Control of Networked Traffic Flow Distribution - A Stochastic Distribution System Perspective

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ABSTRACT

Networked traffic flow is a common scenario for urban transportation, where the distribution of vehicle queues either at controlled intersections or highway segments reflect the smoothness of the traffic flow in the network. At signalized intersections, the traffic queues are controlled by traffic signal control settings and effective traffic lights control would realize both smooth traffic flow and minimize fuel consumption. Funded by the Energy Efficient Mobility Systems (EEMS) program of the Vehicle Technologies Office of the US Department of Energy, we performed a preliminary investigation on the modelling and control framework in context of urban network of signalized intersections. In specific, we developed a recursive input-output traffic queueing models. The queue formation can be modeled as a stochastic process where the number of vehicles entering each intersection is a random number. Further, we proposed a preliminary B-Spline stochastic model for a one-way single-lane corridor traffic system based on theory of stochastic distribution control.. It has been shown that the developed stochastic model would provide the optimal probability density function (PDF) of the traffic queueing length as a dynamic function of the traffic signal setting parameters. Based upon such a stochastic distribution model, we have proposed a preliminary closed loop framework on stochastic distribution control for the traffic queueing system to make the traffic queueing length PDF follow a target PDF that potentially realizes the smooth traffic flow distribution in a concerned corridor.

KEYWORDS:

Traffic queueing length modelling, stochastic distribution control, probability density function.

1 INTRODUCTION

In the transportation systems field, traffic signal control is a challenging problem with the objective to produce a smooth day-by-day traffic flow distribution with minimum congestions everywhere in a traffic network. An urban signalized traffic network consists of traffic signals at road intersections along with sensors such as loop detectors and cameras that are capable of providing feedback signals to traffic controllers. It is well-known that the traffic flow in the network is randomly distributed both in space and time. Therefore, the traffic network can be modeled as a large scale multivariable stochastic distribution control system and its dynamics is very complex in nature. Indeed, the complex traffic network system has the following characteristics:

- The system is of a multiple scale in both the time and the space;
- The system is time-driven and event-driven by nature;
- The system modeling is generally difficult to perform due to nonlinear, stochastic, and strong coupling nature of the system components;
- The performance analysis of the system is difficult to realize;
- The system optimization is both multi-objective and stochastic.

Figure 1: Traffic Network.

Major challenges with such a complex traffic system are: (a) operational quality - that is how a uniform traffic flow distribution can be maintained in the networked traffic flow area with minimum energy consumption such as gas and electricity, and (b) how the operational safety can be ensured for road users such as drivers, passengers and pedestrians.

The operation and control of a complex traffic network system can be framed as a three-layered structure as shown in Figure 2. The traffic operation control monitors the traffic flow distribution across the network and the intersection controls the local traffic flow for approaching vehicles through signal control settings.

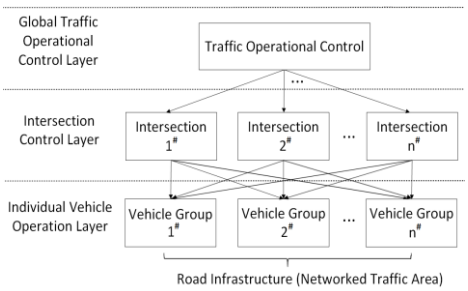

Figure 2: 3-Layer Structure of the Traffic Network System.

This 3-layered complex system is dynamic in nature having interactions among mass, energy, and information flows as shown in Figure 3, where the mass flow represents mass and volume of vehicle moving streams, the energy flow represents gas/electricity consumed by vehicles, and the information flow represents information exchange among vehicles, intersection signal controllers, and traffic operational monitoring and management. Data such as number of vehicles approaching an intersection are available from various sensors such as probe detectors, loop detectors and camera images, etc. Sometimes individual vehicles such as connected and automated vehicles (CAVs) also provide information such as vehicle-to-vehicle (V2V) and vehicle-to-infrastructure (V2I) that can be used for control operations.

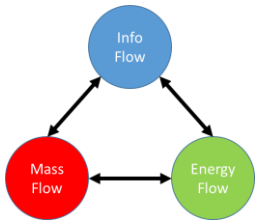

Figure 3: Mass-Energy-Information Flow of the Traffic Network System.

In this paper, we consider the problem of controlling traffic signal timings of a given traffic network such that the traffic distribution in the network can be made uniform and the traffic queueing length can be minimized. The traffic control system related with this problem is shown in Figure 4, where the traffic system consisting of a road network with installed loop detectors and traffic signals is represented as the plant T in Figure 4. The plant T has traffic signal timing sequences that are red, green and yellow signal timings as the input and traffic distribution (traffic queueing length and densities) as the output. The traffic signals controller at each intersection of a given network is represented as the controller K in Figure 4. The goal of the traffic control system is to control traffic signals at each intersection of the network using the controller K such that the traffic distribution in terms of traffic queueing length at each intersection of the network is minimized and uniformly distributed, namely this means that the distributions of the traffic queueing are uniform in their length.

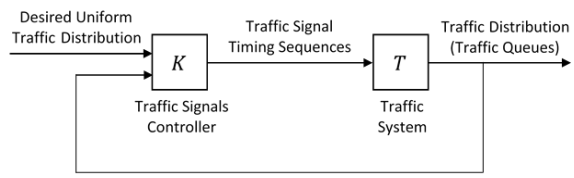

Figure 4: Traffic Control System.

2 Traffic Signal Control – Brief Review

Over the past decades, the traffic signal control for an intersection has been well developed, where fixed timing control and adaptive timing control are the common ones that are used nowadays. In this section, a brief review will be made and further details have been given in our recent report [1].

2.1 Fixed Timing Control: In fixed timing control, the red, green, and yellow traffic signal timings and cycles at each intersection of a given traffic network are fixed regardless of the actual traffic conditions. The actual timings and cycles are designed based upon historic traffic data in line with the traffic demand at different times of the day and at different intersections of the traffic network. Therefore, fixed timing control is an open loop control infrastructure that cannot cope with real-time traffic conditions in the network which causes uneven traffic flow distribution, albeit such a strategy has been widely adopted nowadays.

2.2 Adaptive Timing Control: In adaptive timing control, the red, green, and yellow traffic signal timings and cycles at each intersection of a given traffic network are tuned adaptively using the measurements of the real-time approaching traffic flow near intersections. Therefore, adaptive timing control is a closed loop control infrastructure that can cope with real-time traffic conditions in the network. There are different theories/methods such as fuzzy systems, artificial neural networks, evolutionary computing, swarm intelligence, reinforced learning and adaptive dynamic programming, etc., that have been developed for the realization adaptive timing control strategies. However only a limited use (less than 300 implementations) has been reported.

Therefore, it is imperative to further develop real-time signal control for the networked traffic flow control so as to realize

maximized mobility with minimized energy use in the concerned networked traffic flow areas. This forms the main purpose of this research where both modelling and control aspects will be addressed.

