)$ triangular on [0.6, 1.4] with mode = 1.0 for SL 1, and $\bar{D}_{SL2}$ = 14.0 and $d_G(\gamma)$ triangular on [0.5, 1.5] with mode = 1.0 for SL 2: (a) CDFs $CDF_{D,WL1}(t)$, $CDF_{D,WL2}(t)$, $CDF_{D,SL1}(t)$ and $CDF_{D,SL2}(t)$ for link failure time, and (b) CDFs $pF_{Di}(t), i = 1,2,3,4,$ for PLOAS.

The PLOAS results obtained with the inclusion of the indicated aleatory variation in the delay times for the individual links are presented in Fig. 4 for each of the four cases in Table 1 (i.e., for $pF_{Di}(t)$, $i$ = 1, 2, 3, 4). The corresponding results for no delay in link failure time are presented in Fig. 2a,c.

As in Table 3 of Sect. 4, results obtained with two different sampling-based procedures for the estimation of PLOAS are compared in Table 4 with the quadrature-based results in Fig. 4b. The sampling-based procedures differ only in how the vector $\mathbf{t}_l$ in Eq. (4.14) is sampled. For the first sampling-based procedure (SB1), the link failure times are sampled directly from the failure time CDFs in Fig. 4a (i.e., from the final CDFs for link failure time that have incorporated the aleatory variability in the failure delay times). For the second sampling-based procedure (SB2), (i) values for $\alpha$ and $\beta$ are sampled for each link and used to determine precursor failure times, (ii) delay times are sampled from the distributions characterizing aleatory uncertainty in delay time for the individual links, and (iii) each determined precursor failure time (i.e., for a sampled $\alpha$, $\beta$ pair) is added to a sampled delay time to obtain the vectors $\mathbf{t}_l$ in Eq. (4.14). As shown in Table 4, the quadrature procedure and the two sampling procedures give what are effectively the same values for PLOAS.

The verification procedure indicated in Table 1 and discussed in conjunction with Eqs. (4.17) and (4.18) is also applicable to PLOAS determined with aleatory variability in link failure delay times. As shown in Fig. 5, the verification condition $pF_i(t) \rightarrow pF_i(\infty)$ as $t \rightarrow \infty$ described in Eqs. (4.17) and (4.18) is satisfied for $pF_{Di}(t)$ determined with (i) aleatory variability in link failure delay times, (ii) $nWL = 2$ and $nSL = 3$, (iii) the properties of SL 1 in Table 2 and Fig. 4 used for all links, and

$$pF_i(\infty) = \begin{cases} pF_1(\infty) = nWL!\,nSL!/(nWL + nSL)! = 1/10 = 0.1 \text{ for } i = 1 \\ pF_2(\infty) = nSL/(nWL + nSL) = 3/5 = 0.6 \text{ for } i = 2 \\ pF_3(\infty) = nWL/(nWL + nSL) = 2/5 = 0.4 \text{ for } i = 3 \\ pF_4(\infty) = 1 - nWL!\,nSL!/(nWL + nSL)! = 9/10 = 0.9 \text{ for } i = 4 \end{cases} \tag{6.1}$$

as stated in Table 1.

The results presented in Fig. 4, Table 4 and Fig. 5 were obtained with use of the CPLOAS program [25; 35] and thus provide verification results that indicate aleatory variability in link failure delay times in the calculation of the PLOAS values $pF_{Di}(t), i = 1,2,3,4,$ is implemented correctly in CPLOAS.

Table 4 Comparison of PLOAS results at $t$ = 200 with inclusion of aleatory variability in link failure delay times obtained with (i) quadrature procedures in Eqs. (4.1)-(4.4) with $nSD = 10^4$, and (ii) sampling-based procedures SB1 and SB2 with $nR = 10^6$.

Case 1: Failure of all SLs before failure of any WL

$$pF_1(200) = \begin{cases} 0.0104 & \text{Quadrature} \\ 0.0105 & \text{Sampling SB1, 95\% CI=[0.0103, 0.0107]} \\ 0.0104 & \text{Sampling SB2, 95\% CI=[0.0102, 0.0106]} \end{cases}$$

Case 2: Failure of any SL before failure of any WL

$$pF_2(200) = \begin{cases} 0.1263 & \text{Quadrature} \\ 0.1271 & \text{Sampling SB1, 95\% CI=[0.1265, 0.1278]} \\ 0.1263 & \text{Sampling SB2, 95\% CI=[0.1256, 0.1269]} \end{cases}$$

Case 3: Failure of all SLs before failure of all WLs

$$pF_3(200) = \begin{cases} 0.0865 & \text{Quadrature} \\ 0.0865 & \text{Sampling SB1, 95\% CI=[0.0859, 0.0870]} \\ 0.0864 & \text{Sampling SB2, 95\% CI=[0.0859, 0.0869]} \end{cases}$$

Case 4: Failure of any SL before failure of all WLs

$$pF_4(200) = \begin{cases} 0.4226 & \text{Quadrature} \\ 0.4226 & \text{Sampling SB1, 95\% CI=[0.4216, 0.4235]} \\ 0.4230 & \text{Sampling SB2, 95\% CI=[0.4221, 0.4240]} \end{cases}$$

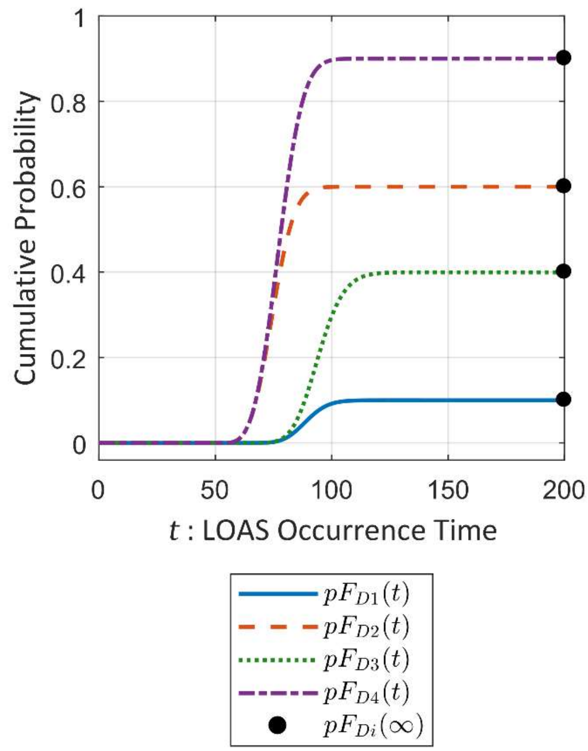


Fig. 5 Verification test for $pF_{Di}(t), i = 1,2,3,4,$ with (i) aleatory variability in link failure delay times, (ii) $nWL = 2$ and $nSL = 3$, and (iii) the properties of SL 1 in Table 2 and Fig. 4 used for all links.

