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# The nonconforming virtual element method for eigenvalue problems

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## Abstract

We analyse the nonconforming Virtual Element Method (VEM) for the approximation of elliptic eigenvalue problems. The nonconforming VEM allow to treat in the same formulation the two- and three-dimensional case. We present two possible formulations of the discrete problem, derived respectively by the nonstabilized and stabilized approximation of the  $L^2$ -inner product, and we study the convergence properties of the corresponding discrete eigenvalue problems. The proposed schemes provide a correct approximation of the spectrum and we prove optimal-order error estimates for the eigenfunctions and the usual double order of convergence of the eigenvalues. Finally we show a large set of numerical tests supporting the theoretical results, including a comparison with the conforming Virtual Element choice.

*Key words:* nonconforming virtual element, eigenvalue problem, polygonal mesh

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## 1. Introduction

The Virtual Element Method (in short, VEM) was introduced in [13] as a generalization of the finite element method to arbitrary polygonal and polyhedral meshes and as a variational reformulation of the Mimetic Finite Difference (MFD) method [10, 38]. The main idea behind VEM is that the approximation spaces consist of the usual polynomials and additional nonpolynomial functions that locally solve suitable differential problems. Consequently, the virtual functions are not explicitly known pointwisely (hence the name virtual), but only a limited set of information about them are at disposal. Nevertheless, the available information are sufficient to construct the discrete operators and the right-hand side. In fact, the VEM does not require the evaluation of test and trial functions at the integration points, but uses suitable projections onto the space of piecewise polynomials that are exactly computable from the degrees of freedom. Therefore, the approximated discrete bilinear forms require only integration of polynomials on each polytopal element in order to be computed, without the need to integrate complex non-polynomial functions on the elements and without any loss of accuracy. Moreover the VEM can be easily applied to three dimensional problems and can handle non-convex (even non simply connected) elements [3, 14]. The Virtual Element Method has been developed successfully for a large range of mathematical and engineering problems [16, 11, 26, 18, 46, 29, 44, 6, 45]. Finally, high-order and higher-order continuity schemes have been presented in [28] and [23, 17, 4], respectively.

The present paper focuses on the nonconforming VEM for the approximation of the second-order elliptic eigenvalue problem. The main advantage of nonconforming VEM introduced in [7] is to cover “*in one shot*”, i.e., using the same formulation, the two- and three-dimensional case. We recall that for the nonconforming methods, the approximating space is not a subspace of the solution space. In particular, for second-order elliptic problems we do not require the

$H^1$  regularity of the global discrete space as for the conforming schemes, but we just impose that the moments, up to a certain order, of the jumps of the discrete space functions across all mesh interfaces are zero.

The nonconforming VEM has been applied successfully to the general second-order elliptic problem [27], the Stokes equation [25], and the biharmonic problem [5], and the nonconforming approach has recently extended to the  $h$ - and  $p$ -version of the harmonic VEM [39]. As first observed in [7] and recently investigated in [32], the nonconforming VEM coincides with the high-order MFD method proposed in [37].

In the present paper we study the nonconforming VEM for the approximation of the Laplace eigenvalue problem. Using the nonconforming virtual space introduced in [7], we introduce two approximated bilinear forms, one stands for the discrete grad-grad form and the other one stands for the discrete version of the  $L^2$ -inner product. In particular, for the  $L^2$ -inner product, we consider both a nonstabilized form and a stabilized one, and we study the convergence properties of the corresponding discrete formulations. It is shown that the Virtual Element Method provides optimal convergence rates both in the approximations of eigenfunctions and the eigenvalues.

We remark that the conforming VEM formulation has been proposed for the approximation of the Steklov eigenvalue problem [41, 40], the Laplace eigenvalue problem [34], the acoustic vibration problem [12], and the vibration problem of Kirchhoff plates [42], whereas [24] deals with the Mimetic Finite Difference approximation of the eigenvalue problem in mixed form.

The outline of the paper is as follows. In Section 2 we formulate the model Laplace eigenvalue problem. In Section 3 we introduce the broken Sobolev spaces (with respect to the polygonal decompositions) and we define the conformity error. Moreover we recall the definition of the nonconforming Virtual Element Spaces and their degrees of freedom. In Section 4 we construct the approximated bilinear forms and we state the nonconforming virtual problem. In Section 5 we recall some fundamental results for the spectral approximation of compact operators. In Section 6 we show the optimal rate of convergence of the method by proving the *priori* error estimates for the eigenvalues and eigenfunctions. Section 7 presents several numerical tests. Finally, in Section 8 we offer our final remarks and conclusions.

## 2. The continuous eigenvalue problem

In this section we describe the continuous eigenvalue problem and its associated source problem. Throughout the paper, we use the notation of Sobolev spaces, norms and seminorms detailed in [1]. In particular, the symbols  $|\cdot|_{s,\omega}$  and  $\|\cdot\|_{s,\omega}$  are the seminorm and the norm of the Sobolev space  $H^s(\omega)$  defined on the open bounded subset  $\omega$  of  $\mathbb{R}^d$ , and  $(\cdot, \cdot)_\omega$  is the  $L^2$ -inner product. If  $\omega$  is the whole computational domain  $\Omega$ , the subscript may be omitted and we may denote the Sobolev seminorm and norm by  $|\cdot|_s$  and  $\|\cdot\|_s$ , and the  $L^2$ -inner product by  $(\cdot, \cdot)$ .

Let  $\Omega \subset \mathbb{R}^d$  for  $d = 2, 3$  be an open polytopal domain with Lipschitz boundary  $\Gamma$ . Consider the eigenvalue problem: Find  $\lambda \in \mathbb{R}$ , such that there exists  $u \neq 0$ :

$$-\Delta u = \lambda u \quad \text{in } \Omega, \quad (1)$$

$$u = 0 \quad \text{on } \Gamma. \quad (2)$$

Its variational formulation reads as: Find  $(\lambda, u) \in \mathbb{R} \times V$ ,  $\|u\|_0 = 1$ , such that

$$a(u, v) = \lambda b(u, v) \quad \forall v \in V, \quad (3)$$

where  $V = H_0^1(\Omega)$ , the bilinear form  $a : V \times V \rightarrow \mathbb{R}$  is given by

$$a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v \, d\Omega \quad \forall u, v \in V, \quad (4)$$

and the bilinear form  $b : V \times V \rightarrow \mathbb{R}$  is the  $L^2$ -inner product on  $\Omega$ , i.e.,

$$b(u, v) = (u, v) \quad \forall u, v \in V. \quad (5)$$

The eigenvalues of problem (3) form a positive increasing divergent sequence and the corresponding eigenfunctions are an orthonormal basis of  $V$  with respect both to the  $L^2$ -inner product and the scalar product associated with the bilinear form  $a(\cdot, \cdot)$ . Moreover, each eigenspace has finite dimension [19, 8].

In the spectral convergence analysis, we will need the approximation results of the source problem associated with (3), which we state as follows: Find  $u^s \in V$  such that

$$a(u^s, v) = b(f, v) \quad \forall v \in V, \quad (6)$$

where we assume the forcing term  $f$  (at least) in  $L^2(\Omega)$ . Due to regularity results [2, 35], there exists a constant  $r > 1/2$  that depends only on  $\Omega$  such that the solution  $u^s$  belongs to the space  $H^{1+r}(\Omega)$ . If  $\Omega$  is a convex polytopal domain, then  $r \geq 1$ . Instead,  $r \geq \pi/\omega - \varepsilon$  for any  $\varepsilon > 0$  if  $\Omega$  is a two-dimensional non-convex polygonal domain with maximum interior angle  $\omega < 2\pi$ . A similar result holds in the three-dimensional case,  $\omega$  being the maximum reentrant wedge angle. Eventually, there exists a positive constant  $C$  such that

$$|u^s|_{1+r} \leq C \|f\|_0. \quad (7)$$

### 3. The nonconforming virtual element method

In this section, we introduce the family of mesh decompositions of the computational domain and the mesh regularity assumptions and, then, we define the non-conforming virtual element space (and related approximation properties) that we need for the proper formulation of the virtual element method, cf. [7].

#### 3.1. Mesh definition and regularity assumptions

Let  $\mathcal{T} = \{\Omega_h\}_h$  be a family of decompositions of  $\Omega$  into nonoverlapping polytopal elements  $\mathbf{P}$  with nonintersecting boundary  $\partial\mathbf{P}$ , center of gravity  $\mathbf{x}_\mathbf{P}$ ,  $d$ -dimensional measure  $|\mathbf{P}|$ , and diameter  $h_\mathbf{P} = \sup_{\mathbf{x}, \mathbf{y} \in \mathbf{P}} |\mathbf{x} - \mathbf{y}|$ . The subindex  $h$  that labels each mesh  $\Omega_h$  is the maximum of the diameters  $h_\mathbf{P}$  of the elements of that mesh. The boundary of  $\mathbf{P}$  is formed by straight edges when  $d = 2$  and flat faces when  $d = 3$ . The midpoint and length of each edge  $e$  are denoted by  $\mathbf{x}_e$  and  $h_e$ , respectively. The center of gravity, diameter and area of each face  $f$  are denoted by  $\mathbf{x}_f$ ,  $h_f$ , and  $|f|$ , respectively. Sometimes we may refer to the geometric objects forming the elemental boundary  $\partial\mathbf{P}$  by the term *side* instead of *edge/face*, and adopt a unified notation by using the symbol  $\sigma$  instead of  $e$  or  $f$  regardless of the number of spatial dimensions. Accordingly,  $\mathbf{x}_\sigma$ ,  $h_\sigma$ , and  $|\sigma|$  denote the center of gravity, diameter, and measure of side  $\sigma$ .

We denote the unit normal vector to the elemental boundary  $\partial\mathbf{P}$  by  $\mathbf{n}_\mathbf{P}$ , and the unit normal vector to edge  $e$ , face  $f$  and side  $\sigma$  by  $\mathbf{n}_e$ ,  $\mathbf{n}_f$ ,  $\mathbf{n}_\sigma$ , respectively. Each vector  $\mathbf{n}_\mathbf{P}$  points out of  $\mathbf{P}$  and the orientation of  $\mathbf{n}_e$ ,  $\mathbf{n}_f$ , and  $\mathbf{n}_\sigma$  is fixed *once and for all* in every mesh  $\Omega_h$ . Finally,  $\mathcal{E}_h$ ,  $\mathcal{F}_h$ , and  $\mathcal{S}_h$  denote the set of edges, faces, and sides of the skeleton of  $\Omega_h$ . We may distinguish between *internal* and *boundary* sides by using the superscript 0 and  $\partial$ . Therefore,  $\mathcal{S}_h^0$  is the set of the internal sides,  $\mathcal{S}_h^\partial$  the set of the boundary sides, and, obviously,  $\mathcal{S}_h^0 \cap \mathcal{S}_h^\partial = \emptyset$  and  $\mathcal{S}_h = \mathcal{S}_h^0 \cup \mathcal{S}_h^\partial$ .

Now, we state the mesh regularity assumptions that are required for the convergence analysis. Since the method cannot be used simultaneously for  $d = 2$  and  $d = 3$  and, hence, no ambiguity is possible in such sense, we may refer to the two- and three-dimensional case using the same label **(A0)** and the same symbol  $\rho$  to denote the *mesh regularity constant*. Note that the assumptions for  $d = 2$  can be derived from those for  $d = 3$  by reducing the spatial dimension.

#### **(A0) Mesh regularity assumptions.**

- **d = 3.** There exists a positive constant  $\varrho$  independent of  $h$  (and, hence, of  $\Omega_h$ ) such that for every polyhedral element  $\mathbf{P} \in \Omega_h$  it holds that
  - (i)  $\mathbf{P}$  is star-shaped with respect to a ball with radius  $\geq \varrho h_\mathbf{P}$ ;
  - (ii) every face  $f \in \mathbf{P}$  is star-shaped with respect to a disk with radius  $\geq \varrho h_f$ ;
  - (iii) for every edge  $e \in \partial f$  of every face  $f \in \partial\mathbf{P}$  it holds that  $h_e \geq \varrho h_f \geq \varrho^2 h_\mathbf{P}$ .
- **d = 2.** There exists a positive constant  $\varrho$  independent of  $h$  (and, hence, of  $\Omega_h$ ) such that for every polygonal element  $\mathbf{P} \in \Omega_h$  it holds that
  - (i)  $\mathbf{P}$  is star-shaped with respect to a disk with radius  $\geq \varrho h_\mathbf{P}$ ;
  - (ii) for every edge  $e \in \partial\mathbf{P}$  it holds that  $h_e \geq \varrho h_\mathbf{P}$ .

