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Margins Associated with Loss of Assured Safety for Systems with Multiple Time-Dependent Failure Modes

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Margins Associated with Loss of Assured Safety for Systems with Multiple Time-Dependent Failure Modes

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Abstract

Representations for margins associated with loss of assured safety (LOAS) for weak link (WL)/strong link (SL) systems involving multiple time-dependent failure modes are developed. The following topics are described: (i) defining properties for WLs and SLs, (ii) background on cumulative distribution functions (CDFs) for link failure time, link property value at link failure, and time at which LOAS occurs, (iii) CDFs for failure time margins defined by (time at which SL system fails) − (time at which WL system fails), (iv) CDFs for SL system property values at LOAS, (v) CDFs for WL/SL property value margins defined by (property value at which SL system fails) − (property value at which WL system fails), and (vi) CDFs for SL property value margins defined by (property value of failing SL at time of SL system failure) − (property value of this SL at time of WL system failure). Included in this presentation is a demonstration of a verification strategy based on defining and approximating the indicated margin results with (i) procedures based on formal integral representations and associated quadrature approximations and (ii) procedures based on algorithms for sampling-based approximations.

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NOMENCLATURE

Abbreviation	Definition
CDF	cumulative distribution function
DOE	Department of Energy
LOAS	loss of assured safety
NNSA	National Nuclear Security Administration
PLOAS	probability of loss of assured safety
QMU	quantification of margins and uncertainty
SL	strong link
SNL	Sandia National Laboratories
WL	weak link

1. Introduction

Representations for margins associated with loss of assured safety (LOAS) for weak link (WL)/strong link (SL) systems [1-6] involving multiple time-dependent failure modes are developed. As described in Ref. [7], LOAS occurs under accident conditions (e.g., a fire) when SL failures place the overall system in a potentially operational mode before deactivation of the overall system as a result of WL failures. Specifically, margins defined by the following differences are considered:

(i) (time at which SL system fails) − (time at which WL system fails),

(ii) (property value at which SL system fails) − (property value at which WL system fails),

and

(iii) (property value of failing SL at time of SL system failure) − (property value of this SL at time of WL system failure).

The development of margins associated with LOAS in this presentation builds on previous work in Refs. [7; 8]. Specifically, Ref. [7] develops representations for the probability of loss of assured safety (PLOAS) for WL/ SL systems involving multiple time-dependent failure modes, and Ref. [8] develops representations for property values associated with the failure of individual links in a system with multiple WLs and SLs.

The following topics are described in this presentation: (i) defining properties for WLs and SLs (Sect. 2), (ii) background on cumulative distribution functions (CDFs) for link failure time, link property value at link failure, and time at which LOAS occurs (Sect. 3), (iii) CDFs for failure time margins (Sect. 4), (iv) CDFs for system property values at LOAS (Sect. 5), (v) CDFs for margins based on WL and SL property values (Sect. 6), and (vi) CDFs for margins involving only SL property values (Sect. 7). The presentation then ends with a summary discussion (Sect. 8).

Verification is an important component of any analysis used to support important decisions [9-13], where verification corresponds to "the process of determining that a model implementation accurately represents the developers conceptual description of the model and the solution to the model " ([9], p. 3). An important part of this presentation is the demonstration of a verification strategy based on defining and approximating the margin results of interest with (i) procedures based on formal integral representations and associated quadrature approximations and (ii) procedures based on algorithms for sampling-based approximations. The two procedures have very different mathematical structures and are implemented independently of each other. As a result, agreement between margin results obtained with the two procedures provides a strong verification result that the procedures have both (i) correct mathematical derivations and (ii) correct numerical implementations. Fortunately, the indicated agreement was observed and is extensively illustrated.

The presented work has been performed in support of the National Nuclear Security Administration's (NNSA's) mandate for the quantification of margins and uncertainties (QMU) in analyses of the United States' nuclear stockpile (see Refs. [14-17] for summary discussions of NNSA's mandate for QMU, Refs. [18-28] for additional background on the development of NNSA's mandate for QMU, and Refs. [29-40] for recent work on the implementation of NNSA's mandate for QMU).

2. WL/SL Properties

The failure time CDF for a single WL or SL is based on the following assumed properties of that link for the time interval $t_{mn} \le t \le t_{mx}$, where t_{mn} and t_{mx} define the endpoints of the time interval considered for analysis:

$$\bar{p}(t) = \text{increasing positive function defining nominal link property for } t_{mn} \le t \le t_{mx}, \tag{2.1}$$

$$\bar{q}(t) = \text{decreasing or constant-valued positive function defining nominal failure value for link property for } t_{mn} \le t \le t_{mx}, \tag{2.2}$$

$$d_A(\alpha) = \text{density function for a positive variable } \alpha \text{ used to characterize aleatory uncertainty in link property,} \tag{2.3}$$

$$d_B(\beta) = \text{density function for a positive variable } \beta \text{ used to characterize aleatory uncertainty in link failure value,} \tag{2.4}$$

$$p(t \mid \alpha) = \alpha \bar{p}(t) = \text{ link property value for } t_{mn} \le t \le t_{mx} \text{ given } \alpha, \tag{2.5}$$

and

$$q(t \mid \beta) = \beta \bar{q}(t) = \text{ link failure value for } t_{mn} \le t \le t_{mx} \text{ given } \beta. \tag{2.6}$$

Further, $d_A(\alpha)$ and $d_B(\beta)$ are assumed (i) to be defined on intervals $[\alpha_{mn}, \alpha_{mx}]$ and $[\beta_{mn}, \beta_{mx}]$ and (ii) to equal zero outside these intervals. Although this does not have to be the case, it is anticipated that α and β will be assigned distributions with a mode of 1.0 in most analyses so that $\bar{p}(t)$ and $\bar{q}(t)$ will be the modes (i.e., most likely values) for $p(t|\alpha)$ and $q(t|\beta)$. For given values of α and β, link failure occurs at the time t at which $\alpha \bar{p}(t) = \beta \bar{q}(t)$.

As indicated in Eqs. (2.3) and (2.4), the variables α and β are used for the incorporation of the effects of aleatory uncertainty into the analysis of a WL/SL system. Specifically, aleatory uncertainty is used as a descriptor for random variability in the properties of a system and is distinct from epistemic uncertainty, which results from a lack of knowledge about the value of a quantity that has a fixed (i.e., unique) but poorly known value. Additional discussion of the role of aleatory uncertainty and epistemic uncertainty in the analysis of complex systems is available in Refs. [16; 31; 41-51].

This presentation uses a notional WL/SL system with nWL = 2 WLs and nSL = 2 SLs to illustrate margin calculations. Property definitions for the four links are summarized in Table 1 and illustrated in Fig. 1.

Table 1 Defining properties for two WLs and two SLs used in the illustration of the definition and calculation of margins for WL/SL systems.

General Properties for Links in Fig. 1

$$\bar{p}(\tau) = \frac{\bar{p}(\infty)\bar{p}(0)}{\bar{p}(0) + [\bar{p}(\infty) - \bar{p}(0)]\exp(-r_1\tau)} \quad \text{for all links}$$

$\bar{q}(\tau)$ constant-valued for WL 1 and SL 1

$$\bar{q}(\tau) = \frac{\bar{q}(0)}{1 + k\tau^{r_2}} \quad \text{for WL 2 and SL 2}$$

Additional Properties of WL 1 in Fig. 1

$\bar{p}(\infty) = 950, \bar{p}(0) = 300, r_1 = 0.02, \bar{q}(\tau) = 650$

$d_A(\alpha)$ triangular on [0.88, 1.15] with mode 1.0

$d_B(\beta)$ triangular on [0.8, 1.15] with mode 1.0

Additional Properties of WL 2 in Fig. 1

$\bar{p}(\infty) = 850, \bar{p}(0) = 300, r_1 = 0.02$

$\bar{q}(0) = 650, k = 2.21 \times 10^{-4}, r_2 = 1.5$

$d_A(\alpha)$ triangular on [0.85, 1.2] with mode 1.0

$d_B(\beta)$ triangular on [0.75, 1.2] with mode 1.0

Additional Properties of SL 1 in Fig. 1

$\bar{p}(\infty) = 1025, \bar{p}(0) = 300, r_1 = 0.025, \bar{q}(\tau) = 775$

$d_A(\alpha)$ triangular on [0.9, 1.15] with mode 1.0

$d_B(\beta)$ uniform on [0.8, 1.15]

Additional Properties of SL 2 in Fig. 1

$\bar{p}(\infty) = 950, \bar{p}(0) = 300, r_1 = 0.025$

$\bar{q}(0) = 750, k = 1.41 \times 10^{-4}, r_2 = 1.5$

$d_A(\alpha)$ triangular on [0.8, 1.1] with mode 1.0

$d_B(\beta)$ uniform on [0.85, 1.3]

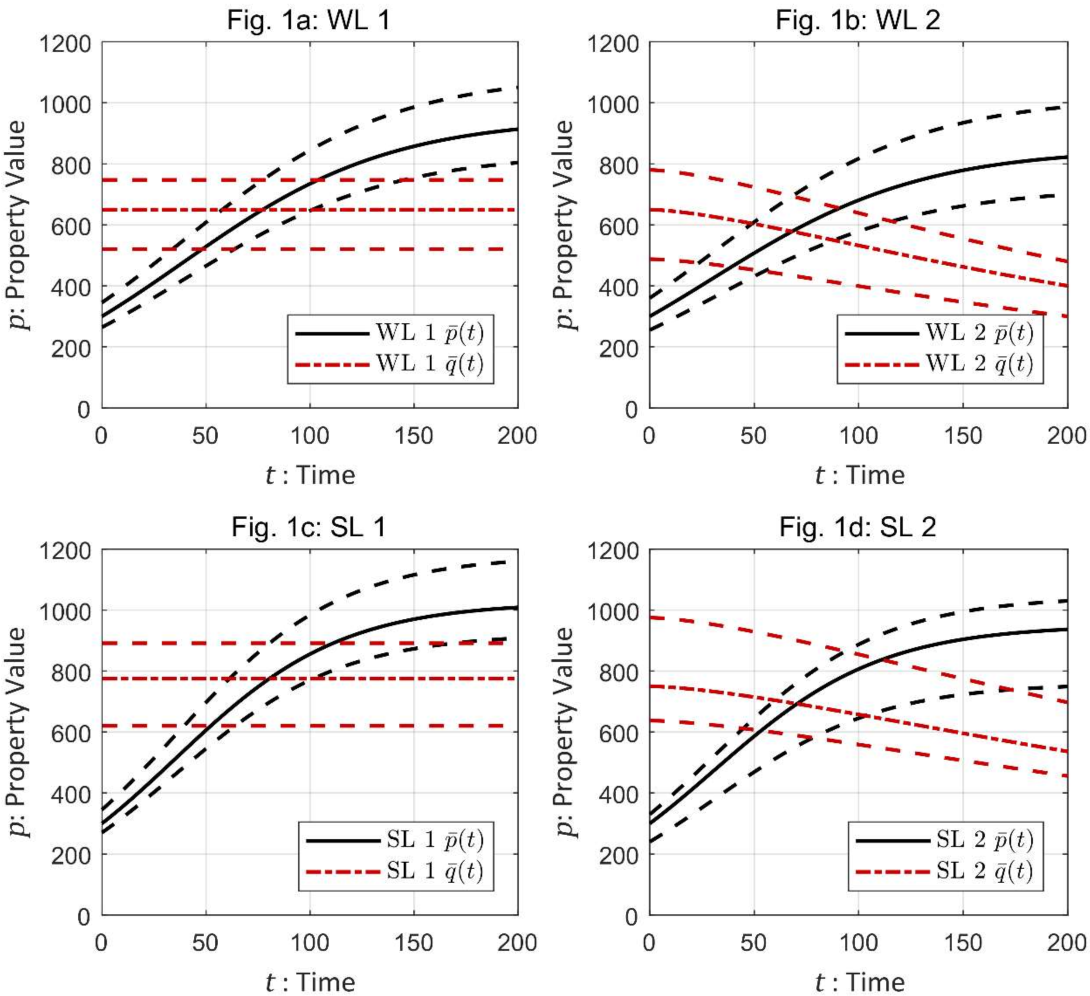

Fig. 1 Summary plots of the properties of two WLs and two SLs used in the illustration of the definition and calculation of margins for WL/SL systems with the dashed lines corresponding to the boundaries defined by $[\alpha_{mn}\bar{p}(t), \alpha_{mx}\bar{p}(t)]$ and $[\beta_{mn}\bar{q}(t), \beta_{mx}\bar{q}(t)]$: (a) WL 1, (b) WL 2, (c) SL 1, and (d) SL2.

3. Background: CDFs for Link Failure Time, Link Property Value at Link Failure, and Time at which LOAS Occurs

The CDF for link failure time is defined by

$$\begin{aligned} CDF_T(t) &= \int_{\alpha_{mn}}^{\alpha_{mx}} \left[\int_{\beta_{mn}}^{F(\alpha,t)} d_B(\beta)\,\mathrm{d}\beta \right] d_A(\alpha)\,\mathrm{d}\alpha \\ &= \int_{\alpha_{mn}}^{\alpha_{mx}} CDF_B[F(\alpha,t)]\,\mathrm{d}CDF_A(\alpha) \end{aligned} \tag{3.1}$$

and extensively discussed in Ref. [7], where (i) the first integral is a Riemann integral with

$$F(\alpha,t) = \alpha\bar{p}(t)/\bar{q}(t) = \alpha / r(t) \text{ and } r(t) = \bar{q}(t)/\bar{p}(t) \tag{3.2}$$

and (ii) the second integral is a Riemann-Stieltjes integral with

$$CDF_A(\alpha) = \int_{\alpha_{mn}}^{\alpha} d_A(\tilde{\alpha})\mathrm{d}\tilde{\alpha} \text{ and } CDF_B(\beta) = \int_{\beta_{mn}}^{\beta} d_B(\tilde{\beta})\mathrm{d}\tilde{\beta}. \tag{3.3}$$

The CDFs for link failure time for the four links defined in Table 1 and illustrated in Fig. 1 are shown in Fig. 2a. Quadrature and sampling-based procedures for the estimation of CDFs for link failure time are discussed and illustrated in Ref. [7].

The CDF for link property at time of link failure is formally defined in Eq. (8.1) of Ref. [8] by

$$\begin{aligned} CDF_P(p\,|\,[t_{mn},t_{mx}]) &= \text{ probability that link fails at a value less than or equal to } p \\ &\quad\; \text{in the time interval } [t_{mn},t_{mx}] \\ &= \int_{t_{mn}}^{t_{mx}} CDF_P(p\,|\,\tau)\mathrm{d}CDF_T(\tau) \\ &= \int_{t_{mn}}^{t_{mx}} CDF_P(p\,|\,\tau)d_T(\tau)\mathrm{d}\tau, \end{aligned} \tag{3.4}$$

where (i) $CDF_P(p\,|\,\tau)$ is the CDF for link property value p at link failure conditional on link failure occurring at time τ, and (ii) $d_T(\tau)$ is the density function for link failure time defined by

$$\begin{aligned} d_T(\tau) &= \mathrm{d}CDF_T(\tau)/\mathrm{d}\tau \\ &= \left[\mathrm{d}\left[\bar{q}(\tau)/\bar{p}(\tau)\right]/\mathrm{d}\tau\right]\int_{\alpha_{mn}}^{\alpha_{mx}} d_B\left[\alpha/r(\tau)\right]\alpha d_A(\alpha)\mathrm{d}\alpha \\ &= \left[\mathrm{d}\left[1/r(\tau)\right]/\mathrm{d}\tau\right]\int_{\alpha_{mn}}^{\alpha_{mx}} d_B\left[\alpha/r(\tau)\right]\alpha d_A(\alpha)\mathrm{d}\alpha \end{aligned} \tag{3.5}$$

as shown in Eq. (8.2) of Ref. [8]. Sampling-based procedures for the estimation of $CDF_P(p\,|\,[t_{mn},t_{mx}])$ are described in Sect. 6 of Ref. [8], and integral-based representations for $CDF_P(p\,|\,[t_{mn},t_{mx}])$ are derived in Sects. 4 and 5 of Ref. [8] for the following two cases: (i) $\bar{p}(t)$

increasing and $\overline{q}(t)$ decreasing, and (ii) $\overline{p}(t)$ increasing and $\overline{q}(t)$ constant-valued. The CDFs for link property at time of link failure for the four links defined in Table 1 and illustrated in Fig. 1 are shown in Fig. 2b.

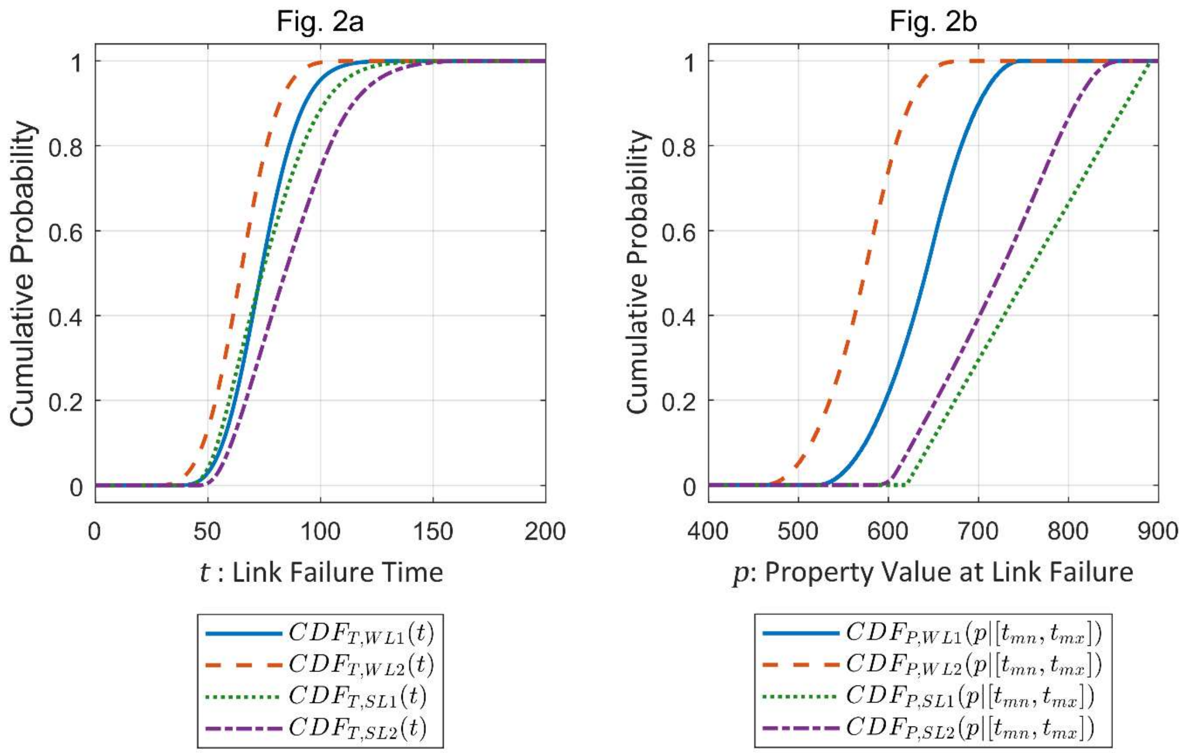

Fig. 2 Illustration of failure properties for links defined in Table 1: (a) CDFs for link failure time (i.e., $CDF_{T,WL1}(t)$, $CDF_{T,WL2}(t)$, $CDF_{T,SL1}(t)$ and $CDF_{T,SL2}(t)$), and (b) CDFs for link property at time of link failure (i.e., $CDF_{P,WL1}(p\,|\,[t_{mn},t_{mx}])$, $CDF_{P,WL2}(p\,|\,[t_{mn},t_{mx}])$, $CDF_{P,SL1}(p\,|\,[t_{mn},t_{mx}])$ and $CDF_{P,SL2}(p\,|\,[t_{mn},t_{mx}])$ with $[t_{mn},t_{mx}]=[0,200]$.

The CDFs for the time at which LOAS occurs are defined by the probabilities summarized in Table 2 for the four indicated failure patterns. Specifically, Table 2 shows representations for PLOAS as functions $pF_i(t)$ of time (i.e., $pF_i(t)$ is the probability that LOAS occurs by time t for failure pattern i). The listed verification tests are the outcomes of assigning the same properties to all links as developed and described in Ref. [52]. The representations in Table 2 are derived and extensively illustrated in Ref. [7]. The CDFs for the time at which LOAS occurs for the four links defined in Table 1 and illustrated in Fig. 1 are shown in Fig. 3.

Table 2 Representation of Time-Dependent Values $pF_i(t)$, i = 1, 2, 3, 4, for PLOAS and Associated Verification Tests for Alternate Definitions of LOAS for WL/SL Systems with (i) *nWL* WLs and *nSL* SLs and (ii) independent distributions for link failure time ([53], Table 10).