2.3 Traffic Flow Modeling

The traffic flow modelling aims to establish a dynamic model between the signal timing and the traffic queueing length at a concerned intersection. In this context, traffic flow can be modeled using the following approaches:

- First principle modeling where relevant mass-balance principle can be used to develop the basic flow dynamics and balance of the traffic flows;
- Data-driven modeling that uses data from loop detector/camera images data and machine learning to obtain traffic flow model;
- Semi-physical modeling which is a combination of the first principle modeling and data-driven modeling that can also be used to develop traffic flow model.

Based on these approaches, in this paper we propose to develop the following two models for the traffic flow, namely the recursive input-output traffic queueing length model and stochastic distribution queueing length model.

2.3.1 Recursive Input-Output Traffic Queue Model

There is a wide spectrum of literature dealing with formulation of a deterministic queueing length model for traffic flow (plant) system. In [2], a mathematical model of queueing length at an intersection has been developed using queueing theory. Similarly based on the classic queueing theory, [3] developed a traffic model and showed that the queueing system is not Ergodic. In [4], an analytical queueing model has been developed to analyze queues and delays at an isolated two-phase intersection control that uses a simple adaptive signal strategy. The work in [5] proposed a method to estimate intersection turning volumes from actuated traffic signal information without using loop detector data. A combined queue model has been proposed in [6] that captures advantages of the existing approaches while deficiencies are avoided. The work in [7] has described queue discharge flow and speed models for signalized intersections. An analytic model for queues at signalized intersections has been introduced in [8] and it is shown to be applicable in dynamic assignment problems. The article of [9] has reported a time-dependent formulation for queue length. Moreover, the book chapter in [10] can be referred for history about the evolution of delay and queue length models for traffic signals.

In this paper, we propose to develop the traffic system model in terms of traffic queue lengths for the timing of red, green and yellow signals for each street of every intersection of the network as a function of red, green and yellow signal timings at the fixed cycle duration as well as number of vehicles entered and left the streets. A recursive input-output model is formulated first for a one-way single-lane street traffic and then stochastic distribution queueing length model is formed. Using these models, a preliminary closed-loop framework is proposed that can be used to control the distribution of queueing length for a smooth traffic flow. An illustrative example is given to reveal the potential of such a closed loop framework.

2.3.2 Stochastic Distribution Traffic Queue Model

As stated previously, the traffic system is a large-scale multivariable stochastic distribution control system because the traffic flows are randomly distributed in the network where the number of the vehicles entering a road is stochastic in nature. There is limited literature [11] - [13] on the real-time control of stochastic distributions of traffic queues in the networked traffic flow area. According to [14], stochastic distribution control can be used to develop distributed traffic flow control to make the probability density functions (PDFs) of the traffic queueing length in the network to approach narrowly uniform distribution. This would reflect a smooth traffic flow. One example of this kind of model is given in [15] as

$$\frac{\partial}{\partial x}\bar{\rho}(x,t) + \frac{\partial \bar{\rho}(x,t)\bar{v}(x,t)}{\partial x} = 0 \tag{1}$$

where $\bar{\rho}(x,t)$ is the mean traffic density at point x and at time t and $\bar{v}(x,t)$ is the average vehicle speed at location x at time t. Considering traffic signal timing sequences as the control input u to the plant with noise, this model can be modified as

$$\frac{\partial}{\partial x}\bar{\rho}(x,t) + \frac{\partial \bar{\rho}(x,t)\bar{v}(x,t)}{\partial x} = Bu(t) + \text{noise}(x,t) \tag{2}$$

which is in fact a stochastic partial differential equation. Using stochastic distribution theory, a controller can be designed to generate the control signal u to realize a uniform distribution of the traffic flow in the concerned networked traffic flow area at any time.

A linear B-Spline model is developed in [16]. In this report, we propose to develop the stochastic distribution traffic queueing length model using the stochastic distribution control theory given in [16]. For this purpose, we will represent the stochastic distribution traffic queueing length model in terms of probability density function of traffic queue length for a one-way single-lane street traffic as a function of red, green, and yellow signal timings as well as data available from loop detectors and camera images. This model will then allow us to obtain a preliminary closed-loop control framework for the traffic signal control.

The paper is organized as follows. The necessary and preliminary definitions are mentioned in section 3. The problem is formulated in section 3, followed by the main results in section 4. Finally, the conclusion is given in section 5.

3 PRELIMINARIES AND A SIMPLE RECURSIVE QUEUE MODEL

In this paper, we consider continuous time signals and systems. The average length of a vehicle is denoted by $\tilde{l}$. The sets of non-negative real numbers and non-negative integers are denoted by $\mathbb{R}_+$ and $\mathbb{Z}_+$ respectively. The transpose is represented by $[\,.\,]'$. A diagonal matrix whose diagonal entries starting in the upper left corner are $a_1, \dots\dots, a_n$ is denoted by $\mathrm{diag}(a_1, \dots a_n)$. A real valued signal x truncated between times τ_1 and τ_2 is denoted by $x_{[\tau_1,\tau_2]}$ such that

$$x_{[\tau_1,\tau_2]}(t) = \begin{cases} x(t), & \forall t \in [\tau_1, \tau_2] \\ 0, & \forall t \notin [\tau_1, \tau_2] \end{cases} \tag{3}$$

Definition 1 (Signal Timing Sequence): Given a traffic signal σ taking values in the set $\{\text{Red}, \text{Green}, \text{and Yellow}\}$ as shown in Figure 5, the signal timing sequence $\tilde{\sigma}$ is defined as the sequence

$$\tilde{\sigma} = \{t_k\}, k \in \mathbb{Z}_+ \tag{4}$$

such that

$$\begin{aligned} \sigma(t) &= \text{Red (R)}, & \forall t \in [t_{k-1}, t_k] \\ \sigma(t) &= \text{Green (G)}, & \forall t \in [t_k, t_{k+1}] \\ \sigma(t) &= \text{Yellow (Y)}, & \forall t \in [t_{k+1}, t_{k+2}] \end{aligned}$$

□

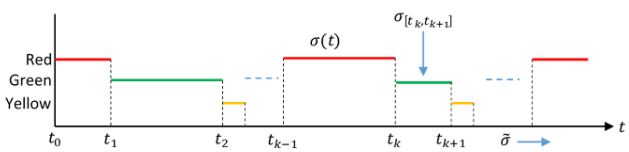

Figure 5: Traffic signal σ and signal timing sequence $\tilde{\sigma}$.