**Remark 3.1** *The star-shapedness property implies that elements and faces are simply connected subsets of  $\mathbb{R}^d$  and  $\mathbb{R}^{d-1}$ , respectively. The scaling assumption implies that the number of edges and faces in each elemental boundary is uniformly bounded over the whole mesh family  $\mathcal{T}$ .*

### 3.2. Basic setting

We introduce the broken Sobolev space for any  $s > 0$

$$H^s(\Omega_h) = \prod_{\mathbf{P} \in \Omega_h} H^s(\mathbf{P}) = \{ v \in L^2(\Omega) : v|_{\mathbf{P}} \in H^s(\mathbf{P}) \},$$

and define the broken  $H^s$ -norm

$$\|v\|_{s,h}^2 = \sum_{\mathbf{P} \in \Omega_h} \|v\|_{s,\mathbf{P}}^2 \quad \forall v \in H^s(\Omega_h), \quad (8)$$

and for  $s = 1$  the broken  $H^1$ -seminorm

$$|v|_{1,h}^2 = \sum_{\mathbf{P} \in \Omega_h} \|\nabla v\|_{0,\mathbf{P}}^2 \quad \forall v \in H^1(\Omega_h). \quad (9)$$

Let  $\sigma \subset \partial\mathbf{P}_\sigma^+ \cap \partial\mathbf{P}_\sigma^-$  be the internal side (edge/face) shared by elements  $\mathbf{P}_\sigma^+$  and  $\mathbf{P}_\sigma^-$ , and  $v$  a function that belongs to  $H^1(\Omega_h)$ . We denote the traces of  $v$  on  $\sigma$  from the interior of elements  $\mathbf{P}_\sigma^\pm$  by  $v_\sigma^\pm$ , and the unit normal vectors to  $\sigma$  pointing from  $\mathbf{P}_\sigma^+$  to  $\mathbf{P}_\sigma^-$  by  $\mathbf{n}_\sigma^\pm$ . Then, we introduce the jump operator  $\llbracket v \rrbracket = v_\sigma^+ \mathbf{n}_\sigma^+ + v_\sigma^- \mathbf{n}_\sigma^-$  at each internal side  $\sigma \in \mathcal{S}_h^0$ , and  $\llbracket v \rrbracket = v_\sigma \mathbf{n}_\sigma$  at each boundary side  $\sigma \in \mathcal{S}_h^\partial$ . The nonconforming space  $H^{1,nc}(\Omega_h; k)$  for any integer  $k \geq 1$  is the subspace of the broken Sobolev space  $H^1(\Omega_h)$  defined as

$$H^{1,nc}(\Omega_h; k) = \left\{ v \in H^1(\Omega_h) : \int_\sigma \llbracket v \rrbracket \cdot \mathbf{n}_\sigma q \, d\sigma = 0 \quad \forall q \in \mathbb{P}_{k-1}(\sigma), \quad \forall \sigma \in \mathcal{S}_h \right\}. \quad (10)$$

Since  $\llbracket v \rrbracket = 0$  on any mesh side whenever  $v$  belongs to  $H^1(\Omega)$ , it is trivial to show that  $H_0^1(\Omega) \subset H^{1,nc}(\Omega_h; k)$ .

Hereafter, we consider the extension of the bilinear form  $a(\cdot, \cdot)$  to the broken Sobolev space  $H^1(\Omega_h)$ , which is given by splitting it as sum of local terms:

$$a : H^1(\Omega_h) \times H^1(\Omega_h) \rightarrow \mathbb{R} \quad \text{such that}$$

$$a(u, v) = \sum_{\mathbf{P} \in \Omega_h} a^\mathbf{P}(u, v) \quad \text{where} \quad a^\mathbf{P}(u, v) = \int_{\mathbf{P}} \nabla u \cdot \nabla v \, d\mathbf{P}, \quad \forall u, v \in H^1(\Omega_h). \quad (11)$$

Clearly, the same definition applies when at least one entry of  $a(\cdot, \cdot)$  belongs to the nonconforming space  $H^{1,nc}(\Omega_h; k)$ , which is a subspace of  $H^1(\Omega_h)$ , and the nonconforming virtual element space, which will be defined in the next section as a subspace of  $H^{1,nc}(\Omega_h; k)$ .

The nonconforming space with  $k = 1$  has the minimal regularity required for the VEM formulation and the convergence analysis. It is straightforward to show that  $|\cdot|_{1,h}$  is a norm on  $H^{1,nc}(\Omega_h; k)$ , although it is only a seminorm for the discontinuous functions of  $H^1(\Omega_h)$ . Moreover, using the Poincaré-Friedrichs inequality  $\|v\|_0^2 \leq C_{PF} |v|_{1,h}^2$ , which holds for every  $v \in H^{1,nc}(\Omega_h; k)$  and some positive constant  $C_{PF}$  independent of  $h$ , cf. [20], we can show that  $|\cdot|_{1,h}$  is equivalent to  $\|\cdot\|_{1,h}$ . Therefore, we may refer to the seminorm  $|\cdot|_{1,h}$  as a norm in  $H^{1,nc}(\Omega_h; k)$ .

According to [7], for  $u \in H^s(\Omega)$  with  $s \geq 3/2$  solution to (6) and  $v \in H^{1,nc}(\Omega_h; k)$  we find that

$$\mathcal{N}_h(u, v) := a(u, v) - b(f, v) = \sum_{\sigma \in \mathcal{S}_h} \int_\sigma \nabla u \cdot \llbracket v \rrbracket \, d\sigma. \quad (12)$$

The quantity  $\mathcal{N}_h(u, v)$  is called the *conformity error*.

We now recall an estimate for the term measuring the nonconformity.

**Lemma 3.2** *Assume (A0) is satisfied. Let  $u \in H^{1+r}(\Omega)$  with  $r \geq 1$  be the solution to (6). Let  $v \in H^{1,nc}(\Omega_h; k)$ ,  $k \geq 1$ , as defined in (10). Then, there exists a constant  $C > 0$  depending only on the polynomial degree and the mesh regularity such that*

$$|\mathcal{N}_h(u, v)| \leq Ch^t |u|_{1+r} |v|_{1,h} \quad (13)$$

where  $t = \min\{k, r\}$  and  $\mathcal{N}_h(u, v)$  is defined in (12).

Throughout the paper,  $\mathbb{P}_\ell(D)$  denotes the space of polynomials of degree up to  $\ell$  for any integer number  $\ell \geq 0$  on the bounded connected subset  $D$  of  $\mathbb{R}^\nu$  with  $\nu = 1, 2, 3$ . The polynomial space  $\mathbb{P}_\ell(D)$  is finite dimensional and we denote its dimension by  $\pi_{\ell,\nu}$ . It holds that  $\pi_{\ell,1} = \ell + 1$  for  $\nu = 1$ ;  $\pi_{\ell,2} = (\ell + 1)(\ell + 2)/2$  for  $\nu = 2$ ;  $\pi_{\ell,3} = (\ell + 1)(\ell + 2)(\ell + 3)/6$  for  $\nu = 3$ . We also conventionally take  $\mathbb{P}_{-1}(D) = \{0\}$  and  $\pi_{-1,\nu} = 0$ . Let  $\mathbf{x}_D$  denote the center of gravity of  $D$  and  $h_D$  its characteristic length, as, for instance, the edge length for  $\nu = 1$ , the face diameter for  $\nu = 2$  and the cell diameter for  $\nu = 3$ . A basis for  $\mathbb{P}_\ell(D)$  is provided by  $\mathcal{M}_\ell(D) = \{((\mathbf{x} - \mathbf{x}_D)/h_D)^\alpha \text{ with } |\alpha| \leq \ell\}$ , the set of the *scaled monomials of degree up to  $\ell$* , where  $\alpha = (\alpha_1, \dots, \alpha_\nu)$  is a  $\nu$ -dimensional multi-index of nonnegative integers  $\alpha_i$  with degree  $|\alpha| = \alpha_1 + \dots + \alpha_\nu$  and such that  $\mathbf{x}^\alpha = x_1^{\alpha_1} \dots x_\nu^{\alpha_\nu}$  for any  $\mathbf{x} \in \mathbb{R}^\nu$ . We will also use the set of *scaled monomials of degree exactly equal to  $\ell$* , denoted by  $\mathcal{M}_\ell^*(D)$  and obtained by setting  $|\alpha| = \ell$  in the definition of  $\mathcal{M}_\ell(D)$ . Finally, we denote by  $\Pi_\ell^{0,\mathbf{P}} : L^2(\mathbf{P}) \rightarrow \mathbb{P}_\ell(\mathbf{P})$  for  $\ell \geq 0$  the  $L^2$ -orthogonal projection onto the polynomial space  $\mathbb{P}_\ell(\mathbf{P})$ , and by  $\Pi_\ell^{0,\sigma} : L^2(\sigma) \rightarrow \mathbb{P}_\ell(\sigma)$  for  $\ell \geq 0$  the  $L^2$ -orthogonal projection onto the polynomial space  $\mathbb{P}_\ell(\sigma)$ .

### 3.3. Local and global nonconforming virtual element space

We construct the local nonconforming virtual element space by resorting to the so-called *enhancement strategy* originally devised in [3] for the conforming VEM and later extended to the nonconforming VEM in [27]. To this end, on every polytopal cell  $\mathbf{P} \in \Omega_h$  and for any integer number  $k \geq 1$  we first define the finite dimensional functional space

$$\tilde{V}_k^h(\mathbf{P}) = \left\{ v_h \in H^1(\mathbf{P}) : \frac{\partial v_h}{\partial \mathbf{n}} \in \mathbb{P}_{k-1}(\sigma) \quad \forall \sigma \subset \partial \mathbf{P}, \Delta v_h \in \mathbb{P}_k(\mathbf{P}) \right\}. \quad (14)$$

We notice that the space  $\tilde{V}_k^h(\mathbf{P})$  clearly contains the polynomials of degree  $k$ .

Then, we introduce the set of continuous linear functionals from  $\tilde{V}_k^h(\mathbf{P})$  to  $\mathbb{R}$  that for every virtual function  $v_h$  of  $\tilde{V}_k^h(\mathbf{P})$  provide:

**(D1)** the moments of  $v_h$  of order up to  $k - 1$  on each  $(d - 1)$ -dimensional side  $\sigma \in \partial \mathbf{P}$ :

$$\frac{1}{|\sigma|} \int_\sigma v_h m \, d\sigma, \quad \forall m \in \mathcal{M}_{k-1}(\sigma), \quad \forall \sigma \in \partial \mathbf{P}; \quad (15)$$

**(D2)** the moments of  $v_h$  of order up to  $k - 2$  on  $\mathbf{P}$ :

$$\frac{1}{|\mathbf{P}|} \int_{\mathbf{P}} v_h m \, d\mathbf{P}, \quad \forall m \in \mathcal{M}_{k-2}(\mathbf{P}). \quad (16)$$

Finally, we introduce the elliptic projection operator  $\Pi_k^{\nabla,\mathbf{P}} : \tilde{V}_k^h(\mathbf{P}) \rightarrow \mathbb{P}_k(\mathbf{P})$  that for any  $v_h \in \tilde{V}_k^h(\mathbf{P})$  is defined by:

$$\int_{\mathbf{P}} \nabla \Pi_k^{\nabla,\mathbf{P}} v_h \cdot \nabla q \, d\mathbf{P} = \int_{\mathbf{P}} \nabla v_h \cdot \nabla q \, d\mathbf{P} \quad \forall q \in \mathbb{P}_k(\mathbf{P}) \quad (17)$$

together with the additional conditions:

$$\int_{\partial \mathbf{P}} (\Pi_k^{\nabla,\mathbf{P}} v_h - v_h) \, d\sigma = 0 \quad \text{if } k = 1, \quad (18)$$

$$\int_{\mathbf{P}} (\Pi_k^{\nabla,\mathbf{P}} v_h - v_h) \, d\mathbf{x} = 0 \quad \text{if } k \geq 2. \quad (19)$$

As proved in [27], the polynomial projection  $\Pi_k^{\nabla,\mathbf{P}} v_h$  is exactly computable using only the values from the linear functionals **(D1)**-**(D2)**. Furthermore,  $\Pi_k^{\nabla,\mathbf{P}}$  is a polynomial-preserving operator, i.e.,  $\Pi_k^{\nabla,\mathbf{P}} q = q$  for every  $q \in \mathbb{P}_k(\mathbf{P})$ .

We are now ready to introduce the *local nonconforming virtual element space of order  $k$*  on the polytopal element  $\mathbf{P}$ , which is the subspace of  $\tilde{V}_k^h(\mathbf{P})$  defined as follow:

$$V_k^h(\mathbf{P}) = \left\{ v \in \tilde{V}_k^h(\mathbf{P}) \quad \text{such that} \quad (v_h - \Pi_k^{\nabla,\mathbf{P}} v_h, m)_{\mathbf{P}} = 0 \quad \forall m \in \mathcal{M}_{k-1}^*(\mathbf{P}) \cup \mathcal{M}_k^*(\mathbf{P}) \right\}. \quad (20)$$

The space  $V_k^h(\mathbf{P})$  has the two important properties that we outline below:

- (i) it still contains the space of polynomials of degree at most  $k$ ;
- (ii) the values provided by the set of continuous linear functionals **(D1)**-**(D2)** uniquely determine every function  $v_h$  of  $V_k^h(\mathbf{P})$  and can be taken as the *degrees of freedom* of  $V_k^h(\mathbf{P})$ .