Failure Pattern 1: Failure of all SLs before failure of any WL (Eqs. (2.1) and (2.5), Ref. [52])

$$pF_1(t) = \sum_{k=1}^{nSL}\left(\int_{t_{mn}}^{t}\left\{\prod_{l=1,l\neq k}^{nSL} CDF_{T,SL,l}(\tau)\right\}\left\{\prod_{j=1}^{nWL}\left[1 - CDF_{T,WL,j}(\tau)\right]\right\}\mathrm{d}CDF_{T,SL,k}(\tau)\right)$$

Verification test: $pF_1(\infty) = nWL!\,nSL!/(nWL + nSL)!$

Failure Pattern 2: Failure of any SL before failure of any WL (Eqs. (3.1) and (3.4), Ref. [52])

$$pF_2(t) = \sum_{k=1}^{nSL}\left(\int_{t_{mn}}^{t}\left\{\prod_{l=1,l\neq k}^{nSL}\left[1 - CDF_{T,SL,l}(\tau)\right]\right\}\left\{\prod_{j=1}^{nWL}\left[1 - CDF_{T,WL,j}(\tau)\right]\right\}\mathrm{d}CDF_{T,SL,k}(\tau)\right)$$

Verification test: $pF_2(\infty) = nSL/(nWL + nSL)$

Failure Pattern 3: Failure of all SLs before failure of all WLs (Eqs. (4.1) and (4.4), Ref. [52])

$$pF_3(t) = \sum_{k=1}^{nSL}\left(\int_{t_{mn}}^{t}\left\{\prod_{l=1,l\neq k}^{nSL} CDF_{T,SL,l}(\tau)\right\}\left\{1 - \prod_{j=1}^{nWL} CDF_{T,WL,j}(\tau)\right\}\mathrm{d}CDF_{T,SL,k}(\tau)\right)$$

Verification test: $pF_3(\infty) = nWL/(nWL + nSL)$

Failure Pattern 4: Failure of any SL before failure of all WLs (Eqs. (5.1) and (5.4), Ref. [52])

$$pF_4(t) = \sum_{k=1}^{nSL}\left(\int_{t_{mn}}^{t}\left\{\prod_{l=1,l\neq k}^{nSL}\left[1 - CDF_{T,SL,l}(\tau)\right]\right\}\left\{1 - \prod_{j=1}^{nWL} CDF_{T,WL,j}(\tau)\right\}\mathrm{d}CDF_{T,SL,k}(\tau)\right)$$

Verification test: $pF_4(\infty) = 1 - \left[nWL!\,nSL!/(nWL + nSL)!\right]$

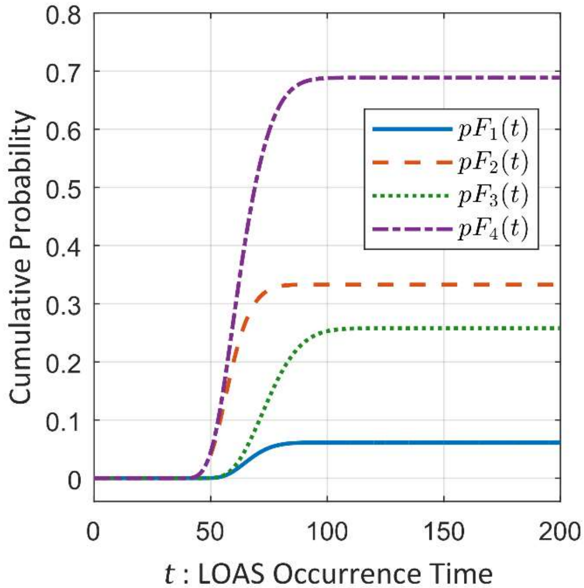

Fig. 3 Illustration of the CDFs $pF_i(t), i = 1,2,3,4,$ defined in Table 2 for the time at which LOAS occurs for the four links defined in Table 1.

4. CDFs for Failure Time Margins

4.1. Preliminaries: CDFs for Failure Time Margins

Suppose *nL* links are under consideration with $CDF_{Tk}(\tau)$, *k* = 1, 2, …, *nL*, the failure time CDF for link *k* (i.e., $CDF_{Tk}(\tau)$ is the probability that link *k* fails at or before time τ). Then, under the assumption that the link failure times are independent,

$$\begin{aligned} pLLF(t) &= \sum_{k=1}^{nL}\left(\int_{t_{mn}}^{t}\left\{\prod_{l=1,l\neq k}^{nL} CDF_{Tl}(\tau)\right\}\mathrm{d}CDF_{Tk}(\tau)\right) \\ &= \prod_{k=1}^{nL} CDF_{Tl}(t) \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} pFLF(t) &= \sum_{k=1}^{nL}\left(\int_{t_{mn}}^{t}\left\{\prod_{l=1,l\neq k}^{nL} \left[1-CDF_{Tl}(\tau)\right]\right\}\mathrm{d}CDF_{Tk}(\tau)\right) \\ &= 1-\prod_{l=1}^{nL}\left[1-CDF_{Tl}(t)\right], \end{aligned} \tag{4.2}$$

where *pLLF*(*t*) is the probability that the last link failure occurs at or before time *t* and *pFLF*(*t*) is the probability that the first link failure occurs at or before time *t*. The initial definitions for $pLLF(t)$ and $pFLF(t)$ are the same as the definitions for $pF_1(t)$ and $pF_2(t)$, respectively, in Table 2 with the WL CDFs assumed to always have a value of 0 (i.e., $CDF_{T,WL,j}(\tau)=0$ for all values of τ). The second definitions follow directly from the definitions of the CDFs $CDF_{Tk}(\tau)$. The equivalence of the pairs of definitions for $pLLF(t)$ and $pFLF(t)$ can be directly established but the notation becomes very complex for $nL > 2$.

Distributions for time margins corresponding to the difference between time when the failure of WLs deactivate a system and the failure of SLs result in LOAS can be calculated for the four WL/SL failure patterns defined in Table 2 with the use of results of the form shown in Eqs. (4.1) and (4.2).

4.2. Formal Representation: CDFs for Failure Time Margins

For a particular WL/SL configuration (i.e., one of the failure patterns defined in Table 2) let (i) $CDF_{T,WL}(t_{WL})$ be the CDF defined on the interval $[t_{mn}, t_{mx}]$ for the time t_{WL} when the failure of the system WLs potentially deactivates the system and (ii) $CDF_{T,SL}(t_{SL})$ be the CDF defined on the interval $[t_{mn}, t_{mx}]$ for the time t_{SL} when the failure of the system SLs potentially results in LOAS. The modifier “potentially” appears in the preceding sentence because the indicated failures may or may not have the indicated effect because of the timing of the WL and SL failures.

The desired distribution for the margin m defined by $t_{SL} - t_{WL}$ is given by

$$\begin{aligned} CDF_{TM}(m \mid [t_{mn}, t_{mx}]) &= prob\left(t_{SL} - t_{WL} \le m\right) \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} CDF_{T,SL}\left(m + t_i\right) \Delta CDF_{T,WL}\left(t_i\right) \\ &= \int_{t_{mn}}^{t_{mx}} CDF_{T,SL}\left(m + t\right) \mathrm{d}CDF_{T,WL}\left(t\right) \\ &= \int_{t_{mn}}^{t_{mx}} CDF_{T,SL}\left(m + t\right) \left[\mathrm{d}CDF_{T,WL}\left(t\right)/\mathrm{d}t\right] \mathrm{d}t, \end{aligned} \tag{4.3}$$

where $t_i, i = 0, 1, \cdots, n,$ is a subdivision of $[t_{mn}, t_{mx}]$. In effect, the result in Eq. (4.3) corresponds to the convolution of two probability distributions (e.g., see ([54], p.53).

The manners in which $CDF_{T,WL}(t_{WL})$ and $CDF_{T,SL}(t)$ are defined for the failure patterns in Table 2 are as follows for a WL/SL system with nWL WLs and nSL SLs:

$$\begin{aligned} \text{Failure Pattern 1: } CDF_{T,WL}(t_{WL}) &= pFLF\left(t_{WL}\right) \text{ in Eq. (4.2)} \\ CDF_{T,SL}(t_{SL}) &= pLLF\left(t_{SL}\right) \text{ in Eq. (4.1),} \end{aligned} \tag{4.4}$$

$$\begin{aligned} \text{Failure Pattern 2: } CDF_{T,WL}(t_{WL}) &= pFLF\left(t_{WL}\right) \text{ in Eq. (4.2)} \\ CDF_{T,SL}(t_{SL}) &= pFLF\left(t_{SL}\right) \text{ in Eq. (4.2),} \end{aligned} \tag{4.5}$$

$$\begin{aligned} \text{Failure Pattern 3: } CDF_{T,WL}(t_{WL}) &= pLLF\left(t_{WL}\right) \text{ in Eq. (4.1)} \\ CDF_{T,SL}(t_{SL}) &= pLLF\left(t_{SL}\right) \text{ in Eq. (4.1),} \end{aligned} \tag{4.6}$$

$$\begin{aligned} \text{Failure Pattern 4: } CDF_{T,WL}(t_{WL}) &= pLLF\left(t_{WL}\right) \text{ in Eq. (4.1)} \\ CDF_{T,SL}(t_{SL}) &= pFLF\left(t_{SL}\right) \text{ in Eq. (4.2).} \end{aligned} \tag{4.7}$$

Examples of the link system failure probabilities $CDF_{T,WL}(t_{WL}) = pFLF(t_{WL})$, $CDF_{T,WL}(t_{WL}) = pLLF(t_{WL})$, $CDF_{T,SL}(t_{SL}) = pFLF(t_{SL})$ and $CDF_{T,SL}(t_{SL}) = pLLF(t_{SL})$ for the two SLs and two WLs defined and illustrated in Table 1 and Fig. 1 are presented in Fig. 4a. The values for $pLLF(t)$ and $pFLF(t)$ were obtained as indicated in Eqs. (4.1) and (4.2) by the multiplication of link failure time CDFs (see Eq. (3.1)) calculated by the CPLOAS program [55; 56].

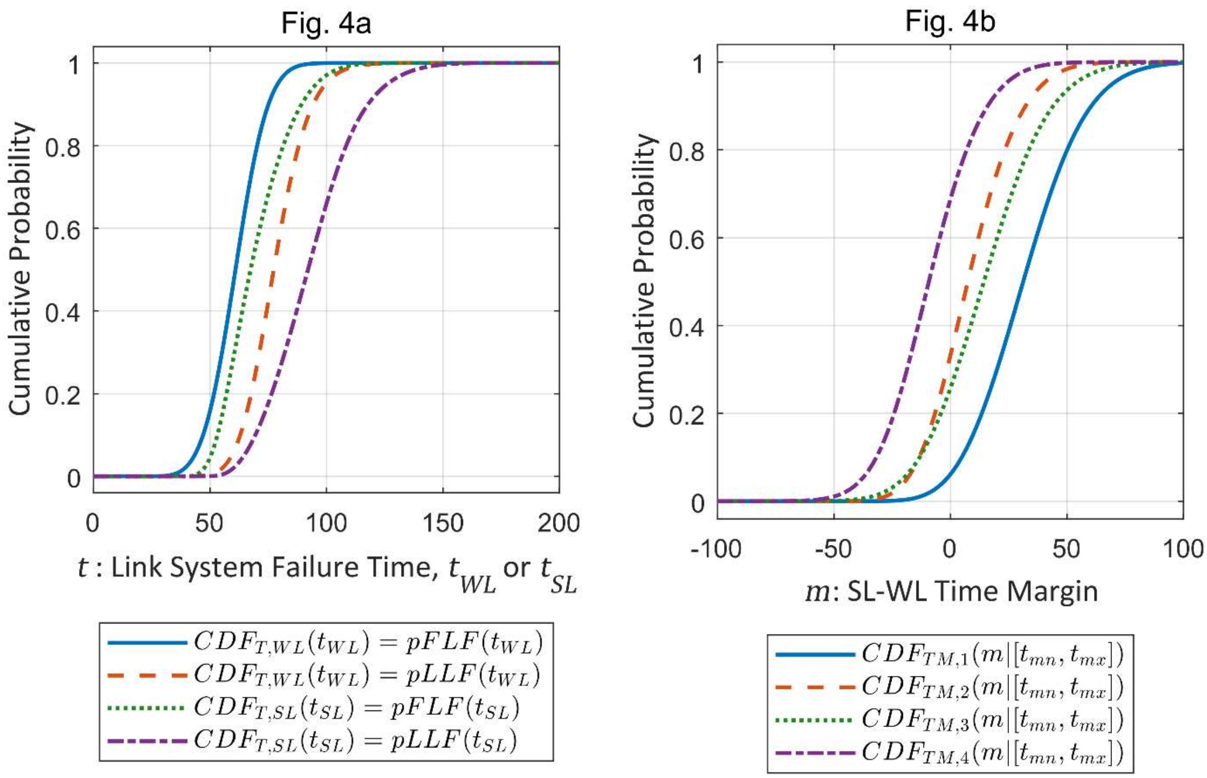

Fig. 4 Failure time margins for (i) the four failure patterns resulting in LOAS indicated in Eqs. (4.4)-(4.7) and (ii) the four links defined and illustrated in Table 1 and Fig. 1 with $[t_{mn}, t_{mx}] = [0, 200]$: (a) link system failure probabilities $CDF_{T,WL}(t_{WL}) = pFLF(t_{WL})$, $CDF_{T,WL}(t_{WL}) = pLLF(t_{WL})$, $CDF_{T,SL}(t_{SL}) = pFLF(t_{SL})$ and $CDF_{T,SL}(t_{SL}) = pLLF(t_{SL})$ with $pLLF(t)$ and $pFLF(t)$ defined in Eqs. (4.1)-(4.2), and (b) failure time margins $CDF_{TM,i}(m \mid [t_{mn}, t_{mx}])$ for failure pattern i, i = 1, 2, 3, 4, defined in Eq. (4.3).

An illustration of the calculation of failure time margins $CDF_{TM,i}(m \mid [t_{mn}, t_{mx}])$, i = 1,2,3,4, for failure pattern i in Table 2 with the use of quadrature-based procedures to evaluate the final integral in Eq. (4.3) is presented in Fig. 4b. Specifically, the margin results in Fig. 4b were obtained for the two SLs and two WLs defined and illustrated in Table 1 and Fig. 1 with use of procedures contained in the MATLAB numerical package [57] as summarized below. For a given failure pattern, the corresponding CDFs $CDF_{T,WL}(t_{WL})$ and $CDF_{T,SL}(t_{SL})$ indicated in Eqs. (4.4)-(4.7) and illustrated in Fig. 4a were approximated with the **spline** and **ppval** functions to obtain the spline approximations $spCDF_{T,SL}(t_{SL})$ and $spCDF_{T,WL}(t_{WL})$. Next, $\mathrm{d}CDF_{T,WL}(t_{WL})/\mathrm{d}t_{WL}$ was obtained by differentiating $spCDF_{T,WL}(t_{WL})$ with the **fnder** function to obtain the derivative $derCDF_{T,WL}(t_{WL})$. Then, the product $spCDF_{T,SL}(m + t_{WL}) derCDF_{T,WL}(t_{WL})$ was integrated over $[t_{mn}, t_{mx}] = [0, 200]$ with the **integral** function to numerically evaluate the final integral in Eq. (4.3)

. In turn, the indicated calculations were performed for multiple values of m to obtain the CDFs $CDF_{TM,i}(m\,|\,[t_{mn},t_{mx}])$, i = 1,2,3,4, for failure pattern i in Fig. 4b.

A brief elaboration on the nature of the results in Fig. 4 follows. As discussed in Ref. [58] and illustrated in Fig. 4, the performance of a WL/SL system under accident conditions can be viewed as a competing failure problem. In essence, there is a race to failure between the WL system and the SL system, with (i) the WL system winning the race if it fails before or at the same time as the SL system fails and (ii) the WL system losing the race if it fails after the SL system fails. The margin m of winning the race by the WL system is defined by

$$\begin{aligned} m &= (\text{time } t_{SL} \text{ of SL system failure}) - (\text{time } t_{WL} \text{ of SL system failure}) \\ &= t_{SL} - t_{WL}, \end{aligned} \tag{4.8}$$

with a nonnegative margin corresponding to the WL system winning the race and a negative margin corresponding to the WL system losing the race. In the problem under consideration, the link system failure times t_{WL} and t_{SL} do not have fixed values but rather enter the analysis as distributions of values as a result of the aleatory uncertainty associated with the property values and failure values for the individual links.

The results in Fig. 4 are for 4 WL/SL systems, each with: (i) 2 WLs and 2 SLs, (ii) one of 2 possible definitions for WL system failure (i.e., first link to fail or last link to fail), and (iii) one of 2 possible definitions for SL system failure (i.e., first link to fail or last link to fail). As illustrated in Fig. 4a, this results in (i) distributions (i.e., CDFs)

$$CDF_{T,WL}(t_{WL}) = pLLF(t_{WL}) \text{ and } CDF_{T,WL}(t_{WL}) = pFLF(t_{WL}) \tag{4.9}$$

for WL system failure time t_{WL} and (ii) distributions

$$CDF_{T,SL}(t_{SL}) = pLLF(t_{SL}) \text{ and } CDF_{T,SL}(t_{SL}) = pFLF(t_{SL}) \tag{4.10}$$

for SL system failure time. The indicated CDFs are obtained as indicated in Sect. 4.1.

In turn as illustrated in Fig. 4b, the 4 WL/SL systems and their associated CDFs for link system failure time result in the following distributions (i.e., CDFs) for the failure time margin m in Eq. (4.8):

$$CDF_{T,WL}(t_{WL}) = pFLF(t_{WL}),\ CDF_{T,SL}(t_{SL}) = pLLF(t_{SL}) \rightarrow CDF_{TM,1}(m\,|\,[t_{mn},t_{mx}]), \tag{4.11}$$

$$CDF_{T,WL}(t_{WL}) = pFLF(t_{WL}),\ CDF_{T,SL}(t_{SL}) = pFLF(t_{SL}) \rightarrow CDF_{TM,2}(m\,|\,[t_{mn},t_{mx}]), \tag{4.12}$$

$$CDF_{T,WL}(t_{WL}) = pLLF(t_{WL}),\ CDF_{T,SL}(t_{SL}) = pLLF(t_{SL}) \rightarrow CDF_{TM,3}(m\,|\,[t_{mn},t_{mx}]), \tag{4.13}$$

$$CDF_{T,WL}(t_{WL}) = pLLF(t_{WL}),\ CDF_{T,SL}(t_{SL}) = pFLF(t_{SL}) \rightarrow CDF_{TM,4}(m\,|\,[t_{mn},t_{mx}]). \tag{4.14}$$

The margin results $CDF_{TM,i}(m\,|\,[t_{mn},t_{mx}]),\ i=1,2,3,4,$ in Eqs. (4.11)-(4.14) summarize the previously indicated "race" to failure for the 4 WL/SL systems under considerations and were obtained by numerical evaluation of the integral in Eq. (4.3). As shown in Sects. 4.3 and 4.4. the CDFs $CDF_{TM,i}(m\,|\,[t_{mn},t_{mx}]),\ i=1,2,3,4,$ can also be obtained with sampling-based procedures.

4.3. Sampling-based Estimation of $CDF_{TM}(m\,|\,[t_{mn},t_{mx}])$ for Failure Time Margins

A sampling-based determination of $CDF_{TM}(m\,|\,[t_{mn},t_{mx}]) = prob(t_{SL}-t_{WL}\le m)$ is now considered. For this determination, the representation for $prob(t_{SL}-t_{WL}\le m)$ can be formulated as

$$\begin{aligned} prob_i\left(t_{SL}-t_{WL}\le m\right) &= \int_{t_{mn}}^{t_{mx}}\int_{t_{mn}}^{t_{mx}} \delta_m\left(t_{SL}-t_{WL}\right)\mathrm{d}CDF_{T,SL}\left(t_{SL}\right)\mathrm{d}CDF_{T,WL}\left(t_{WL}\right) \\ &= \int_{t_{mn}}^{t_{mx}}\int_{t_{mn}}^{t_{mx}} \delta_m\left(t_{SL}-t_{WL}\right)d_{T,SL}\left(t_{SL}\right)d_{T,WL}\left(t_{WL}\right)\mathrm{d}t_{SL}\mathrm{d}t_{WL}, \end{aligned} \tag{4.15}$$

where (i) the subscripts i = 1, 2, 3, 4 indicate the definitions of $CDF_{T,SL}(t_{SL})$ and $CDF_{T,WL}(t_{WL})$ for Patterns 1-4 in Eqs. (4.4)-(4.7) and thus correspondingly for the four WL/SL failure patterns in Table 2, (ii) $d_{T,SL}(t_{SL})$ and $d_{T,WL}(t_{WL})$ are the density functions associated with $CDF_{T,SL}(t_{SL})$ and $CDF_{T,WL}(t_{WL})$, and (iii)

$$\delta_m\left(t_{SL}-t_{WL}\right)=\begin{cases}1 & \text{if } t_{SL}-t_{WL}<m\\ 0 & \text{otherwise.}\end{cases} \tag{4.16}$$

In turn, the approximation

$$prob_i\left(t_{SL}-t_{WL}\le m\right)\cong\sum_{r=1}^{n}\delta_m\left(t_{SL,r}-t_{WL,r}\right)\Big/n \tag{4.17}$$

results for a random sample $[t_{SL,r},t_{WL,r}]$, r = 1,2, …, n, from $[t_{mn},t_{mx}]\times[t_{mn},t_{mx}]$ obtained with use of $CDF_{T,SL}(t_{SL})$ and $CDF_{T,WL}(t_{WL})$.

Another possibility is to generate a sample of individual link failure times

$$\mathbf{s}_r=\left[tWL_{1r},tWL_{2r},\cdots,tWL_{nWL,r},tSL_{1r},tSL_{2r},\cdots,tSL_{nSL,r}\right],\ r=1,2,\cdots,n, \tag{4.18}$$

directly from the corresponding link failure time CDFs and then define margins by

$$m_i(\mathbf{s}_r) = \begin{cases} \max\{tSL_{1r}, tSL_{2r}, \cdots, tSL_{nSL,r}\} - \min\{tWL_{1r}, tWL_{2r}, \cdots, tWL_{nWL,r}\} & \text{for } i = 1 \\ \min\{tSL_{1r}, tSL_{2r}, \cdots, tSL_{nSL,r}\} - \min\{tWL_{1r}, tWL_{2r}, \cdots, tWL_{nWL,r}\} & \text{for } i = 2 \\ \max\{tSL_{1r}, tSL_{2r}, \cdots, tSL_{nSL,r}\} - \max\{tWL_{1r}, tWL_{2r}, \cdots, tWL_{nWL,r}\} & \text{for } i = 3 \\ \min\{tSL_{1r}, tSL_{2r}, \cdots, tSL_{nSL,r}\} - \max\{tWL_{1r}, tWL_{2r}, \cdots, tWL_{nWL,r}\} & \text{for } i = 4 \end{cases}$$

$$= \begin{cases} \overline{tSL}_{k(r),r} - \underline{tWL}_{l(r),r} & \text{for } i = 1 \\ \underline{tSL}_{k(r),r} - \underline{tWL}_{l(r),r} & \text{for } i = 2 \\ \overline{tSL}_{k(r),r} - \overline{tWL}_{l(r),r} & \text{for } i = 3 \\ \underline{tSL}_{k(r),r} - \overline{tWL}_{l(r),r} & \text{for } i = 4 \end{cases} \tag{4.19}$$

with (i) i indicating Patterns 1-4 in Eqs. (4.4)-(4.7) and, correspondingly, the four WL/SL failure patterns in Table 2, (ii) underscores and overscores indicating minimums and maximums, respectively, and (iii) the subscripts $k(r)$ and $l(r)$ identifying the links that failed at the indicated times. Specifically, $k(r)$ identifies the SL whose failure potentially results in LOAS, and $l(r)$ identifies the WL whose failure potentially prevents LOAS. Then,

$$prob_i(t_{SL} - t_{WL} \le m) \cong \sum_{r=1}^{n} \delta_m[m_i(\mathbf{s}_r)]/n \tag{4.20}$$

with $\delta_m[m_i(\mathbf{s}_r)]$ defined as in Eq. (4.16).