Definition 2 (Red Signal Queue): Given a traffic signal σ with signal timing sequence $\tilde{\sigma}$ having $\sigma_{[t_k,t_{k+1}]} = \text{Red}$ for a one-way single-lane street S, finite n number of adjacent standstill vehicles on the street S at time $t \in [t_k, t_{k+1}]$, then the red signal queue $r(t)$ starting from the traffic signal σ location on the street S at time $t \in [t_k, t_{k+1}]$ is defined as

$$r(t) = n\tilde{l} \tag{5}$$

□

Definition 3 (Green Signal Queue): Given a traffic signal σ with signal timing sequence $\tilde{\sigma}$ having $\sigma_{[t_k,t_{k+1}]} = \text{Green}$ for a one-way single-lane street S, the red signal queue $r(t_k)$, finite n number of vehicles joined the queue $r(t_k)$ at time $t \in [t_k, t_{k+1}]$, finite m number of vehicles left the traffic signal σ location at time $t \in [t_k, t_{k+1}]$, then the green signal queue $g(t)$ near the traffic signal σ location on the street S at time $t \in [t_k, t_{k+1}]$ is defined as

$$g(t) = r(t_k) + (n - m)\tilde{l} \tag{6}$$

□

Definition 4 (Yellow Signal Queue): Given a traffic signal σ with signal timing sequence $\tilde{\sigma}$ having $\sigma_{[t_k,t_{k+1}]} = \text{Yellow}$ for a one-way single-lane street S, the green signal queue $g(t_k)$, finite n number of vehicles joined the queue $g(t_k)$ at time $t \in [t_k, t_{k+1}]$, finite m number of vehicles left the traffic signal σ location at time $t \in [t_k, t_{k+1}]$, then the yellow signal queue $y(t)$ near the traffic signal σ location on the street S at time $t \in [t_k, t_{k+1}]$ is defined as

$$y(t) = g(t_k) + (n - m)\tilde{l} \tag{7}$$

□

Definition 5 (Probability Density Function): For a continuous random variable$x$, its probability density function (PDF) is defines as

$$\gamma(c) = \frac{dF(c)}{dc} \tag{8}$$

where $F(c)$ denotes the probability of the random variable x less than c. □

4 PROBLEM FORMULATION

4.1 One-way Single-lane Corridor Traffic

Consider a one-way single-lane street having a traffic signal σ as shown in Figure 6. We consider two loop detectors, namely N and M, as shown in Figure 6. It is assumed that the loop detector or other type of sensor M is placed immediately ahead of the stop line at the traffic signal σ, while the loop detector or other type of sensor N is placed at a distance l from the loop detector M.

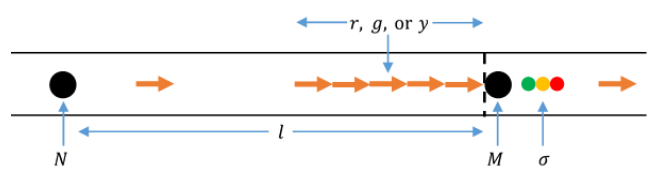

Figure 6: One-way single-lane Street having a traffic signal σ and loop detectors N and M.

We denote the vehicle detection counters for the loop detectors N and M by symbols n and m respectively taking the values in the set $\mathbb{Z}_+$. Detection time of the nth vehicle detected at the loop detector N is denoted by τ_n. The speed of the nth vehicle detected at the loop detector N is denoted by v_n. We use $\tilde{n}$ to denote the number of vehicles, detected by the loop detector N during given signal time interval for a red, green, or yellow signal, which cannot reach a respective existing red, green, or yellow signal queue in the same time interval.

With these notations, we can represent the one-way single-lane corridor shown in Figure 6 as a traffic system T shown in Figure 7. The traffic system T has the traffic signal σ with signal timing sequence $\tilde{\sigma}$ as the inputs and red signal queue r, green signal queue g, and yellow signal queue y as the outputs. The state of the system T at a time t depends on the number of vehicles entered and left the street at the time t which is shown by the loop detector counters n and m.

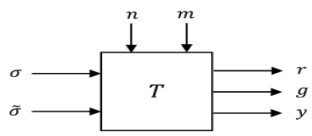

Figure 7: Traffic System T.

4.1.1 Recursive Input-Output Traffic Queue Model

There are several problems that need to be defined for the traffic queueing length model. These are described as follows.

Problem 1 (Input-output model of traffic system T for red signal): Given the traffic signal σ with signal timing sequence $\tilde{\sigma}$ as the inputs to the traffic system T as shown in Figure 7, using the loop detectors N and M data, determine the input-output model of the traffic system T with standstill red signal queue r as the output. □

Problem 2 (Input-output model of traffic system T for green signal): Given the traffic signal σ with signal timing sequence $\tilde{\sigma}$ as the inputs to the traffic system T as shown in Figure 7, using the loop detectors N and M data, determine the input-output model of the traffic system T with moving green signal queue g as the output. □

Problem 3 (Input-output model of traffic system T for yellow signal): Given the traffic signal σ with signal timing sequence $\tilde{\sigma}$ as the inputs to the traffic system T as shown in Figure 7, using the loop detectors N and M data, determine the input-output model of the traffic system T with moving yellow signal queue y as the output. □

4.1.2. Stochastic Distribution Traffic Queueing Length Model and Controller

We consider the traffic system T (one-way single-lane street) shown in Figure 7 as a stochastic system S shown in Figure 8. The stochastic system S has the traffic signal σ with signal timing sequence $\tilde{\sigma}$ as the control input $\psi = [\sigma \quad \tilde{\sigma}]'$ coming from a controller and the output as a queueing length probability density function denoted by $\gamma(\phi, \psi)$, where $\phi = [r \quad g \quad y]'$ is a function of red r, green g, and yellow y signal queues as the output. A random number of vehicles entering and leaving the intersection is denoted as the random inputs to the stochastic system S.

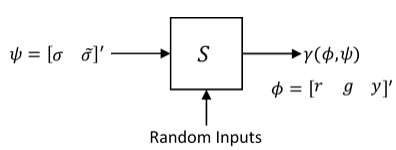

Figure 8: Stochastic System S.

Problem 4 (Stochastic Distribution Traffic Queueing Length Model of Stochastic System S): Given the traffic signal σ with signal timing sequence $\tilde{\sigma}$ as the inputs to the stochastic system S with number of vehicles as random input shown in Figure 8, develop the stochastic distribution traffic queueing length model of the system S using linear B-Spline stochastic model formulation in stochastic distribution control theory.

Problem 5 (Stochastic Distribution Controller for Stochastic System S): Given the stochastic system S having traffic signal control ψ as the input and queueing length probability density function (PDF) $\gamma(\phi, \psi)$ as the output shown in Figure 8, design a stochastic distribution controller K for the closed loop system shown in Figure 9 to generate the control input ψ so that the output PDF (i.e., $\gamma(\phi, \psi)$) is made to follow a target PDF. □

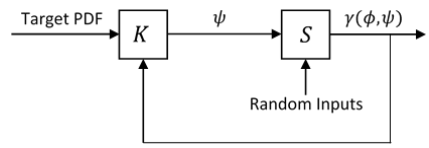

Figure 9: Stochastic Distribution Control System.