Property (i) above is a direct consequence of space definitions (14) and (20), and guarantees the optimal order of approximation, while property (ii) follows from the unisolvency of the degrees of freedom **(D1)**-**(D2)** that was proved in [7, 27]). Additionally, the  $L^2$ -orthogonal projection  $\Pi_k^{0,\mathbf{P}} v_h$  is *exactly* computable using only the degrees of freedom of  $v_h$ , cf. [27], and  $\Pi_k^{0,\mathbf{P}} v_h = \Pi_k^{\nabla,\mathbf{P}} v_h$  for  $k = 1, 2$  as for the conforming VEM, cf. [3].

Finally, the *global nonconforming virtual element space*  $V_k^h$  of order  $k \geq 1$  subordinate to the mesh  $\Omega_h$  is obtained by gluing together the elemental spaces  $V_k^h(\mathbf{P})$  to form a subspace of the nonconforming space  $H^{1,nc}(\Omega_h; k)$ . The formal definition reads as:

$$V_k^h := \left\{ v_h \in H^{1,nc}(\Omega_h; k) : v_h|_{\mathbf{P}} \in V_k^h(\mathbf{P}) \quad \forall \mathbf{P} \in \Omega_h \right\}. \quad (21)$$

A set of degrees of freedom for  $V_k^h$  is given by collecting the values from the linear functionals **(D1)** for all the mesh sides and **(D2)** for all the mesh elements. The unisolvence of such degrees of freedom (15)-(16) for  $V_k^h$  is an immediate consequence of their unisolvence on each local space  $V_k^h(\mathbf{P})$ . Thus, the dimension of  $V_k^h$  is equal to  $N^S \times \pi_{k-1,d-1} + N^P \times \pi_{k-2,d}$ , where  $N^S$  is the total number of sides, and we recall that  $\pi_{\ell,\nu}$  is the dimension of the space of polynomials of degree up to  $\ell$  in  $\mathbb{R}^\nu$ .

**Remark 3.3** *The set of degrees of freedom can be properly redefined by excluding the moments on the  $N^{S,\partial}$  boundary sides, i.e., for  $\sigma \in S_h^\partial$ , which are set to zero to impose the homogeneous Dirichlet boundary condition (2). This reduces the dimension of  $V_k^h$  to  $N^{S,0} \times \pi_{k-1,d-1} + N^P \times \pi_{k-2,d}$ .*

### 3.4. Approximation properties

Both for completeness of exposition and future reference in the paper, we briefly summarize the local approximation properties by polynomial functions and functions in the virtual nonconforming space. We omit here any details about the derivation of these estimate and refer the interested readers to References [30, 22, 13, 3, 7, 15, 21].

*Local polynomial approximations.* On a given element  $\mathbf{P} \in \Omega_h$ , let  $v \in H^s(\mathbf{P})$  with  $1 \leq s \leq k+1$ . Under mesh assumptions **(A0)**, there exists a piecewise polynomial approximation  $v_\pi$  that is of degree  $k$  on each element, such that

$$\|v - v_\pi\|_{0,\mathbf{P}} + h_{\mathbf{P}} |v - v_\pi|_{1,\mathbf{P}} \leq Ch_{\mathbf{P}}^s |v|_{s,\mathbf{P}}, \quad (22)$$

for some constant  $C > 0$  that depends only on the polynomial degree  $k$  and the mesh regularity constant  $\varrho$ .

An instance of such a local polynomial approximation is provided by the  $L^2$ -projection  $\Pi_k^{0,\mathbf{P}} v$  onto the local polynomial space  $\mathbb{P}_k(\mathbf{P})$ , which satisfies the (optimal) error bound (22).

Furthermore, consider the internal side  $\sigma \in S_h^0$  and let  $\mathbf{P}_\sigma^\pm$  be the two elements sharing  $\sigma$ , so that  $\sigma = \partial\mathbf{P}_\sigma^+ \cap \partial\mathbf{P}_\sigma^-$ . We denote  $\Omega_\sigma = \mathbf{P}_\sigma^+ \cup \mathbf{P}_\sigma^-$ . Then, the following trace inequality [22] holds for every  $v \in H^1(\Omega_h)$

$$\|v\|_{0,\sigma} \leq Ch_\sigma^{-\frac{1}{2}} \|v\|_{0,\Omega_\sigma} + Ch_\sigma^{\frac{1}{2}} |v|_{1,\Omega_\sigma}. \quad (23)$$

Moreover, for every  $v \in H^s(\Omega_h)$  with  $1 \leq s \leq k+1$ , combining the  $L^2$ -projection approximation property with (23) the following useful error estimate holds [33]

$$\|v - \Pi_k^{0,\sigma} v\|_{0,\sigma} \leq Ch_\sigma^{s-\frac{1}{2}} |v|_{s,\Omega_\sigma}. \quad (24)$$

The same estimate holds also for the boundary sides  $\sigma \in S_h^\partial$  by taking  $\Omega_\sigma = \mathbf{P}$ , the element to which  $\sigma$  belongs.

*Interpolation error.* Similarly, under mesh regularity assumptions **(A0)**, we can define an interpolation operator in  $V_k^h$  having optimal approximation properties. Therefore, for every  $v \in H^s(\mathbf{P})$  with  $1 \leq s \leq k+1$  we can find the local interpolate  $v^\intercal \in V_k^h(\mathbf{P})$  such that

$$\|v - v^\intercal\|_{0,\mathbf{P}} + h_{\mathbf{P}} |v - v^\intercal|_{1,\mathbf{P}} \leq Ch_{\mathbf{P}}^s |v|_{s,\mathbf{P}}, \quad (25)$$



where  $C > 0$  is a positive constant independent of  $h$ .

#### 4. Virtual element discretization

This section briefly reviews the nonconforming virtual element discretization of the source problem and its extension to the eigenvalue problem.

##### 4.1. The virtual element discretization of the source problem

The goal of the present section is to introduce the virtual element discretization of the source problem (6). According to [7, 34], we define a suitable discrete bilinear form  $a_h(\cdot, \cdot)$  approximating the continuous gradient-gradient form  $a(\cdot, \cdot)$ , whereas for what concerns the bilinear form  $b(\cdot, \cdot)$  we propose two possible discretizations, hereafter denoted by  $b_h(\cdot, \cdot)$  and  $\tilde{b}_h(\cdot, \cdot)$ .

The discrete bilinear form  $a_h(\cdot, \cdot)$  is the sum of elemental contributions

$$a_h(u_h, v_h) = \sum_{\mathbf{P} \in \Omega_h} a_h^{\mathbf{P}}(u_h, v_h), \quad (26)$$

where

$$a_h^{\mathbf{P}}(u_h, v_h) = a^{\mathbf{P}}(\Pi_k^{\nabla, \mathbf{P}} u_h, \Pi_k^{\nabla, \mathbf{P}} v_h) + S^{\mathbf{P}}\left((I - \Pi_k^{\nabla, \mathbf{P}})u_h, (I - \Pi_k^{\nabla, \mathbf{P}})v_h\right), \quad (27)$$

and  $S^{\mathbf{P}}(\cdot, \cdot)$  denotes any symmetric positive definite bilinear form on the element  $\mathbf{P}$  for which there exist two positive uniform constants  $c_*$  and  $c^*$  such that

$$c_* a^{\mathbf{P}}(v_h, v_h) \leq S^{\mathbf{P}}(v_h, v_h) \leq c^* a^{\mathbf{P}}(v_h, v_h) \quad \forall v_h \in V_k^h(\mathbf{P}) \cap \ker(\Pi_k^{\nabla, \mathbf{P}}). \quad (28)$$

This requirement implies that  $S^{\mathbf{P}}(\cdot, \cdot)$  scales like  $a^{\mathbf{P}}(\cdot, \cdot)$ , namely  $S^{\mathbf{P}}(\cdot, \cdot) \simeq h_{\mathbf{P}}^{d-2}$ .

The choice of the discrete form  $a_h(\cdot, \cdot)$  is driven by the need to satisfy the following properties:

- *k-consistency*: for all  $v_h \in V_k^h(\mathbf{P})$  and for all  $q \in \mathbb{P}_k(\mathbf{P})$  it holds

$$a_h^{\mathbf{P}}(v_h, q) = a^{\mathbf{P}}(v_h, q); \quad (29)$$

- *stability*: there exists two positive constants  $\alpha_*$ ,  $\alpha^*$ , independent of  $h$  and of  $\mathbf{P}$ , such that

$$\alpha_* a^{\mathbf{P}}(v_h, v_h) \leq a_h^{\mathbf{P}}(v_h, v_h) \leq \alpha^* a^{\mathbf{P}}(v_h, v_h) \quad \forall v_h \in V_k^h(\mathbf{P}). \quad (30)$$

In particular, the first term in (27) ensures the  $k$ -consistency of the method and the second one its stability, cf. [7].

For what concerns the right hand side  $b(\cdot, \cdot)$  and following the setting in [34], we introduce two possible approximated bilinear forms.

*Non-stabilized bilinear form.* In the first choice, we consider the bilinear form  $b_h(\cdot, \cdot)$ , which satisfies the  $k$ -consistency property but not the stability property. Let us split the right-hand side  $b_h(\cdot, \cdot)$  with the sum of local contributions:

$$b_h(f, v_h) = \sum_{\mathbf{P} \in \Omega_h} b_h^{\mathbf{P}}(f, v_h).$$

Then, we define

$$b_h^{\mathbf{P}}(f, v_h) = b^{\mathbf{P}}(\Pi_k^{0, \mathbf{P}} f, v_h) \quad \forall v_h \in V_k^h(\mathbf{P}). \quad (31)$$

We observe that each local term is fully computable for any functions  $f \in L^2(\Omega)$  and  $v_h$  in  $V_k^h$  since

$$b_h(f, v_h) = \sum_{\mathbf{P} \in \Omega_h} \int_{\mathbf{P}} (\Pi_k^{0, \mathbf{P}} f) v_h \, d\mathbf{P} = \sum_{\mathbf{P} \in \Omega_h} \int_{\mathbf{P}} f \Pi_k^{0, \mathbf{P}} v_h \, d\mathbf{P},$$

and  $\Pi_k^{0, \mathbf{P}} v_h$  is computable (exactly) from the degrees of freedom of  $v_h$  (cf. Property (i)). Moreover, by definition of  $L^2$ -projection  $\Pi_k^{0, \mathbf{P}}$ , it is straightforward to check that

$$b_h^{\mathbf{P}}(f, v_h) = b^{\mathbf{P}}(\Pi_k^{0, \mathbf{P}} f, \Pi_k^{0, \mathbf{P}} v_h) \quad \forall v_h \in V_k^h(\mathbf{P}). \quad (32)$$

We estimate the approximation error of the right-hand side by using the Cauchy-Schwarz inequality twice and estimate (22), after noting that  $v_h|_P \in H^1(P)$  and  $\Pi_0^{0,P} v_h$  is orthogonal to  $f - \Pi_k^{0,P} f$ :

$$\begin{aligned} |b(f, v_h) - b_h(f, v_h)| &\leq \sum_{P \in \Omega_h} |b^P(f - \Pi_k^{0,P} f, v_h)| = \sum_{P \in \Omega_h} |b^P((I - \Pi_k^{0,P})f, (I - \Pi_0^{0,P})v_h)| \\ &\leq \sum_{P \in \Omega_h} \|(I - \Pi_k^{0,P})f\|_{0,P} \|(I - \Pi_0^{0,P})v_h\|_{0,P} \\ &\leq \sum_{P \in \Omega_h} Ch_P \|(I - \Pi_k^{0,P})f\|_{0,P} |v_h|_{1,P} \leq Ch \|(I - \Pi_k^{0,P})f\|_0 |v_h|_{1,h}. \end{aligned} \quad (33)$$

Then, the virtual element discretization of source problem (6) reads as : Find  $u_h^s \in V_k^h$  such that

$$a_h(u_h^s, v_h) = b_h(f, v_h) \quad \forall v_h \in V_k^h. \quad (34)$$

*Stabilized bilinear form.* The second approximation of the bilinear form in (5) is inspired by the definition of the virtual bilinear form  $a_h$ . In order to distinguish the two formulations, we denote the stabilized bilinear form by  $\tilde{b}_h(\cdot, \cdot)$ . As usual, we first decompose  $\tilde{b}_h(\cdot, \cdot)$  into the sum of local contributions:

$$\tilde{b}_h(f, v_h) = \sum_{P \in \Omega_h} \tilde{b}_h^P(f, v_h),$$

and, then, we define

$$\tilde{b}_h^P(f, v_h) = b^P(\Pi_k^{0,P} f, \Pi_k^{0,P} v_h) + \tilde{S}^P((I - \Pi_k^{0,P})f, (I - \Pi_k^{0,P})v_h), \quad (35)$$

where  $\tilde{S}^P$  is any positive definite bilinear form on the element  $P$  such that there exist two uniform positive constants  $\tilde{c}_*$  and  $\tilde{c}^*$  such that

$$\tilde{c}_* b^P(v, v) \leq \tilde{S}^P(v, v) \leq \tilde{c}^* b(v, v) \quad \forall v \in V_k^h(P) \cap \ker(\Pi_k^{0,P}). \quad (36)$$

We notice that the bilinear form  $\tilde{b}_h^P$  defined above satisfies both the  $k$ -consistency and the stability property.