Yet another sampling approach is to (i) sample the defining parameters for the individual links (i.e., the α's and β's) in consistency with their specified distributions as defined by the density functions $d_A(\alpha)$ and $d_B(\beta)$ for the individual links and (ii) then determine the resultant margins. Specifically, a sample

$$\mathbf{p}_r = [\mathbf{pWL}_{1r}, \mathbf{pWL}_{2r}, \cdots, \mathbf{pWL}_{nWL,r}, \mathbf{pSL}_{1r}, \mathbf{pSL}_{1r}, \cdots, \mathbf{pSL}_{nSL,r}], \; r = 1, 2, \cdots, n \tag{4.21}$$

with

$$\mathbf{pWL}_{lr} = [\alpha_{WL,lr}, \beta_{WL,lr}] \;\; \text{for} \;\; l = 1, 2, \cdots, nWL \tag{4.22}$$

and

$$\mathbf{pSL}_{lr} = [\alpha_{SL,lr}, \beta_{SL,lr}] \;\; \text{for} \;\; l = 1, 2, \cdots, nSL \tag{4.23}$$

is generated from the distributions of the α's and β's for the individual links. Next, the link failure times in Eq. (4.18) are determined as functions

$$tWL_{lr} = tWL_{lr}(\mathbf{pWL}_{lr}) = tWL_{lr}(\alpha_{WL,lr}, \beta_{WL,lr}) \;\; \text{for} \;\; l = 1, 2, \cdots, nWL \tag{4.24}$$

and

$$tSL_{lr} = tSL_{lr}\left(\mathbf{pSL}_{lr}\right) = tSL_{lr}\left(\alpha_{SL,lr}, \beta_{SL,lr}\right) \text{ for } l = 1, 2, \cdots, nSL \tag{4.25}$$

of elements of the sampled vectors $\mathbf{p}_r$ in Eq. (4.21). At this point, $prob_i(t_{SL} - t_{WL} \le m)$ can be determined as indicated in Eqs. (4.19)-(4.20) with the redefinition of the elements of $\mathbf{p}_r$ described in Eqs. (4.24)-(4.25).

A more mathematically explicit description of the use of sampling-based procedures in WL/SL analyses is given in Sect. 5 of Ref. [7]. This section also describes the use of importance sampling in WL/SL analyses.

4.4. Verification Results for $CDF_{TM}(m \mid [t_{mn}, t_{mx}])$ for Failure Time Margins

As a verification test, Fig. 5 provides a comparison of $CDF_{TM,i}(m \mid [t_{mn}, t_{mx}])$ for failure pattern *i*, *i* = 1, 2, 3, 4, calculated with (i) version 2.10 of the CPLOAS program [55; 56], the second sampling-based procedure indicated in Eqs. (4.18)-(4.20), and a sample of size $n = 10^6$ and (ii) the quadrature-based procedure described in Sect. 4.2. The sampling-based results in Fig. 5 are essentially the same as the quadrature-based results in Fig. 4b. Further, careful comparison of the quadrature-based CDFs in Fig. 4b and the sampling-based CDFs in Fig. 5 shows that corresponding CDFs almost exactly overlay. This level of agreement provides a strong verification that the procedures for obtaining $CDF_{TM}(m \mid [t_{mn}, t_{mx}])$ described in Sects. 4.2 and 4.3 are correct in both (i) mathematical development and (ii) computational implementation.

4.5. Connection Between Failure Time Margins $CDF_{TM,i}(0 \mid [t_{mn}, t_{mx}])$, $i = 1, 2, 3, 4$, and PLOAS

The probabilities $CDF_{TM,i}(0 \mid [t_{mn}, t_{mx}])$, *i* = 1, 2, 3, 4, associated with the failure time margin $m = 0$ defined in conjunction with Eq. (4.3) are equal to the probabilities $pF_i(t_{mx})$, *i* = 1, 2, 3, 4, for LOAS defined in Table 2 and illustrated in Fig. 3. Specifically,

$$pF_i(t_{mx}) = CDF_{TM,i}(0 \mid [t_{mn}, t_{mx}]) = \int_{t_{mn}}^{t_{mx}} CDF_{T,SL}(t_{WL}) \mathrm{d}CDF_{T,WL}(t_{WL}) \tag{4.26}$$

for *i* = 1, 2, 3, 4 with the appropriate values for $CDF_{T,SL}(t_{WL})$ and $CDF_{T,WL}(t_{WL})$ for failure pattern *i* defined in Eqs. (4.4)-(4.7). This connection is (i) illustrated by the equality of $pF_i(200)$ and $CDF_{TM,i}(0 \mid [0, 200])$ in Fig. 3 and Fig. 4b and (ii) provides an additional verification result that the failure time margins in Figs. 4 and 5 have been calculated correctly.

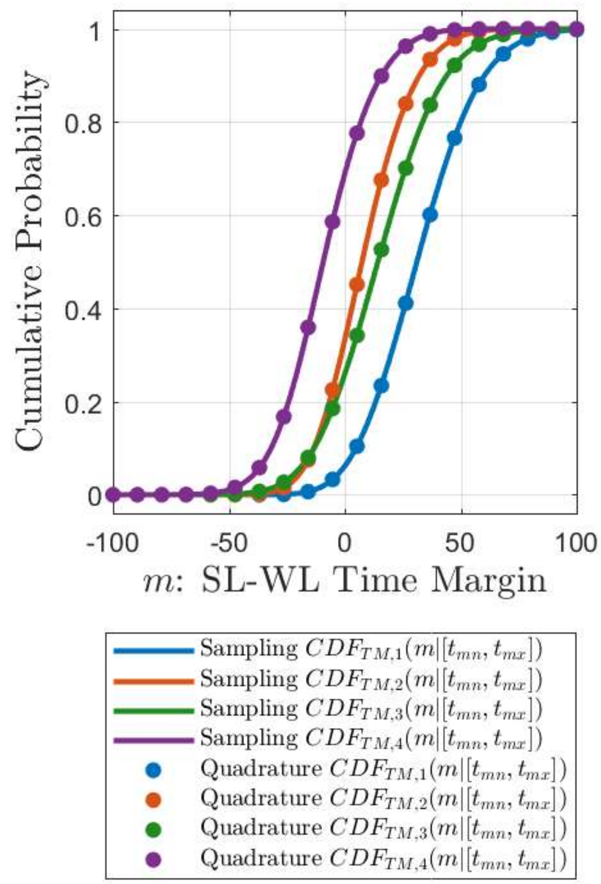

Fig. 5 Verification results for failure time margins $CDF_{TM,i}(m\,|\,[t_{mn}, t_{mx}]), i = 1,2,3,4,$ obtained with quadrature-based procedures (see Sect. 4.2) and sampling-based procedures (see Eqs. (4.21) -(4.25) in Sect. 4.3) for (i) the four failure patterns resulting in LOAS indicated in Eqs. (4.4)-(4.7) and (ii) the four links defined and illustrated in Table 1 and Fig. 1 with $[t_{mn}, t_{mx}] = [0, 200]$.

4.6. Connection Between Failure Time Margins $CDF_{TM,i}(0\,|\,[t_{mn}, t_{mx}])$, *i* = 1, 2, 3, 4, and PLOAS

The probabilities $CDF_{TM,i}(0\,|\,[t_{mn}, t_{mx}])$, i = 1, 2, 3, 4, associated with the failure time margin $m = 0$ defined in conjunction with Eq. (4.3) are equal to the probabilities $pF_i(t_{mx})$, i = 1, 2, 3, 4, for LOAS defined in Table 2 and illustrated in Fig. 3. Specifically,

$$pF_i(t_{mx}) = CDF_{TM,i}(0\,|\,[t_{mn}, t_{mx}]) = \int_{t_{mn}}^{t_{mx}} CDF_{T,SL}(t_{WL})\mathrm{d}CDF_{T,WL}(t_{WL}) \quad (4.27)$$

for i = 1, 2, 3, 4 with the appropriate values for $CDF_{T,SL}(t_{WL})$ and $CDF_{T,WL}(t_{WL})$ for failure pattern i defined in Eqs. (4.4)-(4.7). This connection is (i) illustrated by the equality of $pF_i(200)$ and

$CDF_{TM,i}(0 \mid [0,200])$ in Fig. 3 and Fig. 4b and (ii) provides an additional verification result that the failure time margins in Figs. 4 and 5 have been calculated correctly.

5. CDFs for System Property Values at LOAS

5.1. Formal Representation: System Involving Only One Type of Link

A system involving only one type of link is considered in this section (i.e., only WLs or only SLs). Further, the links comprising the indicated system must have the same type of failure property. For example, all links failing on the basis of temperature or all links failing on the basis of pressure are acceptable possibilities. However, some links failing on the basis of temperature and other links failing on the basis of pressure is not acceptable. Consistent with the failure modes in Table 2, two failure possibilities are considered: (i) system failure occurs at the time of the last link failure, and (ii) system failure occurs at the time of the first link failure.

As in Sect. 4, nL links are under consideration with failure time CDFs $CDF_{Tk}(\tau)$, $k = 1, 2, \ldots, nL$. Further, as developed in Sect. 9 of Ref. [8],

$$\begin{aligned} CDF_{Pk}(p \mid \tau) &= \text{probability that link } k \text{ fails at a property value } \le p \text{ conditional on} \\ &\quad\ \text{failure of link } k \text{ at time } \tau \qquad\qquad (5.1) \\ &= \int_{p_{mn,k}(\tau)}^{p_{mx,k}(p,\tau)} d_{Pk}(\tilde{p} \mid \tau)\mathrm{d}\tilde{p}, \end{aligned}$$

where (i) $d_{Pk}(\tilde{p} \mid \tau)$ is the density function corresponding to $CDF_{Pk}(p \mid \tau)$ defined in Eq. (9.63) and Table 6 of Ref. [8], (ii) $[p_{mn,k}(\tau), p_{mx,k}(\tau)]$ is the interval of property values at which link k could fail at time τ (see Table 3) and thus corresponds to the sample space associated with $d_{Pk}(\tilde{p} \mid \tau)$, and (iii) $p_{mx,k}(p,\tau) = \min\{p, p_{mx,k}(\tau)\}$. Given the preceding, replacement of $\mathrm{d}CDF_{Tk}(\tau)$ in Eqs. (4.1) and (4.2) by $CDF_{Pk}(p \mid \tau)\mathrm{d}CDF_{Tk}(\tau)$ produces

$$\begin{aligned} CDF_{P,LLF}(p \mid [t_{mn}, t_{mx}]) &= \text{probability that last link failure occurs at} \\ &\quad\ \text{a property value } \le p \text{ at a time } \tau \le t_{mx} \\ &= \sum_{k=1}^{nL}\left(\int_{t_{mn}}^{t_{mx}}\left\{\prod_{l=1, l\ne k}^{nL} CDF_{Tl}(\tau)\right\} CDF_{Pk}(p \mid \tau)\mathrm{d}CDF_{Tk}(\tau)\right) \qquad (5.2) \\ &= \sum_{k=1}^{nL}\left(\int_{\tau_{mn,k}(p)}^{\tau_{mx,k}(t_{mx},p)}\left\{\prod_{l=1, l\ne k}^{nL} CDF_{Tl}(\tau)\right\}\left\{\int_{p_{mn,k}(\tau)}^{p_{mx,k}(p,\tau)} d_{Pk}(\tilde{p} \mid \tau)\mathrm{d}\tilde{p}\right\} d_{Tk}(\tau)\mathrm{d}\tau\right) \\ &= \sum_{k=1}^{nL}\left(\int_{\tau_{mn,k}(p)}^{\tau_{mx,k}(t_{mx},p)}\left\{\prod_{l=1, l\ne k}^{nL} CDF_{Tl}(\tau)\right\}\left\{\int_{p_{mn,k}(\tau)}^{p_{mx,k}(p,\tau)} d_{Pk}(\tilde{p} \mid \tau) d_{Tk}(\tau)\mathrm{d}\tilde{p}\right\}\mathrm{d}\tau\right) \end{aligned}$$

and, in like manner,

$$CDF_{P,FLF}(p \mid [t_{mn}, t_{mx}]) = \text{probability that first link failure occurs at a property value } \le p \text{ at a time } \tau \le t_{mx}$$

$$= \sum_{k=1}^{nL} \left(\int_{t_{mn}}^{t_{mx}} \left\{ \prod_{l=1,l\neq k}^{nL} \left[1 - CDF_{Tl}(\tau)\right] \right\} CDF_{Pk}(p \mid \tau) \mathrm{d}CDF_{Tk}(\tau) \right) \tag{5.3}$$

$$= \sum_{k=1}^{nL} \left(\int_{\tau_{mn,k}(p)}^{\tau_{mx,k}(t_{mx},p)} \left\{ \prod_{l=1,l\neq k}^{nL} \left[1 - CDF_{Tl}(\tau)\right] \right\} \left\{ \int_{p_{mn,k}(\tau)}^{p_{mx,k}(p,\tau)} d_{Pk}(\tilde{p} \mid \tau) d_{Tk}(\tau) \mathrm{d}\tilde{p} \right\} \mathrm{d}\tau \right)$$

with (i) $CDF_{Pk}(p \mid \tau)$ replaced by the representation in Eq. (5.1), (ii) $\mathrm{d}CDF_{Tk}(\tau)$ replaced by $d_{Tk}(\tau)\mathrm{d}\tau$ with $d_{Tk}(\tau) = \mathrm{d}CDF_k(\tau)/\mathrm{d}\tau$ as defined in Eq. (8.2) of Ref. [8] and subsequently defined in more detail in Eq. (9.64) of Ref. [8], (iii) $\tau_{mx,k}(t_{mx}, p) = \min\{t_{mx}, \tau_{mx,k}(p)\}$ with $\tau_{mx,k}(p)$ = last time at which link *k* can fail at a property value $\le p$, and (iv) the integral over $[t_{mn}, t_{mx}]$ equal to 0 for link *k* and $t_{mx} \le \tau_{mn,k}(p)$ with $\tau_{mn,k}(p)$ = first time at which link *k* can fail at a property value $\le p$.

The definitions of the density functions $d_{Pk}(\tilde{p} \mid \tau)$ and $d_{Tk}(\tau)$ are complicated (see Eqs. (9.63) and (9.64) in Ref. [8]). However, some simplification occurs when the product $d_{Pk}(\tilde{p} \mid \tau) d_{Tk}(\tau)$ is involved, as is the case in Eqs. (5.2) and (5.3). Specifically,

$$d_{Pk}(\tilde{p} \mid \tau) d_{Tk}(\tau) = \left\{ \mathrm{d}\left[\frac{1}{r_k(\tau)}\right] / \mathrm{d}\tau \right\} \left\{ \frac{\tilde{p}}{\bar{p}_k^2(\tau)} \right\} d_{Ak}\left[\tilde{p} / \bar{p}_k(\tau)\right] d_{Bk}\left[\tilde{p} / \bar{q}_k(\tau)\right] \tag{5.4}$$

as developed in conjunction with Eqs. (9.63), (9.64) and (9.65) of Ref. [8]. In turn, replacement of $d_{Pk}(\tilde{p} \mid \tau) d_{Tk}(\tau)$ in Eqs. (5.2) and (5.3) with the preceding form produces the representations

$$\begin{aligned} CDF_{P,LLF}(p \mid [t_{mn}, t_{mx}]) &= \sum_{k=1}^{nL} \left(\int_{\tau_{mn,k}(p)}^{\tau_{mx,k}(t_{mx},p)} \left\{ \prod_{l=1,l\neq k}^{nL} CDF_{Tl}(\tau) \right\} \left\{ \mathrm{d}\left[\frac{1}{r_k(\tau)}\right] / \mathrm{d}\tau \right\} \right. \\ &\quad \left. \times \left\{ \int_{p_{mn,k}(\tau)}^{p_{mx,k}(p,\tau)} \left\{ \frac{\tilde{p}}{\bar{p}_k^2(\tau)} \right\} d_{Ak}\left[\tilde{p} / \bar{p}_k(\tau)\right] d_{Bk}\left[\tilde{p} / \bar{q}_k(\tau)\right] \mathrm{d}\tilde{p} \right\} \mathrm{d}\tau \right) \end{aligned} \tag{5.5}$$

and

$$\begin{aligned} CDF_{P,FLF}(p \mid [t_{mn}, t_{mx}]) &= \sum_{k=1}^{nL} \left(\int_{\tau_{mn,k}(p)}^{\tau_{mx,k}(t_{mx},p)} \left\{ \prod_{l=1,l\neq k}^{nL} \left[1 - CDF_{Tl}(\tau)\right] \right\} \left\{ \mathrm{d}\left[\frac{1}{r_k(\tau)}\right] / \mathrm{d}\tau \right\} \right. \\ &\quad \left. \times \left\{ \int_{p_{mn,k}(\tau)}^{p_{mx,k}(p,\tau)} \left\{ \frac{\tilde{p}}{\bar{p}_k^2(\tau)} \right\} d_{Ak}\left[\tilde{p} / \bar{p}_k(\tau)\right] d_{Bk}\left[\tilde{p} / \bar{q}_k(\tau)\right] \mathrm{d}\tilde{p} \right\} \mathrm{d}\tau \right) \end{aligned} \tag{5.6}$$

for $CDF_{P,LLF}(p\,|\,[t_{mn},t_{mx}])$ and $CDF_{P,FLF}(p\,|\,[t_{mn},t_{mx}])$ with the integral over $[t_{mn},t_{mx}]$ equal to 0 for link k and $t_{mx} \le \tau_{mn,k}(p)$. Although the representations for $CDF_{P,LLF}(p\,|\,[t_{mn},t_{mx}])$ and $CDF_{P,FLF}(p\,|\,[t_{mn},t_{mx}])$ in Eqs. (5.5) and (5.6) are complicated, they are numerically simpler than the representations in Eqs. (5.2) and (5.3) due to the significant complexity in the definitions of $CDF_{Pk}(p\,|\,\tau)$ and $d_{Tk}(\tau)$.

Table 3 Integration limits associated with a link in Eqs. (5.1)-(5.6) with $\bar{p}(\tau)$ increasing and $\bar{q}(\tau)$ either decreasing or constant-valued (adapted from Ref. [8], Table 7).