## 7. PLOAS with Delay Times a Function of Link Property at Time that Precursor Failure Occurs: Formal Representation

Another possibility is that the delay time $D(p)$ from precursor occurrence to link failure is a function of the link property value $p$ that exists at the time of precursor occurrence. In this situation, the CDF for link failure time is given by

$$\begin{aligned} CDF_D(t) = &\text{ probability that link failure occurs at or before time } t \text{ for} \\ &\text{ a delay } D(p) \text{ that is a function of the link property value } p \\ &\text{ at the time of precursor occurrence} \qquad (7.1) \\ = &\int_0^t \left( \int_{p_{mn}(\tau)}^{p_{mx}(\tau)} \delta\{t - [\tau + D(p)]\} \, d_P(p \mid \tau) \mathrm{d}p \right) \mathrm{d}CDF_P(\tau), \end{aligned}$$

where (i) the interval $[p_{mn}(\tau), p_{mx}(\tau)]$ contains the link property values that could potentially result in precursor occurrence at time $\tau$, (ii) $d_P(p \mid \tau)$ defined on $[p_{mn}(\tau), p_{mx}(\tau)]$ is the density function for link property values $p$ that could result in precursor occurrence conditional on the assumption that precursor occurrence has taken place at time $\tau$ (i.e., conditional on precursor occurrence at time $\tau$, the density function $d_P(p \mid \tau)$ corresponds to the distribution of link property values $p$ at which the precursor occurrence could have taken place) and (iii) $\delta\{\sim\}$ is defined as in Eq. (3.2).

The density function $d_P(p \mid \tau)$ and the corresponding interval of definition $[p_{mn}(\tau), p_{mx}(\tau)]$ are defined in Table 5. As indicated in Table 5, the definitions of $d_P(p \mid \tau)$ and $[p_{mn}(\tau), p_{mx}(\tau)]$ are conditional on the ratio $r(\tau) = \bar{q}(\tau) / \bar{p}(\tau)$, which results in four possibilities (i.e., $d_{pi}(p \mid \tau \in \mathcal{P}_i)$, $i = 1, 2, 3, 4$) for (i) the definition of $d_P(p \mid \tau)$ and (ii) the corresponding definition of the interval $[p_{mn}(\tau), p_{mx}(\tau)]$ for $d_P(p \mid \tau)$. A summary of the sets $\mathcal{P}_i$ and associated intervals $[p_{mn}(\tau), p_{mx}(\tau)]$ is provided in Table 6.

Table 5 Summary of density functions $d_P(p \mid \tau) = d_{Pi}(p \mid \tau \in \mathcal{P}_i)$ for link property values $p$ that could potentially result in precursor occurrence at time $\tau \in \mathcal{P}_i$ ([36], Table 6).

$$d_{P1}(p \mid \tau \in \mathcal{P}_1) = \frac{\left(p / \overline{q}^2(\tau)\right) d_\alpha\left[p / \overline{p}(\tau)\right] d_\beta\left[p / \overline{q}(\tau)\right]}{\int_{r(\tau)\beta_{mn}}^{\alpha_{mx}} \alpha d_\alpha(\alpha) d_\beta(\alpha / r(\tau)) \mathrm{d}\alpha}$$

for $\mathcal{P}_1 = \{\tau : \alpha_{mn} \le r(\tau)\beta_{mn} \le \alpha_{mx} \le r(\tau)\beta_{mx}\}$ and $p \in \mathcal{S}_1(p \mid \tau \in \mathcal{P}_1) = \left[\beta_{mn}\overline{q}(\tau), \alpha_{mx}\overline{p}(\tau)\right]$

---

$$d_{P2}(p \mid \tau \in \mathcal{P}_2) = \frac{\left(p / \overline{q}^2(\tau)\right) d_\alpha\left[p / \overline{p}(\tau)\right] d_\beta\left[p / \overline{q}(\tau)\right]}{\int_{r(\tau)\beta_{mn}}^{r(\tau)\beta_{mx}} \alpha d_\alpha(\alpha) d_\beta(\alpha / r(\tau)) \mathrm{d}\alpha}$$

for $\mathcal{P}_2 = \{\tau : \alpha_{mn} \le r(\tau)\beta_{mn} \le r(\tau)\beta_{mx} \le \alpha_{mx}\}$ and $p \in \mathcal{S}_2(p \mid \tau \in \mathcal{P}_2) = \left[\beta_{mn}\overline{q}(\tau), \beta_{mx}\overline{q}(\tau)\right]$

---

$$d_{P3}(p \mid \tau \in \mathcal{P}_3) = \frac{\left(p / \overline{q}^2(\tau)\right) d_\alpha\left[p / \overline{p}(\tau)\right] d_\beta\left[p / \overline{q}(\tau)\right]}{\int_{\alpha_{mn}}^{\alpha_{mx}} \alpha d_\alpha(\alpha) d_\beta(\alpha / r(\tau)) \mathrm{d}\alpha}$$

for $\mathcal{P}_3 = \{\tau : r(\tau)\beta_{mn} \le \alpha_{mn} \le \alpha_{mx} \le r(\tau)\beta_{mx}\}$ and $p \in \mathcal{S}_3(p \mid \tau \in \mathcal{P}_3) = \left[\alpha_{mn}\overline{p}(\tau), \alpha_{mx}\overline{p}(\tau)\right]$

