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Model reduction for nonlinear dynamical systems: discrete optimality and time parallelism

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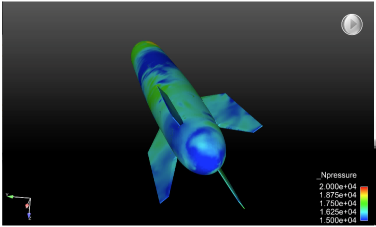
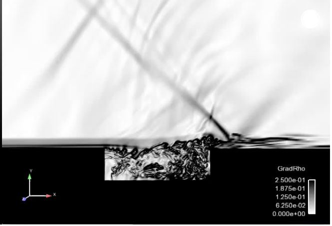
Computational barrier at Sandia



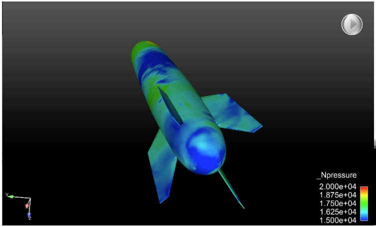


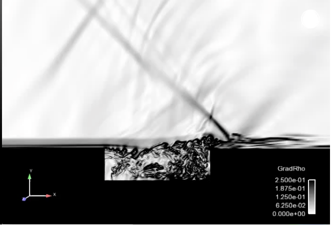
- CFD model
  - 100 million cells
  - 200,000 time steps
- High simulation costs
  - 6 weeks, 5000 cores
  - 6 runs maxes out Cielo

Barrier

- Fast-turnaround design
- Uncertainty quantification

Computational barrier at Sandia





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Objective: break barrier via nonlinear model reduction

ROM: state of the art [Benner et al., 2015]

- Linear time-invariant systems: mature [Antoulas, 2005]
  - Balanced truncation [Moore, 1981]
  - Empirical balanced truncation [Willcox and Peraire, 2002, Rowley, 2005, Or and Speyer, 2010, Ma et al., 2011]
  - Moment matching [Bai, 2002, Freund, 2003, Gallivan et al., 2004, Gugercin et al., 2008, Baur et al., 2011]
  - Loewner framework [Lefteriu and Antoulas, 2010, Ionita and Antoulas, 2014]
  - \+ *Reliable*: guaranteed stability, moment matching, optimality
  - \+ *Certified*: *a priori* error bounds

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- Elliptic/parabolic PDEs (FEM): mature [Rozza et al., 2008]
  - Reduced-basis method

    [Prud'Homme et al., 2001, Veroy et al., 2003, Barrault et al., 2004]
  - Subsystem-based reduced-basis method

    [Maday and Rønquist, 2002, Phuong Huynh et al., 2013, Eftang and Patera, 2013]

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- Nonlinear dynamical systems: unproven
  - Proper orthogonal decomposition (POD)–Galerkin
  - \- *Not reliable*: Stability and accuracy not guaranteed
  - \- *Not certified*: error bounds not sharp

Our research goal

Nonlinear model-reduction methods that are
accurate, low cost, certified, and reliable.

+ Accuracy
  - Improve projection technique [C. et al., 2011a, C. et al., 2015a]
  - Preserve problem structure [C. et al., 2012, C. et al., 2015c]
+ Low cost
  - Sample-mesh approach [C. et al., 2011b, C. et al., 2013]
  - Leverage time-domain data [C. et al., 2015b]
+ Certification
  - Error bounds [C. et al., 2015a]
  - Statistical error modeling [Drohmann and C., 2015]
+ Reliability
  - *A posteriori* h-refinement [C., 2015]

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Collaborators: M. Barone (Sandia), H. Antil (GMU)

POD–Galerkin: offline data collection

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{f}(\boldsymbol{x}; t, \boldsymbol{\mu}); \quad \boldsymbol{x}(0, \boldsymbol{\mu}) = \boldsymbol{x}^0(\boldsymbol{\mu}), \quad t \in [0, T], \quad \boldsymbol{\mu} \in \mathcal{D}$$

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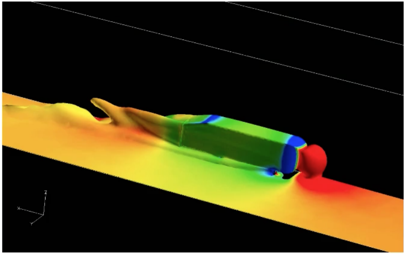
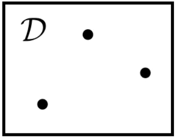
1 Collect 'snapshots' of the state

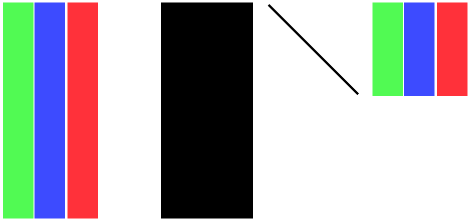




POD–Galerkin: offline data collection

2 Data compression

- Compute SVD: $[\,\mathbf{X}_1\ \mathbf{X}_2\ \mathbf{X}_3\,] = \quad \mathbf{U} \quad \mathbf{\Sigma} \quad \mathbf{V}^T$

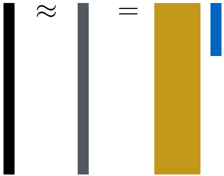


- Truncate: $\mathbf{\Phi} = [\boldsymbol{u}_1 \ \cdots \ \boldsymbol{u}_p]$

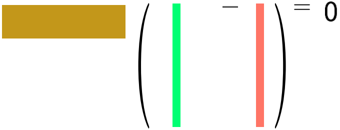
POD–Galerkin: online projection

Full-order model: $$\boxed{\frac{d\boldsymbol{x}}{dt} = \boldsymbol{f}(\boldsymbol{x}; t, \boldsymbol{\mu}), \quad \boldsymbol{x}(0, \boldsymbol{\mu}) = \boldsymbol{x}^0(\boldsymbol{\mu})}$$

1 $\boldsymbol{x}(t) \approx \tilde{\boldsymbol{x}}(t) = \boldsymbol{\Phi}\hat{\boldsymbol{x}}(t)$