4.2 Two-way Single-lane Street Traffic Network

Consider a traffic network of two-way single-lane streets shown in Figure 10. Consider an intersection of ith horizontal street with j^{th}vertical street and an east bound lane i^e approaching it as shown in Figure 10. For this particular lane and intersection, we denote the traffic signal and signal timing sequence by $\sigma_{i,j}^e$ and $\tilde{\sigma}_{i,j}^e$ respectively. Two loop detectors are denoted by $N_{i,j-1}^e$ and $M_{i,j}^e$. The loop detector $N_{i,j-1}^e$ is placed at a distance $l_{i,j}^e$ from the loop detector $M_{i,j}^e$.

We denote the vehicle detection counters for the loop detectors $N_{i,j-1}^e$ and $M_{i,j}^e$ by symbols $n_{i,j-1}^e$ and $m_{i,j}^e$ respectively taking the values in the set $\mathbb{Z}_+$. Detection time of the nth vehicle detected at the loop detector $N_{i,j-1}^e$ is denoted by $\tau_{n_{i,j-1}^e}$. Speed of the nth vehicle detected at the loop detector $N_{i,j-1}^e$ is denoted by $v_{n_{i,j-1}^e}$. We use $\tilde{n}_{i,j-1}^e$ to denote the number of vehicles, detected by the loop detector $N_{i,j-1}^e$ during given signal time interval for a red, green, or yellow signal, which cannot reach a respective existing red, green, or yellow signal queue in the same time interval. We denote the red, green, and yellow signal queues by $r_{i,j}^e$, $g_{i,j}^e$, and $y_{i,j}^e$ respectively. The same setup of traffic signal and loop detectors along with notation is applicable for other streets of the traffic network shown in Figure 10.

We represent the two-way single-lane street traffic network shown in Figure 10 as a traffic system T as follows

$$T = \mathrm{diag}(\ldots, \cdots T_{i-1,j-1}, T_{i-1,j}, T_{i-1,j+1} \cdots, \cdots T_{i,j-1}, T_{i,j}, T_{i,j+1} \cdots \cdots T_{i+1,j-1}, T_{i+1,j}, T_{i+1,j+1} \cdots, \cdots) \quad (9)$$

where

$$T_{i,j} = \mathrm{diag}\left(T_{i,j}^n, T_{i,j}^e, T_{i,j}^w, T_{i,j}^s\right) \quad (10)$$

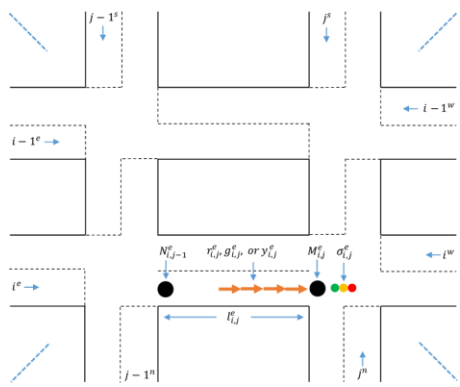

Figure 10: Two-way Single-lane Street Traffic Network.

The system $T_{i,j}$ represents the traffic system consisting of four one-way single-lane streets namely j^n, j^s, i^e and i^w approaching the (i,j) intersection. The sub-system $T_{i,j}^e$ represents the traffic system consisting the east-bound one-way single-lane street i^e between the intersections $(i,j-1)$ and (i,j). As shown in Figure 11, the traffic sub-system $T_{i,j}^e$ has the traffic signals $\sigma_{i,j}^e, \sigma_{i,j-1}^e, \sigma_{i,j-1}^s, \sigma_{i,j-1}^n$ with signal timing sequence $\tilde{\sigma}_{i,j}^e$ as the inputs. The red signal queue $r_{i,j}^e$ due to all traffic, red signal queue $\overleftrightarrow{r}_{i,j}^e$ due to through traffic of street i^e, red signal queue $\overleftarrow{r}_{i,j}^e$ due to left turning traffic of street $(j-1)^s$, and red signal queue $\overrightarrow{r}_{i,j}^e$ due to right turning traffic of street $(j-1)^n$ coming from the intersection $(i,j-1)$ are some of the outputs of the system $T_{i,j}^e$. Similarly, $g_{i,j}^e, \overleftrightarrow{g}_{i,j}^e, \overleftarrow{g}_{i,j}^e, \overrightarrow{g}_{i,j}^e$ and $y_{i,j}^e$ are the remaining outputs of the system $T_{i,j}^e$. The state of the system $T_{i,j}^e$ at a time t depends on the number of vehicles entered and left the street at the time t which is shown by the loop detector counters $n_{i,j}^e$ and $m_{i,j}^e$.

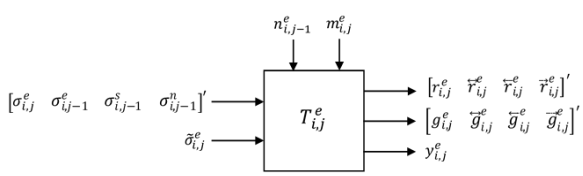

Figure 11: Traffic system $T_{i,j}^e$.

Problem 6 (Input-output model of traffic system $T_{i,j}^e$ for red signal): Given the traffic signals $\sigma_{i,j}^e, \sigma_{i,j-1}^e, \sigma_{i,j-1}^s$, and $\sigma_{i,j-1}^n$ with signal timing sequence $\tilde{\sigma}_{i,j}^e$ as the inputs to the traffic system $T_{i,j}^e$ as shown in Figure 11, using the loop detectors $N_{i,j-1}^e$ and $M_{i,j}^e$ data, determine the input-output model of the traffic system $T_{i,j}^e$ with red signal queue $r_{i,j}^e$ due to all traffic, red signal queue $\overleftrightarrow{r}_{i,j}^e$ due to through traffic of street i^e, red signal queue $\overleftarrow{r}_{i,j}^e$ due to left turning traffic of street $(j-1)^s$, and red signal queue $\overrightarrow{r}_{i,j}^e$ due to right turning traffic of street $(j-1)^n$ coming from the intersection $(i,j-1)$ as the outputs. □

Problem 7 (Input-output model of traffic system $T_{i,j}^e$ for green signal): Given the traffic signal $\sigma_{i,j}^e, \sigma_{i,j-1}^e, \sigma_{i,j-1}^s$, and $\sigma_{i,j-1}^n$ with signal timing sequence $\tilde{\sigma}_{i,j}^e$ as the inputs to the traffic system $T_{i,j}^e$ as shown in Figure 11, using the loop detectors $N_{i,j-1}^e$ and $M_{i,j}^e$ data, determine the input-output model of the traffic system $T_{i,j}^e$ with moving green signal queue $g_{i,j}^e$ due to all traffic, accumulating green signal queue $\overleftrightarrow{g}_{i,j}^e$ due to through traffic of street i^e, accumulating green signal queue $\overleftarrow{g}_{i,j}^e$ due to left turning traffic of $(j-1)^s$, and accumulating green signal queue $\overrightarrow{g}_{i,j}^e$ due to right turning traffic of street $(j-1)^n$ coming from the intersection street $(i,j-1)$ as the outputs. □