---

$$d_{P4}\left(p \mid \tau \in \mathcal{P}_4\right) = \frac{\left(p / \overline{q}^2(\tau)\right) d_\alpha\left[p / \overline{p}(\tau)\right] d_\beta\left[p / \overline{q}(\tau)\right]}{\int_{\alpha_{mn}}^{r(\tau)\beta_{mx}} \alpha d_\alpha(\alpha) d_\beta(\alpha / r(\tau)) \mathrm{d}\alpha}$$

for $\mathcal{P}_4 = \{\tau : r(\tau)\beta_{mn} \le \alpha_{mn} \le r(\tau)\beta_{mx} \le \alpha_{mx}\}$ and $p \in \mathcal{S}_4(p \mid \tau \in \mathcal{P}_4) = \left[\alpha_{mn}\overline{p}(\tau), \beta_{mx}\overline{q}(\tau)\right]$

Table 6 Summary of the intervals of definition $[p_{mn}(\tau), p_{mx}(\tau)]$ for the density functions $d_{Pi}(p \mid \tau \in \mathcal{P}_i)$, $i$ = 1, 2, 3, 4, defined in Table 5 with $\bar{p}(\tau)$ increasing and $\bar{q}(\tau)$ either decreasing or constant-valued (adapted from Ref.[36], Table 7).

---

$[p_{mn}(\tau), p_{mx}(\tau)] =$ interval of link failure values $p$ at time $\tau$ for $\alpha_{mn}/\beta_{mn} < \alpha_{mx}/\beta_{mx}$

$$= \begin{cases} [\beta_{mn}\bar{q}(\tau), \alpha_{mx}\bar{p}(\tau)] \text{ for } \tau \in \mathcal{P}_1 = \{\tau : \tau_f = r^{-1}(\alpha_{mx}/\beta_{mn}) \le \tau \le r^{-1}(\alpha_{mx}/\beta_{mx}) = \tau_{mx}\} \\ [\beta_{mn}\bar{q}(\tau), \beta_{mx}\bar{q}(\tau)] \text{ for } \tau \in \mathcal{P}_2 = \{\tau : \tau_{mx} = r^{-1}(\alpha_{mx}/\beta_{mx}) \le \tau \le r^{-1}(\alpha_{mn}/\beta_{mn}) = \tau_{mn}\} \\ [\alpha_{mn}\bar{p}(\tau), \beta_{mx}\bar{q}(\tau)] \text{ for } \tau \in \mathcal{P}_4 = \{\tau : \tau_{mn} = r^{-1}(\alpha_{mn}/\beta_{mn}) \le \tau \le r^{-1}(\alpha_{mn}/\beta_{mx}) = \tau_l\} \end{cases}$$

$[p_{mn}(\tau), p_{mx}(\tau)] =$ interval of link failure values $p$ at time $\tau$ for $\alpha_{mx}/\beta_{mx} < \alpha_{mn}/\beta_{mn}$

$$= \begin{cases} [\beta_{mn}\bar{q}(\tau), \alpha_{mx}\bar{p}(\tau)] \text{ for } \tau \in \mathcal{P}_1 = \{\tau : \tau_f = r^{-1}(\alpha_{mx}/\beta_{mn}) \le \tau \le r^{-1}(\alpha_{mn}/\beta_{mn}) = \tau_{mn}\} \\ [\alpha_{mn}\bar{p}(\tau), \alpha_{mx}\bar{p}(\tau)] \text{ for } \tau \in \mathcal{P}_3 = \{\tau : \tau_{mn} = r^{-1}(\alpha_{mn}/\beta_{mn}) \le \tau \le r^{-1}(\alpha_{mx}/\beta_{mx}) = \tau_{mx}\} \\ [\alpha_{mn}\bar{p}(\tau), \beta_{mx}\bar{q}(\tau)] \text{ for } \tau \in \mathcal{P}_4 = \{\tau : \tau_{mx} = r^{-1}(\alpha_{mx}/\beta_{mx}) \le \tau \le r^{-1}(\alpha_{mn}/\beta_{mx}) = \tau_l\} \end{cases}$$

$[p_{mn}(\tau), p_{mx}(\tau)] =$ interval of link failure values $p$ at time $\tau$ for $\alpha_{mn}/\beta_{mn} = \alpha_{mx}/\beta_{mx}$

$$= \begin{cases} [\beta_{mn}\bar{q}(\tau), \alpha_{mx}\bar{p}(\tau)] \text{ for } \tau \in \mathcal{P}_1 = \{\tau : \tau_f = r^{-1}(\alpha_{mx}/\beta_{mn}) \le \tau \le r^{-1}(\alpha_{mx}/\beta_{mx}) = \tau_{mx}\} \\ [\alpha_{mn}\bar{p}(\tau), \beta_{mx}\bar{q}(\tau)] \text{ for } \tau \in \mathcal{P}_4 = \{\tau : \tau_{mx} = r^{-1}(\alpha_{mx}/\beta_{mx}) \le \tau \le r^{-1}(\alpha_{mn}/\beta_{mx}) = \tau_l\} \end{cases}$$