2 $\boldsymbol{\Phi}^T\left(\boldsymbol{f}(\tilde{\boldsymbol{x}}; t, \boldsymbol{\mu}) - \frac{d\tilde{\boldsymbol{x}}}{dt}\right) = 0$



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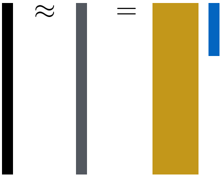

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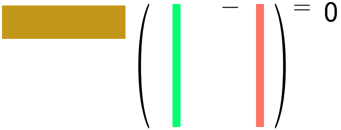
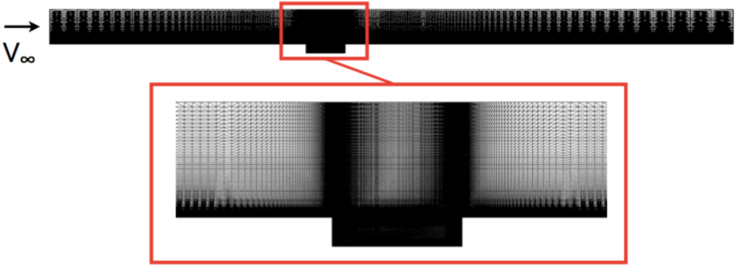


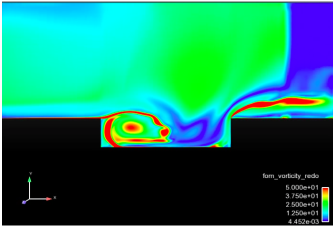
Galerkin ROM: $$\boxed{\frac{d\hat{\boldsymbol{x}}}{dt} = \boldsymbol{\Phi}^T\boldsymbol{f}(\boldsymbol{\Phi}\hat{\boldsymbol{x}}; t, \boldsymbol{\mu}), \quad \hat{\boldsymbol{x}}(0, \boldsymbol{\mu}) = \boldsymbol{\Phi}^T\boldsymbol{x}^0(\boldsymbol{\mu})}$$

Cavity-flow problem. Collaborator: M. Barone (SNL)

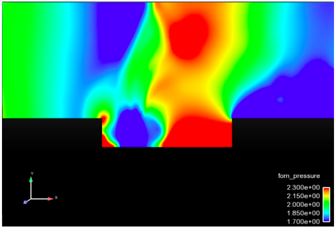


- Unsteady Navier–Stokes
- DES turbulence model
- 1.2 million degrees of freedom
- $\text{Re} = 6.3 \times 10^6$
- $M_\infty = 0.6$
- CFD code: AERO-F [Farhat et al., 2003]

Full-order model responses

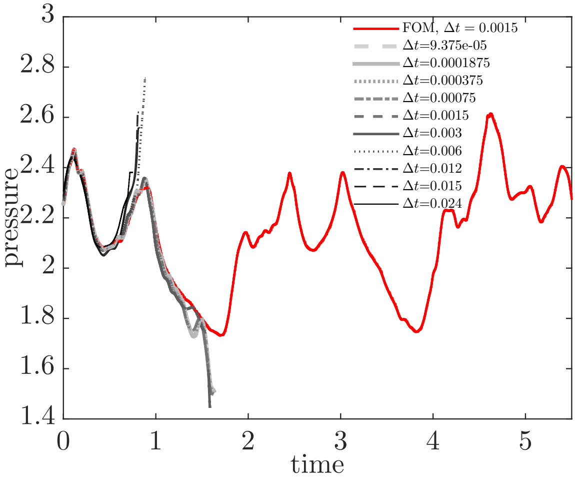


vorticity field



pressure field

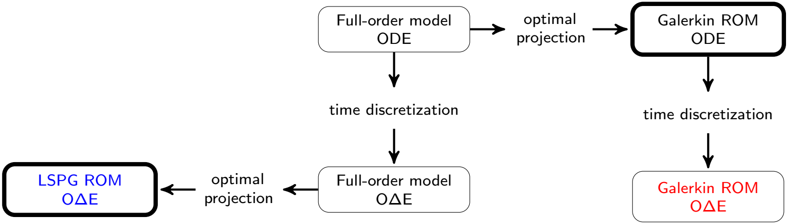
POD–Galerkin failure (basis dimension 204)



- Galerkin ROMs unstable

How to construct a ROM for nonlinear dynamical systems?

- Optimize then discretize? (Galerkin)
- Discretize then optimize? (Least-squares Petrov–Galerkin)



- Related investigation (LTI) [Lattimer, 2016]

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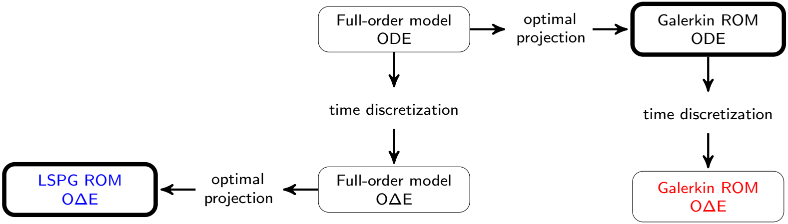

- Outstanding questions:
  1. Which notion of optimality is better in practice?
  2. Are the two techniques ever equivalent?
  3. Discrete-time error bounds?
  4. Time step selection?
- Related investigation (LTI) [Lattimer, 2016]

Full-order model
ODE

↓

time discretization

↓

Full-order model
OΔE

Full-order model (FOM)

- ODE: time continuous

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{f}(\boldsymbol{x}, t), \quad \boldsymbol{x}(0) = \boldsymbol{x}^0, \quad t \in [0, T]$$

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$$\boldsymbol{r}^n(\boldsymbol{x}) := \alpha_0 \boldsymbol{x} - \Delta t \beta_0 \boldsymbol{f}(\boldsymbol{x}, t^n) + \sum_{j=1}^{k} \alpha_j \boldsymbol{x}^{n-j} - \Delta t \sum_{j=1}^{k} \beta_j \boldsymbol{f}\left(\boldsymbol{x}^{n-j}, t^{n-j}\right)$$