$\tau_{mn}(p) =$ first time that link failure could occur at a property value $\tilde{p} \le p$

$$= \begin{cases} \tau_f = r^{-1}(\alpha_{mx}/\beta_{mn}) \text{ for } p_f \le p \le p_{mx} \\ \tau_f(p) = \bar{q}^{-1}(p/\beta_{mn}) \text{ for } p_{mn} \le p < p_f \text{ (not relevant for } \bar{q}(\tau)=c \text{ because } p_{mn}=p_f) \end{cases}$$

$\tau_{mx}(p) =$ last time that link failure could occur at a property value $\tilde{p} \le p$

$$= \begin{cases} \tau_l = r^{-1}(\alpha_{mn}/\beta_{mx}) \text{ for } p_l < p \le p_{mx} \text{ (not relevant for } \bar{q}(\tau)=c \text{ because } p_l = p_{mx}) \\ \tau_l(p) = \bar{p}^{-1}(p/\alpha_{mn}) \text{ for } p_{mn} \le p \le p_l \end{cases}$$

$$\tau_{mx}(t,p) = \min\{t, \tau_{mx}(p)\}$$

$[p_{mn}(\tau), p_{mx}(\tau)] =$ interval of link failure values p at time τ for $\alpha_{mn}/\beta_{mn} < \alpha_{mx}/\beta_{mx}$

$$= \begin{cases} [\beta_{mn}\bar{q}(\tau), \alpha_{mx}\bar{p}(\tau)] \text{ for } \tau \in \mathcal{P}_1 = \{\tau : \tau_f = r^{-1}(\alpha_{mx}/\beta_{mn}) \le \tau \le r^{-1}(\alpha_{mx}/\beta_{mx}) = \tau_{mx}\} \\ [\beta_{mn}\bar{q}(\tau), \beta_{mx}\bar{q}(\tau)] \text{ for } \tau \in \mathcal{P}_2 = \{\tau : \tau_{mx} = r^{-1}(\alpha_{mx}/\beta_{mx}) \le \tau \le r^{-1}(\alpha_{mn}/\beta_{mn}) = \tau_{mn}\} \\ [\alpha_{mn}\bar{p}(\tau), \beta_{mx}\bar{q}(\tau)] \text{ for } \tau \in \mathcal{P}_4 = \{\tau : \tau_{mn} = r^{-1}(\alpha_{mn}/\beta_{mn}) \le \tau \le r^{-1}(\alpha_{mn}/\beta_{mx}) = \tau_l\} \end{cases}$$

$[p_{mn}(\tau), p_{mx}(\tau)] =$ interval of link failure values p at time τ for $\alpha_{mx}/\beta_{mx} < \alpha_{mn}/\beta_{mn}$

$$= \begin{cases} [\beta_{mn}\bar{q}(\tau), \alpha_{mx}\bar{p}(\tau)] \text{ for } \tau \in \mathcal{P}_1 = \{\tau : \tau_f = r^{-1}(\alpha_{mx}/\beta_{mn}) \le \tau \le r^{-1}(\alpha_{mn}/\beta_{mn}) = \tau_{mn}\} \\ [\alpha_{mn}\bar{p}(\tau), \alpha_{mx}\bar{p}(\tau)] \text{ for } \tau \in \mathcal{P}_3 = \{\tau : \tau_{mn} = r^{-1}(\alpha_{mn}/\beta_{mn}) \le \tau \le r^{-1}(\alpha_{mx}/\beta_{mx}) = \tau_{mx}\} \\ [\alpha_{mn}\bar{p}(\tau), \beta_{mx}\bar{q}(\tau)] \text{ for } \tau \in \mathcal{P}_4 = \{\tau : \tau_{mx} = r^{-1}(\alpha_{mx}/\beta_{mx}) \le \tau \le r^{-1}(\alpha_{mn}/\beta_{mx}) = \tau_l\} \end{cases}$$

$[p_{mn}(\tau), p_{mx}(\tau)] =$ interval of link failure values p at time τ for $\alpha_{mn}/\beta_{mn} = \alpha_{mx}/\beta_{mx}$

$$= \begin{cases} [\beta_{mn}\bar{q}(\tau), \alpha_{mx}\bar{p}(\tau)] \text{ for } \tau \in \mathcal{P}_1 = \{\tau : \tau_f = r^{-1}(\alpha_{mx}/\beta_{mn}) \le \tau \le r^{-1}(\alpha_{mx}/\beta_{mx}) = \tau_{mx}\} \\ [\alpha_{mn}\bar{p}(\tau), \beta_{mx}\bar{q}(\tau)] \text{ for } \tau \in \mathcal{P}_4 = \{\tau : \tau_{mx} = r^{-1}(\alpha_{mx}/\beta_{mx}) \le \tau \le r^{-1}(\alpha_{mn}/\beta_{mx}) = \tau_l\} \end{cases}$$

$$p_{mx}(p,\tau) = \min\{p, p_{mx}(\tau)\}$$

For future use, it is noted that a derivation for the equality

$$
\begin{aligned}
CDF_{Pk}(p \mid \tau) d_{Tk}(\tau)\mathrm{d}\tau &= \left\{\int_{p_{mn,k}(\tau)}^{p_{mx,k}(p,\tau)} d_{Pk}(\tilde{p} \mid \tau) d_{Tk}(\tau)\mathrm{d}\tilde{p}\right\}\mathrm{d}\tau \\
&= \left\{\int_{p_{mn,k}(\tau)}^{p_{mx,k}(p,\tau)} \left\{\mathrm{d}\left[\frac{1}{r_k(\tau)}\right] / \mathrm{d}\tau\right\}\left\{\frac{\tilde{p}}{\overline{p}_k^2(\tau)}\right\} d_{Ak}\left[\tilde{p} / \overline{p}_k(\tau)\right] d_{Bk}\left[\tilde{p} / \overline{q}_k(\tau)\right]\mathrm{d}\tilde{p}\right\}\mathrm{d}\tau \qquad (5.7) \\
&= \left\{\mathrm{d}\left[\frac{1}{r_k(\tau)}\right] / \mathrm{d}\tau\right\}\left\{\int_{p_{mn,k}(\tau)}^{p_{mx,k}(p,\tau)} \left\{\frac{\tilde{p}}{\overline{p}_k^2(\tau)}\right\} d_{Ak}\left[\tilde{p} / \overline{p}_k(\tau)\right] d_{Bk}\left[\tilde{p} / \overline{q}_k(\tau)\right]\mathrm{d}\tilde{p}\right\}\mathrm{d}\tau
\end{aligned}
$$

is embedded in the derivations leading to Eqs. (5.5) and (5.6).

As examples, CDFs for property value at link system failure (i.e., $CDF_{P,LLF}(p \mid [t_{mn}, t_{mx}])$ and $CDF_{P,FLF}(p \mid [t_{mn}, t_{mx}])$ with $[t_{mn}, t_{mx}] = [0, 200]$) for systems with two WLs and systems with two SLs links defined with the WLs and SLs described and illustrated in Table 1 and Fig. 1 are presented in Fig. 6. With respect to notation,

$$CDF_{P,WL,LLF}(p \mid [t_{mn}, t_{mx}]) = CDF_{P,LLF}(p \mid [t_{mn}, t_{mx}]) \text{ for WL systems,} \qquad (5.8)$$

$$CDF_{P,WL,FLF}(p \mid [t_{mn}, t_{mx}]) = CDF_{P,FLF}(p \mid [t_{mn}, t_{mx}]) \text{ for WL systems,} \qquad (5.9)$$

$$CDF_{P,SL,LLF}(p \mid [t_{mn}, t_{mx}]) = CDF_{P,LLF}(p \mid [t_{mn}, t_{mx}]) \text{ for SL systems,} \qquad (5.10)$$

$$CDF_{P,SL,FLF}(p \mid [t_{mn}, t_{mx}]) = CDF_{P,FLF}(p \mid [t_{mn}, t_{mx}]) \text{ for SL systems.} \qquad (5.11)$$

Quadrature-based evaluation of the defining integrals in Eqs. (5.5) and (5.6) for $CDF_{P,LLF}(p \mid [t_{mn}, t_{mx}])$ and $CDF_{P,FLF}(p \mid [t_{mn}, t_{mx}])$ was performed with the MATLAB program **TwoD** [59]. In the development of the integrand, (i) representations for the density functions $d_{Ak}(\alpha)$ and $d_{Bk}(\beta)$ were obtained with use of the **makedist** and **pdf** functions, (ii) the representation for the derivative was obtained with use of the **diff** and **matlabFunction** functions, and (iii) representations for the expressions involving products of CDFs were obtained with use of the **spline** function.

The double integrals in Eqs. (5.5) and (5.6) have time τ as the outer variable of integration and property value p as the inner variable of integration. If desired, the integrals in Eqs. (5.5) and (5.6) can be rewritten with p as the outer variable of integration and τ as the inner variable of integration.

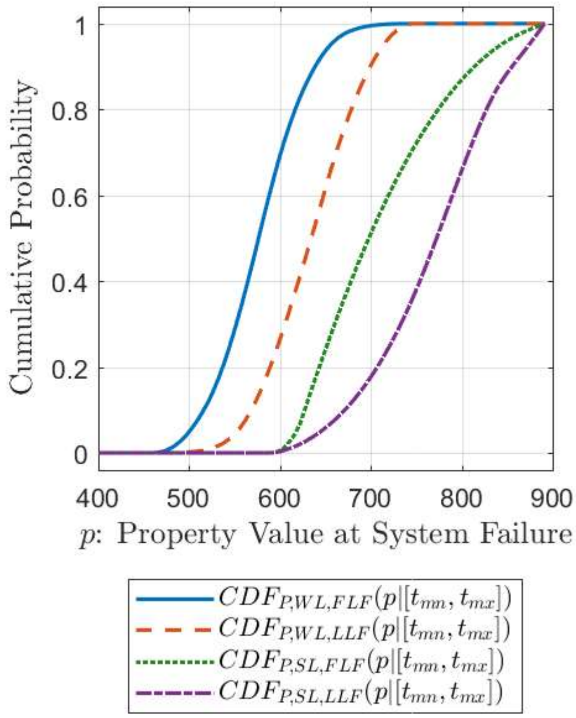

Fig. 6 Property value at link system failure CDFs (i.e., $CDF_{P,WL,LLF}(p\,|\,[t_{mn}, t_{mx}])$, $CDF_{P,WL,FLF}(p\,|\,[t_{mn}, t_{mx}])$, $CDF_{P,SL,LLF}(p\,|\,[t_{mn}, t_{mx}])$ and $CDF_{P,SL,FLF}(p\,|\,[t_{mn}, t_{mx}])$ with $[t_{mn}, t_{mx}] = [0, 200]$) obtained by quadrature-based evaluation of the integrals in Eqs. (5.5) and (5.6) for systems with two WLs and systems with two SLs links defined with the WLs and SLs described and illustrated in Table 1 and Fig. 1.

Another possibility is to remove the term $\tilde{p} / \overline{p}_k^2(\tau)$ in Eqs. (5.5) and (5.6) through the change of variables $\alpha_k(\tilde{p}) = \tilde{p} / \overline{p}_k(\tau)$. This produces the representations

$$\begin{aligned} CDF_{P,LLF}(p\,|\,[t_{mn}, t_{mx}]) \quad &= \sum_{k=1}^{nL} \left(\int_{\tau_{mn,k}(p)}^{\tau_{mx,k}(t_{mx}, p)} \left\{ \prod_{l=1, l \neq k}^{nL} CDF_{Tl}(\tau) \right\} \left\{ \mathrm{d}\left[\frac{1}{r_k(\tau)} \right] / \, \mathrm{d}\tau \right\} \right. \\ &\quad \left. \times \left\{ \int_{p_{mn,k}(\tau)/\overline{p}_k(\tau)}^{p_{mx,k}(p,\tau)/\overline{p}_k(\tau)} \alpha\, d_{Ak}(\alpha)\, d_{Bk}\left[\alpha / r_k(\tau)\right] \mathrm{d}\alpha \right\} \mathrm{d}\tau \right) \end{aligned} \tag{5.12}$$

and

$$CDF_{P,FLF}(p\,|\,[t_{mn},t_{mx}]) = \sum_{k=1}^{nL}\left(\int_{\tau_{mn,k}(p)}^{\tau_{mx,k}(t_{mx},p)}\left\{\prod_{l=1,l\neq k}^{nL}\left[1-CDF_{Tl}(\tau)\right]\right\}\left\{\mathrm{d}\left[\frac{1}{r_k(\tau)}\right]/\mathrm{d}\tau\right\} \times\left\{\int_{p_{mn,k}(\tau)/\bar{p}_k(\tau)}^{p_{mx,k}(p,\tau)/\bar{p}_k(\tau)}\alpha d_{Ak}(\alpha)d_{Bk}\left[\alpha/r_k(\tau)\right]\mathrm{d}\alpha\right\}\mathrm{d}\tau\right) \tag{5.13}$$

for $CDF_{P,LLF}(p\,|\,[t_{mn},t_{mx}])$ and $CDF_{P,FLF}(p\,|\,[t_{mn},t_{mx}])$ with the integral over $[t_{mn},t_{mx}]$ equal to 0 for link k and $t_{mx} \le \tau_{mn,k}(p)$.

5.2. Sampling-based Estimation of $CDF_{P,LLF}(p\,|\,[t_{mn},t_{mx}])$ and $CDF_{P,FLF}(p\,|\,[t_{mn},t_{mx}])$

Another possibility is to use a sampling-based procedure to estimate $CDF_{P,LLF}(p\,|\,[t_{mn},t_{mx}])$ and $CDF_{P,FLF}(p\,|\,[t_{mn},t_{mx}])$. This procedure involves (i) generating a sample from the defining parameters for the individual links (i.e., the α's and β's) in consistency with their specified distributions as defined by the density functions $d_A(\alpha)$ and $d_B(\beta)$ for the individual links and (ii) then determining the resultant failure times and associated failure values. Specifically, a sample

$$\mathbf{p}_r = \left[\mathbf{pL}_{1r},\mathbf{pL}_{2r},\cdots,\mathbf{pL}_{nL,r}\right], r = 1,2,\cdots,n \tag{5.14}$$

with

$$\mathbf{pL}_{lr} = \left[\alpha_{lr},\beta_{lr}\right] \text{ for } l = 1,2,\cdots,nL \tag{5.15}$$

is generated from the distributions of the α's and β's for the individual links. Next, the link failure times tL_{lr} and failure values pL_{lr} are determined as functions

$$tL_{lr} = tL_{lr}\left(\mathbf{pL}_{lr}\right) = tL_{lr}\left(\alpha_{lr},\beta_{lr}\right) \tag{5.16}$$

and

$$pL_{lr} = pL_{lr}\left(\mathbf{pL}_{lr}\right) = pL_{lr}\left(\alpha_{lr},\beta_{lr}\right) = \alpha_{lr}\bar{p}_l[tL_{lr}\left(\alpha_{lr},\beta_{lr}\right)] \tag{5.17}$$

of elements of the sampled vectors $\mathbf{p}_r$ in Eq. (5.14).

The following additional expressions are now introduced for use in estimating $CDF_{P,LLF}(p\,|\,[t_{mn},t_{mx}])$ and $CDF_{P,FLF}(p\,|\,[t_{mn},t_{mx}])$:

$$f(r) = \text{ link designator for } tL_{f(r),r} = \min\{tL_{1r},tL_{2r},\cdots,tL_{nL,r}\}, \tag{5.18}$$

$$l(r) = \text{ link designator for } tL_{l(r),r} = \max\{tL_{1r},tL_{2r},\cdots,tL_{nL,r}\} \tag{5.19}$$

$$pL_{f(r),r} = \alpha_{f(r),r}\bar{p}_{f(r)}(tL_{f(r),r}),\ pL_{l(r),r} = \alpha_{l(r),r}\bar{p}_{l(r)}(tL_{l(r),r}) \tag{5.20}$$

and

$$\delta_{tp}(\tilde{t},\tilde{p}) = \begin{cases} 1 & \text{for } \tilde{t} \le t_{mx} \text{ and } \tilde{p} \le p \\ 0 & \text{otherwise.} \end{cases} \tag{5.21}$$

In turn,

$$CDF_{P,LLF}(p \mid [t_{mn}, t_{mx}]) \cong \sum_{r=1}^{n} \delta_{tp}(tL_{l(r),r}, pL_{l(r),r}) / n \tag{5.22}$$

and

$$CDF_{P,FLF}(p \mid [t_{mn}, t_{mx}]) \cong \sum_{r=1}^{n} \delta_{tp}(tL_{f(r),r}, pL_{f(r),r}) / n. \tag{5.23}$$

5.3. Verification Results for $CDF_{P,LLF}(p \mid [t_{mn}, t_{mx}])$ and $CDF_{P,FLF}(p \mid [t_{mn}, t_{mx}])$

As a verification test, Fig. 7 presents a comparison of CDFs for property value at link system failure (i.e., $CDF_{P,LLF}(p \mid [t_{mn}, t_{mx}])$ and $CDF_{P,FLF}(p \mid [t_{mn}, t_{mx}])$) for systems with two WLs and systems with two SLs links obtained by (i) quadrature-based evaluation of the integrals in Eqs. (5.5) and (5.6) as described in Sect. 5.1 and (ii) a sampling-based calculation as defined in Eqs. (5.22) and (5.23) performed with the CPLOAS program [55; 56] and a sample of size $n = 10^6$. The CDFs $CDF_{P,LLF}(p \mid [t_{mn}, t_{mx}])$ and $CDF_{P,FLF}(p \mid [t_{mn}, t_{mx}])$ are not defined outputs of the CPLOAS program but (i) these CDFs are calculated as indicated in Sect. 5.2 as part of the SL-SL margin analysis described in Sect. 7.3 and (ii) were obtained for this analysis by adding a few lines of code to CPLOAS to write them to a saved output file. The results obtained with the two evaluation procedures are essentially identical. This level of agreement provides a strong verification that the two procedures for obtaining $CDF_{P,LLF}(p \mid [t_{mn}, t_{mx}])$ and $CDF_{P,FLF}(p \mid [t_{mn}, t_{mx}])$ are correct in both (i) mathematical development and (ii) computational implementation.

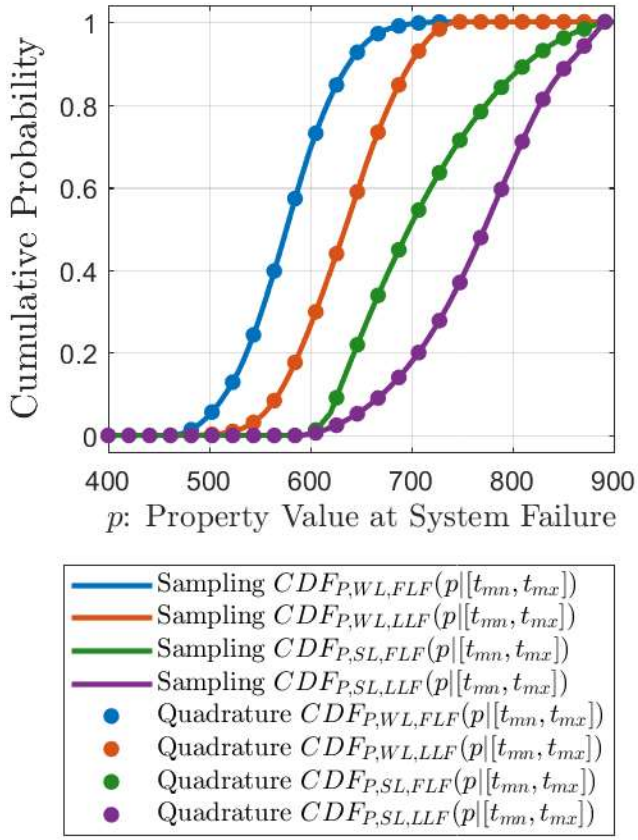

Fig. 7 Verification results for property value at link system failure (i.e., $CDF_{P,WL,LLF}(p\,|\,[t_{mn},t_{mx}])$, $CDF_{P,WL,FLF}(p\,|\,[t_{mn},t_{mx}])$, $CDF_{P,SL,LLF}(p\,|\,[t_{mn},t_{mx}])$ and $CDF_{P,SL,FLF}(p\,|\,[t_{mn},t_{mx}])$ with $[t_{mn},t_{mx}]=[0,200]$) obtained with quadrature-based procedures (see Sect. 5.1) and sampling-based procedures (see Sect. 5.2) for the four links defined and illustrated in Table 1 and Fig. 1.

5.4. Formal Representation: System Involving Multiple WLs and SLs

Distributions (i.e., CDFs) for SL system failure values (i.e., property value at link failure for SL whose failure results in LOAS) for systems involving multiple WLs and SLs have representations that are similar to the representations for $CDF_{P,LLF}(p\,|\,[t_{mn},t_{mx}])$ and $CDF_{P,FLF}(p\,|\,[t_{mn},t_{mx}])$ developed in Sect. 5.1. As in Sect. 5.1, all SLs must have the same type of failure property (e.g., temperature for all SLs or pressure for all SLs). However, WLs do not have this restriction (e.g., some WLs could fail on the basis of temperature while other WLs fail on the basis of pressure). The starting points for these representations are the probabilities for LOAS in Table 2. For notational convenience, let

$CDF_{P,SL,i}(p\,|\,[t_{mn},t_{mx}])=$ probability that LOAS occurs at a SL value $\le p$ at a time $\tau \le t_{mx}$ for failure pattern $i=1,2,3,4$ defined in Table 2. (5.24)

Representations for $CDF_{P,SL,i}(p\,|\,[t_{mn},t_{mx}])$, $i=1,2,3,4,$ that are analogous to the representations for $CDF_{P,LLF}(p\,|\,[t_{mn},t_{mx}])$ and $CDF_{P,FLF}(p\,|\,[t_{mn},t_{mx}])$ developed in in Eqs. (5.2)-(5.6) of Sect. 5.1 result by replacing $\mathrm{d}CDF_{T,SL,k}(\tau)$ in the representations for $pF_1(t_{mx})$, $pF_2(t_{mx})$, $pF_3(t_{mx})$ and $pF_4(t_{mx})$ in Table 2 with

$$CDF_{P,SL,k}(p\,|\,\tau)\mathrm{d}CDF_{T,SL,k}(\tau), \tag{5.25}$$

or equivalently, with

$$\begin{aligned} &CDF_{P,SL,k}(p\,|\,\tau)d_{T,SL,k}\mathrm{d}\tau \\ &\quad=\left\{\mathrm{d}\left[\frac{1}{r_k(\tau)}\right]/\,\mathrm{d}\tau\right\}\left\{\int_{p_{mn,k}(\tau)}^{p_{mx,k}(p,\tau)}\left\{\frac{\tilde{p}}{\overline{p}_k^2(\tau)}\right\}d_{Ak}\left[\tilde{p}\,/\,\overline{p}_k(\tau)\right]d_{Bk}\left[\tilde{p}\,/\,\overline{q}_k(\tau)\right]\mathrm{d}\tilde{p}\right\}\mathrm{d}\tau \end{aligned} \tag{5.26}$$

as defined in Eq. (5.7). Specifically, replacement of $\mathrm{d}CDF_{T,SL,k}(\tau)$ with the expressions in Eqs. (5.25) and (5.26) produces the representations for $CDF_{P,SL,i}(p\,|\,[t_{mn},t_{mx}])$, $i=1,2,3,4,$ summarized in Table 4.

Table 4 Representation of $CDF_{P,SL,i}(p\,|\,[t_{mn},t_{mx}])$, i = 1, 2, 3, 4, for (i) SL system property at time of LOAS, (ii) nWL WLs and nSL SLs, (iii) independent distributions for link failure time, and (iv) the integral over $[t_{mn},t_{mx}]$ equal to 0 for SL k and $t_{mx}\le\tau_{mn,k}(p)$ with $\tau_{mn,k}(p)$ and other integration limits defined in Table 3.