---

The representation for $CDF_{PT}(t)$ in Eq. (7.1) can also be expressed in the equivalent form

$$\begin{aligned} CDF_P(t) &= \int_{t_{mn}}^{t} \left( \int_{p_{mn}(\tau)}^{p_{mx}(\tau)} \delta\{t - [\tau + D(p)]\} d_P(p \mid \tau) \mathrm{d}p \right) \mathrm{d}CDF_P(\tau) \\ &= \int_{t_{mn}}^{t} \left( \int_{p_{mn}(\tau)}^{p_{mx}(\tau)} \delta\{t - [\tau + D(p)]\} d_P(p \mid \tau) \mathrm{d}p \right) \left( \mathrm{d}CDF_P(\tau)/\mathrm{d}\tau \right) \mathrm{d}\tau \\ &= \int_{t_{mn}}^{t} \left( \int_{p_{mn}(\tau)}^{p_{mx}(\tau)} \delta\{t - [\tau + D(p)]\} d_P(p \mid \tau) \mathrm{d}p \right) d_P(\tau) \mathrm{d}\tau \\ &= \int_{t_{mn}}^{t} \left( \int_{p_{mn}(\tau)}^{p_{mx}(\tau)} \delta\{t - [\tau + D(p)]\} d_P(p \mid \tau) d_P(\tau) \mathrm{d}p \right) \mathrm{d}\tau, \end{aligned} \tag{7.2}$$

where $d_P(\tau)$ is the density function associated with $CDF_P(\tau)$. As shown in Eq. (9.65) of Ref. [36],

$$d_P(p|\tau) d_P(\tau) = \left\{ \mathrm{d}\left[ \frac{1}{r(\tau)} \right] / \mathrm{d}\tau \right\} \left\{ \frac{p}{\bar{p}^2(\tau)} \right\} d_A[p/\bar{p}(\tau)] d_B[p/\bar{q}(\tau)], \tag{7.3}$$

with $r(\tau) = \overline{q}(\tau) / \overline{p}(\tau)$. In turn, substitution of the preceding representation for $d_P(p|\tau)d_P(\tau)$ into Eq. (7.2) produces the following representation for $CDF_D(t)$:

$$
\begin{aligned}
CDF_D(t) &= \int_{t_{mn}}^{t} \left( \int_{p_{mn}(\tau)}^{p_{mx}(\tau)} \delta\left\{t - [\tau + D(p)]\right\} \right. \\
&\qquad \left. \times \left\{ \mathrm{d}\left[\frac{1}{r(\tau)}\right] / \mathrm{d}\tau \right\} \left\{ p / \overline{p}^2(\tau) \right\} d_A\left[p / \overline{p}(\tau)\right] d_B\left[p / \overline{q}(\tau)\right] \mathrm{d}p \right) \mathrm{d}\tau \qquad (7.4) \\
&= \int_{t_{mn}}^{t} \left( \left\{ \mathrm{d}\left[\frac{1}{r(\tau)}\right] / \mathrm{d}\tau \right\} \int_{p_{mn}(\tau)}^{p_{mx}(\tau)} \delta\left\{t - [\tau + D(p)]\right\} \left\{ p / \overline{p}^2(\tau) \right\} d_A\left[p / \overline{p}(\tau)\right] d_B\left[p / \overline{q}(\tau)\right] \mathrm{d}p \right) \mathrm{d}\tau.
\end{aligned}
$$

Although it looks complicated, the preceding representation for $CDF_D(t)$ is actually simpler from a numerical implementation perspective than the representation for $CDF_D(t)$ in Eq. (7.1) because it removes the complex structure associated with the definition of $d_P(p \mid \tau)$ summarized in Table 5. Except for the limits of integration and the indicator function $\delta\{t - [\tau + D(p)]\}$, the integral in Eq. (7.4) is the same as the integral in Eq. (10.10) of Ref. [36].

Some simplification of the inner integral in Eq. (7.4) is possible if $D(p)$ is an increasing function of $p$ or a decreasing function of $p$. Specifically,

$$
\begin{aligned}
&\delta\left[t - (\tau + D(p))\right] = 0 \ \text{ for } \ t - (\tau + D(p)) < 0 \\
&\qquad \Rightarrow t - \tau < D(p) \Rightarrow \begin{cases} D^{-1}(t-\tau) < p & \text{for } D(p) \text{ increasing} \\ p < D^{-1}(t-\tau) & \text{for } D(p) \text{ decreasing} \end{cases}
\end{aligned} \qquad (7.5)
$$

and

$$
\begin{aligned}
&\delta\left[t - (\tau + D(p))\right] = 1.0 \ \text{ for } \ 0 \le t - (\tau + D(p)) \\
&\qquad \Rightarrow D(p) \le (t - \tau) \Rightarrow \begin{cases} p \le D^{-1}(t-\tau) & \text{for } D(p) \text{ increasing} \\ D^{-1}(t-\tau) \le p & \text{for } D(p) \text{ decreasing.} \end{cases}
\end{aligned} \qquad (7.6)
$$

In turn, the inner integral in Eq. (7.4) can be expressed as

$$
\int_{p_{mn}(\tau)}^{p_{mx}(\tau)} F(\tau, p) \left\{ p / \overline{p}^2(\tau) \right\} d_A\left[p / \overline{p}(\tau)\right] d_B\left[p / \overline{q}(\tau)\right] \mathrm{d}p \qquad (7.7)
$$

with

$$
F(\tau, p) = \begin{cases} 0 & \text{for } D^{-1}(t-\tau) < p \\ 1 & \text{for } p \le D^{-1}(t-\tau) \end{cases} \quad \text{for } D(p) \text{ increasing} \qquad (7.8)
$$

and

$$F(\tau, p) = \begin{cases} 0 \ \text{ for } \ p < D^{-1}(t-\tau) \\ 1 \ \text{ for } \ D^{-1}(t-\tau) \le p \end{cases} \ \text{ for } \ D(p) \ \text{ decreasing.} \tag{7.9}$$

Once the CDFs for the individual links developed in Eqs. (7.1)-(7.4) are available, they can be used with the approximations in Eqs. (4.1)-(4.4) in the determination of PLOAS. Another possibility is to numerically approximate the corresponding integrals in Table 1 with procedures contained in the MATLAB numerical package [37]. However, approximation of the defining integrals for $CDF_D(t)$ can be a numerically challenging undertaking.