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$$\boldsymbol{x}^n = \boldsymbol{x}^{n-1} + \Delta t \sum_{i=1}^{s} b_i \boldsymbol{x}_i^n \text{ (explicit state update)}$$

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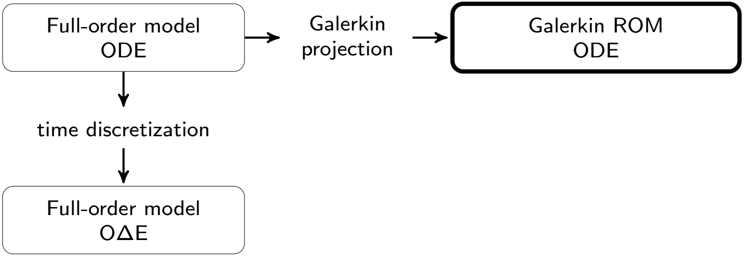
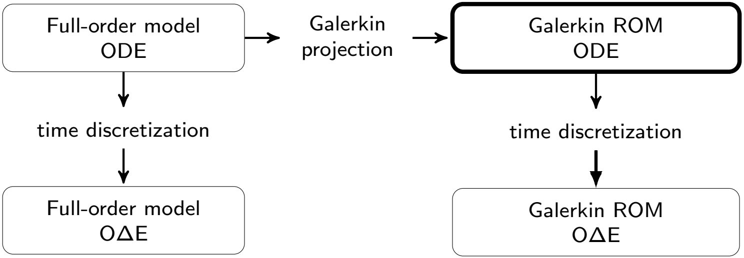
This talk focuses on linear multistep schemes.

Galerkin ROM: first optimize



Galerkin: first optimize, then discretize



Galerkin ROM

- ODE

$$\frac{d\hat{\boldsymbol{x}}}{dt} = \boldsymbol{\Phi}^T \boldsymbol{f}(\boldsymbol{\Phi}\hat{\boldsymbol{x}}, t), \quad \hat{\boldsymbol{x}}(0) = \boldsymbol{\Phi}^T \boldsymbol{x}^0, \quad t \in [0, T]$$

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+ Continuous velocity $\frac{d\hat{\boldsymbol{x}}}{dt}$ is optimal

Theorem (Galerkin ROM: continuous optimality)

The Galerkin ROM velocity minimizes the time-continuous FOM residual:

$$\frac{d\tilde{\boldsymbol{x}}}{dt}(\boldsymbol{x}, t) = \arg \min_{\boldsymbol{v} \in range(\boldsymbol{\Phi})} \|\boldsymbol{v} - \boldsymbol{f}(\boldsymbol{x}, t)\|_2^2$$

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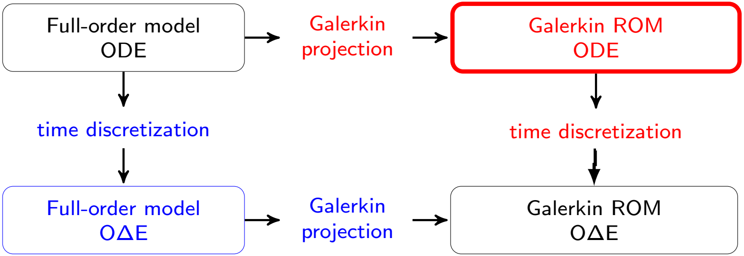
Can we fix this? Will doing so help?

Galerkin ROM: Commutativity

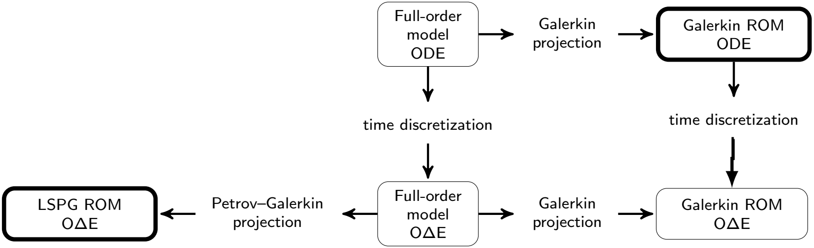
Theorem

Projection and time discretization are commutative for Galerkin ROMs:

$$\hat{\boldsymbol{r}}^n(\hat{\boldsymbol{x}}) = \boldsymbol{\Phi}^T \boldsymbol{r}^n(\boldsymbol{\Phi}\hat{\boldsymbol{x}})$$



LSPG ROM: first discretize, then optimize



LSPG ROM

- FOM OΔE

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$$\hat{\boldsymbol{x}}^n = \arg\min_{\hat{\boldsymbol{z}} \in \mathbb{R}^p} \|\boldsymbol{A}\boldsymbol{r}^n(\boldsymbol{\Phi}\hat{\boldsymbol{z}})\|_2^2.$$

$$\Updownarrow$$

$$\boldsymbol{\Psi}^n(\hat{\boldsymbol{x}}^n)^T \boldsymbol{r}^n(\boldsymbol{\Phi}\hat{\boldsymbol{x}}^n) = 0, \quad \boldsymbol{\Psi}^n(\hat{\boldsymbol{x}}) := \boldsymbol{A}^T\boldsymbol{A}\frac{\partial \boldsymbol{r}^n}{\partial \boldsymbol{x}}(\boldsymbol{\Phi}\hat{\boldsymbol{x}})\boldsymbol{\Phi}$$

  - $\boldsymbol{A} = \boldsymbol{I}$: LSPG [LeGresley, 2006, Bui-Thanh et al., 2008, C. et al., 2011a]