Failure Pattern 1: Failure of all SLs before failure of any WL

$$\begin{aligned} &CDF_{P,SL,1}(p\,|\,[t_{mn},t_{mx}]) \\ &\quad=\sum_{k=1}^{nSL}\left(\int_{t_{mn}}^{t_{mx}}\left\{\prod_{l=1,l\ne k}^{nSL}CDF_{T,SL,l}(\tau)\right\}\left\{\prod_{j=1}^{nWL}\left[1-CDF_{T,WL,j}(\tau)\right]\right\}CDF_{P,SL,k}(p\,|\,\tau)\mathrm{d}CDF_{T,SL,k}(\tau)\right) \\ &\quad=\sum_{k=1}^{nSL}\left(\int_{\tau_{mn,k}(p)}^{\tau_{mx,k}(t_{mx},p)}\left\{\prod_{l=1,l\ne k}^{nSL}CDF_{T,SL,l}(\tau)\right\}\left\{\prod_{j=1}^{nWL}\left[1-CDF_{T,WL,j}(\tau)\right]\right\}\left\{\mathrm{d}\left[\frac{1}{r_k(\tau)}\right]/\,\mathrm{d}\tau\right\}\right. \\ &\qquad\left.\times\left\{\int_{p_{mn,k}(\tau)}^{p_{mx,k}(p,\tau)}\left\{\frac{\tilde{p}}{\overline{p}_k^2(\tau)}\right\}d_{Ak}\left[\tilde{p}\,/\,\overline{p}_k(\tau)\right]d_{Bk}\left[\tilde{p}\,/\,\overline{q}_k(\tau)\right]\mathrm{d}\tilde{p}\right\}\mathrm{d}\tau\right) \end{aligned}$$

Table 4 Continued

Failure Pattern 2: Failure of any SL before failure of any WL

$$
\begin{aligned}
&CDF_{P,SL,2}(p\,|\,[t_{mn},t_{mx}]) \\
&= \sum_{k=1}^{nSL}\left(\int_{t_{mn}}^{t_{mx}}\left\{\prod_{l=1,l\neq k}^{nSL}\left[1-CDF_{T,SL,l}(\tau)\right]\right\}\left\{\prod_{j=1}^{nWL}\left[1-CDF_{T,WL,j}(\tau)\right]\right\}CDF_{P,SL,k}(p\,|\,\tau)\mathrm{d}CDF_{T,SL,k}(\tau)\right) \\
&= \sum_{k=1}^{nSL}\left(\int_{\tau_{mn,k}(p)}^{\tau_{mx,k}(t_{mx},p)}\left\{\prod_{l=1,l\neq k}^{nSL}\left[1-CDF_{T,SL,l}(\tau)\right]\right\}\left\{\prod_{j=1}^{nWL}\left[1-CDF_{T,WL,j}(\tau)\right]\right\}\left\{\mathrm{d}\left[\frac{1}{r_k(\tau)}\right]/\,\mathrm{d}\tau\right\}\right. \\
&\qquad \left.\times\left\{\int_{p_{mn,k}(\tau)}^{p_{mx,k}(p,\tau)}\left\{\frac{\tilde{p}}{\bar{p}_k^2(\tau)}\right\}d_{Ak}\left[\tilde{p}/\bar{p}_k(\tau)\right]d_{Bk}\left[\tilde{p}/\bar{q}_k(\tau)\right]\mathrm{d}\tilde{p}\right\}\mathrm{d}\tau\right)
\end{aligned}
$$

Failure Pattern 3: Failure of all SLs before failure of all WLs

$$
\begin{aligned}
&CDF_{P,SL,3}(p\,|\,[t_{mn},t_{mx}]) \\
&= \sum_{k=1}^{nSL}\left(\int_{t_{mn}}^{t_{mx}}\left\{\prod_{l=1,l\neq k}^{nSL}CDF_{T,SL,l}(\tau)\right\}\left\{1-\prod_{j=1}^{nWL}CDF_{T,WL,j}(\tau)\right\}CDF_{P,SL,k}(p\,|\,\tau)\mathrm{d}CDF_{T,SL,k}(\tau)\right) \\
&= \sum_{k=1}^{nSL}\left(\int_{\tau_{mn,k}(p)}^{\tau_{mx,k}(t_{mx},p)}\left\{\prod_{l=1,l\neq k}^{nSL}CDF_{T,SL,l}(\tau)\right\}\left\{1-\prod_{j=1}^{nWL}CDF_{T,WL,j}(\tau)\right\}\left\{\mathrm{d}\left[\frac{1}{r_k(\tau)}\right]/\,\mathrm{d}\tau\right\}\right. \\
&\qquad \left.\times\left\{\int_{p_{mn,k}(\tau)}^{p_{mx,k}(p,\tau)}\left\{\frac{\tilde{p}}{\bar{p}_k^2(\tau)}\right\}d_{Ak}\left[\tilde{p}/\bar{p}_k(\tau)\right]d_{Bk}\left[\tilde{p}/\bar{q}_k(\tau)\right]\mathrm{d}\tilde{p}\right\}\mathrm{d}\tau\right)
\end{aligned}
$$

Failure Pattern 4: Failure of any SL before failure of all WLs

$$
\begin{aligned}
&CDF_{P,SL,4}(p\,|\,[t_{mn},t_{mx}]) \\
&= \sum_{k=1}^{nSL}\left(\int_{t_{mn}}^{t_{mx}}\left\{\prod_{l=1,l\neq k}^{nSL}\left[1-CDF_{T,SL,l}(\tau)\right]\right\}\left\{1-\prod_{j=1}^{nWL}CDF_{T,WL,j}(\tau)\right\}CDF_{P,SL,k}(p\,|\,\tau)\mathrm{d}CDF_{T,SL,k}(\tau)\right) \\
&= \sum_{k=1}^{nSL}\left(\int_{\tau_{mn,k}(p)}^{\tau_{mx,k}(t_{mx},p)}\left\{\prod_{l=1,l\neq k}^{nSL}\left[1-CDF_{T,SL,l}(\tau)\right]\right\}\left\{1-\prod_{j=1}^{nWL}CDF_{T,WL,j}(\tau)\right\}\left\{\mathrm{d}\left[\frac{1}{r_k(\tau)}\right]/\,\mathrm{d}\tau\right\}\right. \\
&\qquad \left.\times\left\{\int_{p_{mn,k}(\tau)}^{p_{mx,k}(p,\tau)}\left\{\frac{\tilde{p}}{\bar{p}_k^2(\tau)}\right\}d_{Ak}\left[\tilde{p}/\bar{p}_k(\tau)\right]d_{Bk}\left[\tilde{p}/\bar{q}_k(\tau)\right]\mathrm{d}\tilde{p}\right\}\mathrm{d}\tau\right)
\end{aligned}
$$

As an example, the CDFs $CDF_{P,SL,i}(p\,|\,[t_{mn},t_{mx}])$, $i=1,2,3,4$, for SL property value at LOAS determined over all possible times of link failure as indicated in Table 4 are illustrated in Fig. 8. The results in Fig. 8 were obtained by use of numerical (i.e., quadrature) procedures to evaluate the integrals in Table 4 with the MATLAB program **TwoD** [59] in a manner similar to that described in Sect. 5.1.

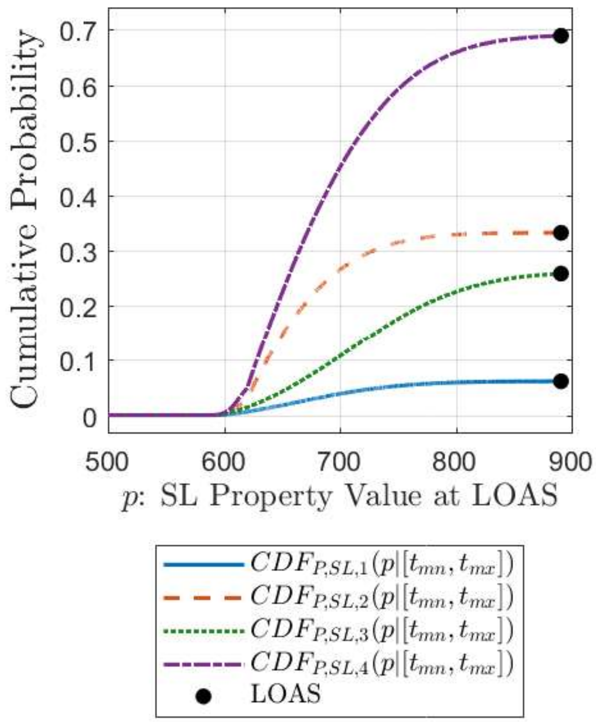

Fig. 8 Property value of SL at LOAS CDFs $CDF_{P,SL,i}(p\,|\,[t_{mn}, t_{mx}])$ determined over all possible times of link failure as indicated in Table 4 for (i) failure pattern i, $i = 1, 2, 3, 4$, resulting in LOAS as indicated in Eqs. (4.4)-(4.7) and Table 2, (ii) the four links defined and illustrated in Table 1 and Fig. 1 with $[t_{mn}, t_{mx}] = [0, 200]$, and (iii) quadrature-based evaluation of the integrals in Table 4.

As should be the case, the CDFs in Fig. 8 have structures that are similar to the corresponding CDFs in Fig. 3 for PLOAS.

5.5. Sampling-based Estimation of $CDF_{P,SL,i}(p\,|\,[t_{mn}, t_{mx}])$ for System Involving Multiple WLs and SLs

Another possibility is to use a sampling-based procedure to estimate $CDF_{P,SL,i}(p\,|\,[t_{mn}, t_{mx}])$. Similarly to the sampling-based procedure to estimate $CDF_{P,LLF}(p\,|\,[t_{mn}, t_{mx}])$ and $CDF_{P,FLF}(p\,|\,[t_{mn}, t_{mx}])$ described in Sect. 5.3, this procedure involves (i) generating a sample from the defining parameters for the individual links (i.e., the α's and β's) in consistency with their specified distributions as defined by the density functions $d_A(\alpha)$ and $d_B(\beta)$ for the individual links and (ii) then determining the resultant failure times and associated failure values. Specifically, a sample

$$\mathbf{p}_r = \left[\mathbf{pWL}_{1r}, \mathbf{pWL}_{2r}, \cdots, \mathbf{pWL}_{nWL,r}, \mathbf{pSL}_{1r}, \mathbf{pSL}_{1r}, \cdots, \mathbf{pSL}_{nSL,r}\right], \; r = 1, 2, \cdots, n \quad (5.27)$$

with

$$\mathbf{pWL}_{lr} = \left[\alpha_{WL,lr}, \beta_{WL,lr}\right] \text{ for } l = 1, 2, \cdots, nWL \quad (5.28)$$

and

$$\mathbf{pSL}_{lr} = \left[\alpha_{SL,lr}, \beta_{SL,lr}\right] \text{ for } l = 1, 2, \cdots, nSL \quad (5.29)$$

is generated from the distributions of the α's and β's for the individual links. Next, the link failure times (i.e., tWL_{lr} and tSL_{lr}) and SL failure values (i.e., pSL_{lr}) are determined as functions of elements of the sampled vectors $\mathbf{p}_r$ in Eq. (5.27) as indicated below:

$$tWL_{lr} = tWL_{lr}\left(\mathbf{pWL}_{lr}\right) = tWL_{lr}\left(\alpha_{WL,lr}, \beta_{WL,lr}\right), \quad (5.30)$$

$$tSL_{lr} = tSL_{lr}\left(\mathbf{pSL}_{lr}\right) = tSL_{lr}\left(\alpha_{SL,lr}, \beta_{SL,lr}\right), \quad (5.31)$$

and

$$pSL_{lr} = pSL_{lr}(\mathbf{pSL}_{lr}) = pSL_{lr}(\alpha_{SL,lr}, \beta_{SL,lr}) = \alpha_{SL,lr}\,\overline{p}_{SL,l}[tSL_{lr}(\alpha_{SL,lr}, \beta_{SL,lr})] \quad (5.32)$$

with $l = 1, 2, \cdots, nWL$ for WLs and $l = 1, 2, \cdots, nSL$ for SLs.

Once the results in Eqs. (5.30)-(5.32) are available, the next step is to define the following quantities:

$$\underline{WL}(r) = \text{ link designator for } tWL_{\underline{WL}(r),r} = \min\{tWL_{1r}, tWL_{2r}, \cdots, tWL_{nL,r}\}, \quad (5.33)$$

$$\overline{WL}(r) = \text{ link designator for } tWL_{\overline{WL}(r),r} = \max\{tWL_{1r}, tWL_{2r}, \cdots, tWL_{nL,r}\}, \quad (5.34)$$

$$\underline{SL}(r) = \text{ link designator for } tSL_{\underline{SL}(r),r} = \min\{tSL_{1r}, tSL_{2r}, \cdots, tSL_{nL,r}\}, \quad (5.35)$$

$$\overline{SL}(r) = \text{ link designator for } tSL_{\overline{SL}(r),r} = \max\{tSL_{1r}, tSL_{2r}, \cdots, tSL_{nL,r}\}, \quad (5.36)$$

and

$$pSL_{\underline{SL}(r),r} = \alpha_{\underline{SL}(r),r}\,\overline{p}_{\underline{SL}(r)}(tSL_{\underline{SL}(r),r}), \; pSL_{\overline{SL}(r),r} = \alpha_{\overline{SL}(r),r}\,\overline{p}_{\overline{SL}(r)}(tSL_{\overline{SL}(r),r}). \quad (5.37)$$

Further, the indicator function $\delta_{tp}(tWL, tSL, pSL)$ is defined by

$$\delta_{tp}(tWL, tSL, pSL) = \begin{cases} 1 \text{ for } tSL \le \min\{tWL, t_{mx}\} \text{ and } pSL \le p \\ 0 \text{ otherwise,} \end{cases} \tag{5.38}$$

where (i) p is a WL property value, (ii) tWL is a time at which WL failure potentially prevents LOAS, (iii) tSL is a time at which SL failure potentially results in LOAS, and (iv) pSL is the property value at time tSL of the SL whose failure at time tSL potentially results in LOAS.

Given the results in Eqs. (5.33)-(5.38), $CDF_{P,SL,i}(p\,|\,[t_{mn}, t_{mx}])$, $i = 1,2,3,4$, can be approximated by

$$CDF_{P,SL,1}(p\,|\,[t_{mn}, t_{mx}]) = \sum_{r=1}^{n} \delta_{tp}(tWL_{\underline{WL}(r),r}, tSL_{\overline{SL}(r),r}, pSL_{\overline{SL}(r),r})\,/\,n, \tag{5.39}$$

$$CDF_{P,SL,2}(p\,|\,[t_{mn}, t_{mx}]) = \sum_{r=1}^{n} \delta_{tp}(tWL_{\underline{WL}(r),r}, tSL_{\underline{SL}(r),r}, pSL_{\underline{SL}(r),r})\,/\,n, \tag{5.40}$$

$$CDF_{P,SL,3}(p\,|\,[t_{mn}, t_{mx}]) = \sum_{r=1}^{n} \delta_{tp}(tWL_{\overline{WL}(r),r}, tSL_{\overline{SL}(r),r}, pSL_{\overline{SL}(r),r})\,/\,n \tag{5.41}$$

and

$$CDF_{P,SL,4}(p\,|\,[t_{mn}, t_{mx}]) = \sum_{r=1}^{n} \delta_{tp}(tWL_{\overline{WL}(r),r}, tSL_{\underline{SL}(r),r}, pSL_{\underline{SL}(r),r})\,/\,n. \tag{5.42}$$

5.6. Verification Results for $CDF_{P,SL,i}(p\,|\,[t_{mn}, t_{mx}])$

As a verification test, Fig. 9 presents a comparison of CDFs $CDF_{P,SL,i}(p\,|\,[t_{mn}, t_{mx}])$, i = 1, 2, 3, 4, for SL property value at LOAS for a system with two WLs and two SLs links obtained by (i) quadrature-based evaluation of the integrals in Table 4 as described in Sect. 5.4 and (ii) a sampling-based calculation as described in Sect. 5.5 and performed with the CPLOAS program [55; 56] and a sample of size $n = 10^6$. The CDFs $CDF_{P,SL,i}(p\,|\,[t_{mn}, t_{mx}])$, i = 1, 2, 3, 4, are not defined outputs of the CPLOAS program but (i) these CDFs are calculated as indicated in Sect. 5.5 as part of the SL-SL margin analysis described in Sect. 7.3 and (ii) were obtained for the present analysis by adding a few lines of code to CPLOAS to write them to a saved output file. The results obtained with the two evaluation procedures are essentially identical. This level of agreement provides a strong verification that the two procedures for obtaining $CDF_{P,LLF}(p\,|\,[t_{mn}, t_{mx}])$ and $CDF_{P,FLF}(p\,|\,[t_{mn}, t_{mx}])$ are correct in both (i) mathematical development and (ii) computational implementation.

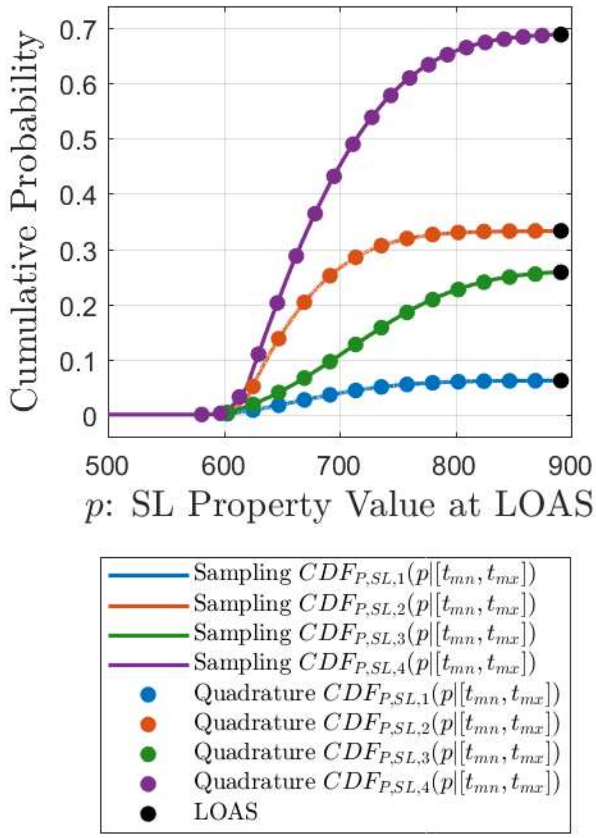

Fig. 9 Verification results for property value of SL at LOAS CDFs $CDF_{P,SL,i}(p\,|\,[t_{mn}, t_{mx}])$, $i = 1$, 2, 3, 4, obtained with quadrature-based procedures (see Sect. 5.4) and sampling-based procedures (see Sect. 5.5) for (i) the four failure patterns resulting in LOAS indicated in Eqs. (4.4)-(4.7) and (ii) the four links defined and illustrated in Table 1 and Fig. 1 with $[t_{mn}, t_{mx}] = [0, 200]$.

6. CDFs for Margins Based on WL and SL Property Values

6.1. Formal Representation: CDFs for Margins Based on WL and SL Property Values

For a particular WL/SL configuration (i.e., one of the failure patterns defined in Table 2), let (i) $CDF_{P,WL}(p_{WL} \mid [t_{mn}, t_{mx}])$ represent the CDF defined on the interval $[pWL_{mn}, pWL_{mx}]$ for the property value p_{WL} at which the failure of the system WLs in the time interval $[t_{mn}, t_{mx}]$ potentially deactivates the system and (ii) $CDF_{P,SL}(p_{SL} \mid [t_{mn}, t_{mx}])$ represent the CDF defined on the interval $[pSL_{mn}, pSL_{mx}]$ for the property value p_{SL} at which the failure of the system SLs in the time interval $[t_{mn}, t_{mx}]$ potentially results in LOAS. The modifier "potentially" appears in the preceding sentence because the indicated failures may or may not have the indicated effect because of the timing of both WL and SL failures. The CDFs $CDF_{P,WL}(p_{WL} \mid [t_{mn}, t_{mx}])$ and $CDF_{P,SL}(p_{SL} \mid [t_{mn}, t_{mx}])$ are defined and illustrated in Sect. 5.1.

The desired CDF $CDF_{PM}(m \mid [t_{mn}, t_{mx}])$ for property value margin m defined by $p_{SL} - p_{WL} = m$ is defined similarly to the time margin CDF $CDF_{TM}(m \mid [t_{mn}, t_{mx}])$ in Eq. (4.3). Specifically,

$$
\begin{aligned}
CDF_{PM}(m \mid [t_{mn}, t_{mx}]) &= prob\left(p_{SL} - p_{WL} \le m \mid [t_{mn}, t_{mx}]\right) \\
&= \lim_{n \to \infty} \sum_{i=1}^{n} CDF_{P,SL}\left(m + p_i \mid [t_{mn}, t_{mx}]\right) \Delta CDF_{P,WL}\left(p_i \mid [t_{mn}, t_{mx}]\right) \\
&= \int_{pWL_{mn}}^{pWL_{mx}} CDF_{P,SL}\left(m + p \mid [t_{mn}, t_{mx}]\right) \mathrm{d}CDF_{P,WL}\left(p \mid [t_{mn}, t_{mx}]\right) \\
&= \int_{pWL_{mn}}^{pWL_{mx}} CDF_{P,SL}\left(m + p \mid [t_{mn}, t_{mx}]\right) \left[\mathrm{d}CDF_{P,WL}\left(p \mid [t_{mn}, t_{mx}]\right) / \mathrm{d}p\right] \mathrm{d}p
\end{aligned}
\tag{6.1}
$$

with $p_i, i = 0, 1, \cdots, n,$ a subdivision of $[pWL_{mn}, pWL_{mx}]$. However, for this margin to be meaningful, the WLs and SLs must be defined on the basis of the same type of system property and with use of the same units (e.g., temperature in degrees Kelvin).