**Remark 4.1** In analogy with the condition on the form  $S^P(\cdot, \cdot)$ , we require that the form  $\tilde{S}^P(\cdot, \cdot)$  scales like  $b^P(\cdot, \cdot)$ , that is  $\tilde{S}^P(\cdot, \cdot) \simeq h^d$ .

By definition (35) and from inequalities in (36), using similar computations as in (33), for all  $f \in L^2(P)$  and for all  $v \in V_k^h(P)$  it holds that

$$\begin{aligned} |b(f, v_h) - \tilde{b}_h(f, v_h)| &\leq \sum_{P \in \Omega_h} \left( |b^P(f - \Pi_k^{0,P} f, v_h)| + \tilde{S}^P((I - \Pi_k^{0,P})f, (I - \Pi_k^{0,P})v_h) \right) \\ &\leq \sum_{P \in \Omega_h} \left( |b^P((I - \Pi_k^{0,P})f, (I - \Pi_0^{0,P})v_h)| + \tilde{c}^* \|(I - \Pi_k^{0,P})f\|_0 \|(I - \Pi_k^{0,P})v_h\|_0 \right) \\ &\leq \sum_{P \in \Omega_h} (1 + \tilde{c}^*) \|(I - \Pi_k^{0,P})f\|_{0,P} \|(I - \Pi_0^{0,P})v_h\|_{0,P} \\ &\leq \sum_{P \in \Omega_h} Ch_P \|(I - \Pi_k^{0,P})f\|_{0,P} |v_h|_{1,P} \leq Ch \sum_{P \in \Omega_h} \|(I - \Pi_k^{0,P})f\|_0 |v_h|_{1,h}. \end{aligned} \quad (37)$$

The second formulation reads as: Find  $\tilde{u}_h^s \in V_k^h$  such that

$$a_h(\tilde{u}_h^s, v_h) = \tilde{b}_h(f, v_h) \quad \forall v_h \in V_k^h. \quad (38)$$

The well-posedness of the discrete formulation (38) stems from that of the discrete formulation (34), since the bilinear form  $a_h(\cdot, \cdot)$  is coercive due to the *stability* property.

Following the same arguments as in [7] but with  $\Pi_k^{0,P} f$  for  $k \geq 1$  instead of  $\Pi_{k-2}^{0,P} f$ , we derive the error estimates in the  $H^1$ -norm and  $L^2$ -norm that we summarize in the following theorem for next reference in the paper.

We can state the following optimal error estimates between the solution of the continuous and discrete source problems.

**Theorem 4.2** *Under the assumptions (A0), let  $u^s \in H^{1+r}(\Omega)$  with regularity index  $r \geq 1$  be the solution to (6) with  $f \in L^2(\Omega)$ .*

*Let  $u_h^s$  and  $\tilde{u}_h^s \in V_k^h$  be respectively the solutions to the nonconforming virtual element method (34) and (38). Let  $v_h \in \{u_h^s, \tilde{u}_h^s\}$ , then we have the following error estimates*

–  $H^1$ -error estimate:

$$|u^s - v_h|_{1,h} \leq C \left( h^t |u^s|_{1+r} + h \|(I - \Pi_k^{0,P})f\|_0 \right)$$

–  $L^2$ -error estimate (for a convex  $\Omega$ ):

$$\|u^s - v_h\|_0 \leq C \left( h^{t+1} |u^s|_{1+r} + h^2 \|(I - \Pi_k^{0,P})f\|_0 \right)$$

with  $t = \min(k, r)$ .

The proofs of these estimates are omitted as they are almost identical to those of [7, Theorem 4.3, Theorem 4.5]. The only difference is that here the forcing term  $f$  is approximated in each element by the orthogonal projection onto polynomials of degree  $k$  instead of  $\bar{k} = \max(k - 2, 0)$  (see estimates (33) and (37)). Note, indeed, that in the  $L^2$ -estimate of [7, Theorem 4.5], the term that depends on the approximation of  $f$  is given by

$$(h^2 + h^{\bar{k}+1}) \|(I - \Pi_k^{0,P})f\|_0 = \begin{cases} (h^2 + h) \|(I - \Pi_k^{0,P})f\|_0 & \text{for } k = 1, 2, \\ h^2 \|(I - \Pi_k^{0,P})f\|_0 & \text{for } k \geq 3. \end{cases}$$

The difference in the coefficients is a consequence of the orthogonality of  $(I - \Pi_k^{0,P})f$  to the polynomials of degree  $k$  instead of  $\bar{k}$ .

**Remark 4.3** *We observe that if the load term  $f$  is an eigenfunction of problem (3), then  $f$  solves the continuous source problem (6) with datum  $\lambda f$  and thus, thanks to the regularity result (7), it belongs to  $H^{1+r}(\Omega)$  and  $|f|_{1+r} \leq C \|f\|_0$ . Then, the a priori error estimates in Theorem 4.2 reduce to*

–  $H^1$ -error estimate:

$$|u^s - v_h|_{1,h} \leq C \left( h^t |u^s|_{1+r} + h \|(I - \Pi_k^{0,P})f\|_0 \right) \leq h^t |u^s|_{1+r} \leq Ch^t \|f\|_0 \leq Ch^t,$$

–  $L^2$ -error estimate:

$$\|u^s - v_h\|_0 \leq C \left( h^{t+1} |u^s|_{1+r} + h^2 \|(I - \Pi_k^{0,P})f\|_0 \right) \leq Ch^{t+1} |u^s|_{1+r} \leq Ch^{t+1} \|f\|_0 \leq Ch^{t+1},$$

since

$$\|(I - \Pi_k^{0,P})f\|_0 \leq Ch^{\min\{k+1, 1+r\}} |f|_{1+r} \leq Ch^{t+1} \|f\|_0.$$

**Remark 4.4** *In [7] it is proved that the discrete problem (34) is well-posed by taking, instead of  $\Pi_k^{0,P} f$ ,  $\Pi_{k-2}^{0,P} f$  for  $k \geq 2$  and a first-order approximation of  $\Pi_0^{0,P} f$  for  $k = 1$  in the definition of the discrete bilinear form  $b_h(\cdot, \cdot)$ . However, in the definition of the discrete bilinear forms  $b_h(\cdot, \cdot)$  and  $\tilde{b}_h(\cdot, \cdot)$ , we project onto the space  $\mathbb{P}_k(\mathcal{P})$ . This choice does not provide a better convergence rate, due to the  $k$ -consistency property, but it has been numerically observed that it gives more accurate results.*

#### 4.2. The virtual element discretization of the eigenvalue problem.

In the light of the nonconforming virtual element method (34) and (38) introduced in the previous section, following [34], we consider two different discretizations of the eigenvalue problem (3). The first formulation is inspired to the source problem (34), and uses definition (31). We formulate the virtual element approximation of (3) as: Find  $(\lambda_h, u_h) \in \mathbb{R} \times V_k^h$ ,  $\|u_h\|_0 = 1$ , such that

$$a_h(u_h, v_h) = \lambda_h b_h(u_h, v_h) \quad \forall v_h \in V_k^h. \quad (39)$$

The second formulation is inspired to the virtual source problem (38) and uses definition (35). We formulate the second approximation of problem (3) as: *Find*  $(\tilde{\lambda}_h, \tilde{u}_h) \in \mathbb{R} \times V_k^h$ ,  $\|\tilde{u}_h\|_0 = 1$ , *such that*

$$a_h(\tilde{u}_h, v_h) = \tilde{\lambda}_h \tilde{b}_h(\tilde{u}_h, v_h) \quad \forall v_h \in V_k^h. \quad (40)$$

## 5. Spectral approximation for compact operators

In this section, we briefly recall some spectral approximation results that can be deduced from [8, 19, 36]. For more general results, we refer to the original papers. Before stating the spectral approximation results, we introduce a natural compact operator associated with problem (3) and its discrete counterpart, and we recall their connection with the eigenmode convergence.

We associate problem (3) with its solution operator  $T \in \mathcal{L}(L^2(\Omega))$ , which is the bounded linear operator  $T : L^2(\Omega) \rightarrow L^2(\Omega)$  mapping the forcing term  $f$  to  $u^s =: Tf$ :

$$\begin{cases} Tf \in V \text{ such that} \\ a(Tf, v) = b(f, v) \quad \forall v \in V. \end{cases}$$

Operator  $T$  is self-adjoint and positive definite with respect to the inner products  $a$  and  $b$  on  $V$ , and compact due to the compact embedding of  $H^1(\Omega)$  in  $L^2(\Omega)$ .

Similarly, we associate problem (34) with its solution operator  $T_h \in \mathcal{L}(L^2(\Omega))$  and problem (38) with its solution operator  $\tilde{T}_h \in \mathcal{L}(L^2(\Omega))$ . The former is the bounded linear operator mapping the forcing term  $f$  to  $u_h^s =: T_h f$  and satisfies:

$$\begin{cases} T_h f \in V_k^h \text{ such that} \\ a_h(T_h f, v_h) = b_h(f, v_h) \quad \forall v_h \in V_k^h. \end{cases}$$

The latter is the bounded linear operator mapping the forcing term  $f$  to  $\tilde{u}_h^s =: \tilde{T}_h f$  and satisfies:

$$\begin{cases} \tilde{T}_h f \in V_k^h \text{ such that} \\ a_h(\tilde{T}_h f, v_h) = \tilde{b}_h(f, v_h) \quad \forall v_h \in V_k^h. \end{cases}$$

Both operators  $T_h$  and  $\tilde{T}_h$  are self-adjoint and positive definite with respect to the inner products  $a_h$ ,  $b_h$  and  $a_h$ ,  $\tilde{b}_h$ . They are also compact since their ranges are finite dimensional.

The eigensolutions of the continuous problem (3) and the discrete problems (39) and (40) are respectively related to the eigenmodes of the operators  $T$ ,  $T_h$ , and  $\tilde{T}_h$ . In particular,  $(\lambda, u)$  is an eigenpair of problem (3) if and only if  $Tu = (1/\lambda)u$ , i.e.  $(\frac{1}{\lambda}, u)$  is an eigenpair for the operator  $T$ , and analogously for problems (39) and (40) and operators  $T_h$  and  $\tilde{T}_h$ . By virtue of this correspondence, the convergence analysis can be derived from the spectral approximation theory for compact operators. In the rest of this section we refer only to operators  $T$  and  $\tilde{T}_h$ . Identical considerations hold for operators  $T$  and  $T_h$  and we omit them for brevity.

A sufficient condition for the correct spectral approximation of a compact operator  $T$  is the uniform convergence to  $T$  of the family of discrete operators  $\{\tilde{T}_h\}_h$  (see [19, Proposition 7.4], cf. also [8]):

$$\|T - \tilde{T}_h\|_{\mathcal{L}(L^2(\Omega))} \rightarrow 0, \quad \text{as } h \rightarrow 0, \quad (41)$$

or, equivalently,

$$\|(T - \tilde{T}_h)f\|_0 \leq C\rho(h)\|f\|_0 \quad \forall f \in L^2(\Omega), \quad (42)$$

with  $\rho(h)$  tending to zero as  $h$  goes to zero. Condition (42) usually follows by a-priori estimates with no additional regularity assumption on  $f$ . Besides the convergence of the eigenmodes, condition (41), or the equivalent condition (42), implies that no spurious eigenvalues may pollute the spectrum. In fact,

- (i) each continuous eigenvalue is approximated by a number of discrete eigenvalues (counted with their multiplicity) that corresponds exactly to its multiplicity;

(ii) each discrete eigenvalue approximates a continuous eigenvalue.

Condition (41) does not provide any indication on the approximation rate. It is common to split the convergence analysis for eigenvalue problems into two steps: first, the convergence and the absence of spurious modes is studied; then, suitable convergence rates are proved. We now report the main results about the spectral approximation for compact operators. (cf. [8, Theorems 7.1–7.4]; see also [19, Theorem 9.3–9.7]), which deal with the order of convergence of eigenvalues and eigenfunctions.