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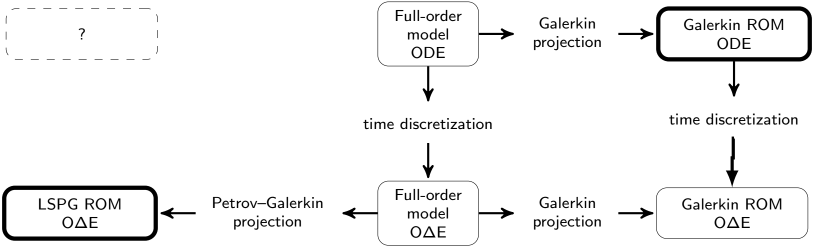
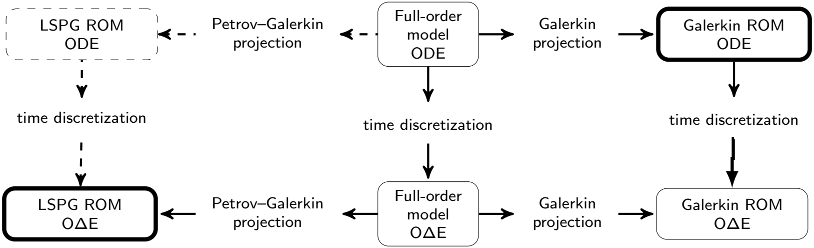
\+ Discrete solution is optimal

Does the LSPG ROM have a time-continuous representation?



Does the LSPG ROM have a time-continuous representation?

Sometimes.



LSPG ROM: continuous representation

Theorem

The LSPG ROM is equivalent to applying a Petrov–Galerkin projection to the FOM ODE with test basis

$$\boldsymbol{\Psi}(\hat{\boldsymbol{x}}, t) = \boldsymbol{A}^T\boldsymbol{A}\left(\alpha_0\boldsymbol{I} - \Delta t\beta_0\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}(\boldsymbol{x}^0 + \boldsymbol{\Phi}\hat{\boldsymbol{x}}, t)\right)\boldsymbol{\Phi}$$

if

1. $\beta_j = 0$, $j \geq 1$ *(e.g., a single-step method),*
2. *the velocity* $\boldsymbol{f}$ *is linear in the state, or*
3. $\beta_0 = 0$ *(i.e., explicit schemes).*

LSPG ROM: continuous representation

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The LSPG ROM is equivalent to applying a Petrov–Galerkin projection to the FOM ODE with test basis

$$\boldsymbol{\Psi}(\hat{\boldsymbol{x}}, t) = \boldsymbol{A}^T\boldsymbol{A}\left(\alpha_0\boldsymbol{I} - \Delta t\beta_0\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}(\boldsymbol{x}^0 + \boldsymbol{\Phi}\hat{\boldsymbol{x}}, t)\right)\boldsymbol{\Phi}$$

if

1. $\beta_j = 0$, $j \geq 1$ *(e.g., a single-step method),*
2. *the velocity* $\boldsymbol{f}$ *is linear in the state, or*
3. $\beta_0 = 0$ *(i.e., explicit schemes).*

Time-continuous test basis depends on time-discretization parameters!

Are the two approaches ever equivalent?

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- Galerkin: $$\boldsymbol{\Phi}^T \boldsymbol{r}^n(\boldsymbol{\Phi}\hat{\boldsymbol{x}}^n) = 0$$
- LSPG: $$\boldsymbol{\Psi}^n(\hat{\boldsymbol{x}}^n)^T \boldsymbol{r}^n(\boldsymbol{\Phi}\hat{\boldsymbol{x}}^n) = 0$$

Does $\boldsymbol{\Psi}^n(\hat{\boldsymbol{x}}^n) = \boldsymbol{\Phi}$ ever?

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Does $\boldsymbol{\Psi}^n(\hat{\boldsymbol{x}}^n) = \boldsymbol{\Phi}$ ever?

Yes.

$$\boldsymbol{\Psi}^n(\hat{\boldsymbol{x}}) := \boldsymbol{A}^T\boldsymbol{A}\frac{\partial \boldsymbol{r}^n}{\partial \boldsymbol{x}}(\boldsymbol{\Phi}\hat{\boldsymbol{x}})\boldsymbol{\Phi} = \boldsymbol{A}^T\boldsymbol{A}\left(\alpha_0\boldsymbol{I} - \Delta t\beta_0\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}(\boldsymbol{\Phi}\hat{\boldsymbol{x}}, t^n)\right)\boldsymbol{\Phi}$$

Theorem

The two approaches are equivalent ($\boldsymbol{\Psi}^n(\hat{\boldsymbol{x}}) = \boldsymbol{\Phi}$)

1. *in the limit of* $\Delta t \to 0$ *with* $\boldsymbol{A} = 1/\sqrt{\alpha_0}\boldsymbol{I}$,
2. *if the scheme is explicit* ($\beta_0 = 0$) *with* $\boldsymbol{A} = 1/\sqrt{\alpha_0}\boldsymbol{I}$, *or*
3. *if* $\frac{\partial \boldsymbol{r}^n}{\partial \boldsymbol{x}}$ *is positive definite with* $[\frac{\partial \boldsymbol{r}^n}{\partial \boldsymbol{x}}]^{-1} = \boldsymbol{A}^T\boldsymbol{A}$.

Discrete-time error bound

Theorem

If the following conditions hold:

1. $\boldsymbol{f}(\cdot, t)$ *is Lipschitz continuous with Lipschitz constant* κ*, and*
2. Δt *is such that* $0 < h := |\alpha_0| - |\beta_0|\kappa\Delta t$*,*

then

$$\|\delta\boldsymbol{x}_G^n\| \le \frac{\Delta t}{h}\sum_{\ell=0}^{k}|\beta_\ell|\left\|\left(\boldsymbol{I} - \mathbb{V}\right)\boldsymbol{f}\left(\boldsymbol{x}_0 + \boldsymbol{\Phi}\hat{\boldsymbol{x}}_G^{n-\ell}\right)\right\| + \frac{1}{h}\sum_{\ell=1}^{k}\left(|\beta_\ell|\kappa\Delta t + |\alpha_\ell|\right)\|\delta\boldsymbol{x}_G^{n-\ell}\|$$

$$\|\delta\boldsymbol{x}_L^n\| \le \frac{\Delta t}{h}\sum_{\ell=0}^{k}|\beta_\ell|\left\|\left(\boldsymbol{I} - \mathbb{P}^n\right)\boldsymbol{f}\left(\boldsymbol{x}_0 + \boldsymbol{\Phi}\hat{\boldsymbol{x}}_L^{n-\ell}\right)\right\| + \frac{1}{h}\sum_{\ell=1}^{k}\left(|\beta_\ell|\kappa\Delta t + |\alpha_\ell|\right)\|\delta\boldsymbol{x}_L^{n-\ell}\|,$$