At this point, the analysis is conceptually the same as the analysis in Sect. 4.3 for failure time margins. However, the challenge is to determine the CDFs $CDF_{P,WL}(p \mid [t_{mn}, t_{mx}])$ and $CDF_{P,SL}(p \mid [t_{mn}, t_{mx}])$ in Eq. (6.1). If only one WL (i.e., nWL = 1) and one SL (i.e., nSL = 1) are involved, then the desired CDFs can be determined as indicated in conjunction with Eqs. (3.4) and (3.5). However, the situation is more complex for nWL > 1 and/or nSL > 1. Integral-based and sampling-based procedures to determine $CDF_{P,WL}(p \mid [t_{mn}, t_{mx}])$ and $CDF_{P,SL}(p \mid [t_{mn}, t_{mx}])$ for nWL > 1 and nSL > 1 are described in Sects. 5.1 and 5.2. If $CDF_{P,WL,i}(p \mid [t_{mn}, t_{mx}])$ and $CDF_{P,SL,i}(p \mid [t_{mn}, t_{mx}])$ can be determined for each of the four failure patterns in Eqs. (4.4)-(4.7) and Table 2, then the corresponding margin CDFs $CDF_{PM,i}(m\,|\,[t_{mn},t_{mx}])$ can be obtained by evaluating the integral

$$\begin{aligned} CDF_{PM,i}(m\,|\,[t_{mn},t_{mx}]) &= \int_{pWL_{mn,i}}^{pWL_{mx,i}} CDF_{P,SL,i}\left(m+p\,|\,[t_{mn},t_{mx}]\right)\mathrm{d}CDF_{P,WL,i}\left(p\,|\,[t_{mn},t_{mx}]\right) \\ &= \int_{pWL_{mn,i}}^{pWL_{mx,i}} CDF_{P,SL,i}\left(m+p\,|\,[t_{mn},t_{mx}]\right)\left[\mathrm{d}CDF_{P,WL,i}\left(p\,|\,[t_{mn},t_{mx}]\right)/\,\mathrm{d}p\right]\mathrm{d}p, \end{aligned} \tag{6.2}$$

which is the integral in Eq. (6.1) with $CDF_{P,WL,i}(p\,|\,[t_{mn},t_{mx}])$ and $CDF_{P,SL,i}(p\,|\,[t_{mn},t_{mx}])$ replacing $CDF_{P,WL}(p\,|\,[t_{mn},t_{mx}])$ and $CDF_{P,SL}(p\,|\,[t_{mn},t_{mx}])$.

As an example, property margin CDFs $CDF_{PM,i}(m\,|\,[t_{mn},t_{mx}])$, i = 1, 2, 3, 4, for systems with two WLs and two SLs links defined with the WLs and SLs described and illustrated in Table 1 and Fig. 1 are presented in Fig. 10. The associated CDFs $CDF_{P,SL,i}(p\,|\,[t_{mn},t_{mx}])$ and $CDF_{P,WL,i}(p\,|\,[t_{mn},t_{mx}])$ in Eq. (6.2) are defined and illustrated in conjunction with Fig. 6. Quadrature-based evaluations of the defining integrals for $CDF_{PM,i}(m\,|\,[t_{mn},t_{mx}])$ in Eq. (6.2) were performed with the MATLAB numerical package [57] as summarized below in a manner similar to that described in Sect. 4.2 for time margins. Specifically, (i) $\mathrm{d}CDF_{P,WL,i}(p\,|\,[t_{mn},t_{mx}])/\,\mathrm{d}p$ was initially approximated with use of the **diff** function and then represented as $ftderCDF_{P,WL,i}(p)$ with use of the **fit** function and the 'gauss1' model option, (ii) $CDF_{P,SL,i}(p\,|\,[t_{mn},t_{mx}])$ was represented as $ftCDF_{P,SL,i}(p)$ with use of the **fit** function and the 'gauss1' model option, and (iii) the integral of $ftCDF_{P,SL,i}(p)\times ftderCDF_{P,WL,i}(p)$ over $[pWL_{mn,i},pWL_{mx,i}]$ was evaluated with the **integral** function.

6.2. Sampling-based Estimation of $CDF_{MP}(m\,|\,[t_{mn},t_{mx}])$ Based on WL and SL Property Values

The CDFs $CDF_{P,WL,i}(p\,|\,[t_{mn},t_{mx}])$ and $CDF_{P,SL,i}(p\,|\,[t_{mn},t_{mx}])$ can also be used in a sampling-based procedure to estimate $CDF_{PM,i}(m\,|\,[t_{mn},t_{mx}])$ that is analogous to the sampling-based procedure in Eq. (4.17) to estimate $CDF_{TM,i}(m\,|\,[t_{mn},t_{mx}])$. Specifically,

$$prob_i\left(p_{SL}-p_{WL}\le m\,|\,[t_{mn},t_{mx}]\right)\cong\sum_{r=1}^{n}\delta_m\left(p_{SL,r}-p_{WL,r}\right)\Big/n \tag{6.3}$$

for (i) a random sample $[p_{SL,r},p_{WL,r}]$, r = 1,2, …, n, from $[pSL_{mn,i},pSL_{mx,i}]\times[pWL_{mn,i},pWL_{mx,i}]$ obtained with use of $CDF_{P,SL,i}(p\,|\,[t_{mn},t_{mx}])$ and $CDF_{P,WL,i}(p\,|\,[t_{mn},t_{mx}])$, and (ii) $\delta_m(\sim)$ defined as in Eq. (4.16).

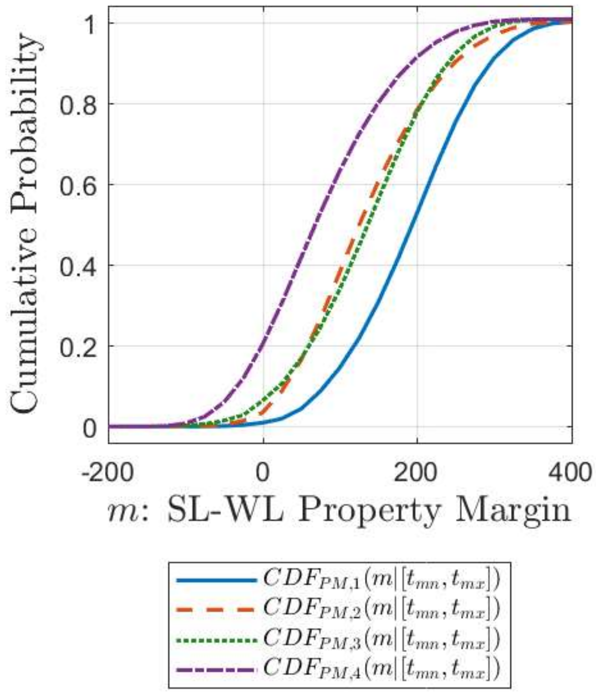

Fig. 10 Property margin CDFs $CDF_{PM,i}(m\,|\,[t_{mn},t_{mx}])$ for (i) failure pattern i, $i = 1, 2, 3, 4$, resulting in LOAS as indicated in Table 2, (ii) the four links defined and illustrated in Table 1 and Fig. 1 with $[t_{mn},t_{mx}] = [0, 200]$, and (iii) quadrature-based evaluation of the integrals in Eq. (6.2).

However, given the complexity of $CDF_{P,WL,i}(p\,|\,[t_{mn},t_{mx}])$ and $CDF_{P,SL,i}(p\,|\,[t_{mn},t_{mx}])$, their direct use in the determination of the distribution for the margin m defined by $p_{SL} - p_{WL}$ may not be computationally convenient. Instead, a procedure based on sampling the defining parameters for the individual links (i.e., the α's and β's) and then determining the resultant margins may be more practicable. Specifically, a sampling-based procedure similar to the procedure described in conjunction with Eqs. (4.21)-(4.25) to determine $prob_i(t_{SL} - t_{WL} \le m)$ can also be used to determine $prob_i(p_{SL} - p_{WL} \le m\,|\,[t_{mn},t_{mx}])$ for the margin m defined by $p_{SL} - p_{WL}$ with i indicating Patterns 1-4 described in Eqs. (4.4)-(4.7) and correspondingly the four WL/SL failure patterns in Table 2.

The indicated procedure to determine $prob_i(p_{SL} - p_{WL} \le m\,|\,[t_{mn},t_{mx}])$ begins with (i) a sample $\mathbf{s}_r, r = 1, 2, \cdots, n$ of the form indicated in Eqs. (4.21)-(4.23), (ii) the associated link failure times

$$tWL_{lr},\ l = 1, 2, \cdots, nWL, \text{ and } tSL_{lr},\ l = 1, 2, \cdots, nSL, \tag{6.4}$$

defined in Eqs. (4.24)-(4.25), and (iii) the resultant link system failure times

$$\overline{tSL}_{k(r),r}, \underline{tSL}_{k(r),r}, \overline{tWL}_{l(r),r}, \underline{tWL}_{l(r),r} \tag{6.5}$$

defined in Eq. (4.19). Once the link system failure times are available, the desired property margins are given by

$$m_i(\mathbf{s}_r) = \begin{cases} \alpha_{SL,k(r),r}\,\bar{p}_{SL,k(r)}\left(\overline{tSL}_{k(r),r}\right) - \alpha_{WL,l(r),r}\,\bar{p}_{WL,l(r)}\left(\underline{tWL}_{l(r),r}\right) & \text{for } i=1 \\ \alpha_{SL,k(r),r}\,\bar{p}_{SL,k(r)}\left(\underline{tSL}_{k(r),r}\right) - \alpha_{WL,l(r),r}\,\bar{p}_{WL,l(r)}\left(\underline{tWL}_{l(r),r}\right) & \text{for } i=2 \\ \alpha_{SL,k(r),r}\,\bar{p}_{SL,k(r)}\left(\overline{tSL}_{k(r),r}\right) - \alpha_{WL,l(r),r}\,\bar{p}_{WL,l(r)}\left(\overline{tWL}_{l(r),r}\right) & \text{for } i=3 \\ \alpha_{SL,k(r),r}\,\bar{p}_{SL,k(r)}\left(\underline{tSL}_{k(r),r}\right) - \alpha_{WL,l(r),r}\,\bar{p}_{WL,l(r)}\left(\overline{tWL}_{l(r),r}\right) & \text{for } i=4 \end{cases} \tag{6.6}$$

and the resultant approximation to $prob_i(p_{SL} - p_{WL} \le m \mid [t_{mn}, t_{mx}])$ is obtained as in Eq. (4.20) for $prob_i(t_{SL} - t_{WL} \le m)$.

6.3. Verification Results for $CDF_{PM,i}(p \mid [t_{mn}, t_{mx}])$

As a verification test, Fig. 11 presents a comparison of CDFs $CDF_{PM,i}(p \mid [t_{mn}, t_{mx}])$, $i =$ 1, 2, 3, 4, for property value margins for a system with two WLs and two SLs obtained by (i) quadrature-based evaluation of the integrals in Eq. (6.2) as described in Sect. 6.1 and (ii) a sampling-based calculation as described in Sect. 6.2 and performed with the CPLOAS program [55; 56] and a sample of size $n = 10^6$. The CDFs $CDF_{PM,i}(p \mid [t_{mn}, t_{mx}])$, $i = 1, 2, 3, 4$, are not defined outputs of the CPLOAS program but (i) these CDFs are calculated as indicated in Sect. 6.2 as part of the SL-SL margin analysis described in Sect. 7.3 and (ii) were obtained for the present analysis by adding a few lines of code to CPLOAS to write them to a saved output file. The results obtained with the two evaluation procedures are essentially identical. This level of agreement provides a strong verification that the two procedures for obtaining $CDF_{PM,i}(p \mid [t_{mn}, t_{mx}])$ are correct in both (i) mathematical development and (ii) computational implementation.

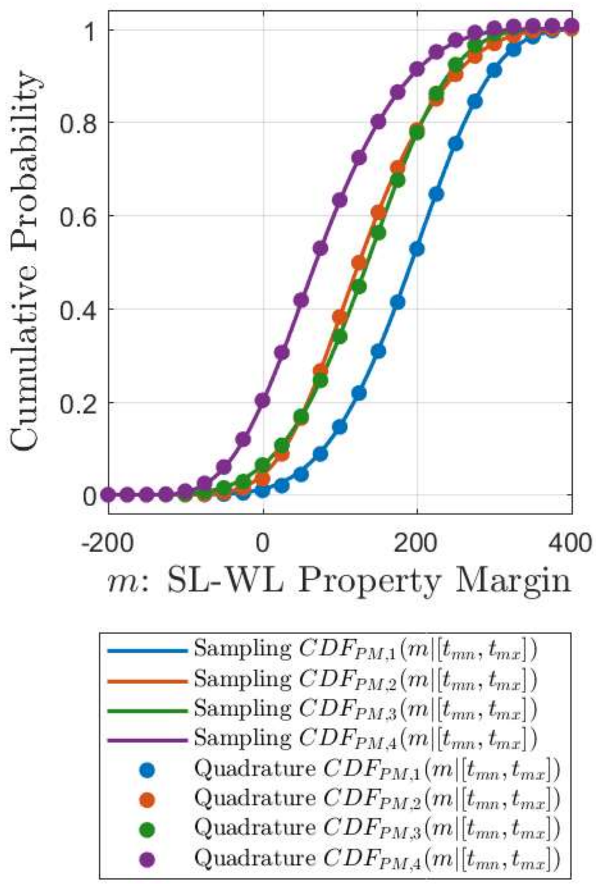

Fig. 11 Verification results for property value margin CDFs $CDF_{PM,i}(p\,|\,[t_{mn}, t_{mx}])$, i = 1, 2, 3, 4, obtained with quadrature-based procedures (see Sect. 6.1) and sampling-based procedures (see Sect. 6.2) for (i) the four failure patterns resulting in LOAS indicated in Table 2 and (ii) the four links defined and illustrated in Table 1 and Fig. 1 with $[t_{mn}, t_{mx}] = [0, 200]$.

7. CDFs for Margins Involving Only SL Property Values

7.1. Preliminaries: CDFs for Margins Involving Only SL Property Values

Margins involving only SL property values for a WL/SL configuration involving *nWL* WLs and *nSL* SLs are now considered. For the situation in which WL failures potentially deactivate the system at time t_{WL} and SL failures potentially result in LOAS at time t_{SL}, the margin of interest is defined to be the difference

$$p_k(t_{SL} \mid \alpha) - p_k(t_{WL} \mid \alpha) = \alpha \bar{p}_k(t_{SL}) - \alpha \bar{p}_k(t_{WL}) = m_k\left(\alpha, t_{SL}, t_{WL}\right) \tag{7.1}$$

between the property value $p_k(t_{SL} \mid \alpha) = \alpha \bar{p}_k(t_{SL})$ of the SL (i.e., SL *k*) whose time of failure corresponds to t_{SL} and the property value $p_k(t_{WL} \mid \alpha) = \alpha \bar{p}_k(t_{WL})$ of this SL at the time that the WL failure potentially deactivates the system (i.e., t_{WL}). The following entities are used in the derivation of the CDF $CDF_{PM,SL}(m \mid [t_{mn}, t_{mx}])$ for the SL property value margin *m* defined in Eq. (7.1) for link failures in the time interval $[t_{mn}, t_{mx}]$:

$CDF_{T,WL}(t) =$ CDF defined on $[t_{mn}, t_{mx}]$ for time at which WL failure potentially averts LOAS (i.e., at last WL failure or first WL failure as defined in Eqs. (4.1) and (4.2)), (7.2)

$CDF_{T,SL,k}(t) =$ CDF defined on $[t_{mn}, t_{mx}]$ for time at which SL k fails with corresponding density function $d_{T,SL,k}(t)$, (7.3)

and

$CDF_{Ak}(\alpha \mid t_F) =$ CDF for α values for SL k that could result in failure of SL k at time t_F conditional on SL k having failed at time t_F with corresponding density function $d_{Ak}(\alpha \mid t_F)$. (7.4)

The CDF $CDF_{T,WL}(t)$ is defined in Eq. (4.1) or (4.2) as appropriate. A derivation for $CDF_{Ak}(\alpha \mid t_F)$ is presented in Sect. 9.4 of Ref. [8], and the resultant values for $CDF_{Ak}(\alpha \mid t_F)$ and $d_{Ak}(\alpha \mid t_F)$ are summarized in Table 5 of Ref. [8]. Further, the ranges of α values associated with $CDF_{Ak}(\alpha \mid t_F)$ and $d_{Ak}(\alpha \mid t_F)$ are summarized in Tables 4 and 5 of Ref. [8]and reproduced in this presentation as Table 5.

Table 5 Intervals $[\alpha_{mn}(\tau), \alpha_{mx}(\tau)]$ of values for α resulting in link failure conditional on link failure at time τ with $\bar{p}(\tau)$ increasing and $\bar{q}(\tau)$ either decreasing or constant-valued (adapted from Ref. [8], Tables 4 and 5).

$[\alpha_{mn}(\tau), \alpha_{mx}(\tau)] =$ interval of α values resulting in link failure conditional on link failure at time τ for $\alpha_{mn}/\beta_{mn} < \alpha_{mx}/\beta_{mx}$

$$= \begin{cases} [\beta_{mn} r(\tau), \alpha_{mx}] \text{ for } \tau \in \mathcal{P}_1 = \{\tau : \tau_f = r^{-1}(\alpha_{mx}/\beta_{mn}) \le \tau \le r^{-1}(\alpha_{mx}/\beta_{mx}) = \tau_{mx}\} \\ [\beta_{mn} r(\tau), \beta_{mx} r(\tau)] \text{ for } \tau \in \mathcal{P}_2 = \{\tau : \tau_{mx} = r^{-1}(\alpha_{mx}/\beta_{mx}) \le \tau \le r^{-1}(\alpha_{mn}/\beta_{mn}) = \tau_{mn}\} \\ [\alpha_{mn}, \beta_{mx} r(\tau)] \text{ for } \tau \in \mathcal{P}_4 = \{\tau : \tau_{mn} = r^{-1}(\alpha_{mn}/\beta_{mn}) \le \tau \le r^{-1}(\alpha_{mn}/\beta_{mx}) = \tau_l\} \end{cases}$$

$[\alpha_{mn}(\tau), \alpha_{mx}(\tau)] =$ interval of α values resulting in link failure conditional on link failure at time τ for $\alpha_{mx}/\beta_{mx} < \alpha_{mn}/\beta_{mn}$

$$= \begin{cases} [\beta_{mn} r(\tau), \alpha_{mx}] \text{ for } \tau \in \mathcal{P}_1 = \{\tau : \tau_f = r^{-1}(\alpha_{mx}/\beta_{mn}) \le \tau \le r^{-1}(\alpha_{mn}/\beta_{mn}) = \tau_{mn}\} \\ [\alpha_{mn}, \alpha_{mx}] \text{ for } \tau \in \mathcal{P}_3 = \{\tau : \tau_{mn} = r^{-1}(\alpha_{mn}/\beta_{mn}) \le \tau \le r^{-1}(\alpha_{mx}/\beta_{mx}) = \tau_{mx}\} \\ [\alpha_{mn}, \beta_{mx} r(\tau)] \text{ for } \tau \in \mathcal{P}_4 = \{\tau : \tau_{mx} = r^{-1}(\alpha_{mx}/\beta_{mx}) \le \tau \le r^{-1}(\alpha_{mn}/\beta_{mx}) = \tau_l\} \end{cases}$$

$[\alpha_{mn}(\tau), \alpha_{mx}(\tau)] =$ interval of α values resulting in link failure conditional on link failure at time τ for $\alpha_{mn}/\beta_{mn} = \alpha_{mx}/\beta_{mx}$

$$= \begin{cases} [\beta_{mn} r(\tau), \alpha_{mx}] \text{ for } \tau \in \mathcal{P}_1 = \{\tau : \tau_f = r^{-1}(\alpha_{mx}/\beta_{mn}) \le \tau \le r^{-1}(\alpha_{mx}/\beta_{mx}) = \tau_{mx}\} \\ [\alpha_{mn}, \beta_{mx} r(\tau)] \text{ for } \tau \in \mathcal{P}_4 = \{\tau : \tau_{mx} = r^{-1}(\alpha_{mx}/\beta_{mx}) \le \tau \le r^{-1}(\alpha_{mn}/\beta_{mx}) = \tau_l\} \end{cases}$$

7.2. Formal Representation: CDFs for Margins Involving Only SL Property Values

The integral representations for $CDF_{PM,SL}(m \mid [t_{mn}, t_{mx}])$ are complex. As a result of α, t_{SL} and t_{WL} each having a distribution, the integral representations for $CDF_{PM,SL}(m \mid [t_{mn}, t_{mx}])$ will involve triple integrals. As indicated in Table 2, definition of LOAS involves two possible definitions of WL system failure and two possible definitions of SL system failure, with the two possible definitions for each type of link corresponding to (i) time of first link failure and (ii) time of last link failure. In the following, the two possible definitions of WL failure are assumed to have been appropriately incorporated into the definition of $CDF_{T,WL}(t)$ as indicated in Eqs. (4.1) and (4.2). The effects of the two link failure definitions for SLs are introduced into the representations for $CDF_{PM,SL}(m \mid [t_{mn}, t_{mx}])$ through the notation

$$P_k(\tau) = \prod_{l=1, l \ne k}^{nSL} \left[1 - CDF_{T,SL,l}(\tau)\right] \text{ for LOAS associated with first SL failure} \qquad (7.5)$$

and

$$P_k(\tau) = \prod_{l=1, l\neq k}^{nSL} CDF_{T,SL,l}(\tau) \text{ for LOAS associated with last SL failure.} \tag{7.6}$$

In addition, (i) possible values $m < 0$, $m = 0$, and $0 < m$ are considered for m in the definition of $CDF_{PM,SL}(m\,|\,[t_{mn}, t_{mx}])$, (ii) the notation

$$d_k(\tau, t) = \bar{p}_k(\tau) - \bar{p}_k(t) \tag{7.7}$$

is used to simplify some of the following expressions, and (iii) a subdivision $t_i, i = 0, 1, \cdots, n,$ of the time interval $[t_{mn}, t_{mx}]$ with

$$\begin{aligned} t_{\underline{n}(k)} &= \text{ first time at which SL } k \text{ fails} \\ &= r_k^{-1}(\alpha_{mx} / \beta_{mn}) \\ &= \tau_{fk} \end{aligned} \tag{7.8}$$

and

$$\begin{aligned} t_{\bar{n}(k)} &= \text{ last time at which SL } k \text{ fails} \\ &= \begin{cases} r_k^{-1}(\alpha_{mn} / \beta_{mx}) & \text{if SL } k \text{ always fails in time interval } [t_{mn}, t_{mx}] \\ t_{mx} & \text{if SL } k \text{ may not fail in time interval } [t_{mn}, t_{mx}] \end{cases} \\ &= \tau_{lk} \end{aligned} \tag{7.9}$$

is under consideration.