**Theorem 5.1** *Let the uniform convergence (41) holds true. Let  $\mu$  be an eigenvalue of  $T$ , with multiplicity  $m$ , and denote the corresponding eigenspace by  $E_\mu$ . Then, exactly  $m$  discrete eigenvalues  $\tilde{\mu}_{1,h}, \dots, \tilde{\mu}_{m,h}$  (repeated according to their respective multiplicities) converges to  $\mu$ . Moreover, let  $\tilde{E}_{\mu,h}$  be the direct sum of the eigenspaces corresponding to the discrete eigenvalues  $\tilde{\mu}_{1,h}, \dots, \tilde{\mu}_{m,h}$  converging to  $\mu$ . Then*

$$\delta(E_\mu, \tilde{E}_{\mu,h}) \leq C \|(T - \tilde{T}_h)|_{E_\mu}\|_{\mathcal{L}(L^2(\Omega))}, \quad (43)$$

where

$$\delta(E_\mu, \tilde{E}_{\mu,h}) = \max(\hat{\delta}(E_\mu, \tilde{E}_{\mu,h}), \hat{\delta}(\tilde{E}_{\mu,h}, E_\mu)), \quad \hat{\delta}(E_\mu, \tilde{E}_{\mu,h}) = \sup_{u \in E_\mu, \|u\|_0=1} \inf_{\tilde{u}_h \in \tilde{E}_{\mu,h}} \|u - \tilde{u}_h\|_0$$

denotes the gap between  $E_\mu$  and  $\tilde{E}_{\mu,h}$ .

Concerning the eigenvalue approximation error, we recall the following result.

**Theorem 5.2** *Let the uniform convergence (41) holds true. Let  $\phi_1, \dots, \phi_m$  be a basis of the eigenspace  $E_\mu$  of  $T$  corresponding to the eigenvalue  $\mu$ . Then, for  $i = 1, \dots, m$*

$$|\mu - \tilde{\mu}_{i,h}| \leq C \left( \sum_{j,k=1}^m |((T - \tilde{T}_h)\phi_k, \phi_j)| + \|(T - \tilde{T}_h)|_{E_\mu}\|_{\mathcal{L}(L^2(\Omega))}^2 \right), \quad (44)$$

where  $\tilde{\mu}_{1,h}, \dots, \tilde{\mu}_{m,h}$  are the  $m$  discrete eigenvalues converging to  $\mu$  repeated according to their multiplicities.

## 6. Convergence analysis and error estimates

In this section we study the convergence of the discrete eigenmodes provided by the VEM approximation to the continuous ones. We will consider the stabilized discrete formulation (40). The analysis can be easily applied to the non-stabilized one (39).

### 6.1. Convergence analysis for the stabilized formulation

In the case of the first VEM approximation of problem (3), which uses the stabilized form  $\tilde{b}_h(\cdot, \cdot)$ , the uniform convergence of the sequence of operators  $\tilde{T}_h$  to  $T$  directly stems from the  $L^2$ -a priori error estimate of Theorem 4.2.

**Theorem 6.1** *The family of operators  $\tilde{T}_h$  associated with problem (38) converges uniformly to the operator  $T$  associated with problem (6), that is,*

$$\|T - \tilde{T}_h\|_{\mathcal{L}(L^2(\Omega))} \rightarrow 0 \quad \text{for } h \rightarrow 0. \quad (45)$$

*Proof.* Let  $u^s$  and  $\tilde{u}_h^s$  be the solutions to the continuous and the discrete source problems (6) and (38). The  $L^2$ -estimate of Theorem 4.2 with  $f \in L^2(\Omega)$  and the stability condition (7) imply that

$$\|u^s - \tilde{u}_h^s\|_0 \leq Ch^{\min(t+1,2)} \|f\|_0$$

with  $t = \min(k, r)$ ,  $k \geq 1$  being the order of the method and  $r$  at least in  $(1/2, 1]$  being the regularity index of the solution  $u^s \in H^{1+r}(\Omega)$  to the continuous source problem in equation (7). From this inequality it follows that

$$\|T - \tilde{T}_h\|_{\mathcal{L}(L^2(\Omega))} = \sup_{f \in L^2(\Omega)} \frac{\|Tf - \tilde{T}_h f\|_0}{\|f\|_0} = \sup_{f \in L^2(\Omega)} \frac{\|u^s - \tilde{u}_h^s\|_0}{\|f\|_0} \leq Ch^{\min(t+1,2)}.$$

□

**Remark 6.2** We observe that if  $f \in \mathcal{E}_\mu$  then, thanks to the  $L^2$  a priori error estimate in Remark (4.3), it holds

$$\|(T - \tilde{T}_h)|_{E_\mu}\|_{\mathcal{L}(L^2(\Omega))} = \sup_{f \in \mathcal{E}_\mu} \frac{\|Tf - \tilde{T}_h f\|_0}{\|f\|_0} = \sup_{f \in \mathcal{E}_\mu} \frac{\|u^s - \tilde{u}_h^s\|_0}{\|f\|_0} \leq Ch^{t+1}.$$

Putting together Theorem 5.1, Theorem 6.1, and Remark 6.2, we can state the following result.

**Theorem 6.3** Let  $\mu$  be an eigenvalue of  $T$ , with multiplicity  $m$ , and denote the corresponding eigenspace by  $E_\mu$ . Then, exactly  $m$  discrete eigenvalues  $\tilde{\mu}_{1,h}, \dots, \tilde{\mu}_{m,h}$  (repeated according to their respective multiplicities) converges to  $\mu$ . Moreover, let  $\tilde{E}_{\mu,h}$  be the direct sum of the eigenspaces corresponding to the discrete eigenvalues  $\tilde{\mu}_{1,h}, \dots, \tilde{\mu}_{m,h}$  converging to  $\mu$ . Then

$$\delta_0(E_\mu, \tilde{E}_{\mu,h}) \leq Ch^{t+1}. \quad (46)$$

A direct consequence of the previous result (cf. [8, 19]) is the following one.

**Theorem 6.4** Let  $u$  be a unit eigenfunction associated with the eigenvalue  $\lambda$  of multiplicity  $m$  and let  $\tilde{w}_h^{(1)}, \dots, \tilde{w}_h^{(m)}$  denote linearly independent eigenfunctions associated with the  $m$  discrete eigenvalues converging to  $\lambda$ . Then there exists  $\tilde{u}_h \in \text{span}\{\tilde{w}_h^{(1)}, \dots, \tilde{w}_h^{(m)}\}$  such that

$$\|u - \tilde{u}_h\|_0 \leq Ch^{t+1},$$

where  $t = \min\{k, r\}$ , being  $k$  the order of the method and  $r$  the regularity index of  $u$ .

Using Theorem 4.2 we can obtain an estimate of the conformity error (12) better than the one in Lemma (3.2) when its argument is the solution to the continuous problem (6) and the discrete problem (38). It is worth noting that the solution to the discrete problem does not need to be an approximation of the solution to the continuous problem, cf. [19].

**Lemma 6.5** Consider  $u \in H^{1+r}(\Omega)$ ,  $v \in L^2(\Omega)$  and let  $Tu, Tv \in H^{1+r}(\Omega)$  with  $r > 1/2$  be, respectively, the solutions to problem (6) with load term  $u$  and  $v$ . Assume that **(A0)** is satisfied and let  $\tilde{T}_h u \in V_k^h \subset H^{1,nc}(\Omega_h; k)$  for some integer  $k \geq 1$  be the virtual element approximation of  $Tu$  that solves problem (38). Then, there exists a constant  $C > 0$ , independent of  $h$ , such that

$$|\mathcal{N}_h(Tv, \tilde{T}_h u)| \leq Ch^{2t} |Tv|_{1+r} |Tu|_{1+r} \quad (47)$$

where  $t = \min\{k, r\}$  and  $\mathcal{N}_h(u, v)$  is the conformity error defined in (12).

*Proof.* We start the following chain of developments from the definition of the conformity error given in (12), note that  $\llbracket Tu \rrbracket = 0$  on every mesh side and that the moments up to order  $k-1$  of  $\llbracket \tilde{T}_h u \rrbracket$  across all mesh interfaces are zero, and apply the Cauchy-Schwarz inequality in the last two steps:

$$\begin{aligned} \mathcal{N}_h(Tv, \tilde{T}_h u) &= \sum_{\sigma \in \mathcal{S}_h} \int_{\sigma} \nabla T v \cdot \llbracket \tilde{T}_h u \rrbracket d\sigma \\ &= \sum_{\sigma \in \mathcal{S}_h} \int_{\sigma} (I - \Pi_{k-1}^{0,\sigma}) \nabla T v \cdot (I - \Pi_0^{0,\sigma}) \llbracket \tilde{T}_h u \rrbracket d\sigma \\ &= \sum_{\sigma \in \mathcal{S}_h} \int_{\sigma} (I - \Pi_{k-1}^{0,\sigma}) \nabla T v \cdot \left( (I - \Pi_0^{0,\sigma}) \llbracket \tilde{T}_h u \rrbracket - (I - \Pi_0^{0,\sigma}) \llbracket Tu \rrbracket \right) d\sigma \\ &= \sum_{\sigma \in \mathcal{S}_h} \int_{\sigma} (I - \Pi_{k-1}^{0,\sigma}) \nabla T v \cdot (I - \Pi_0^{0,\sigma}) \llbracket (\tilde{T}_h - T)u \rrbracket d\sigma \\ &\leq \sum_{\sigma \in \mathcal{S}_h} \|(I - \Pi_{k-1}^{0,\sigma}) \nabla T v \cdot \mathbf{n}_\sigma\|_{0,\sigma} \|(I - \Pi_0^{0,\sigma}) \llbracket (\tilde{T}_h - T)u \rrbracket \cdot \mathbf{n}_\sigma\|_{0,\sigma} \\ &\leq \left[ \sum_{\sigma \in \mathcal{S}_h} \|(I - \Pi_{k-1}^{0,\sigma}) \nabla T v \cdot \mathbf{n}_\sigma\|_{0,\sigma}^2 \right]^{\frac{1}{2}} \times \left[ \sum_{\sigma \in \mathcal{S}_h} \|(I - \Pi_0^{0,\sigma}) \llbracket (\tilde{T}_h - T)u \rrbracket \cdot \mathbf{n}_\sigma\|_{0,\sigma}^2 \right]^{\frac{1}{2}} \\ &= \mathcal{N}_1 \times \mathcal{N}_2. \end{aligned}$$

Trace inequality (24) yields

$$\|(I - \Pi_{k-1}^{0,\sigma}) \nabla T v \cdot \mathbf{n}_\sigma\|_{0,\sigma} \leq C h_\sigma^{t-\frac{1}{2}} |T v|_{1+r,\Omega_\sigma},$$

and summing over all the mesh sides, noting that  $h_\sigma \leq h$ , the number of sides per element is uniformly bounded due to **(A0)** and using definition (9) yield

$$|\mathcal{N}|_1^2 = \sum_{\sigma \in \mathcal{S}_h} \|(I - \Pi_{k-1}^{0,\sigma}) \nabla T v \cdot \mathbf{n}_\sigma\|_{0,\sigma}^2 \leq C (h^{t-\frac{1}{2}})^2 \sum_{\sigma \in \mathcal{S}_h} |T v|_{1+r,\Omega_\sigma}^2 \leq C (h^{t-\frac{1}{2}})^2 \sum_{P \in \Omega_h} |T v|_{1+r,P}^2,$$

and finally

$$|\mathcal{N}_1| \leq C h^{t-\frac{1}{2}} |T v|_{1+r}.$$

Similarly, trace inequality (24) and the jump definition yield

$$\|(I - \Pi_0^{0,\sigma}) \llbracket (\tilde{T}_h - T) u \rrbracket \cdot \mathbf{n}_\sigma\|_{0,\sigma} \leq C h_\sigma^{\frac{1}{2}} |(\tilde{T}_h - T) u|_{1,\Omega_\sigma}$$

and using the same arguments as above we have that

$$\begin{aligned} |\mathcal{N}_2|^2 &= \sum_{\sigma \in \mathcal{S}_h} \|(I - \Pi_0^{0,\sigma}) \llbracket (\tilde{T}_h - T) u \rrbracket \cdot \mathbf{n}_\sigma\|_{0,\sigma}^2 \leq C h \sum_{\sigma \in \mathcal{S}_h} |(\tilde{T}_h - T) u|_{1,\Omega_\sigma}^2 \leq C h \sum_{P \in \Omega_h} |(\tilde{T}_h - T) u|_{1,P}^2 \\ &= C h |(\tilde{T}_h - T) u|_{1,h}^2. \end{aligned}$$

Using this relation and the *a priori* error estimate in the energy norm of Remark 4.3 finally yield:

$$|\mathcal{N}|_2 \leq h^{t+\frac{1}{2}} |T u|_{1+r}.$$

The assertion of the lemma follows by collecting the above estimates together.  $\square$

We now prove the usual double order convergence of the eigenvalues.