## 8. PLOAS with Delay Times a Function of Link Property at Time that Precursor Failure Occurs: Approximation and Illustration

As in Sects. 4 and 6, the example defined and illustrated in Table 2 and Fig. 1 is used for illustration. Further, the delay time $D(p)$ from precursor occurrence to link failure is a function of the system property $p$ that exists at the time of precursor occurrence and is assumed to be of the form

$$D(p) = \begin{cases} 10,000 / p = D_{WL1}(p) & \text{for WL1} \\ 10,500 / p = D_{WL2}(p) & \text{for WL2} \\ 11,000 / p = D_{SL1}(p) & \text{for SL1} \\ 11,500 / p = D_{SL2}(p) & \text{for SL2.} \end{cases} \tag{8.1}$$

The evaluation of the integral in Eq. (7.4) for the four links described and illustrated in Table 2 and Fig. 1 is considered first. Then, the evaluation of the four integrals in Table 1 for PLOAS for the indicated four links is considered.

Two approaches for the evaluation of the integral in Eq. (7.4) are considered: (i) a quadrature-based procedure using the MATLAB numerical package [37] and (ii) a sampling-based procedure.

The quadrature-based procedure using the MATLAB numerical package is described first. Evaluation of the integral in Eq. (7.4) for each of the four links requires the incorporation of the indicator function $\delta\{t-[\tau+D(p)]\}$ into the integration process. As indicated in Eqs. (7.5)-(7.9), some simplification of the inner integral is possible when $D(p)$ is an increasing or decreasing function of $p$. In this example,

$$D(p) = k / p \text{ with } k = 10,000; 10,500; 11,000 \text{ or } 11,500 \tag{8.2}$$

is a decreasing function of $p$. Thus, consistent with Eq. (7.9), $\delta\{t-[\tau+D(p)]\}$ can be replaced by the function

$$F(\tau, p) = \begin{cases} 0 & \text{for } p < D^{-1}(t-\tau) = k / (t-\tau) \\ 1 & \text{for } k / (t-\tau) = D^{-1}(t-\tau) \le p \end{cases} \tag{8.3}$$

in the inner integral in Eq. (7.4) to produce an inner integral of the form shown in Eq. (7.7). To the best of our knowledge, MATLAB does not have an option for the inclusion of functions of the form $\delta\{t-[\tau+D(p)]\}$ as part of the integrand in the numerical evaluation of an integral. Thus, as described in the next paragraph, some additional development is required rather than simply evaluating the integral in Eq. (7.4) with a MATLAB integration routine such as TwoD [38].

The computational strategy used is to initially consider the integral in Eq. (7.7) with $F(\tau, p)$ defined in Eq. (8.3) as a function

$$I(\tau \mid t) = \int_{p_{mn}(\tau)}^{p_{mx}(\tau)} F(\tau, p)\left\{p / \overline{p}^2(\tau)\right\} d_A\left[p / \overline{p}(\tau)\right] d_B\left[p / \overline{q}(\tau)\right] \mathrm{d}p \tag{8.4}$$

of $\tau$ conditional on *t*. For each of the four links under consideration, the values for $\alpha_{mn}, \alpha_{mx}$, $\beta_{mn}$ and $\beta_{mx}$ satisfy the inequality $\alpha_{mx} / \beta_{mx} < \alpha_{mn} / \beta_{mn}$. Thus, integration limits $p_{mn}(\tau)$ and $p_{mx}(\tau)$ are defined by the second possibility in Table 6. When the integration limits $p_{mn}(\tau)$ and $p_{mx}(\tau)$ are combined with the effects of $F(\tau, p)$, the form of $I(\tau \mid t)$ becomes

$$I(\tau \mid t) = \begin{cases} 0 \text{ for } p_{mx}(\tau) \le k / (t-\tau) \\ \int_{k/(t-\tau)}^{p_{mx}(\tau)} \left\{p / \overline{p}^2(\tau)\right\} d_A\left[p / \overline{p}(\tau)\right] d_B\left[p / \overline{q}(\tau)\right] \mathrm{d}p \text{ for } p_{mn}(\tau) < k / (t-\tau) < p_{mx}(\tau) \\ \int_{p_{mn}(\tau)}^{p_{mx}(\tau)} \left\{p / \overline{p}^2(\tau)\right\} d_A\left[p / \overline{p}(\tau)\right] d_B\left[p / \overline{q}(\tau)\right] \mathrm{d}p \text{ for } k / (t-\tau) \le p_{mn}(\tau). \end{cases} \tag{8.5}$$

At this point, the integral in Eq. (7.4) can be viewed as being of the form

$$CDF_D(t) = \int_{t_{mn}}^{t} \left\{ \mathrm{d}\left[\frac{1}{r(\tau)}\right] / \mathrm{d}\tau \right\} I(\tau \mid t) \mathrm{d}\tau \tag{8.6}$$

with $I(\tau \mid t)$ defined in Eq. (8.5).

The representation for $CDF_D(t)$ in Eq. (8.6) can be evaluated with the **integral** function in MATLAB provided the two terms in the integrand can be defined in an appropriate manner. The derivative $\mathrm{d}[1/r(\tau)]/\mathrm{d}\tau$ can be defined with use of the MATLAB functions **diff** and **matlabFunction**. The definition of $I(\tau \mid t)$ is more complex and involves a spline fit to $I(\tau \mid t)$. Specifically, a subdivision $\tau_i, i = 0, 1, ..., nS = 200,$ of $[t_{mn}, t]$ with $\Delta\tau_i = (t - t_{mn}) / nS$ is defined, and $I(\tau_i \mid t), i = 1, 2, ..., nS,$ is obtained by evaluating the integrals in Eq. (8.5) with MATLAB procedures. For these evaluations, the density functions in the integrand are defined with the **makedist** and **pdf** functions and the integrations to obtain $I(\tau_i \mid t), i = 1, 2, ..., nS,$ are performed with the **integral** function. Then, to obtain $I(\tau \mid t)$ for use in the evaluation of the integral defining $CDF_{DT}(t)$ in Eq. (8.6), the function **spline** is used to fit a function to the values $I(\tau_i \mid t), i = 1, 2, ..., nS$. At this point, the function **integral** is used to evaluate $CDF_{DT}(t)$ with the indicated representations for $\mathrm{d}[1/r(\tau)]/\mathrm{d}\tau$ and $I(\tau \mid t)$. The outcome of this calculation is shown in Fig. 6 for the four links defined in Table 2 and illustrated in Fig. 1.