For $m < 0$, $CDF_{PM,SL}(m\,|\,[t_{mn}, t_{mx}])$ can be approximated by

$$\begin{aligned} &CDF_{PM,SL}(m\,|\,[t_{mn}, t_{mx}] \text{ with } m < 0) \\ &= \lim_{n\to\infty} \sum_{k=1}^{nSL} \sum_{i=\underline{n}(k)+2}^{n} \sum_{j=\underline{n}(k)+1}^{\min\{i-1, \bar{n}(k)\}} \left[prob_{Ak}\left(\alpha : \alpha\bar{p}_k(t_j) - \alpha\bar{p}_k(t_i) \le m\right)\right] \\ &\qquad \times \left[P_k(t_j) \times \Delta CDF_{T,SL,k}(t_j)\right] \Delta CDF_{T,WL}(t_i) \\ &= \lim_{n\to\infty} \sum_{k=1}^{nSL} \sum_{i=\underline{n}(k)+2}^{n} \sum_{j=\underline{n}(k)+1}^{\min\{i-1, \bar{n}(k)\}} \left[prob_{Ak}\left(\alpha : \alpha\bar{p}_k(t_j) - \alpha\bar{p}_k(t_i) \le m\right)\right] \\ &\qquad \times \left[P_k(t_j) \times d_{T,SL,k}(t_j)\Delta t_j\right] \Delta CDF_{T,WL}(t_i), \end{aligned} \tag{7.10}$$

with

$prob_{Ak}\left(\alpha : \alpha\overline{p}_k(t_j) - \alpha\overline{p}_k(t_i) \le m\right)$ = probability of a negative SL time margin $\le m$ conditional on (i) failure of SL k at time t_j resulting in LOAS and (ii) failure of the WL system at time $t_i > t_j$, (7.11)

$$\Delta CDF_{T,SL,k}(t_j) = \text{ probability that SL } k \text{ fails in time interval } \Delta t_j, \tag{7.12}$$

$$\begin{aligned} P_k(t_j) \times \Delta CDF_{t,SL,k}(t_j) &\cong P_k(t_j) \times d_{T,SL,k}(t_j)\Delta t_j \\ &\cong \text{probability that failure of SL } k \text{ in time interval } \Delta t_j \\ &\quad \text{could result in LOAS,} \end{aligned} \tag{7.13}$$

$$\Delta CDF_{T,WL}(t_i) = \text{ probability that WL system fails in time interval } \Delta t_i, \tag{7.14}$$

and the indicated offsets involving i and j defined to produce negative margins and also to avoid the potential for a later division by zero. Part of the preceding approximation to $CDF_{PM,SL}(m \mid [t_{mn}, t_{mx}]$ with $m < 0)$ can be simplified as follows:

$$\begin{aligned} &prob_{Ak}\left(\alpha : \alpha\overline{p}_k(t_j) - \alpha\overline{p}_k(t_i) \le m\right) d_{T,SL,k}(t_j) \\ &= prob_{Ak}\left(\alpha : \alpha d_k(t_j, t_i) \le m\right) d_{T,SL,k}(t_j) \\ &= prob_{Ak}\left(\alpha : m / d_k(t_j, t_i) \le \alpha\right) d_{T,SL,k}(t_j) \\ &= CCDF_{Ak}(m / d_k(t_j, t_i) \mid t_j) d_{T,SL,k}(t_j) \\ &= \int_{m/d_k(t_j,t_i)}^{\alpha_{mx,k}(t_j)} d_{Ak}(\alpha \mid t_j) d_{T,SL,k}(t_j) \mathrm{d}\alpha \\ &= \left[\mathrm{d}\left[\overline{p}_k(\tau) / \overline{q}_k(\tau)\right] / \mathrm{d}\tau\right]_{\tau = t_j} \left[\int_{\alpha_{mn,k}(t_j,t_i)}^{\alpha_{mx,k}(t_j)} \alpha d_{Ak}(\alpha) d_{Bk}\left(\alpha / r_k(t_j)\right) \mathrm{d}\alpha\right], \end{aligned} \tag{7.15}$$

where (i) the equality

$$d_{Ak}(\alpha \mid t_j) d_{T,SL,k}(t_j) = \left[\mathrm{d}\left[\overline{p}_k(\tau) / \overline{q}_k(\tau)\right] / \mathrm{d}\tau\right]_{\tau = t_j} \alpha d_{Ak}(\alpha) d_{Bk}\left(\alpha / r_k(t_j)\right) \tag{7.16}$$

is derived in Sect. 9.6 of Ref. [8], (ii)

$$\alpha_{mn,k}(t_j, t_i) = \max\left\{\alpha_{mn,k}(t_j), m / d_k(t_j, t_i)\right\} \text{ and } \alpha_{mx,k}(t_j) \tag{7.17}$$

together define the intervals over which the associated integral is nonzero provided the inequality

$$m / d_k(t_j, t_i) < \alpha_{mx,k}(t_j) \tag{7.18}$$

is valid, and (iii) the interval $[\alpha_{mn,k}(t_j),\alpha_{mx,k}(t_j)]$ for each SL is defined in Table 5. Additional details on the definitions of the intervals $[\alpha_{mn,k}(\tau,t),\alpha_{mx,k}(\tau)]$ are given in Eqs. (7.21) and (7.22).

In turn, substitution of the result in Eq. (7.15) into the representation for $CDF_{PM}(m\,|\,[t_{mn},t_{mx}]$ with $m<0)$ in Eq. (7.10) produces the result

$$
\begin{aligned}
&CDF_{PM,SL}(m\,|\,[t_{mn},t_{mx}]\text{ with } m<0)\\
&\quad=\lim_{n\to\infty}\sum_{k=1}^{nSL}\sum_{i=\underline{n}(k)+2}^{n}\sum_{j=\underline{n}(k)+1}^{\min\{i-1,\overline{n}(k)\}} P_k(t_j)\left[\mathrm{d}\left[\overline{p}_k(\tau)/\overline{q}_k(\tau)\right]/\mathrm{d}\tau\right]_{\tau=t_j}\\
&\qquad\times\left[\int_{\alpha_{mn,k}(t_j,t_i)}^{\alpha_{mx,k}(t_j)}\alpha d_{Ak}(\alpha)d_{Bk}\left(\alpha/r_k(t_j)\right)\mathrm{d}\alpha\right]\Delta\tau_j\Delta CDF_{T,WL}(t_i)\\
&\quad=\sum_{k=1}^{nSL}\int_{\tau_{fk}}^{t_{mx}}\left\{\int_{\tau_{fk}}^{\min\{t,\tau_{lk}\}}\Big(P_k(\tau)\left[\mathrm{d}\left[\overline{p}_k(\tau)/\overline{q}_k(\tau)\right]/\mathrm{d}\tau\right]\right.\\
&\qquad\left.\times\left[\int_{\alpha_{mn,k}(\tau,t)}^{\alpha_{mx,k}(\tau)}\alpha d_{Ak}(\alpha)d_{Bk}\left(\alpha/r_k(\tau)\right)\mathrm{d}\alpha\right]\Big)\mathrm{d}\tau\right\}\mathrm{d}CDF_{T,WL}(t)\\
&\quad=\sum_{k=1}^{nSL}\int_{\tau_{fk}}^{t_{mx}}\int_{\tau_{fk}}^{\min\{t,\tau_{lk}\}}\int_{\alpha_{mn,k}(\tau,t)}^{\alpha_{mx,k}(\tau)} f_k(\alpha,\tau,t)\,\mathrm{d}\alpha\,\mathrm{d}\tau\,\mathrm{d}t
\end{aligned}
\tag{7.19}
$$

with

$$
f_k(\alpha,\tau,t)=\left[\alpha d_{Ak}(\alpha)d_{Bk}\left(\alpha/r_k(\tau)\right)\right]P_k(\tau)\left[\mathrm{d}\left[\overline{p}_k(\tau)/\overline{q}_k(\tau)\right]/\mathrm{d}\tau\right]\left[\mathrm{d}CDF_{T,WL}(t)/\mathrm{d}t\right]
\tag{7.20}
$$

and the second equality resulting in the limit as $n\to\infty$.

With use of the definitions of the intervals $[\alpha_{mn}(\tau),\alpha_{mx}(\tau)]$ in Table 5, it follows that the integration limits for the inner integral over α in Eq. (7.19) are given by

$$
[\alpha_{mn,k}(\tau,t),\alpha_{mx,k}(\tau)]=\begin{cases}
[\max\{\beta_{mn,k}r_k(\tau),m/d_k(\tau,t)\},\alpha_{mx,k}] & \text{for } \tau\in\mathcal{P}_1\\
[\max\{\beta_{mn,k}r_k(\tau),m/d_k(\tau,t)\},\beta_{mx,k}r_k(\tau)] & \text{for } \tau\in\mathcal{P}_2\\
[\max\{\alpha_{mn,k},m/d_k(\tau,t)\},\alpha_{mx,k}] & \text{for } \tau\in\mathcal{P}_3\\
[\max\{\alpha_{mn,k},m/d_k(\tau,t)\},\beta_{mx,k}r_k(\tau)] & \text{for } \tau\in\mathcal{P}_4
\end{cases}
\tag{7.21}
$$

with the definitions of the sets $\mathcal{P}_i,\ i=1,2,3,4,$ conditional on the relationships

$$
\alpha_{mn}/\beta_{mn}<\alpha_{mx}/\beta_{mx},\ \alpha_{mx}/\beta_{mx}<\alpha_{mn}/\beta_{mn}\ \text{ and }\ \alpha_{mn}/\beta_{mn}=\alpha_{mx}/\beta_{mx}.
\tag{7.22}
$$

Further, for

$$\alpha_{mx,k}(\tau) \le m / d_k(\tau,t), \tag{7.23}$$

the integral over α in Eq. (7.19) is equal to 0.

Because the SL property value functions $\bar{p}_k(\tau)$ are increasing, a SL temperature margin $m \le 0$ can only occur if failure of the SL system occurs before or at the same time as failure of the WL system. As a result, $CDF_{PM,SL}(0 \mid [t_{mn}, t_{mx}])$ is defined by

$$\begin{aligned}
CDF_{PM,SL}(0 \mid [t_{mn}, t_{mx}]) &= \lim_{n\to\infty} \sum_{k=1}^{nSL} \sum_{i=\underline{n}(k)+2}^{n} \sum_{j=\underline{n}(k)+1}^{\min\{i-1,\bar{n}(k)\}} \left[P_k(t_j) \times \Delta CDF_{T,SL,k}(t_j) \right] \Delta CDF_{T,WL}(t_i) \\
&= \sum_{k=1}^{nSL} \int_{\tau_{fk}}^{t_{mx}} \left[\int_{\tau_{fk}}^{\min\{t,\tau_{lk}\}} P_k(\tau) \mathrm{d}CDF_{T,SL,k}(\tau) \right] \mathrm{d}CDF_{T,WL}(t) \\
&= \sum_{k=1}^{nSL} \int_{\tau_{fk}}^{t_{mx}} \int_{\tau_{fk}}^{\min\{t,\tau_{lk}\}} g_k(\tau,t) \mathrm{d}\tau \mathrm{d}t
\end{aligned} \tag{7.24}$$

with (i) the second equality resulting in the limit as $n \to \infty$ and (ii)

$$g_k(\tau,t) = P_k(\tau) \left[\mathrm{d}CDF_{T,SL,k}(\tau) / \mathrm{d}\tau \right] \left[\mathrm{d}CDF_{T,WL}(t) / \mathrm{d}t \right] \tag{7.25}$$

in the third equality. The preceding representation for $CDF_{PM,SL}(0 \mid [t_{mn}, t_{mx}])$ is the same as the representation for $CDF_{PM,SL}(m \mid [t_{mn}, t_{mx}] \text{ with } m < 0)$ in Eq. (7.10) with $prob_{Ak}(\cdots)$ replaced by 1.0.

For $m > 0$, $CDF_{PM,SL}(m \mid [t_{mn}, t_{mx}])$ can be approximated by

$$\begin{aligned}
&CDF_{PM,SL}(m \mid [t_{mn}, t_{mx}] \text{ with } 0 < m) \\
&= CDF_{PM}(0 \mid [t_{mn}, t_{mx}]) + \lim_{n\to\infty} \sum_{k=1}^{nSL} \sum_{i=1}^{\bar{n}(k)-1} \sum_{j=\max\{i+1,\underline{n}(k)\}}^{\bar{n}(k)} \left[prob_{Ak}\left(\alpha : \alpha \bar{p}_k(t_j) - \alpha \bar{p}_k(t_i) \le m \right) \right] \\
&\qquad \times \left[P_k(t_j) \times \Delta CDF_{T,SL,k}(t_j) \right] \Delta CDF_{T,WL}(t_i) \\
&= CDF_{PM}(0 \mid [t_{mn}, t_{mx}]) + \lim_{n\to\infty} \sum_{k=1}^{nSL} \sum_{i=1}^{\bar{n}(k)-1} \sum_{j=\max\{i+1,\underline{n}(k)\}}^{\bar{n}(k)} \left[prob_{Ak}\left(\alpha : \alpha \bar{p}_k(t_j) - \alpha \bar{p}_k(t_i) \le m \right) \right] \\
&\qquad \times \left[P_k(t_j) \times d_{T,SL,k}(t_j) \Delta t_j \right] \Delta CDF_{T,WL}(t_i)
\end{aligned} \tag{7.26}$$

with (i)

$$prob_{Ak}\left(\alpha : \alpha \overline{p}_k(t_j) - \alpha \overline{p}_k(t_i) \le m\right) = \text{probability of a positive SL time margin } \le m \text{ conditional on (i) failure of SL } k \text{ at time } t_j \text{ corresponding to LOAS and (ii) failure of the WL system at time } t_i < t_j, \tag{7.27}$$

(ii) $\Delta CDF_{T,SL,k}(t_j)$, $P_k(t_j) \times \Delta CDF_{T,SL,k}(t_j)$, $P_k(t_j) \times d_{T,SL,k}(t_j)\Delta t_j$ and $\Delta CDF_{T,WL}(t_i)$ defined in Eqs. (7.12)-(7.14), and (iii) the indicated offsets involving i and j defined to produce positive margins and also to avoid the potential for a later division by zero. Similarly to the development in Eq.(7.15), part of the preceding approximation to $CDF_{PM}(m \,|\, 0 < m)$ can be simplified as follows:

$$\begin{aligned}
&prob_{Ak}\left(\alpha : \alpha \overline{p}_k(t_j) - \alpha \overline{p}_k(t_i) \le m\right) d_{Tk}(t_j) \\
&= prob_{Ak}\left(\alpha : \alpha d_k(t_j, t_i) \le m\right) d_{Tk}(t_j) \\
&= prob_{Ak}\left(\alpha : \alpha \le m / d_k(t_j, t_i)\right) d_{Tk}(t_j) \\
&= CDF_{Ak}(m / d_k(t_j, t_i) \,|\, t_j) d_{Tk}(t_j) \\
&= \int_{\alpha_{mn,k}(t_j)}^{m/d_k(t_j,t_i)} d_{Ak}(\alpha \,|\, t_j) d_{Tk}(t_j) \mathrm{d}\alpha \\
&= \left[\mathrm{d}\left[\overline{p}_k(\tau) / \overline{q}_k(\tau)\right] / \mathrm{d}\tau\right]_{\tau = t_j} \left[\int_{\alpha_{mn,k}(t_j)}^{\alpha_{mx,k}(t_j,t_i)} \alpha d_{Ak}(\alpha) d_{Bk}\left(\alpha / r_k(t_j)\right) \mathrm{d}\alpha\right],
\end{aligned} \tag{7.28}$$

where (i) the substitution for $d_{Ak}(\alpha \,|\, t_j) d_{Tk}(t_j)$ is defined in Eq. (7.16), (ii)

$$\alpha_{mn,k}(t_j) \text{ and } \alpha_{mx,k}(t_j, t_i) = \min\left\{\alpha_{mx,k}(t_j), m / d_k(t_j, t_i)\right\} \tag{7.29}$$

together define the intervals over which the associated integral is nonzero provided the inequality

$$\alpha_{mn,k}(t_j) < m / d_k(t_j, t_i) \tag{7.30}$$

is valid, and (iii) the interval $[\alpha_{mn,k}(t_j), \alpha_{mx,k}(t_j)]$ for each SL is defined in Table 5. Additional details on the definitions of the intervals $[\alpha_{mn,k}(\tau, t), \alpha_{mx,k}(\tau)]$ are given in Eqs. (7.32) and (7.22).

In turn, substitution of the result in Eq. (7.28) into the representation for $CDF_{PM,SL}(m \,|\, [t_{mn}, t_{mx}] \text{ with } m < 0)$ in Eq. (7.26) produces the result

$$
\begin{aligned}
&CDF_{PM,SL}(m \mid [t_{mn}, t_{mx}] \text{ with } 0 < m) \\
&\quad = CDF_{PM,SL}(0 \mid [t_{mn}, t_{mx}]) + \lim_{n\to\infty} \sum_{k=1}^{nSL} \sum_{i=1}^{\bar{n}(k)-1} \sum_{j=\max\{i+1,\underline{n}(k)\}}^{\bar{n}(k)} P_k(t_j)\left[\mathrm{d}\left[\bar{p}_k(\tau)/\bar{q}_k(\tau)\right]/\mathrm{d}\tau\right]_{\tau=t_j} \\
&\qquad \times \left[\int_{\alpha_{mn,k}(t_j)}^{\alpha_{mx,k}(t_j,t_i)} \alpha d_{Ak}(\alpha) d_{Bk}\left(\alpha/r_k(t_j)\right)\mathrm{d}\alpha\right]\Delta t_j \Delta CDF_{T,WL}(t_i) \\
&\quad = CDF_{PM,SL}(0 \mid [t_{mn}, t_{mx}]) + \sum_{k=1}^{nSL} \int_{t_{mn}}^{\tau_{lk}} \left\{\int_{\max\{t,\tau_{fk}\}}^{\tau_{lk}} \left(P_k(\tau)\left[\mathrm{d}\left[\bar{p}_k(\tau)/\bar{q}_k(\tau)\right]/\mathrm{d}\tau\right]\right.\right. \\
&\qquad \left.\left. \times \left[\int_{\alpha_{mn,k}(\tau)}^{\alpha_{mx,k}(\tau,t)} \alpha d_{Ak}(\alpha) d_{Bk}\left(\alpha/r_k(\tau)\right)\mathrm{d}\alpha\right]\right)\mathrm{d}\tau\right\}\mathrm{d}CDF_{T,WL}(t) \\
&\quad = CDF_{PM,SL}(0 \mid [t_{mn}, t_{mx}]) + \sum_{k=1}^{nSL} \int_{t_{mn}}^{\tau_{lk}} \int_{\max\{t,\tau_{fk}\}}^{\tau_{lk}} \int_{\alpha_{mn,k}(\tau)}^{\alpha_{mx,k}(\tau,t)} f_k(\alpha,\tau,t)\,\mathrm{d}\alpha\,\mathrm{d}\tau\,\mathrm{d}t
\end{aligned} \tag{7.31}
$$

with the second equality resulting in the limit as $n \to \infty$ and $f_k(\alpha,\tau,t)$ defined the same as in Eq. (7.20).

With use of the definitions of the intervals $[\alpha_{mn}(\tau), \alpha_{mx}(\tau)]$ in Table 5, it follows that the integration limits for the inner integral over α in Eq. (7.31) are given by

$$
[\alpha_{mn,k}(\tau), \alpha_{mx,k}(\tau,t)] = \begin{cases} [\beta_{mn,k} r_k(\tau), \min\{\alpha_{mx,k}, m/d_k(\tau,t)\}] & \text{for } \tau \in \mathcal{P}_1 \\ [\beta_{mn,k} r_k(\tau), \min\{\beta_{mx,k} r_k(\tau, m/d_k(\tau,t)\})] & \text{for } \tau \in \mathcal{P}_2 \\ [\alpha_{mn,k}, \min\{\alpha_{mx,k}, m/d_k(\tau,t)\}] & \text{for } \tau \in \mathcal{P}_3 \\ [\alpha_{mn,k}, \min\{\beta_{mx,k} r_k(\tau), m/d_k(\tau,t)\}] & \text{for } \tau \in \mathcal{P}_4 \end{cases} \tag{7.32}
$$

with the definitions of the sets $\mathcal{P}_i, i = 1,2,3,4,$ conditional on the relationships in Eq. (7.22). Further, for

$$
m/d_k(\tau,t) \le \alpha_{mx,k}(\tau), \tag{7.33}
$$

the integral over α in Eq. (7.31) is equal to 0.