**Theorem 6.6** *Let  $\lambda$  be an eigenvalue of problem (3) with multiplicity  $m$ , and denote by  $\tilde{\lambda}_{1,h}, \dots, \tilde{\lambda}_{m,h}$  the  $m$  discrete eigenvalues converging towards  $\lambda$ . Then the following optimal double order convergence holds:*

$$|\lambda - \tilde{\lambda}_{i,h}| \leq C h^{2t} \quad \forall i = 1, \dots, m, \quad (48)$$

with  $t = \min\{k, r\}$ , being  $k$  the order of the method and  $r$  the regularity index of the eigenfunction corresponding to  $\lambda$ .

*Proof.* We use the result stated in Theorem 5.2. It is clear that the second term in the estimate of Theorem 5.2 is of double order compared to the  $H^1$ -rate of convergence. Hence, we analyse in detail the term  $((T - \tilde{T}_h) \phi_j, \phi_k)$ . Let  $u$  and  $v$  be two eigenfunctions associated with the eigenvalue  $\lambda$ . Then, we note that  $(T - \tilde{T}_h) u \in H^{1,nc}(\Omega_h; k)$  and begin the chain of developments that follows from the definition of the conformity error in (12):

$$\begin{aligned} ((T - \tilde{T}_h) u, v) &= a(T v, (T - \tilde{T}_h) u) - \mathcal{N}_h(T v, (T - \tilde{T}_h) u) && [\text{note that } \mathcal{N}_h(T v, T u) = 0] \\ &= a(T v, (T - \tilde{T}_h) u) + \mathcal{N}_h(T v, \tilde{T}_h u) && [\text{add and subtract } \tilde{T}_h v \text{ in the first term}] \\ &= a((T - \tilde{T}_h) v, (T - \tilde{T}_h) u) + a(\tilde{T}_h v, (T - \tilde{T}_h) u) + \mathcal{N}_h(T v, \tilde{T}_h u) && [\text{split the middle term and use (12)}] \\ &= a((T - \tilde{T}_h) v, (T - \tilde{T}_h) u) + b(\tilde{T}_h v, u) + \mathcal{N}_h(T u, \tilde{T}_h v) \\ &\quad - a(\tilde{T}_h v, \tilde{T}_h u) + \mathcal{N}_h(T v, \tilde{T}_h u) && [\text{add both sides of (38)}] \\ &= a((T - \tilde{T}_h) v, (T - \tilde{T}_h) u) + [b(u, \tilde{T}_h v) - \tilde{b}_h(u, \tilde{T}_h v)] + \\ &\quad [a_h(\tilde{T}_h v, \tilde{T}_h u) - a(\tilde{T}_h v, \tilde{T}_h u)] + [\mathcal{N}_h(T u, \tilde{T}_h v) + \mathcal{N}_h(T v, \tilde{T}_h u)] \\ &= \sum_{i=1}^4 R_i. \end{aligned}$$

Term  $R_1$  is clearly of order  $h^{2t}$ , being  $u$  and  $v$  eigenfunctions (see Remark (4.3)) :

$$|R_1| = |a((T - \tilde{T}_h)v, (T - \tilde{T}_h)u)| \leq |(T - \tilde{T}_h)v|_{1,h} |(T - \tilde{T}_h)u|_{1,h} \leq Ch^{2t}.$$

To bound term  $R_2$  using (36), the definition of  $L^2$ -orthogonal projection and triangular inequality, we get

$$\begin{aligned} |R_2| &\leq |b(u, \tilde{T}_h v) - \tilde{b}_h(u, \tilde{T}_h v)| \leq \sum_{P \in \Omega_h} |b^P(u, \tilde{T}_h v) - \tilde{b}_h^P(u, \tilde{T}_h v)| \\ &\leq \sum_{P \in \Omega_h} \left( |b^P(u - \Pi_k^{0,P} u, \tilde{T}_h v)| + \left| \tilde{S}^P \left( (I - \Pi_k^{0,P})u, (I - \Pi_k^{0,P})\tilde{T}_h v \right) \right| \right) \\ &\leq \sum_{P \in \Omega_h} \left( |b^P((I - \Pi_k^{0,P})u, (I - \Pi_k^{0,P})\tilde{T}_h v)| + \tilde{c}^* \|(I - \Pi_k^{0,P})u\|_0 \|(I - \Pi_k^{0,P})\tilde{T}_h v\|_0 \right) \\ &\leq \sum_{P \in \Omega_h} (1 + \tilde{c}^*) \|(I - \Pi_k^{0,P})u\|_{0,P} \|(I - \Pi_k^{0,P})\tilde{T}_h v\|_{0,P} \\ &\leq C \sum_{P \in \Omega_h} \left( \|(I - \Pi_k^{0,P})u\|_{0,P} \left( \|(I - \Pi_k^{0,P})(T - \tilde{T}_h)v\|_{0,P} + \|(I - \Pi_k^{0,P})Tv\|_{0,P} \right) \right) \end{aligned} \quad (49)$$

Now, note that

$$\|u - \Pi_k^{0,P} u\|_{0,P} \leq Ch^{\min(k+1, r+1)} |u|_{r+1, P} \leq Ch^{t+1} |u|_{r+1, P}. \quad (50)$$

By the continuity of  $L^2$ -projection with respect the  $L^2$ -norm we get

$$\|(I - \Pi_k^{0,P})(T - \tilde{T}_h)v\|_{0,P} \leq \|(T - \tilde{T}_h)v\|_{0,P}. \quad (51)$$

Moreover polynomial approximation estimate (22) yields

$$\|(I - \Pi_k^{0,P})Tv\|_{0,P} \leq Ch^{t+1} |Tv|_{1+r, P}. \quad (52)$$

Hence collecting (50), (51), (52) in (49), and using the  $L^2$  *a priori* error estimate in Remark (4.3) and the stability estimate (7), we obtain

$$|b(u, \tilde{T}_h v) - \tilde{b}_h(u, \tilde{T}_h v)| \leq Ch^{2t+2} |u|_{r+1} |Tv|_{r+1} \leq Ch^{2t+2} \|u\|_0 \|v\|_0 = Ch^{2t+2}.$$

To estimate term  $R_3$ , we first consider the developments:

$$\begin{aligned} a^P(\tilde{T}_h u, \tilde{T}_h v) - a_h^P(\tilde{T}_h u, \tilde{T}_h v) &= a^P(\tilde{T}_h u - (Tu)_\pi, \tilde{T}_h v) + a_h^P((Tu)_\pi, \tilde{T}_h v) - a_h^P(\tilde{T}_h u, \tilde{T}_h v) \\ &= a^P(\tilde{T}_h u - (Tu)_\pi, \tilde{T}_h v - (Tv)_\pi) + a_h^P(\tilde{T}_h u - (Tu)_\pi, (Tv)_\pi) \\ &\quad + a_h^P((Tu)_\pi - \tilde{T}_h u, \tilde{T}_h v) \\ &= a^P(\tilde{T}_h u - (Tu)_\pi, \tilde{T}_h v - (Tv)_\pi) + a_h^P(\tilde{T}_h u - (Tu)_\pi, (Tv)_\pi - \tilde{T}_h v), \end{aligned}$$

where we make use of the consistency condition (29) to introduce  $(Tu)_\pi$  and  $(Tv)_\pi$ , the elemental polynomial approximations of  $Tu$  and  $Tv$  that exist in accordance with (22). The terms on the right-hand side of the previous equation are similar and we can estimate both as follows:

$$|a^P(\tilde{T}_h u - (Tu)_\pi, \tilde{T}_h v - (Tv)_\pi)| \leq \left( |(\tilde{T}_h - T)u|_{1, P} + |Tu - (Tu)_\pi|_{1, P} \right) \left( |(\tilde{T}_h - T)v|_{1, P} + |Tv - (Tv)_\pi|_{1, P} \right),$$

and, using the *a priori* error estimate in the broken  $H^1$ -norm in Remark (4.3), the local approximation properties of the VEM space by polynomials (22), and the stability estimate (7), it holds

$$\begin{aligned} |R_3| &\leq \sum_{P \in \Omega_h} \left| a^P(\tilde{T}_h u, \tilde{T}_h v) - a_h^P(\tilde{T}_h u, \tilde{T}_h v) \right| \\ &\leq C \left( |(\tilde{T}_h - T)u|_{1, h} + |Tu - (Tu)_\pi|_{1, h} \right) \left( |(\tilde{T}_h - T)v|_{1, h} + |Tv - (Tv)_\pi|_{1, h} \right) \leq Ch^{2t} |Tu|_{1+r} |Tv|_{1+r} \leq Ch^{2t}. \end{aligned}$$

Finally, for term  $R_4$  we apply the triangle inequality, Lemma 6.5, and the stability estimate (7) to obtain:

$$|R_4| \leq |\mathcal{N}_h(Tu, T_h v)| + |\mathcal{N}_h(Tv, T_h u)| \leq Ch^{2t} |Tv|_{1+r} |Tu|_{1+r} \leq Ch^{2t}.$$

The assertion of the theorem follows from the above estimates.  $\square$



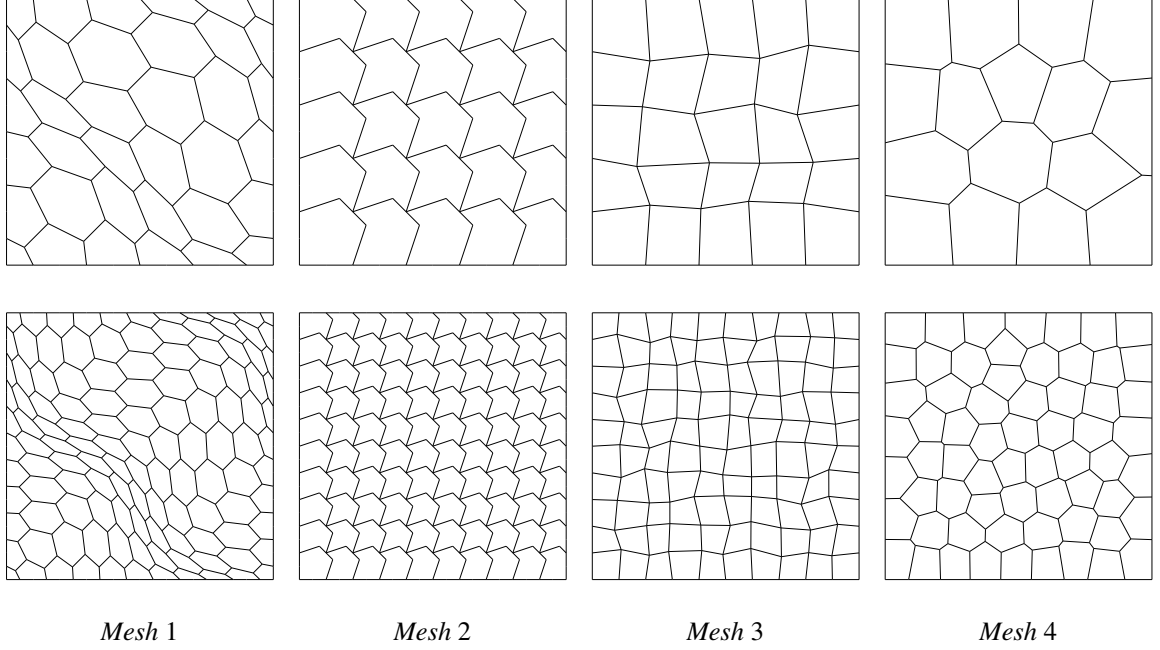


Fig. 1. Base meshes (top row) and first refined meshes (bottom row) of the following mesh families from left to right: mainly hexagonal mesh; nonconvex octagonal mesh; randomized quadrilateral mesh; voronoi mesh.

The proof of the optimal error estimate for the eigenfunctions in the discrete energy norm follows along the same line as the one for the nonconforming finite element method. We briefly report it here for the sake of completeness.

**Theorem 6.7** *With the same notation as in Theorem 6.4, we have*

$$|u - \tilde{u}_h|_{1,h} \leq Ch^t,$$

where  $t = \min(k, r)$ ,  $k$  being the order of the method and  $r$  the regularity index of  $u$ .