with

- $\delta\boldsymbol{x}_G^n := \boldsymbol{x}_\star^n - \boldsymbol{\Phi}\hat{\boldsymbol{x}}_G^n$.
- $\delta\boldsymbol{x}_L^n := \boldsymbol{x}_\star^n - \boldsymbol{\Phi}\hat{\boldsymbol{x}}_L^n$
- $\mathbb{V} := \boldsymbol{\Phi}\boldsymbol{\Phi}^T$
- $\mathbb{P}^n := \boldsymbol{\Phi}\left(\left(\boldsymbol{\Psi}^n\right)^T\boldsymbol{\Phi}\right)^{-1}\left(\boldsymbol{\Psi}^n\right)^T$

LSPG ROM yields a smaller error bound

Theorem (Backward Euler)

If conditions (1) and (2) hold, then

$$\|\delta \boldsymbol{x}_G^n\| \leq \Delta t \sum_{j=0}^{n-1} \frac{1}{(h)^{j+1}} \underbrace{\left\| \left(\boldsymbol{I} - \mathbb{V}\right) \boldsymbol{f}\left(\boldsymbol{x}_0 + \boldsymbol{\Phi}\hat{\boldsymbol{x}}_G^{n-j}\right) \right\|}_{\varepsilon_G^{n-j}}$$

$$\|\delta \boldsymbol{x}_L^n\| \leq \Delta t \sum_{j=0}^{n-1} \frac{1}{(h)^{j+1}} \underbrace{\left\| \left(\boldsymbol{I} - \mathbb{P}^{n-j}\right) \boldsymbol{f}\left(\boldsymbol{x}_0 + \boldsymbol{\Phi}\hat{\boldsymbol{x}}_L^{n-j}\right) \right\|}_{\varepsilon_L^{n-j}}$$

$$\varepsilon_G^k = \left\|\boldsymbol{\Phi}\hat{\boldsymbol{x}}_G^k - \Delta t \boldsymbol{f}\left(\boldsymbol{x}_0 + \boldsymbol{\Phi}\hat{\boldsymbol{x}}_G^k\right) - \boldsymbol{\Phi}\hat{\boldsymbol{x}}_G^{k-1}\right\|$$

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Corollary (LSPG smaller error bound)

If $\hat{\boldsymbol{x}}_L^{k-1} = \hat{\boldsymbol{x}}_G^{k-1}$, *then* $\varepsilon_L^k \leq \varepsilon_G^k$.

LSPG ROM has an interesting time-step dependence

LSPG ROM has an interesting time-step dependence

Corollary (Backward Euler)

Define

- $\Delta\hat{\boldsymbol{x}}_L^j := \hat{\boldsymbol{x}}_L^j - \hat{\boldsymbol{x}}_L^{j-1}$ *and*
- $\Delta\bar{\boldsymbol{x}}^j$*: full-space solution increment from* $\hat{\boldsymbol{x}}_L^{j-1}$*.*

Then, the LSPG error can also be bounded as

$$\|\delta\boldsymbol{x}_L^n\| \leq \Delta t(1+\kappa\Delta t)\sum_{j=0}^{n-1}\frac{\mu^{n-j}}{(h)^{j+1}}\|\boldsymbol{f}(\hat{\boldsymbol{x}}_L^{j-1}+\Delta\bar{\boldsymbol{x}}^{n-j})\|$$

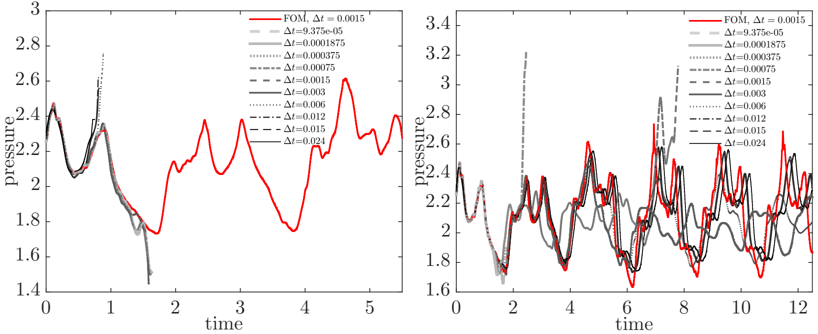
with $\mu^j := \|\boldsymbol{\Phi}\Delta\hat{\boldsymbol{x}}_L^j - \Delta\bar{\boldsymbol{x}}^j\|/\|\Delta\bar{\boldsymbol{x}}^j\|$*.*

Effect of decreasing Δt:

+ The terms $\Delta t(1+\kappa\Delta t)$ and $1/(h)^{j+1}$ decrease
- The number of total time instances n increases

? The term μ^{n-j} may increase or decrease, depending on the spectral content of the basis $\boldsymbol{\Phi}$

Galerkin and LSPG responses for basis dimension $p = 204$

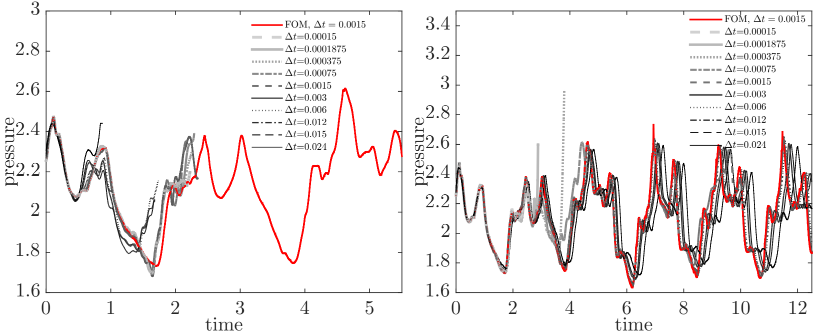


(a) Galerkin (b) LSPG

- Galerkin ROMs unstable for long time intervals

\+ LSPG ROMs accurate and stable (most time steps)