Problem 8 (Input-output model of traffic system $T_{i,j}^e$ for yellow signal): Given the traffic signal $\sigma_{i,j}^e$ with signal timing sequence $\tilde{\sigma}_{i,j}^e$ as the inputs to the traffic system $T_{i,j}^e$ as shown in Figure 11, using the loop detectors $N_{i,j-1}^e$ and $M_{i,j}^e$ data, determine the input-output model of the traffic system $T_{i,j}^e$ with moving yellow signal queue $y_{i,j}^e$ due to all traffic coming from the intersection $(i,j-1)$ as the output. □

To determine red $\overleftrightarrow{r}_{i,j}^e$ or green $\overleftrightarrow{g}_{i,j}^e$ signal queues due to through traffic of street i^e coming from the intersection $(i,j-1)$, we consider two counters namely $\overleftrightarrow{n}_{i,j-1}^e$ and $\overleftrightarrow{\tilde{n}}_{i,j-1}^e$. During red or green signal time interval, the counter $\overleftrightarrow{n}_{i,j-1}^e$ counts the number of vehicles coming from the through traffic of street i^e which can reach an end of respective existing red or green signal queue in the same time interval while the counter $\overleftrightarrow{\tilde{n}}_{i,j-1}^e$ counts the number of vehicles which cannot reach. Similarly, we consider counters $\overleftarrow{n}_{i,j-1}^e, \overleftarrow{\tilde{n}}_{i,j-1}^e$ and $\overrightarrow{n}_{i,j-1}^e, \overrightarrow{\tilde{n}}_{i,j-1}^e$ to determine red $\overleftarrow{r}_{i,j}^e$ or green $\overleftarrow{g}_{i,j}^e$ signal queues due to left turning traffic of street $(j-1)^s$ coming from the intersection $(i,j-1)$ and red $\overrightarrow{r}_{i,j}^e$ or green $\overrightarrow{g}_{i,j}^e$ signal queues due to right turning traffic of street $(j-1)^n$ coming from the intersection $(i,j-1)$ respectively.

5 MAIN RESULTS

In the following, we develop the recursive input-output models of the traffic system T shown in Figure 7 for red, green, and yellow signals as stated in the problem statements 1, 2, and 3 respectively. Due to the space limitation the details of the two-way system will not be given.

5.1 One-way Single-lane Street Traffic

5.1.1 Recursive Input-Output Traffic Queueing Length Models

For the one way system, the following theorem can be obtained that reveals the relationship between the queueing length and the traffic signal timing.

Theorem 1 (Input-output model of traffic system T for red signal): Consider the given traffic signal σ with signal timing sequence $\tilde{\sigma}$ for the traffic system T shown in Figure 7. If $\sigma_{[t_k,t_{k+1}]} = \text{Red}$ then with initial condition $r(t_k) =$

$y(t_k,n,m)+\tilde{n}\tilde{l}$ and re-initialized conditions $n=\tilde{n}=0$ & $\tilde{\tau}_0=t_k$, for all $t\in[t_k,t_{k+1}]$ we have

$$r(t,n) = \begin{cases} r(t_k)+n\tilde{l}, \forall t\in[\tilde{\tau}_n,t_{k+1}] \\ r(t,n-1), \quad \forall t\in[t_k,\tilde{\tau}_n) \\ \quad r(t,n-1), \forall t\in[t_k,t_{k+1}], \text{ otherwise} \end{cases}, \text{ if } \tilde{\tau}_n\in[t_k,t_{k+1}] \tag{11}$$

and it has been denoted that

$$\tilde{n}=\tilde{n}+1 \text{ if } \tilde{\tau}_n\notin[t_k,t_{k+1}]$$

where

$$\tilde{\tau}_n=\frac{l-r(t_k)-(n-1)\tilde{l}}{v_n}+\tau_n$$

Proof: See the appendix A.1. □

Theorem 2 (Input-output model of traffic system T for green signal): Consider the given traffic signal σ with signal timing sequence $\tilde{\sigma}$ for the traffic system T shown in Figure 11. If $\sigma_{[t_k,t_{k+1}]}=\text{Green}$ then with initial condition $g(t_k)=r(t_k,n)+\tilde{n}\tilde{l}$ and re-initialized conditions $n=\tilde{n}=m=0$ & $\tilde{\tau}_0=t_k$, for all $t\in[t_k,t_{k+1}]$ we have

$$g(t,n,m) = \begin{cases} g(t_k)+(n-m)\tilde{l}, \forall t\in[\tilde{\tau}_n,t_{k+1}] \\ g(t,n-1,m), \quad \forall t\in[t_k,\tilde{\tau}_n) \\ \quad g(t,n-1,m), \forall t\in[t_k,t_{k+1}], \text{ otherwise} \end{cases}, \text{ if } \tilde{\tau}_n\in[t_k,t_{k+1} \tag{12}$$

and

$$\tilde{n}=\tilde{n}+1 \text{ if } \tilde{\tau}_n\notin[t_k,t_{k+1}]$$

where

$$\tilde{\tau}_n=\frac{l-[g(t_k)+(n-m-1)\tilde{l}]}{v_n}+\tau_n$$

Proof: See the appendix A.2. □

Theorem 3 (Input-output model of traffic system T for yellow signal): Consider the given traffic signal σ with signal timing sequence $\tilde{\sigma}$ for the traffic system T shown in Figure 7. If $\sigma_{[t_k,t_{k+1}]}=\text{Yellow}$ then with initial condition $y(t_k)=g(t_k,n,m)$ and re-initialized conditions $n=m=0$ & $\tilde{\tau}_0=t_k$, for all $t\in[t_k,t_{k+1}]$ we have

$$y(t,n,m) = \begin{cases} y(t_k)+(n-m)\tilde{l}, \forall t\in[\tilde{\tau}_n,t_{k+1}] \\ y(t,n-1,m), \quad \forall t\in[t_k,\tilde{\tau}_n) \\ \quad y(t,n-1,m), \forall t\in[t_k,t_{k+1}], \text{ otherwise} \end{cases}, \text{ if } \tilde{\tau}_n\in[t_k,t_{k+1} \tag{13}$$

and

$$\tilde{n}=\tilde{n}+1 \text{ if } \tilde{\tau}_n\notin[t_k,t_{k+1}]$$

where

$$\tilde{\tau}_n=\frac{l-[y(t_k)+(n-m-1)\tilde{l}]}{v_n}+\tau_n$$

Proof: See the appendix A.3. □

A first principle model has been developed and tested via simulations and the results are shown in Figure 12, where red, green, and yellow colors stands for the switching on and off for the red, green, and yellow signal lights, respectively.