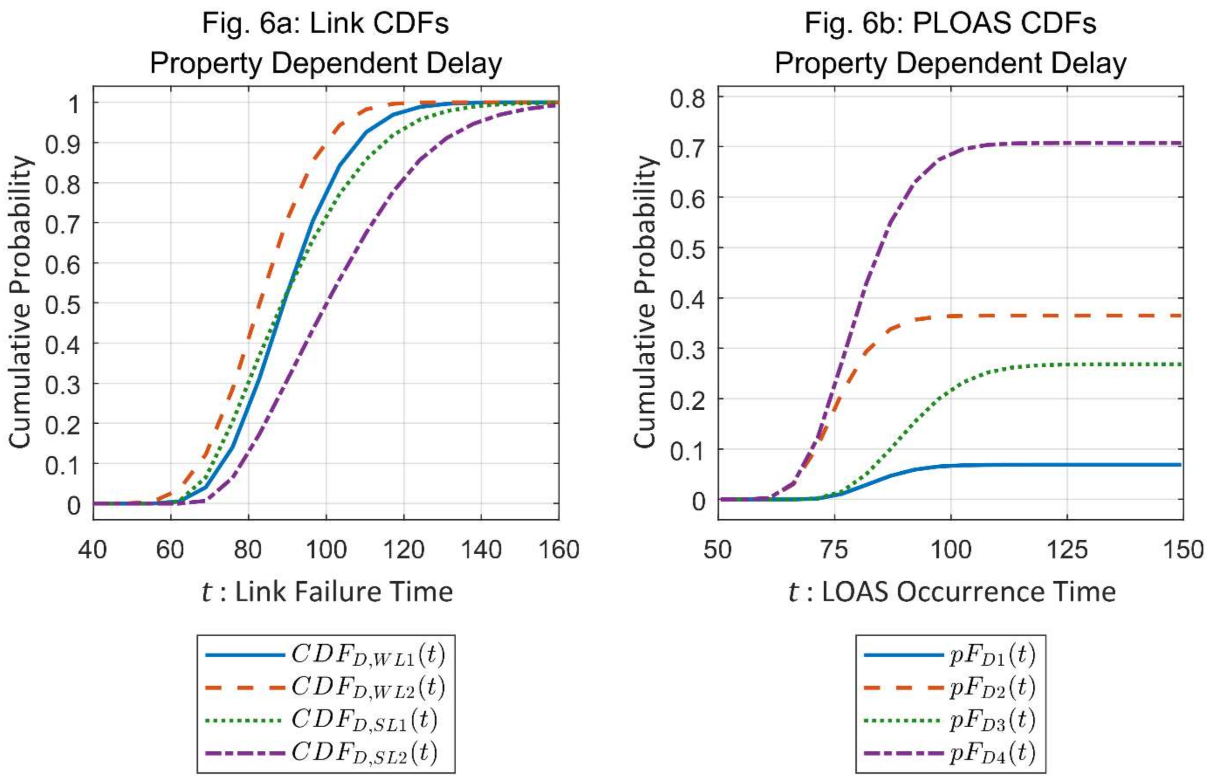


Fig. 6 Time-dependent PLOAS results for delay in link failure time dependent on link property value at precursor failure obtained with (i) quadrature-based procedures and (ii) the failure time delays $D_{WL1}(p)$, $D_{WL2}(p)$, $D_{SL1}(p)$ and $D_{SL2}(p)$ for each link defined in Eq. (8.1): (a) CDFs $CDF_{D,WL1}(t)$, $CDF_{D,WL2}(t)$, $CDF_{D,SL1}(t)$ and $CDF_{D,SL2}(t)$ for link failure time, and (b) CDFs $pF_{Di}(t), i = 1,2,3,4,$ for PLOAS.

Given that the CDFs $CDF_{D,WL1}(t)$, $CDF_{D,WL2}(t)$, $CDF_{D,SL1}(t)$ and $CDF_{D,SL2}(t)$ for link failure time are available, the CDFs $pF_{Di}(t), i = 1,2,3,4,$ for PLOAS can be obtained by evaluating the approximating sums in Eqs. (4.1)-(4.4) as done for the PLOAS results in Sects. 4 and 6. Another possibility is to obtain the CDFs $pF_{Di}(t), i = 1,2,3,4,$ by using procedures in MATLAB to evaluate the integrals in Table 1. For variety and illustration, quadrature procedures based on MATLAB functions will be used to obtain the CDFs $pF_{Di}(t), i = 1,2,3,4,$ for delay in link failure time dependent on link property value at precursor failure. To accomplish this, the function **spline** is used to provide a spline representation for each of the CDFs $CDF_{D,WL1}(t), CDF_{D,WL2}(t)$, $CDF_{D,SL1}(t)$ and $CDF_{D,SL2}(t)$ for link failure time. Then, the spline representations in conjunction with the **ppval** function are used to obtain representations for the integrands in each of the Stieltjes integrals in Table 1. Next, the Stieltjes integrals are converted to Riemann integrals by differentiating $CDF_{D,SL1}(t)$ and $CDF_{D,SL2}(t)$ with the **fnder** function and then applying the **ppval** function to the results of these differentiations. At this point, the original Stieltjes integrals in Table 1 are now Riemann integrals and can be evaluated with the **integral** function. The results of these evaluations to obtain the CDFs $pF_{Di}(t), i = 1,2,3,4,$ for delay in link failure time dependent on link property value at precursor failure are shown in Fig. 6b.