Galerkin and LSPG responses for basis dimension $p = 564$

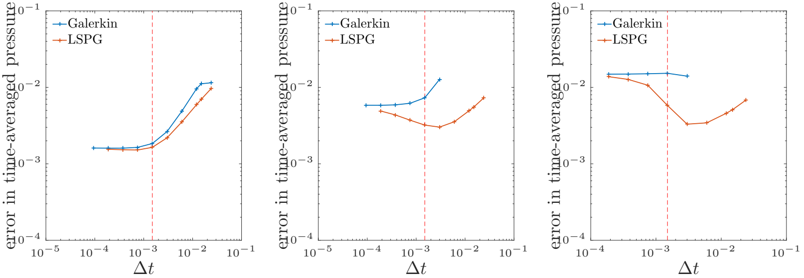


(c) Galerkin (d) LSPG

- Galerkin ROMs remain unstable

+ LSPG ROMs improve

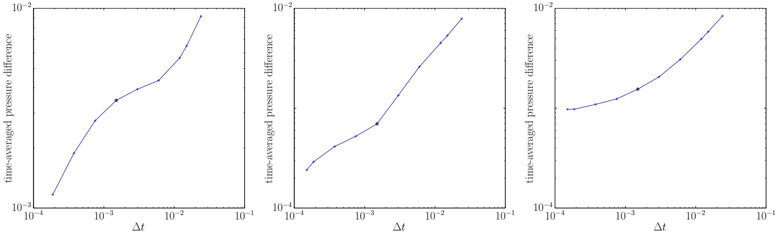
LSPG ROM: superior performance ($p = 204$)



(e) $0 \leq t \leq 0.55$ (f) $0 \leq t \leq 1.1$ (g) $0 \leq t \leq 1.54$

✓ LSPG ROM yields a smaller error for all time intervals and time steps.

Limiting equivalence

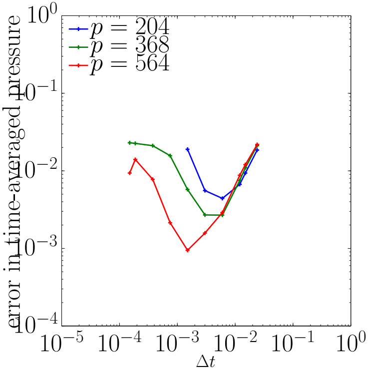


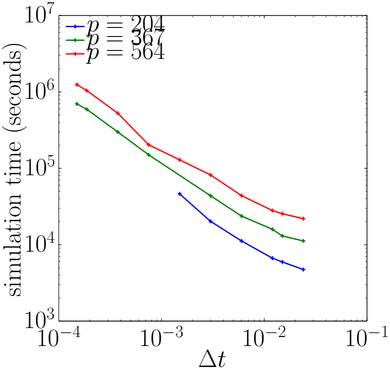
(h) $p = 204$ (i) $p = 368$ (j) $p = 564$

Galerkin/LSPG difference in the stable Galerkin interval $0 \leq t \leq 1.1$.

✓ The LSPG ROM converges to Galerkin as $\Delta t \rightarrow 0$.

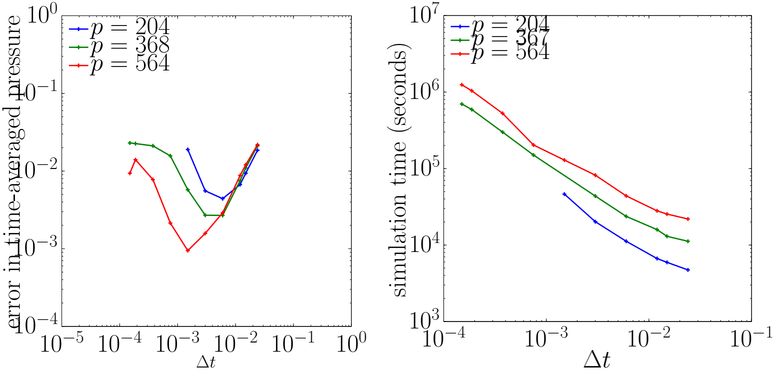
LSPG performance ($t \leq 12.5$ sec)





✓ An intermediate Δt produces the lowest error and better speedup.

LSPG performance ($t \leq 12.5$ sec)



✓ An intermediate Δt produces the lowest error and better speedup.

$p = 564$ case:

- $\Delta t = 1.875 \times 10^{-4}$ sec: relative error $=$ 1.40%, time $=$ 289 hrs
- $\Delta t = 1.5 \times 10^{-3}$ sec: relative error $=$ 0.095%, time $=$ 35.8 hrs

Summary: Improve projection technique

- *Galerkin*: projection and time-discretization are commutative
- *LSPG*: a continuous representation sometimes exists
- Equivalence conditions
  1. Limit of $\Delta t \to 0$
  2. Explicit schemes
  3. Positive definite residual Jacobians
- Discrete-time error bounds
  - LSPG ROM yields smaller error bound than Galerkin
  - Ambiguous role of time step Δt
- Numerical experiments
  - LSPG ROM yields a smaller error than Galerkin
  - Equivalent as $\Delta t \to 0$
  - Error minimized for intermediate Δt
- **Reference**: C., Barone, and Antil. Galerkin v. least-squares Petrov–Galerkin projection in nonlinear model reduction. *arXiv e-print*, (1504.03749), 2015.

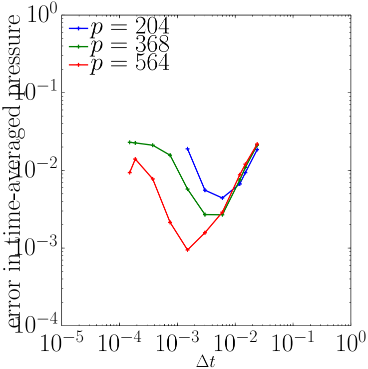
Our research goal

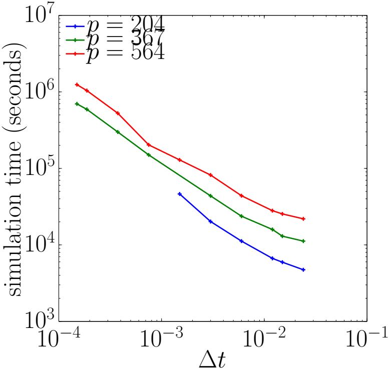
Nonlinear model-reduction methods that are
accurate, low cost, certified, and reliable.