As a reminder, the preceding derivations for

$$
CDF_{PM,SL}(m \mid [t_{mn}, t_{mx}]) = \begin{cases} CDF_{PM,SL}(m \mid [t_{mn}, t_{mx}] \text{ with } m < 0) \\ CDF_{PM,SL}(0 \mid [t_{mn}, t_{mx}]\ ) \\ CDF_{PM,SL}(m \mid [t_{mn}, t_{mx}] \text{ with } 0 < m) \end{cases} \tag{7.34}
$$

define $CDF_{PM,SL}(m\,|\,[t_{mn},t_{mx}])$ for the four failure patterns corresponding to the definition of $pF_i(t), i=1,2,3,4,$ in Table 2. The four definitions for $CDF_{PM,SL}(m\,|\,[t_{mn},t_{mx}])$ denoted by $CDF_{PM,SL,i}(m\,|\,[t_{mn},t_{mx}]), i=1,2,3,4,$ are determined by the definitions used in Eqs. (7.19) and (7.31) for $CDF_{T,WL}(t)$ (see Eq. (7.2)) and $P_k(t)$ (see Eqs. (7.5) and (7.6)). Specifically,

$$CDF_{PM,SL,1}(m\,|\,[t_{mn},t_{mx}]) \sim \text{Failure Pattern 1 in Table 2 with } CDF_{T,WL}(t) \text{ and } P_k(k) \text{ defined in Eqs. (4.2) and (7.6),} \tag{7.35}$$

$$CDF_{PM,SL,2}(m\,|\,[t_{mn},t_{mx}]) \sim \text{Failure Pattern 2 in Table 2 with } CDF_{T,WL}(t) \text{ and } P_k(k) \text{ defined in Eqs. (4.2) and (7.5),} \tag{7.36}$$

$$CDF_{PM,SL,3}(m\,|\,[t_{mn},t_{mx}]) \sim \text{Failure Pattern 3 in Table 2 with } CDF_{T,WL}(t) \text{ and } P_k(k) \text{ defined in Eqs. (4.1) and (7.6),} \tag{7.37}$$

and

$$CDF_{PM,SL,4}(m\,|\,[t_{mn},t_{mx}]) \sim \text{Failure Pattern 4 in Table 2 with } CDF_{T,WL}(t) \text{ and } P_k(k) \text{ defined in Eqs. (4.1) and (7.5).} \tag{7.38}$$

As an example, SL property margin CDFs $CDF_{PM,SL,i}(m\,|\,[t_{mn},t_{mx}])$, i = 1, 2, 3, 4, for systems with two WLs and two SLs links defined with the WLs and SLs described and illustrated in Table 1 and Fig. 1 are presented in Fig. 12. Quadrature-based evaluations of the defining triple integrals for $CDF_{PM,SL,i}(m\,|\,[t_{mn},t_{mx}])$ in Eqs. (7.19) and (7.31) were performed with procedures contained in the MATLAB numerical analysis package [57]. Both integrals have the function $f_k(\alpha,\tau,t)$ defined in Eq. (7.20) as an integrand, the components of which were evaluated as follows: (i) the density functions $d_{Ak}(\alpha)$ and $d_{Bk}(\beta)$ were defined with the **makedist**, **fitdist** and **pdf** functions, (ii) the failure time CDFs associated with the products in $P_k(\tau)$ were defined with the **spline** and **ppval** functions, (iii) the derivative $\mathrm{d}\left[\overline{p}_k(\tau)/\overline{q}_k(\tau)\right]/\mathrm{d}\tau$ was defined with use of the **diff** and **matlabFunction** functions, and (iv) the derivative $\mathrm{d}CDF_{T,WL}(t)/\mathrm{d}t$ was obtained by first constructing a spline representation for $CDF_{WL}(t)$ with the **spline** function and then differentiating this representation with the **fnder** function. The limits of integration were defined with the **min** and **max** functions. The overall calculation was performed for each margin value m by (i) discretizing the time domain into 150 evenly spaced times and (ii) calculating the inner two integrals at each time with the **TwoD** program [59]. This defined a function over time that was approximated with the **spline** function and then integrated with the **integral** function to obtain the value for the triple integral. The process was repeated for multiple values of m to obtain the SL property margin CDFs $CDF_{PM,SL,i}(m\,|\,[t_{mn},t_{mx}])$, i = 1, 2, 3, 4, in Fig. 12. The approximation of the double integral in Eq. (7.24) defining $CDF_{PM,SL}(0\,|\,[t_{mn},t_{mx}])$, i = 1,2, 3, 4, for the four failure patterns can be performed with the **TwoD** program and appropriate representations for the three functions whose product defines the associated integrand $g_k(\tau,t)$ defined in Eq. (7.25).

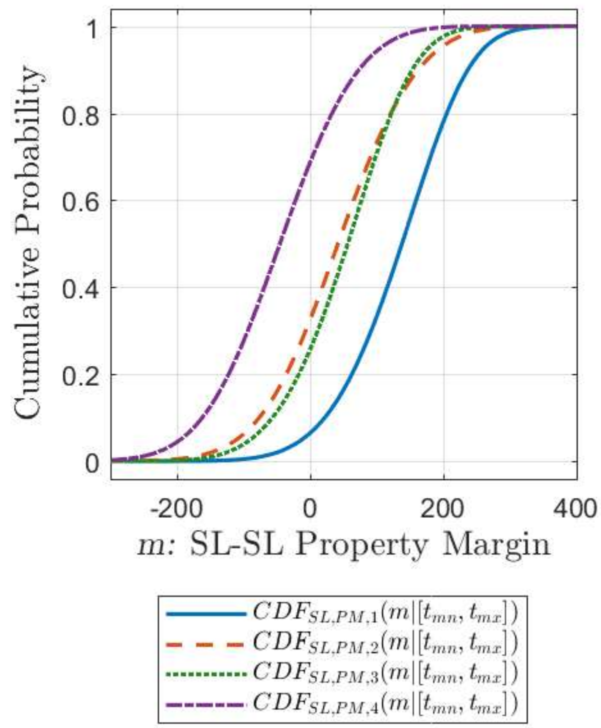

Fig. 12 Strong link (SL) property margin CDFs $CDF_{PM,SL,i}(m\,|\,[t_{mn}, t_{mx}])$ for (i) failure pattern i, i = 1, 2, 3, 4, resulting in LOAS as indicated in Table 2, (ii) the four links defined and illustrated in Table 1 and Fig. 1 with $[t_{mn}, t_{mx}] = [0, 200]$, and (iii) quadrature-based evaluation of the integrals in Eqs. (7.19), (7.24) and (7.31).

7.3. Sampling-based Estimation of CDFs $CDF_{PM,SL,i}(m\,|\,[t_{mn}, t_{mx}])$ for Margins Involving Only SL Property Values

Given the complexity of the integrals in Eqs. (7.19) and (7.31), it is likely to be more practicable to estimate $CDF_{PM,SL,i}(m\,|\,[t_{mn}, t_{mx}])$ by a sampling-based procedure than by numerical evaluation of these integrals. Specifically, the same procedure described in conjunction with Eqs. (6.4)-(6.6) can be used to estimate $CDF_{PM,SL,i}(m\,|\,[t_{mn}, t_{mx}])$ with the only difference being a change in the definition of $m_i(\mathbf{s}_r)$ in Eq. (6.6) to

$$m_i\left(\mathbf{s}_r\right) = \begin{cases} \alpha_{SL,k(r),r}\,\bar{p}_{SL,k(r)}\left(\overline{tSL}_{k(r),r}\right) - \alpha_{SL,k(r),r}\,\bar{p}_{SL,k(r)}\left(\underline{tWL}_{l(r),r}\right) & \text{for } i = 1 \\ \alpha_{SL,k(r),r}\,\bar{p}_{SL,k(r)}\left(\underline{tSL}_{k(r),r}\right) - \alpha_{SL,k(r),r}\,\bar{p}_{SL,k(r)}\left(\underline{tWL}_{l(r),r}\right) & \text{for } i = 2 \\ \alpha_{SL,k(r),r}\,\bar{p}_{SL,k(r)}\left(\overline{tSL}_{k(r),r}\right) - \alpha_{SL,k(r),r}\,\bar{p}_{SL,k(r)}\left(\overline{tWL}_{l(r),r}\right) & \text{for } i = 3 \\ \alpha_{SL,k(r),r}\,\bar{p}_{SL,k(r)}\left(\underline{tSL}_{k(r),r}\right) - \alpha_{SL,k(r),r}\,\bar{p}_{SL,k(r)}\left(\overline{tWL}_{l(r),r}\right) & \text{for } i = 4 \end{cases} \tag{7.39}$$

with the subscripts $k(r)$ and $l(r)$ defined the same as in Eq. (4.19). Specifically, $k(r)$ identifies the SL whose failure potentially results in LOAS, and $l(r)$ identifies the WL whose failure potentially prevents LOAS. The resultant approximations to $CDF_{PM,SL,i}(m\,|\,[t_{mn},t_{mx}])$ for the four indicated cases are then obtained as in Eq. (4.20).

7.4. Verification Results for $CDF_{PM,SL,i}(m\,|\,[t_{mn},t_{mx}])$

As a verification test, Fig. 13 presents a comparison of CDFs $CDF_{PM,SL,i}(m\,|\,[t_{mn},t_{mx}])$, $i = 1$, 2, 3, 4, for SL property value margins for a system with two WLs and two SLs obtained by (i) quadrature-based evaluation of the integrals in Eqs. (7.19), (7.24) and (7.31) as described in Sect. 7.2 and (ii) a sampling-based calculation as described in Sect. 7.3 and performed with the CPLOAS program [55; 56] and a sample of size $n = 10^6$. The results obtained with the two evaluation procedures are essentially identical. This level of agreement provides a strong verification that the

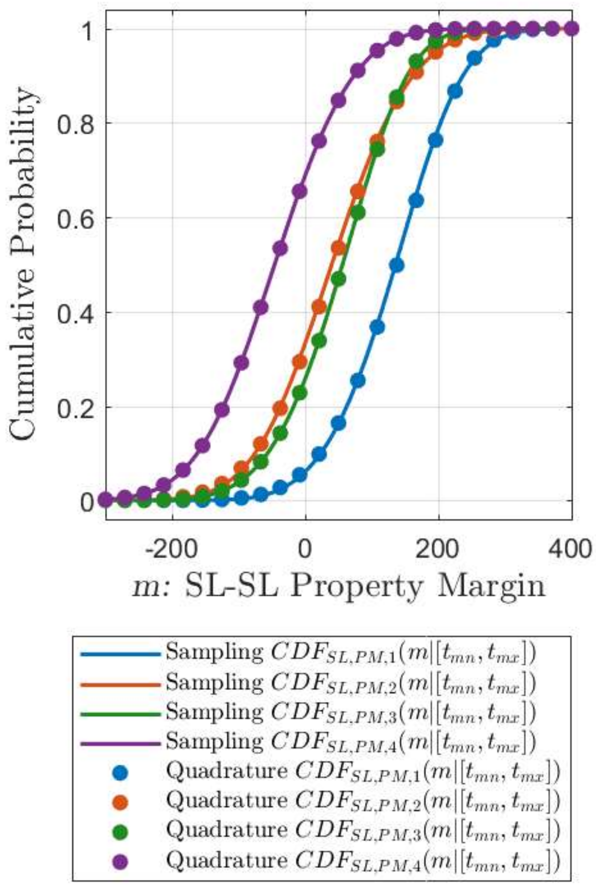

Fig. 13 Verification results for SL property value margin CDFs $CDF_{PM,SL,i}(m\,|\,[t_{mn},t_{mx}])$, $i = 1$, 2, 3, 4, obtained with quadrature-based procedures (see Sect. 7.2) and sampling-based procedures (see Sect. 7.3) for (i) the four failure patterns resulting in LOAS indicated in Table 2 and (ii) the four links defined and illustrated in Table 1 and Fig. 1 with $[t_{mn},t_{mx}] = [0, 200]$.

two procedures for obtaining $CDF_{PM,SL,i}(m\,|\,[t_{mn},t_{mx}])$ are correct in both (i) mathematical development and (ii) computational implementation.

7.5. Connection Between Failure Property Margins $CDF_{PM,SL,i}(0\,|\,[t_{mn},t_{mx}])$, $i = 1,2,3,4$, and PLOAS

The probabilities $CDF_{PM,SL,i}(0\,|\,[t_{mn},t_{mx}])$, i = 1, 2, 3, 4, associated with the failure property margin $m = 0$ defined in conjunction with Eq. (7.24) are equal to the probabilities $pF_i(t_{mx})$, i = 1, 2, 3, 4, for LOAS defined in Table 2. Establishing the indicated equalities is based on the following change in order of integration for a function f and CDFs CDF_i, i = 1, 2, defined on $[t_{mn},t_{mx}]$:

$$\begin{aligned}\int_{t_{mn}}^{t_{mx}}\left[\int_{t_{mn}}^{\tau_1} f(\tau_2)\mathrm{d}CDF_1(\tau_2)\right]\mathrm{d}CDF_2(\tau_1) &= \int_{t_{mn}}^{t_{mx}}\left[\int_{\tau_1}^{t_{mx}} \mathrm{d}CDF_2(\tau_2)\right] f(\tau_1)\mathrm{d}CDF_1(\tau_1)\\ &= \int_{t_{mn}}^{t_{mx}}\left[1-CDF_2(\tau)\right] f(\tau)\mathrm{d}CDF_1(\tau).\end{aligned} \tag{7.40}$$

With use of the preceding relationships, the representation for $CDF_{PM,SL}(0\,|\,[t_{mn},t_{mx}])$ in Eq. (7.24) becomes

$$\begin{aligned}CDF_{PM,SL,i}(0\,|\,[t_{mn},t_{mx}]) &= \sum_{k=1}^{nSL}\int_{t_{mn}}^{t_{mx}}\left[\int_{t_{mn}}^{\tau_1} P_k(\tau_2)\mathrm{d}CDF_{Tk}(\tau_2)\right]\mathrm{d}CDF_{WL}(\tau_1)\\ &= \sum_{k=1}^{nSL}\int_{t_{mn}}^{t_{mx}}\left[1-CDF_{WL}(\tau)\right]P_k(\tau)\mathrm{d}CDF_{Tk}(\tau)\end{aligned} \tag{7.41}$$

and thus corresponds to failure patterns $i = 1,2,3,4$ in Table 2 with

$$1-CDF_{T,WL}(\tau) = \begin{cases} 1-\left(1-\prod_{l=1}^{nWL}\left[1-CDF_{T,WL,l}(\tau)\right]\right) = \prod_{l=1}^{nWL}\left[1-CDF_{T,WL,l}(\tau)\right] & \text{for } i = 1,2\\ 1-\prod_{l=1}^{nWL} CDF_{T,WL,l}(\tau) & \text{for } i = 3,4\end{cases} \tag{7.42}$$

and

$$P_k(\tau) = \begin{cases} \prod_{l=1,l\neq k}^{nSL} CDF_{T,SL,l} & \text{for } i = 1,3\\ \prod_{l=1,l\neq k}^{nSL}\left[1-CDF_{T,SL,l}(\tau)\right] & \text{for } i = 2,4.\end{cases} \tag{7.43}$$

This correspondence is (i) illustrated by the equality of $pF_i(200)$ and $CDF_{PM,SL,i}(0\,|\,[0,200])$ in Fig. 3 and Figs. 12 and 13 and (ii) provides an additional verification result that the SL property value margins in Figs. 12 and 13 have been calculated correctly.

8. Summary Discussion

The results of many complex analyses can be reduced to a single number (e.g., probability of loss of assured safety (PLOAS) in analyses of LOAS for WL/SL systems). However, it is very unwise to rely on a single number as the sole summary result of a complex analysis. Specifically, unless this number is egregiously wrong (e.g., a probability that is > 1.0 or < 0), it is often not possible to tell if this single number is a reasonable analysis result. Therefore, it is prudent to employ procedures that provide perspective on the reasonableness of summary results for complex analyses. Such procedures include (i) examination of intermediate results that underlie a final summary result, (ii) performance of verification analyses to establish that results are being calculated correctly, and (iii) performance of uncertainty and sensitivity analyses to determine both the uncertainty in analysis results and the sources of this uncertainty. The indicated procedures are discussed below in the context of the present analysis

Examination of intermediate results. The primary motivation for the present study was to define and illustrate intermediate and summary results that can be examined to provide perspective on PLOAS results in analyses of LOAS for WL/SL systems. To this end, the following results associated with the analysis of LOAS are defined and illustrated: (i) CDFs for link failure time, link property value at link failure, link system failure time, property value at which a link system fails, and time at which LOAS occurs, (ii) CDFs for failure time margins defined by (time at which SL system fails) − (time at which WL system fails), (iii) CDFs for SL system property values at LOAS, (iv) CDFs for WL/SL property value margins defined by (property value at which SL system fails) − (property value at which WL system fails), and (v) CDFs for SL property value margins defined by (property value of failing SL at time of SL system failure) − (property value of this SL at time of WL system failure). Examination of these results can help provide perspectives on the nature and reasonableness of the final outcomes of an analysis of LOAS for a WL/SL system.

Performance of verification analyses. An additional motivation for the present study was to obtain verification results for the margins associated with LOAS for WL/SL systems calculated with sampling-based procedures implemented as part the CPLOAS program [55; 56]. As is the case with most sampling-based analyses, verification is difficult because the correct result is not known. Specifically, if a closed form solution was known, then there would be little reason to perform a sampling-based analysis.

Tedious checking of coding line by line and examination of intermediate results that are combined in various ways to produce the final result of interest is one way to provide "verification" for a sampling-based analysis. However, with this approach, it is not possible to be completely confident that an error in model structure or implementation has not been overlooked.

For the preceding reasons, it was decided to try to derive closed-form integral representations for the probabilities defining the CDFs for the three margin results calculated in CPLOAS: (i) failure time margin defined by (time SL system fails) − (time WL system fails), (ii) property value margin defined by (property value at which SL system fails) − (property value at which WL system fails), and (iii) SL property value margin defined by (property value of failing SL at time of SL system failure) − (property value of this SL at time of WL system failure). In addition, integral representations were also sought for the probabilities defining the CDFs for two additional results related to margins associated with LOAS: (iv) property value at which a system of WLs or SLs fails and (v) SL property value at which LOAS occurs. After significant effort, the desired integral representations were obtained.

However, the closed-form integral representations required numerical (i.e., quadrature-based) evaluation to obtain values that could be compared with the sampling-based results produced by CPLOAS. These evaluations were obtained with use of the MATLAB numerical package [57]. In some cases, implementation of the desired integral evaluations with MATLAB turned out to be quite complex and computationally inefficient compared to sampling-based evaluations.

Fortunately, the quadrature-based calculations using MATLAB and the sampling-based calculations using CPLOAS produced CDFs for the 5 cases under consideration that matched with high precision for a test problem with 2 SLs and 2 WLs (see Figs. 5, 7, 9, 11, 13). This level of agreement provides a strong verification that the two procedures for obtaining CDFs for the five indicated results are correct in both (i) mathematical development and (ii) computational implementation.

With respect to ease of implementation, the sampling-based implementations in CPLOAS were easier to derive and implement than was the case for deriving and numerically implementing the integral-based representations. Specifically, the sampling-based procedures were developed and implemented before the possibility of developing integral-based procedures was even considered.

Another positive aspect of the verification results in this study is the independence of the development of the three components of the verification results by the three study authors: (i) Development of integral representations for margin results (performed by Jon Helton), (ii) Development and implementation of quadrature-based procedures for the evaluation of integral representations for margin results (performed by Dusty Brooks), and (iii) Development and implementation of sampling-based procedures for the calculation of margin results (performed by Cédric Sallaberry). Such independence is always desirable, but not always possible, in a verification study.

Performance of uncertainty and sensitivity analyses. Although not performed as part of the present study, uncertainty analysis and sensitivity analysis are important components of analyses for complex systems [60-67]. Uncertainty analysis provides an assessment of the uncertainty in analysis results, which is an essential part of an appropriately supported decision. Sensitivity analysis provides an assessment of the contribution of the uncertainty associated with individual analysis inputs to the uncertainty in analysis results. Specifically, sensitivity analysis provides information that supports (i) decisions on where to invest resources to reduce the uncertainty in the analysis outcomes of greatest interest (e.g., PLOAS) and (ii) verification of analysis results. Sensitivity analysis can be used as an important part of analysis verification because an analysis error is revealed when an individual analysis input is shown to have an effect that it should not have.

The uncertainty associated with the analysis of complex systems is usually divided into aleatory uncertainty and epistemic uncertainty, with (i) aleatory uncertainty corresponding to an inherent variability in the behavior of the system under study and (ii) epistemic uncertainty corresponding to a lack of knowledge with respect to the actual value of a quantity that has a fixed value rather than a randomly varying value [16; 31; 41-44; 46-51]. Uncertainty analysis and sensitivity analysis are typically used in the investigation of the effects of epistemic uncertainty; in addition, uncertainty analysis and sensitivity analysis are also used to investigate the combined effects of aleatory uncertainty and epistemic uncertainty when both are included in an analysis (e.g., Refs. [41; 68-71]).

With respect to terminology, the descriptors aleatory uncertainty and epistemic uncertainty came into wide use in the mid 1990's. The alternative descriptors stochastic uncertainty and subjective uncertainty (e.g., as used in Refs. [42; 44; 70; 72; 73]) for aleatory uncertainty and epistemic uncertainty are no longer widely used.

The distributions for the variables α and β defined in Eqs. (2.3) and (2.4) were originally introduced into the representation of WLs and SLs to characterize random variability (i.e., aleatory uncertainty) that arises from random variability in manufactured components or possibly in environmental conditions. In an analysis in which the α's and β's characterize aleatory uncertainty, their distributions must be defined and most likely there will be epistemic uncertainty with respect to how these distributions should be defined (e.g., epistemic uncertainty in the definition of the three parameters in a triangular distribution used to characterize aleatory uncertainty; see Sects. 8-11 of Ref. [7] for notional examples of analyses of the type indicated in this paragraph). In a real analysis for a WL/SL system with the structure defined in this presentation, it is also likely that there will be multiple epistemically uncertainty quantities involved in the determination of the property value and failure value functions defined in Eqs. (2.1) and (2.2). Thus, an uncertainty/sensitivity analysis would be appropriate to determine the effects of the indicated epistemically uncertain quantities. In turn, an assessment of the extent to which the sensitivity analysis results are consistent with the known effects of the individual variables would constitute part of a verification analysis.

If desired, the α's and β's together with their associated distributions could also be used to represent epistemic uncertainty in the functions defined in Eqs. (2.1) and (2.2). However, when a probability distribution is used to represent uncertainty in an input to an analysis, it is very important to be clear on whether the distribution is intended to represent aleatory uncertainty or epistemic uncertainty. Further, when an analysis involves both aleatory uncertainty and epistemic uncertainty, it is important that the analysis be designed in a way that maintains a clear distinction between the effects of aleatory uncertainty and the effects of epistemic uncertainty (e.g., Refs. [41; 70; 71; 73-75]).

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