As a verification check, the results in Fig. 6 can be obtained with sampling-based calculations that are independent of the quadrature-based calculations used to obtain the results presented in Fig. 6. The sampling-based procedure used is similar to the second sampling-based procedure (SB2) used in the verification of quadrature-based results in Sects. 4 and 6. Specifically, (i) values for $\alpha_l$ and $\beta_l$ , $l = 1,2,...,nR,$ are randomly sampled for each link and used to determine precursor failure times $t_l$ and corresponding property values $p_l$ at precursor failure, (ii) delay times $D(p_l)$ are determined individual links, and (iii) each determined precursor delay time $D(p_l)$ is added to the corresponding sampled delay time $t_l$ to obtain the vectors

$$\mathbf{t}_l = \left[tWL_{1l}, tWL_{2l}, tSL_{1l}, tSL_{2l}\right], l = 1,2,...,nR, \tag{8.7}$$

for link failure time as indicated in Eq. (4.14). Then, $CDF_{D,WL1}(t)$ can be approximated by

$$CDF_{D,WL1}(t) \cong \sum_{l=1}^{nR} \delta_t(tWL_{1l}) / nR \text{ with } \delta_t(tWL_{1l}) = \begin{cases} 1 \text{ for } tWL_{1l} \le t \\ 0 \text{ for } t < tWL_{1l} \end{cases} \tag{8.8}$$

and $CDF_{D,WL2}(t)$, $CDF_{D,SL1}(t)$ and $CDF_{D,SL2}(t)$ can be approximated similarly. Further, the CDFs $pF_{Di}(t), i = 1,2,3,4,$ for delay in link failure time dependent on link property value at precursor failure can be approximated as shown in Eq. (4.5).

For comparison, the results of the quadrature-based calculations and the just discussed sampling-based calculations are shown in Fig. 7. As examination of Fig. 7 shows, the results obtained with the two procedures are essentially identical. Given the independence of the implementations of both procedures, this provides a strong verification result that both procedures are correct in both mathematical structure and computational implementation.

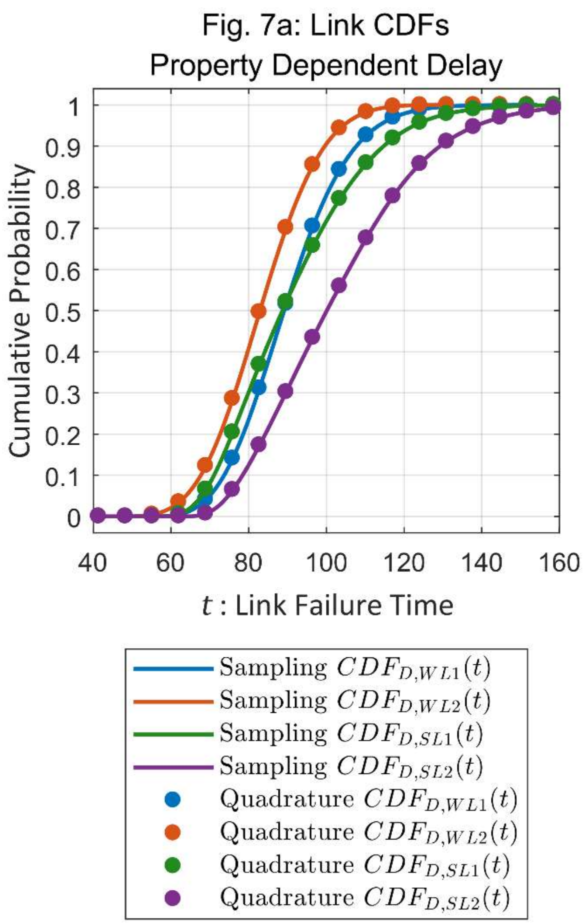


Fig. 7 Time-dependent PLOAS results for delay in link failure time dependent on link property value at precursor failure obtained with (i) quadrature-based procedures, (ii) sampling-based procedures, and (iii) the failure time delays $D_{WL1}(p)$, $D_{WL2}(p)$, $D_{SL1}(p)$ and $D_{SL2}(p)$ for each link defined in Eq. (8.1): (a) CDFs $CDF_{D,WL1}(t)$, $CDF_{D,WL2}(t)$, $CDF_{D,SL1}(t)$ and $CDF_{D,SL2}(t)$ for link failure time, and (b) CDFs $pF_{Di}(t), i = 1, 2, 3, 4$, for PLOAS.

The verification procedure indicated in Table 1 and discussed in conjunction with Eqs. (4.17) and (4.18) is also applicable to PLOAS determined with the delay in link failure time dependent on link property value at precursor occurrence. As shown in Fig. 8, the verification condition $pF_i(t) \to pF_i(\infty)$ as $t \to \infty$ described in Eqs. (4.17) and (4.18) is satisfied for $pF_{Di}(t)$ determined with (i) the delay in link failure time dependent on link property value at precursor occurrence, (ii) $nWL = 3$ and $nSL = 2$, and (iii) the properties of SL 2 in Table 2 and Fig. 6 used for all links. For this example, the asymptotic values for $pF_{Di}(t), i = 1, 2, 3, 4,$ are the same as shown in Eq. (4.17).

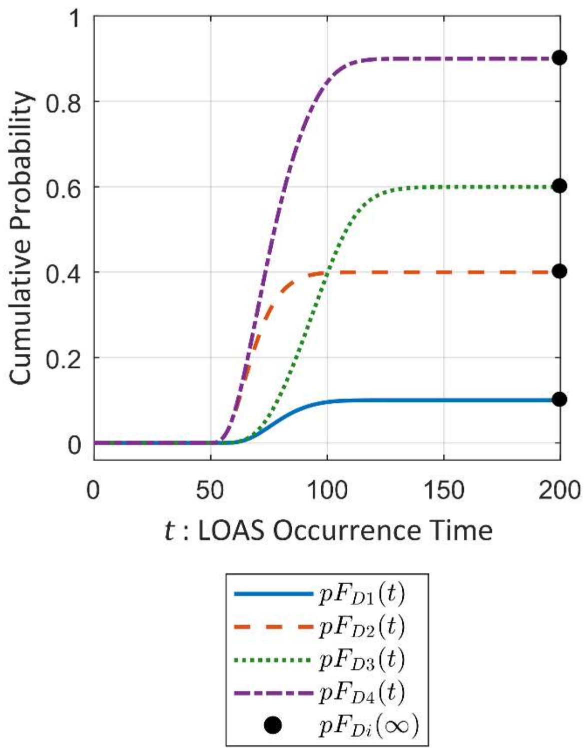


Fig. 8 Verification test for $pF_{Di}(t), i = 1, 2, 3, 4,$ with (i) the delay in link failure time dependent on link property value at precursor occurrence, (ii) $nWL = 3$ and $nSL = 2$, and (iii) the properties of SL 2 in Table 2 and Fig. 6 used for all links.