Figure 12. Simulation results for signaled corridor traffic queue length

5.2. Stochastic Distribution Based Traffic Queueing Length Model

As stated in the problem statement 4, we want to develop the stochastic distribution traffic queue model of the stochastic system S shown in Figure 8 using linear B-Spline model. For this purpose, the linear B-Spline stochastic model formulation will be formulated as follows.

At any time instant, the queueing length (output) PDF $\gamma(\phi,\psi)$ of the stochastic traffic flow system S shown in Figure 8 is approximated by

$$\gamma(\phi,\psi)=\sum_{i=1}^{n} w_i(\psi)B_i(\phi)+e(\phi,\psi), \qquad \phi\in[a,b] \tag{14}$$

where it has been denoted that

- $w_i(\psi),(i=1,2,\dots,n)$ are the weights and they are functions of the input ψ;
- $B_i(\phi),(i=1,2,\dots,n)$ are the pre-specified basis functions defined on the interval $\phi\in[a,b]$;
- $e(\phi,\psi)$ represents the approximation error which satisfies $\|e(\phi,\psi)\|\le\delta$, where δ is a pre-specified arbitrary small positive number.

This approximation is represented graphically in the following Figure 13.

Since $\gamma(\phi,\psi)$ is a PDF defined on the interval $[a,b]$, the following condition should be satisfied.

$$\int_a^b \gamma(\phi,\psi)d\phi=1 \tag{15}$$

Let

$$V(t)=[w_1 \;\; w_2 \;\; \dots \;\; w_{n-1}]'\in\mathbb{R}^{n-1} \tag{16}$$

$$C_0(\phi)=[B_1(\phi) \;\; B_2(\phi) \;\; \dots \;\; B_{n-1}(\phi)]\in\mathbb{R}^{1\times(n-1)} \tag{17}$$

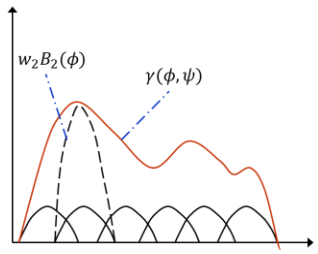

Figure 13: Probability Density Function Approximation for Queueing Length.

Then using (14), (15), (16), and (17) and ignoring the approximation error, we have

$$\int_a^b \gamma(\phi,\psi)d\phi = \left(\int_a^b C_0(\phi)d\phi\right)V(t) + \left(\int_a^b B_n(\phi)d\phi\right)w_n(\psi) = 1$$

This means that the following is true

$$w_n(\psi) = \frac{1-\left(\int_a^b C_0(\phi)d\phi\right)V(t)}{\int_a^b B_n(\phi)d\phi} \tag{18}$$

Therefore, by substituting (18) in (14) and using (16) and (17), we have

$$\gamma(\phi,\psi) = C_0(\phi)V(t) + B_n(\phi)\frac{1-\left(\int_a^b C_0(\phi)d\phi\right)V(t)}{\int_a^b B_n(\phi)d\phi}$$

$$\therefore \quad \gamma(\phi,\psi) = C_0(\phi)V(t) - \frac{B_n(\phi)\left(\int_a^b C_0(\phi)d\phi\right)V(t)}{\int_a^b B_n(\phi)d\phi} + \frac{B_n(\phi)}{\int_a^b B_n(\phi)d\phi} \tag{19}$$

Let

$$C(\phi) = C_0(\phi) - \frac{B_n(\phi)\left(\int_a^b C_0(\phi)d\phi\right)}{\int_a^b B_n(\phi)d\phi} \tag{20}$$

$$L(\phi) = \frac{B_n(\phi)}{\int_a^b B_n(\phi)d\phi} \tag{21}$$

Then by substituting (20) and (21) in (19), we obtain the following linear B-Spline stochastic model formulation as

$$\dot{V}(t) = AV(t) + B\psi$$
$$\gamma(\phi,\psi) = C(\phi)V(t) + L(\phi), \qquad \forall\phi \in [a,b] \tag{22}$$

This means that the stochastic distribution traffic queueing length model of the stochastic system S shown in Figure 8 can be developed using the linear B-Spline stochastic model formulation (22) by specifying weights (w_i), basis functions (B_i), and matrices (A,B) based on a number of vehicles data available from loop detectors.

5.3 Stochastic Distribution Signal Controller – a Preliminary Feedback Framework

Under the assumption that the signal control is realized via digital control framework**,** we can employ the following procedure to design a controller for the stochastic model developed using the linear B-Spline model. In this context, the discrete-time input-output version of the linear B-Spline model (22) is given by

$$\gamma(\phi,\psi_k) = a_1\gamma(\phi,\psi_{k-1}) + \cdots + a_n\gamma(\phi,\psi_{k-n}) + C(\phi)D_0\psi_k + C(\phi)D_1\psi_{k-1} + \cdots + C(\phi)D_m\psi_{k-m} \tag{23}$$

To design a control law ψ_k, we use the following cost function which considers the difference between the actual queue length PDF and its target PDF.

$$J = \int_a^b (\gamma(\phi,\psi_k) - g(\phi))^2 d\phi + R\psi_k^2 \tag{24}$$

where $g(\phi)$ is a target queueing length PDF and $R > 0$ is a pre-specified weight.

It can be seen that since the PDF of the queueing length is a function both in the time and the space, the first term in the performance function (24) is the distance of the two functions in the well-known functional space theory. By minimizing this performance function, we expect to obtain a preliminary closed loop framework for the control of the queueing length distribution using the timing of the traffic signals.

The purpose of such a control design is therefore to select the optimal signal timing ψ_k as the control input so that the distance between the actual and the targeted PDFs of the queueing length is made as close as possible. This would lead to a feedback control using the PDFs of the actual queueing length as the real-time feedback signal.

Assuming that all the information before sample number k is available as feedback signals to be used by the controller, we can define

$$\pi(\phi,k) = \sum_{i=1}^{n} a_i\gamma(\phi,\psi_{i-1}) + \sum_{j=1}^{m} C(\phi)D_j\psi_{k-j} - g(\phi) \tag{25}$$

Then using (23) and (25), we can have the PDF of the actual queueing length at sample number k as follows

$$\gamma(\phi,\psi_k) - g(\phi) = \pi(\phi,k) + C(\phi)D_0\psi_k \tag{26}$$

By substituting (26) in the cost function (24), it can be obtained that

$$J = \int_a^b (\pi(\phi,k) + C(\phi)D_0\psi_k)^2 d\phi + R\psi_k^2$$

$$\therefore \quad J = \int_a^b \pi(\phi,k)^2 d\phi + 2\left[\int_a^b \pi(\phi,k)C(\phi)D_0 d\phi\right]\psi_k + \left[\int_a^b (C(\phi)D_0)^2 d\phi + R\right]\psi_k^2 \tag{27}$$

Therefor the signal timing control law ψ_k at sample number k can be obtained by minimizing the quadratic cost function J in (27) as follows

$$\frac{dJ}{d\psi_k} = 0$$
$$\psi_k = -\frac{\int_a^b \pi(\phi,k)C(\phi)D_0 d\phi}{R+\int_a^b (C(\phi)D_0)^2 d\phi} \tag{28}$$

It has been shown that (28) constitutes a preliminary closed loop feedback control framework, where for the simple corridor traffic flow as shown in Figure 6 an illustrative response of the PDF of the actual queue length can be shown in the following figure.