- + Accuracy
  - Improve projection technique [C. et al., 2011a, C. et al., 2015a]
  - Preserve problem structure [C. et al., 2012, C. et al., 2015c]
- + Low cost
  - Sample-mesh approach [C. et al., 2011b, C. et al., 2013]
  - Leverage time-domain data [C. et al., 2015b]
- + Certification
  - Error bounds [C. et al., 2015a]
  - Statistical error modeling [Drohmann and C., 2015]
- + Reliability
  - *A posteriori* h-refinement [C., 2015]

Collaborators: C. Farhat, J. Cortial (Stanford)

LSPG performance ($t \leq 2.5$ sec)





+ Always sub-3% errors
- More expensive than the FOM
  - FOM simulation: 1 hour, 48 CPU
  - LSPG ROM simulation (fastest): 1.3 hours, 48 CPU

Hyper-reduction via Gappy POD [Everson and Sirovich, 1995]

$$\hat{\boldsymbol{x}}^n = \arg\min_{\hat{\boldsymbol{z}} \in \mathbb{R}^p} \|\boldsymbol{A}\boldsymbol{r}^n(\boldsymbol{\Phi}\hat{\boldsymbol{z}})\|_2^2.$$

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Can we select $\boldsymbol{A}$ *to make this inexpensive?*

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1. $\boldsymbol{r}^n(\boldsymbol{x}) \approx \tilde{\boldsymbol{r}}^n(\boldsymbol{x}) = \boldsymbol{\Phi}_R\hat{\boldsymbol{r}}^n(\boldsymbol{x})$
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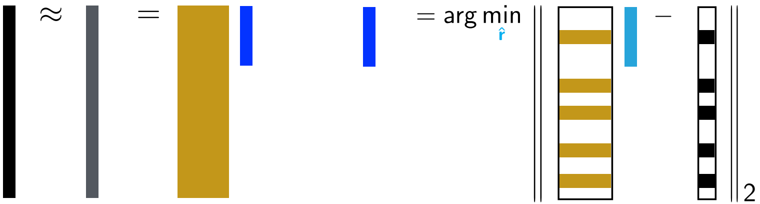

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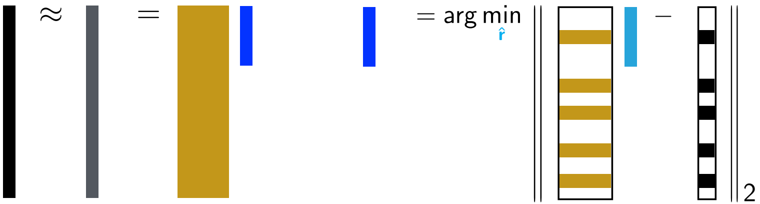

$$\hat{\boldsymbol{x}}^n = \arg\min_{\hat{\boldsymbol{z}}\in\mathbb{R}^p} \left\|\tilde{\boldsymbol{r}}^n\left(\boldsymbol{\Phi}\hat{\boldsymbol{z}}\right)\right\|_2^2 = \arg\min_{\hat{\boldsymbol{z}}\in\mathbb{R}^p} \left\|\boldsymbol{\Phi}_R\hat{\boldsymbol{r}}^n\left(\boldsymbol{\Phi}\hat{\boldsymbol{z}}\right)\right\|_2^2 = \arg\min_{\hat{\boldsymbol{z}}\in\mathbb{R}^p} \left\|\hat{\boldsymbol{r}}^n\left(\boldsymbol{\Phi}\hat{\boldsymbol{z}}\right)\right\|_2^2$$

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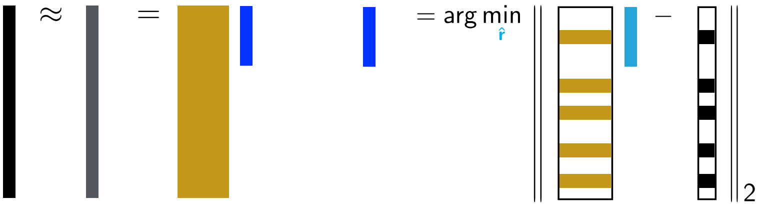

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+ GNAT: $\boldsymbol{A} = (\boldsymbol{P}\boldsymbol{\Phi}_R)^+\boldsymbol{P}$ leads to low-cost
  - *Offline*: Construct $\boldsymbol{\Phi}_R$ (POD) and $\boldsymbol{P}$ (greedy method)

Sample mesh: HPC implementation

$$\hat{\boldsymbol{x}}^n = \arg\min_{\hat{\boldsymbol{z}}\in\mathbb{R}^p} \| (\boldsymbol{P}\boldsymbol{\Phi}_R)^+ \boldsymbol{P}\boldsymbol{r}^n (\boldsymbol{\Phi}\hat{\boldsymbol{z}}) \|_2^2$$

- *Key*: GNAT samples only a few entries of the residual $\boldsymbol{P}\boldsymbol{r}^n$
- *Idea*: Extract minimal subset of the mesh

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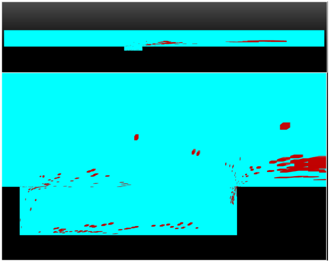