The sampling-based results in Fig. 7 were calculated with the CPLOAS program [25; 35]. Similarly, the verification results in Fig. 8 were also calculated with sampling-based procedures in CPLOAS. At present, the use of property-dependent delay times in the calculation of the PLOAS values $pF_{Di}(t), i = 1, 2, 3, 4,$ is not a formally defined capability of CPLOAS. However, the sampling-based results in Fig. 7 and Fig. 8 were obtained by making a few simple coding additions within the overall structure of CPLOAS. Thus, the verification results in Fig. 7 and Fig. 8 indicate that (i) these additions to CPLOAS are producing correct values for property-dependent link failure delay times in the calculation of the PLOAS values $pF_{Di}(t), i = 1, 2, 3, 4,$ and (ii) if desired at some point in the future, it would be possible to make calculations of this type a formal option within CPLOAS. Further, the agreement of the quadrature-based and sampling-based results in Fig. 7 provides a strong indication that the very complex derivations that produced Eq. (7.4) are correct.

## 9. Summary Discussion

Two primary topics are addressed in this presentation: (i) Implementation and illustration of three possible definitions for delays between the occurrence of link precursor conditions and link failure in LOAS analyses for WL/SL systems, and (ii) Verification of the numerical correctness of the implementation of the three indicated delay definitions in the CPLOAS program.

The three delay definitions are (i) a constant delay between the occurrence of link precursor conditions and link failure, (ii) aleatory variability in the delay between the occurrence of link precursor conditions and link failure, and (iii) delay in link failure time dependent on link property value at time that link precursor conditions occur.

Verification tests for constant delays and aleatory delays in link failure in the calculation pf PLOAS values are formal parts of the CPLOAS program. Specifically, the following independent options for the calculation of the four definitions of PLOAS in Table 1 are available:

(i) Quadrature evaluations that determine PLOAS as indicated in Eqs. (4.1)-(4.4),

(ii) Sampling-based evaluations that involve sampling link failure times from the link failure time CDFs after incorporation of the delays and then determining PLOAS as indicated in Eqs, (4.5)-(4.14), and

(iii) Sampling-based evaluations that involve sampling the variables $\alpha$ and $\beta$ that characterize aleatory uncertainty in the time of link precursor occurrence and the variables $\gamma$ that characterize aleatory uncertainty in link failure delay time, determining the resultant failure times for each link based on the sampled values for $\alpha$, $\beta$ and $\gamma$, and then determining PLOAS as indicated in Eqs. (4.5)-(4.14).

As shown in Table 3 and Table 4, all three numerical procedures provide what is effectively (i.e., within a small amount of numerical variability) the same PLOAS values for constant delays and aleatory delays in link failure, which is a strong verification result that CPLOAS is implementing these delays and associated PLOAS calculations correctly.

The third delay definition (i.e., delay in link failure time dependent on link property value at time that link precursor conditions occur) is not a formally implemented option in CPLOAS. However, it was possible to implement this delay definition with small programming additions in CPLOAS to what is indicated above as the second sampling approach (i.e., $\alpha$ and $\beta$ are sampled, property value $p$ at time precursor occurrence and delay $D(p)$ are obtained, and link failure time is determined). With these additions, the CDFs for link failure after delay are obtained as indicated in Eq. (8.8) and PLOAS values are obtained as indicated in Eqs. (4.5)-(4.14).

However, this calculation with the modified version of CPLOAS does not supply any additional results for verification of calculations with delay in link failure time dependent on link property value at the time that link precursor conditions occur. To obtain verification results for this delay definition, Eq. (7.4) for link failure time delay dependent on link property value at precursor failure was (i) derived, (ii) approximated with a quadrature procedures contained in the

MATLAB numerical package, and (iii) used to defined CDFs for link failure time. Next, the link failure CDFs were used in conjunction with MATLAB quadrature procedures to evaluate the four PLOAS integrals in Table 1 corresponding to different definitions of LOAS. As shown in Fig. 7, sampling-based results obtained from modified CPLOAS and the quadrature results for the integral in Eq. (7.4) and the integrals in Table 1 produce essentially identical results for both the link failure time CDFs and the PLOAS CDFs the four definitions of LOAS. This is a very strong verification result that both sets of calculations are correct.

The primary emphasis of this presentation is on the description and verification of results that have been implemented in the CPLOAS program. However, agreement of quadrature results obtained for the integral in Eq. (7.4) with sampling-based results obtained with the modified CPLOAS program are also relevant to the work that underlies the derivation of this integral. The integral in Eq. (7.4) is based on very complicated mathematical derivations in Ref. [36] that are tedious and difficult to check. Thus, the agreement of the results in Fig. 7 also provides an important verification result for derivations in Ref. [36].

An additional verification test was applied to PLOAS results calculated with CPLOAS for each of the three link failure delay definitions. This test involves the unphysical and counter intuitive assigning of the same properties to all links. However, as shown in Ref. [24], summarized in Table 1, and illustrated in Eqs. (4.17) and (6.1), this assignment results in the WL/SL system having a limiting (i.e., asymptotic) PLOAS value that is a function of the numbers $nWL$ and $nSL$ of WLs and SLs comprising the system. Although not physically realistic, the indicated assignments provide a problem that both (i) uses the mathematical and algorithmic structures involved in the calculation of PLOAS and (ii) has a known solution. As shown in Figs. 3, 5 and 8 for the three link failure delay definitions, these assignments result in the correct limiting PLOAS values and thus provide another verification result indicating that PLOAS values are calculated correctly in CPLOAS.

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