□

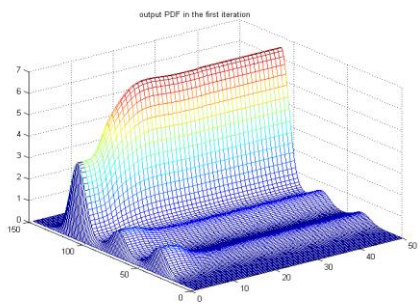

Figure 14. The typical response of the queue length PDF for a signalized corridor.

Indeed, such a 3D response of queueing length PDF will be a typical example of how the response will be like when the traffic signal is controlled using (28), where it is expected that the PDF of the queueing length will become narrow and narrow along with the progress of the time. This would indicate that the traffic flow is getting smother and smoother.

Indeed, traffic flow can also be modeled as stochastic distribution process using kinematic wave theory [17], where we can denote traffic flow as q and its density as k at a point in space x and time t, then it can be obtained that

$$q(x,t) = \frac{\partial N(x,t)}{\partial t}$$
$$k(x,t) = \frac{-\partial N(x,t)}{\partial x} \tag{29}$$

Using the above notations, the following model can be readily obtained

$$\frac{\partial k(x,t)}{\partial t} + \frac{\partial q(x,t)}{\partial x} = 0 \tag{30}$$

Let $w(q, x)$ be the average speed of a traffic stream/flow, then we have

$$w(q,x) = \frac{\partial k}{\partial q}$$

As in [18], the following figure would show the relationship between traffic flow, density, and speed profiles.

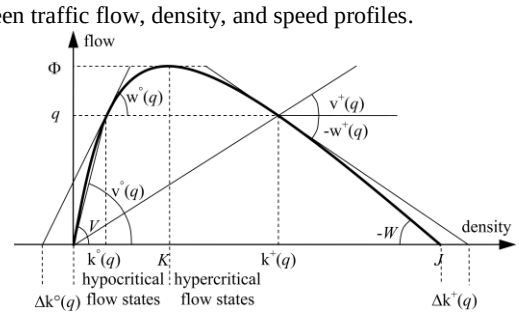

Figure 15. The relationship between flow, density and speed.

It can be seen that such a model can be modified to include the timing of the traffic signal as the control inputs, where stochastic distribution control can also be developed to control the probability density functions of the flow, density and the speed. This belongs to the scope of future study.

6 CONCLUSIONS

In this forward-looking paper, we have developed a recursive input-output traffic queue models and the stochastic distribution control model by taking the timing of red, green and yellow signals as the control input and the queue length as the output for traffic flow corridor by considering number of vehicles entered and left the signaled corridor as both deterministic and stochastic processes. Based on such models, we have proposed a real-time framework that can have potential to smooth the traffic with minimized energy consumption. In particular, we have developed a novel concept for the stochastic distribution control of the timing of the traffic signals which realizes the uniform distribution of the queue length over the concerned networked traffic flow area.

It can also been seen that queueing length has been used as a feedback signal to constitute a closed loop control. This requires that the queueing length be measurable. However, this can be a challenging issue albeit the image processing using camera capture picture can be used as a measure for the queueing length. In this case, the estimation of the queueing length distribution using speed profile can be considered. This belongs to the scope of our future work.

Appendix Main Results

A.1 One-way Single-lane Street Traffic

Proof of Theorem 1: Consider the given traffic signal σ with signal timing sequence $\tilde{\sigma}$ for the one-way single-lane street having the

loop detectors N and M shown in Figure 6. Suppose that $\sigma_{[t_k,t_{k+1}]} = \text{Red}$. Then we derive the red signal queue $r(t,n)$ as a function of time $t \in [t_k, t_{k+1}]$ and counter n. We initialize $r(t,n)$ using the values of yellow signal queue $y(t,n,m)$ and $\tilde{n}$ at the time t_k as

$$r(t_k) = y(t_k, n, m) + \tilde{n}\tilde{l}$$

and re-initialize $n = \tilde{n} = 0$.

Case 1 ($n = 0$): If no vehicle is detected at the loop detector N for all $t \in [t_k, t_{k+1}]$ then

$$r(t,n) = r(t_k), \qquad \forall t \in [t_k, t_{k+1}]$$

Case 2 ($n = 1$): If only one vehicle is detected at the loop detector N for all $t \in [t_k, t_{k+1}]$ then we use its detection time τ_1 and speed v_1 to determine whether it can reach the end of $r(t_k)$ in the interval $[t_k, t_{k+1}]$ or not. Therefore

$$r(t,n) = \begin{cases} r(t_k) + \tilde{l}, \forall t \in [\tilde{\tau}_1, t_{k+1}] \\ r(t_k), \qquad \forall t \in [t_k, \tilde{\tau}_1) \end{cases}, \text{if } \underbrace{\frac{l - r(t_k)}{v_1} + \tau_1}_{\tilde{\tau}_1} \in [t_k, t_{k+1}]$$

$$r(t_k), \qquad \forall t \in [t_k, t_{k+1}], \qquad \text{otherwise}$$

and $\tilde{n} = \tilde{n} + 1$ if $\tilde{\tau}_1 \notin [t_k, t_{k+1}]$.

Case 3 ($n > 1$): If more than one vehicles are detected at the loop detector N for all $t \in [t_k, t_{k+1}]$ then we use their detection times and speeds to determine whether they can reach the end of $r(t,n)$ in the interval $[t_k, t_{k+1}]$ or not. Therefore

$$r(t,n) = \begin{cases} r(t_k) + n\tilde{l}, & \forall t \in [\tilde{\tau}_n, t_{k+1}] \\ r(t, n-1), & \forall t \in [t_k, \tilde{\tau}_n) \end{cases}, \qquad \text{if} \quad \tilde{\tau}_n \in [t_k, t_{k+1}]$$

$$r(t, n-1), \qquad \forall t \in [t_k, t_{k+1}], \qquad \text{otherwise}$$

and $\tilde{n} = \tilde{n} + 1$ if $\tilde{\tau}_n \notin [t_k, t_{k+1}]$

where $\tilde{\tau}_n = \frac{l - r(t_k) - (n-1)\tilde{l}}{v_n} + \tau_n$.

Hence the claim follows by initializing $\tilde{\tau}_0 = t_k$ and combining the cases 1, 2, and 3.