*Proof.*

$$u - \tilde{u}_h = \lambda Tu - \tilde{\lambda}_h \tilde{T}_h \tilde{u}_h = (\lambda - \tilde{\lambda}_h) Tu + \tilde{\lambda}_h (T - \tilde{T}_h) u + \tilde{\lambda}_h \tilde{T}_h (u - \tilde{u}_h),$$

then

$$|u - \tilde{u}_h|_{1,h} \leq |\lambda - \tilde{\lambda}_h| |Tu|_{1,h} + \tilde{\lambda}_h |(T - \tilde{T}_h) u|_{1,h} + \tilde{\lambda}_h |\tilde{T}_h (u - \tilde{u}_h)|_{1,h}.$$

The first term at the right-hand side of the previous equation is of order  $h^{2t}$ , while the second one is of order  $h^t$ . Finally, for the last term, using (30), the continuity of the operator  $\tilde{T}_h$ , and Theorem 6.4, we obtain

$$\begin{aligned} |\tilde{T}_h (u - \tilde{u}_h)|_{1,h} &\leq \frac{1}{\alpha_*} a_h(\tilde{T}_h (u - \tilde{u}_h), \tilde{T}_h (u - \tilde{u}_h)) \\ &= \frac{1}{\alpha_*} \tilde{b}_h(u - \tilde{u}_h, \tilde{T}_h (u - \tilde{u}_h)) \leq C \|u - \tilde{u}_h\|_0^2 \leq Ch^{2t+2}. \end{aligned}$$

□

## 7. Numerical experiments

In this section, we aim to confirm the optimal convergence rate of the numerical approximation of the eigenvalue problem (3) predicted by Theorem 6.6 for the nonconforming virtual element method. In particular, we present the performance of the nonconforming VEM applied to the eigenvalue problem on a two-dimensional square domain (Test Case 1) and on the L-shaped domain (Test Case 2). The convergence of the numerical approximation is shown through the relative error quantity

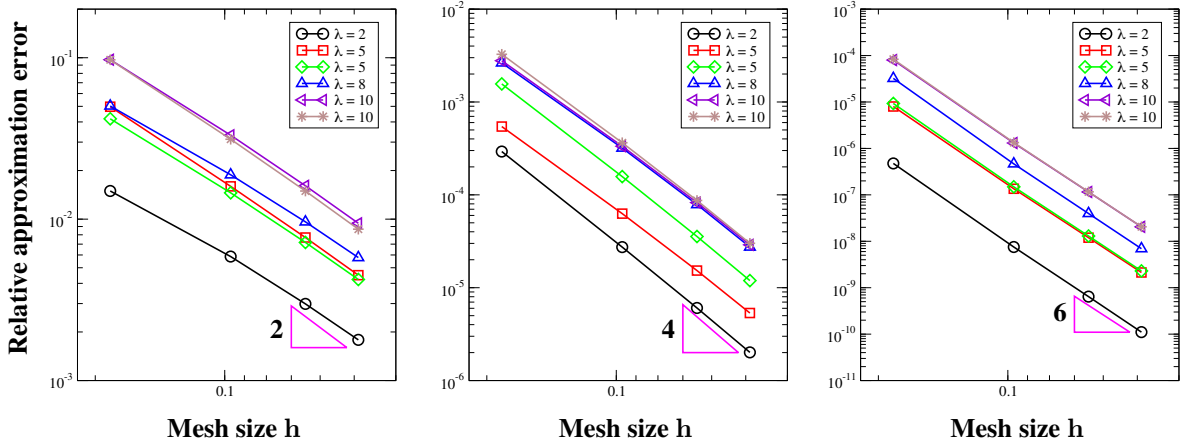


Fig. 2: Test Case 1: Convergence plots for the approximation of the first six eigenvalues using the mainly hexagonal mesh and the nonconforming spaces:  $V_1^h$  (left panel);  $V_2^h$  (mid panel);  $V_3^h$  (right panel). The generalized eigenvalue problem uses the nonstabilized bilinear form  $b_h(\cdot, \cdot)$ .

$$\epsilon_{h,\lambda} := \frac{|\lambda - \lambda_h|}{\lambda}, \quad (53)$$

where  $\lambda$  denotes an eigenvalue of the continuous problem and  $\lambda_h$  its virtual element approximation. For Test Case 1, we also compare the error curves for the nonconforming and the conforming VEM of Reference [34]. For both test cases, we use the scalar stabilization for the bilinear form  $a^P(\cdot, \cdot)$  and  $b^P(\cdot, \cdot)$ , which reads as follows:

$$\begin{aligned} S^P(v_h, w_h) &= \sigma_P \mathbf{v}_h^T \mathbf{w}_h, \\ \tilde{S}^P(v_h, w_h) &= \tau_P h^2 \mathbf{v}_h^T \mathbf{w}_h. \end{aligned}$$

where  $\mathbf{v}_h, \mathbf{w}_h$  denote the vector containing the values of the local DoFs associated to  $v_h, w_h \in V_k^h(\mathbf{P})$  and the stability parameters  $\sigma_P$  and  $\tau_P$  are two positive constants independent of  $h$ . In the numerical tests, when  $k = 1$ , constant  $\sigma_P$  is the mean value of the eigenvalues of the matrix stemming from the consistency part of the local bilinear form  $a^P$ , i.e.,  $a^P(\Pi_1^{\nabla, P}, \Pi_1^{\nabla, P})$ . For  $k = 2, 3$ , we set  $\sigma_P$  to the maximum eigenvalue of  $a^P(\Pi_k^{\nabla, P}, \Pi_k^{\nabla, P})$ . Likewise, when  $k = 1$ , constant  $\tau_P$  is set to the mean value of the eigenvalues of the matrix stemming from the consistency part of the local bilinear form  $b^P(\cdot, \cdot)$ , i.e.,  $b^P(\Pi_1^{0, P}, \Pi_1^{0, P})$ . For  $k = 2, 3$ , we set  $\tau_P$  to the maximum eigenvalue of  $b^P(\Pi_k^{0, P}, \Pi_k^{0, P})$ .

### 7.1. Test 1.

In this test case, we numerically solve the standard eigenvalue problem with homogeneous Dirichlet boundary conditions on the square domain  $\Omega = ]0, 1[ \times ]0, 1[$ . In this case, the eigenvalues of the problem are known and are given by:

$$\lambda = \pi^2(n^2 + m^2) \quad n, m \in \mathbb{N}, \text{ with } n, m \neq 0.$$

On this domain, we consider four different mesh partitionings, denoted by:

- *Mesh 1*, mainly hexagonal mesh with continuously distorted cells;
- *Mesh 2*, nonconvex octagonal mesh;
- *Mesh 3*, randomized quadrilateral mesh;
- *Mesh 4*, central Voronoi tessellation.

The base mesh and the first refined mesh of each mesh sequence is shown in Figure 1. These mesh sequences have been widely used in the mimetic finite difference and virtual element literature, and a detailed description of their construction can be found, for example, in [9]. We just mention that the last mesh sequence of central Voronoi tessellation is generated by the code PolyMesher [43]. The convergence curves for the four mesh sequences above are reported in Figures 2, 3, 4, and 5.

The expected rate of convergence is shown in each panel by the triangle closed to the error curve and indicated by an explicit label. For these calculations, we used the VEM approximation based on the nonconforming space

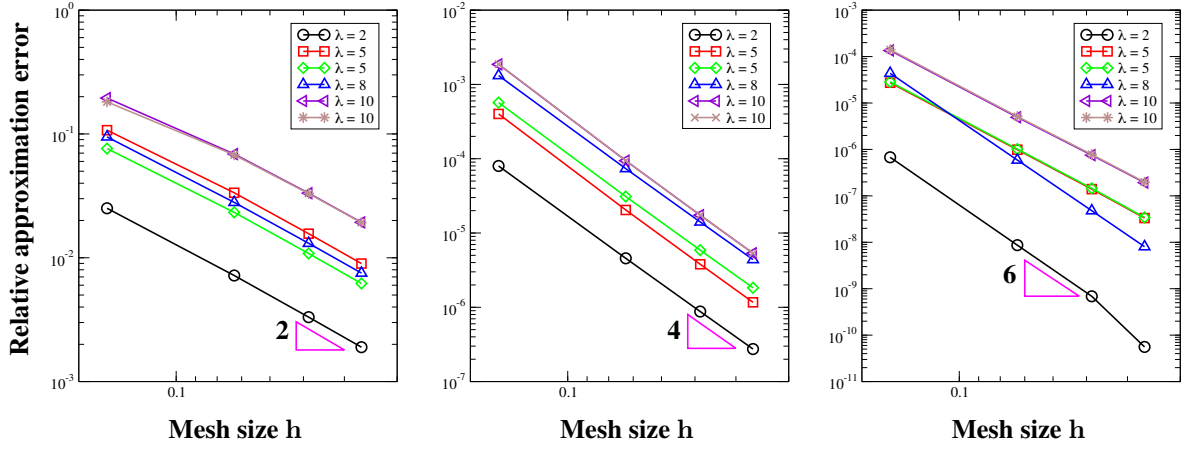


Fig. 3. Test Case 1: Convergence plots for the approximation of the first six eigenvalues using the nonconvex octagon mesh and the nonconforming spaces:  $V_1^h$  (left panel);  $V_2^h$  (mid panel);  $V_3^h$  (right panel). The generalized eigenvalue problem uses the nonstabilized bilinear form  $b_h(\cdot, \cdot)$ .

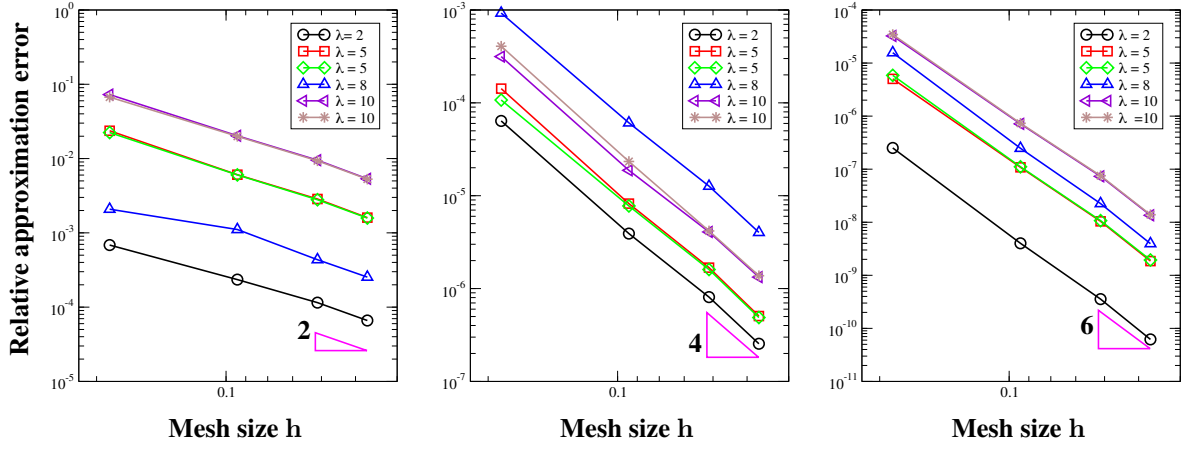


Fig. 4. Test Case 1: Convergence plots for the approximation of the first six eigenvalues using the randomized quadrilateral mesh and the nonconforming spaces:  $V_1^h$  (left panel);  $V_2^h$  (mid panel);  $V_3^h$  (right panel). The generalized eigenvalue problem uses the nonstabilized bilinear form  $b_h(\cdot, \cdot)$ .

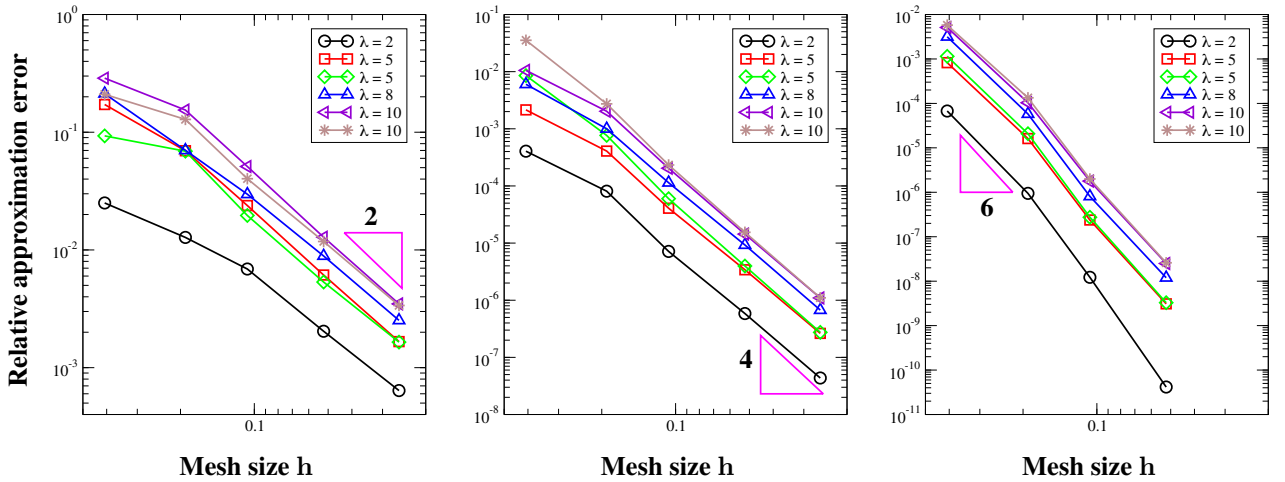


Fig. 5. Test Case 1: Convergence plots for the approximation of the first six eigenvalues using the Voronoi mesh and the nonconforming spaces:  $V_1^h$  (left panel);  $V_2^h$  (mid panel);  $V_3^h$  (right panel). The generalized eigenvalue problem uses the nonstabilized bilinear form  $b_h(\cdot, \cdot)$ .