□

A.2 Proof of Theorem 2: Consider the given traffic signal σ with signal timing sequence $\tilde{\sigma}$ for the one-way single-lane street having the loop detectors N and M shown in Figure 6. Suppose $\sigma_{[t_k,t_{k+1}]} = \text{Green}$. Then we derive the green signal queue $g(t,n,m)$ as a function of time $t \in [t_k, t_{k+1}]$ and counters n and m. We initialize $g(t,n,m)$ using the values of red signal queue $r(t,n)$ and $\tilde{n}$ at the time t_k as

$$g(t_k) = r(t_k, n) + \tilde{n}\tilde{l}$$

and re-initialize $n = \tilde{n} = m = 0$.

Case 1 ($n = 0$): If no vehicle is detected at the loop detector N for all $t \in [t_k, t_{k+1}]$ then

$$g(t,n,m) = g(t_k) - m\tilde{l}, \qquad \forall t \in [t_k, t_{k+1}]$$

Case 2 ($n = 1$): If only one vehicle is detected at the loop detector N for all $t \in [t_k, t_{k+1}]$ then we use its detection time τ_1 and speed v_1 to determine whether it can reach the end of $g(t,n,m)$ in the interval $[t_k, t_{k+1}]$ or not. Therefore

$$g(t,n,m) = \begin{cases} g(t_k) - (m-1)\tilde{l}, & \forall t \in [\tilde{\tau}_1, t_{k+1}] \\ g(t_k) - m\tilde{l}, & \forall t \in [t_k, \tilde{\tau}_1) \end{cases}, \qquad \text{if} \quad \tilde{\tau}_1 \in [t_k, t_{k+1}]$$

$$g(t_k) - m\tilde{l}, \qquad \forall t \in [t_k, t_{k+1}], \qquad \text{otherwise}$$

and $\tilde{n} = \tilde{n} + 1$ if $\tilde{\tau}_1 \notin [t_k, t_{k+1}]$

where $\tilde{\tau}_1 = \frac{l - [g(t_k) - m\tilde{l}]}{v_1} + \tau_1$.

Case 3 ($n > 1$): If more than one vehicles are detected at the loop detector N for all $t \in [t_k, t_{k+1}]$ then we use their detection times and speeds to determine whether they can reach the end of $g(t,n,m)$ in the interval $[t_k, t_{k+1}]$ or not. Therefore

$$g(t,n,m) = \begin{cases} g(t_k) + (n-m)\tilde{l}, & \forall t \in [\tilde{\tau}_n, t_{k+1}] \\ g(t, n-1, m), & \forall t \in [t_k, \tilde{\tau}_n) \end{cases}, \qquad \text{if} \quad \tilde{\tau}_n \in [t_k, t_{k+1}]$$

$$g(t, n-1, m), \qquad \forall t \in [t_k, t_{k+1}], \qquad \text{otherwise}$$

and $\tilde{n} = \tilde{n} + 1$ if $\tilde{\tau}_n \notin [t_k, t_{k+1}]$

where $\tilde{\tau}_n = \frac{l - [g(t_k) + (n-m-1)\tilde{l}]}{v_n} + \tau_n$.

Hence the claim follows by initializing $\tilde{\tau}_0 = t_k$ and combining the cases 1, 2, and 3. □

A.3 Proof of Theorem 3: The proof is similar to the Theorem 2 proof. We initialize $y(t,n.m)$ as $y(t_k) = g(t_k, n, m)$ and reinitialize $n = m = 0$. Since the time interval for yellow signal is very small in practice, we do not reinitialize the counter $\tilde{n}$. □

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REFERENCES

[1] H M Abdul Aziz, Hong Wang, Stanley Young, Joshua Sperling and John M. Beck, Synthesis Study on Transitions in Signal Infrastructure and Control Algorithms for Connected and Automated Transportation, Oak Ridge National Laboratory Report, ORNL/TM-2017/280

[2] Yang, S., & Yang, X. (2014). The Application of the Queuing Theory in the Traffic Flow of Intersection. International Journal of Mathematical, Computational, Physical, Electrical and Computer Engineering, 8(6), 986-989.

[3] Liu, Y., & Lee, K. (2009). Modeling Signalized Intersection using Queueing Theory. University of Florida.

[4] Mirchandani, P., & Zou, N. (2007, March). Queuing Models for Analysis of Traffic Adaptive Signal Control. IEEE Transactions on Intelligent Transportation Systems, 8(1), 50-59.

[5] Gholami, A., Andalibian, R., & Tian, Z. (2016, November). Estimating Intersection Turning Volumes from Actuated Traffic Signal Information. Journal of Traffic and Transportation Engineering, 3(6), 507-519.

[6] Grether, D., Neumann, A., & Nagel, K. (2012). Simulation of Urban Traffic Control: A Queue Model Approach. Procedia Computer Science, 10, 808-814.

[7] Akcelik, R., & Besley, M. (2002). Queue Discharge Flow and Speed Models for Signalised Intersections. Proceedings of the 15th International Symposium on Transportation and Traffic Theory (pp. 99-118). Adelaide, Australia: Transportation and Traffic Theory in the 21st Century.

[8] Viti, F., & Zuylen, H. (2004). Modeling Queues at Signalized Intersections. Transportation Research Record: Journal of the Transportation Research Board, 1883, 68-77.

[9] Akcelik, R. (2011). Time-dependent Expressions for Delay, Stop Rate, and Queue Length at Traffic Signals. Victoria, Australia: Australian Road Research Board.

[10] Rouphail, N., Tarko, A., & Li, J. (1992). Traffic Flow at Signalized Intersections.

[11] Yu, X., & Recher, W. (2006,). Stochastic Adaptive Control Model for Traffic Signal Systems. Transportation Research Part C: Emerging Technologies, 14(4), 263-282.

[12] Wey, W. (2000). Model Formulation and Solution Algorithm of Traffic Signal Control in an Urban Network. Computers, Environment and Urban Systems, 24, 355-377.

[13] List, G., & Mashayekhi, M. (2016). A Modular Colored Stochastic Petri Net for Modeling and Analysis of Signalized Intersections. IEEE Transactions on Intelligent Transportation Systems, 17(3), 701-713.

[14] Wang, H. (2015). Potentials of using Stochastic Distribution Control for Traffic Flows. International Workshop on Transportation Research. Shandong Jiaotong University.

[15] Jabari, S., & Liu, H. (2012). A stochastic Model of Traffic Flow: Theoretical Foundations. Transportation Research Part B: Methodological, 46(1), 156-174.

[16] Wang, H. (2000). Bounded Dynamic Stochastic Systems. Springer.

[17] Lighthill, M. J., G. B. Whitham. 1955. A theory of traffic flow on long crowded roads. Proc. Roy. Soc. A229 317–345.

[18] Gentile, Guido (2010). The General Link Transmission Model for dynamic network loading and a comparison with the DUE algorithm, New developments in transport planning: advances in Dynamic Traffic Assignment 178: 153