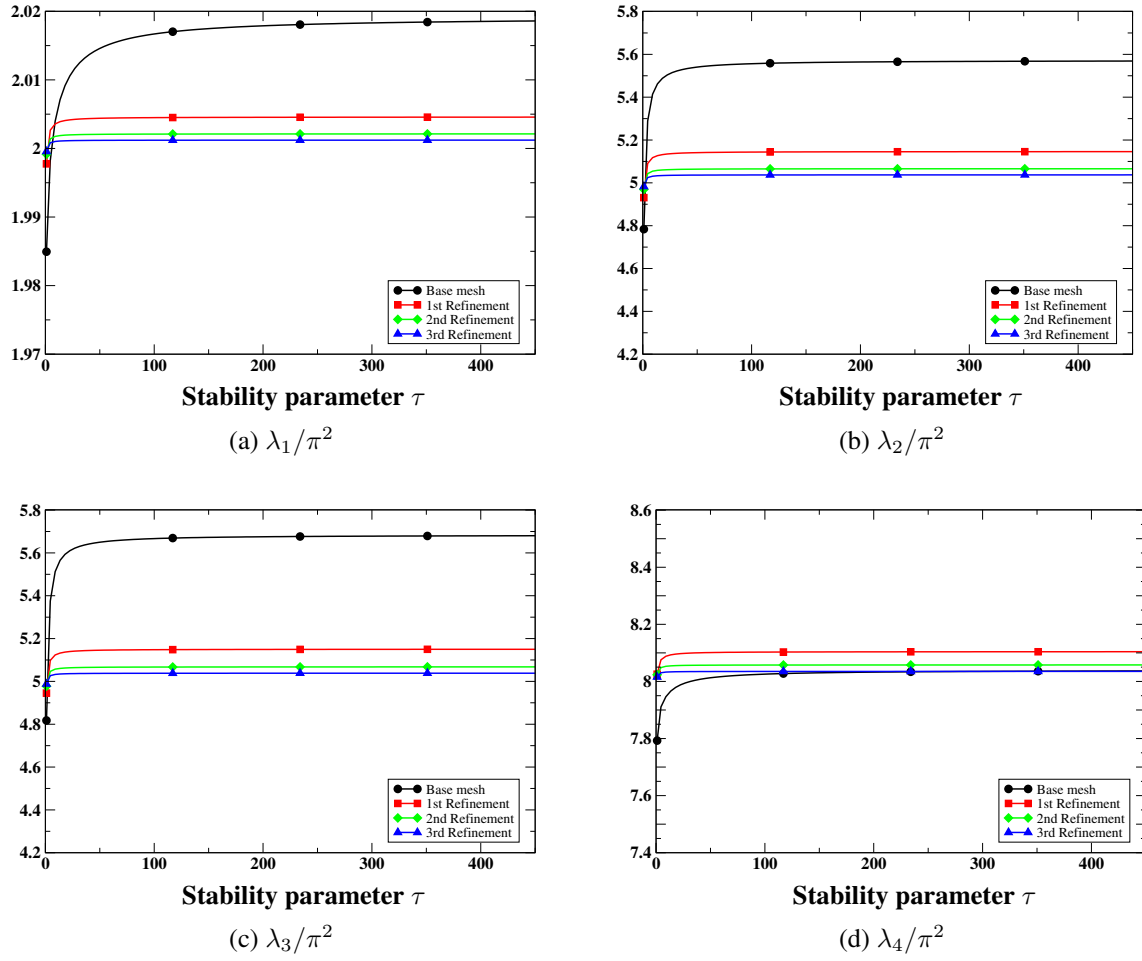


Fig. 6. Test Case 1: Eigenvalue curves versus the stability parameter  $\tau$  using the virtual element space  $V_1^h$  on the first four meshes of the mainly hexagonal mesh family.

$V_k^h$ ,  $k = 1, 2, 3$ , and the VEM formulation (39) using the nonstabilized bilinear form  $b_h(\cdot, \cdot)$ . As already observed in [34] for the conforming VEM approximation, the same computations using formulation (40) and the stabilized bilinear  $\tilde{b}_h(\cdot, \cdot)$  produce almost identical results, which, for this reason, are not shown here. These plots confirm that the nonconforming VEM formulations proposed in this work provide a numerical approximation with optimal convergence rate on a set of representative mesh sequences, including deformed and nonconvex cells.

In the four plots of Figure 6 we show the dependence on the stabilization parameter  $\tau$  of the value of the first four eigenvalues  $\lambda_1/\pi^2 = 2$ ,  $\lambda_2/\pi^2 = 5$ ,  $\lambda_3/\pi^2 = 5$ ,  $\lambda_4/\pi^2 = 8$ . From these plots, it is clear that the eigenvalue approximation is stable in a reasonable range of values of the parameter  $\tau_P$ , and that the curves of the numerical eigenvalue converge to the corresponding eigenvalue.

Finally, in Figure 7 we compare the approximation of the first eigenvalue using the conforming and nonconforming VEM on the four mesh sequences *Mesh 1-Mesh 4*. For all these meshes, we see that the two approximations are very closed.

## 7.2. Test 2.

In this test case, we solve the eigenvalue problem with Neumann boundary conditions on the nonconvex L-shaped domain  $\Omega = \Omega_1 \setminus \Omega_0$ , where  $\Omega_1 = ]-1, 1[ \times ]-1, 1[$ , and  $\Omega_0 = ]0, 1[ \times ]-1, 0[$ . This test problem is taken from

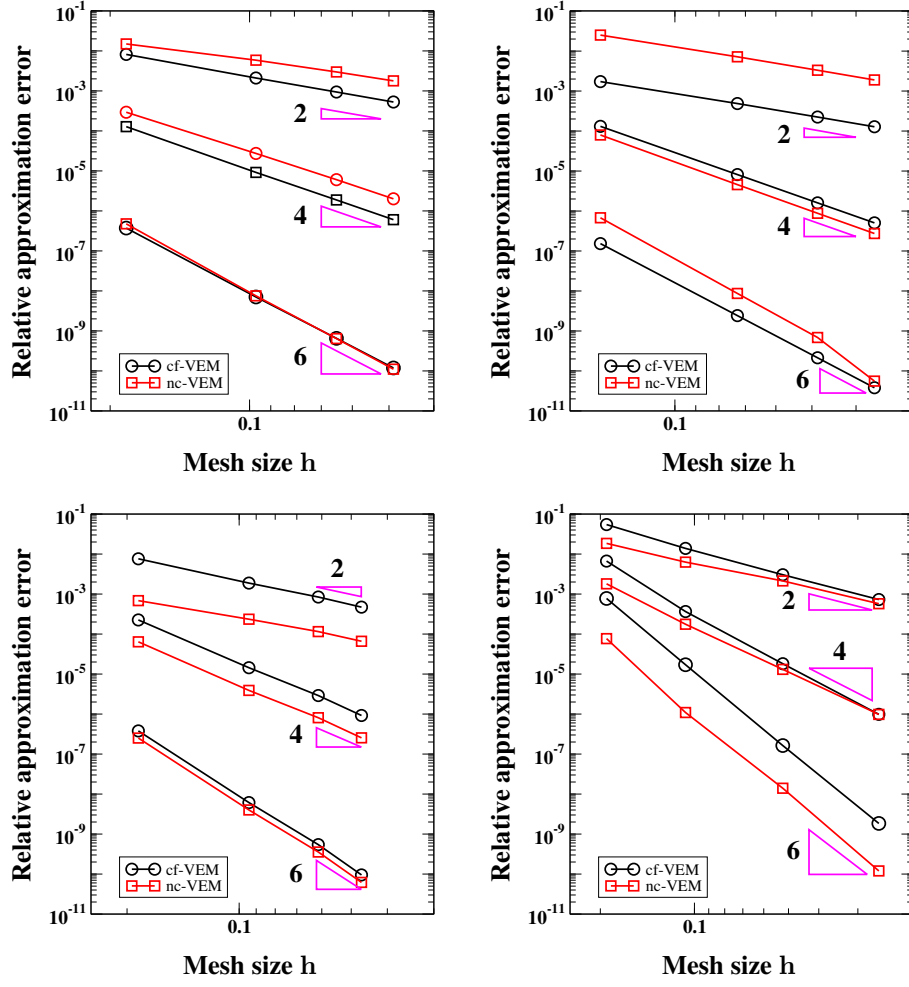


Fig. 7. Test Case 1: Comparison between the approximation of the first eigenvalue using the conforming and nonconforming VEM spaces  $V_k^h$ ,  $k = 1, 2, 3$ , and the mainly hexagonal mesh (top-left panel); the nonconvex octagonal mesh (top-right panel); the randomized quadrilateral mesh (bottom-left panel); the Voronoi mesh (bottom-right panel).

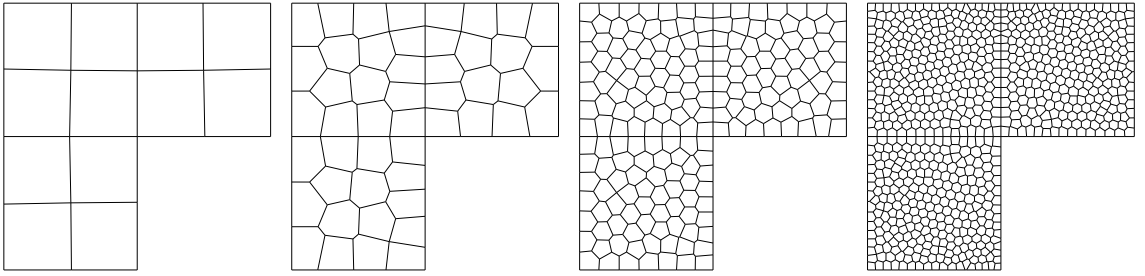


Fig. 8. Test Case 2: Mesh sequence used in the L-shaped domain test.

the benchmark suite of Reference [31]. For these calculation we use the Voronoi decompositions of Figure 8. The convergence results relative to the first and third eigenvalue are shown in Figure 9. For the first eigenvalue, we observe a lower rate of convergence that is related to the fact the corresponding eigenfunction belongs to  $H^{1+r}(\Omega)$ , with  $r = 2/3 - \epsilon$  for any  $\epsilon > 0$  (see [31]). Instead, the third eigenvalue is analytical and the optimal order of convergence is obtained, which can be seen by comparing the slopes of the error curves and the corresponding theoretical slopes reported on the plot. These results confirm the convergence analysis of the previous section and the optimality of the

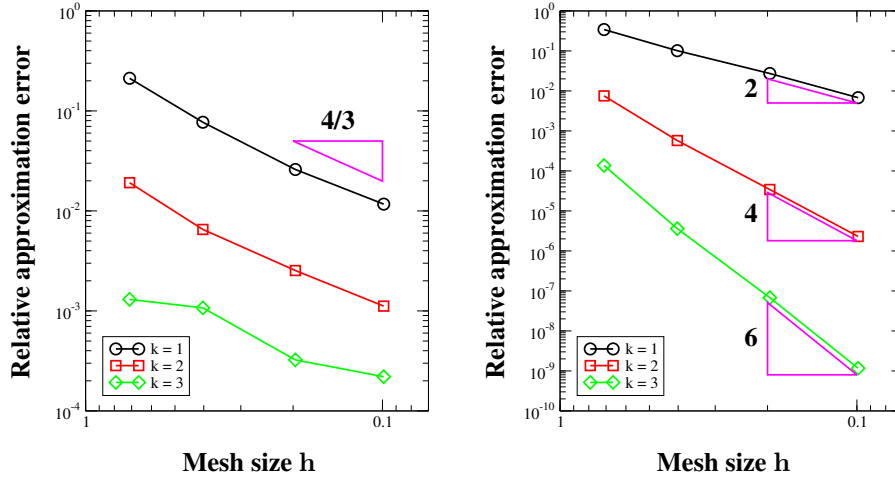


Fig. 9. Test Case 2: convergence curves of the first eigenvalue (left panel) and the third eigenvalue (right panel) using the nonconforming VEM space  $V_k^h$ ,  $k = 1, 2, 3$ ,

method also on nonconvex domains using polygonal meshes.

## 8. Conclusions

We analysed the nonconforming VEM for the approximation of elliptic eigenvalue problems. The nonconforming scheme, contrary to the conforming one, allows to use the same formulation both for the two- and the three-dimensional case. We proposed two different discrete formulations, which differ for the discrete form approximating the  $L^2$ -inner product. In particular, we considered both a nonstabilized form and a stabilized one. We showed that both formulations provide a correct approximation of the spectrum and we proved optimal *a priori* error estimates for the approximations of eigenfunctions both in the  $L^2$ -norm and the discrete energy norm, and the usual double order of convergence of the eigenvalues. Eventually, we presented a wide set of numerical tests which confirm the theoretical results.

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