- Sample mesh: 4.1% nodes, 3.0% cells

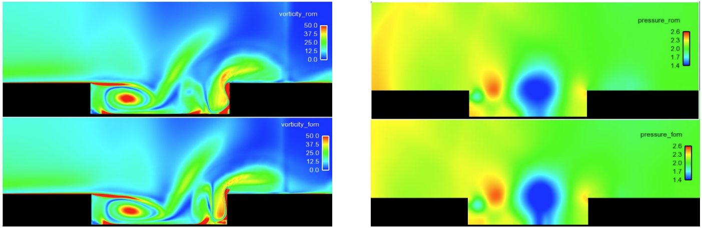
\+ Small problem size: can run on many fewer cores

GNAT performance ($t \leq 12.5$ sec)

vorticity field | pressure field

GNAT ROM

FOM



+ $< 1\%$ error in time-averaged drag
+ 229x CPU-hour savings
  - FOM: 5 hour x 48 CPU
  - GNAT ROM: 32 min x 2 CPU

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  - Improve projection technique [C. et al., 2011a, C. et al., 2015a]
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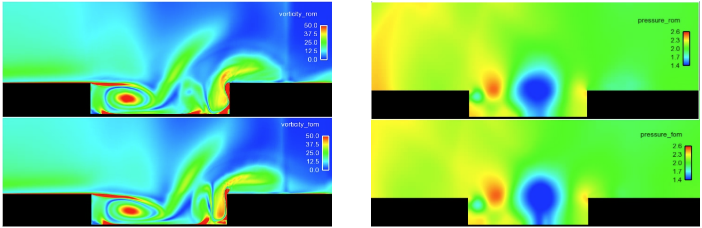
Collaborators: L. Brencher, B. Haasdonk, A. Barth (U Stuttgart)

GNAT performance

vorticity field | pressure field

GNAT ROM

FOM



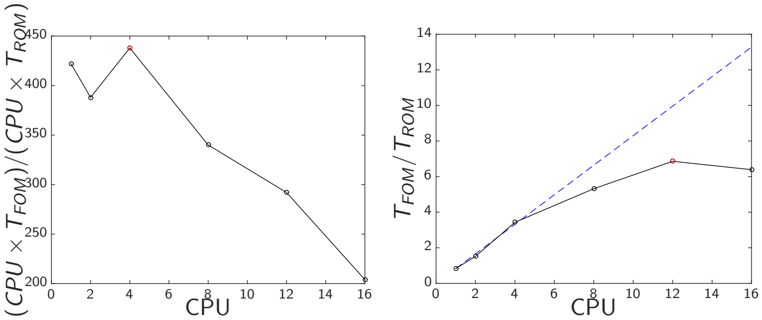
- FOM: 5 hour x 48 CPU
- GNAT ROM: 32 min x 2 CPU.

+ 229x CPU-hour savings. Good for many query.

- 9.4x walltime savings. Bad for real time.

Why?

GNAT: strong scaling (Ahmed body) [C., 2011]



(g) CPU-hour savings (h) Walltime savings

+ Significant CPU-hour savings (max: 438 for 4 CPU)
- Modest walltime savings (max: 7 for 12 CPU)

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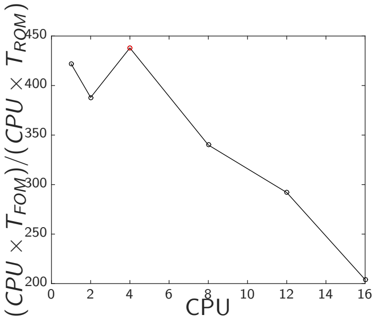
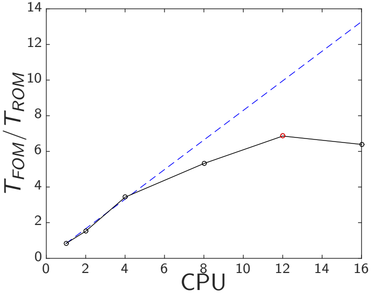


(i) CPU-hour savings



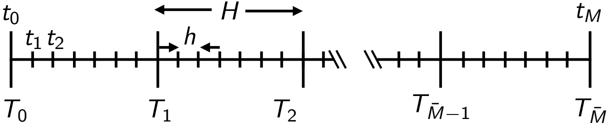
(j) Walltime savings

+ Significant CPU-hour savings (max: 438 for 4 CPU)
- Modest walltime savings (max: 7 for 12 CPU)

Spatial parallelism is quickly saturated!

Time-parallel algorithms [Lions et al., 2001a, Farhat and Chandesris, 2003]

Goal: expose more parallelism to reduce walltime



- Fine propagator: time step Δt

$$\mathcal{F}(\boldsymbol{x}; \tau_1, \tau_2)$$

- Coarse propagator: time step ΔT

$$\mathcal{G}(\boldsymbol{x}; \tau_1, \tau_2)$$

- Parareal iteration k (sequential and parallel steps):

$$\boldsymbol{x}_{k+1}^{m+1} = \mathcal{G}(\boldsymbol{x}_{k+1}^{m}; T_m, T_{m+1}) + \mathcal{F}(\boldsymbol{x}_{k}^{m}; T_m, T_{m+1}) - \mathcal{G}(\boldsymbol{x}_{k}^{m}; T_m, T_{m+1})$$

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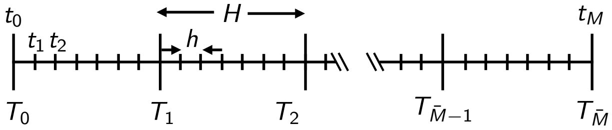

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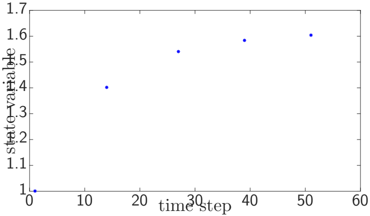
- Interpretations [Gander and Vandewalle, 2007, Falgout et al., 2014]:
  - Deferred/residual-correction scheme $\mathcal{B}(\boldsymbol{x}_{k+1}) = \mathcal{B}(\boldsymbol{x}_k) - \mathcal{A}(\boldsymbol{x}_k)$
  - Multiple shooting method with FD Jacobian approximation
  - Two-level multigrid

Parareal: sequential and parallel steps [Lions et al., 2001a]



$$\boldsymbol{x}_0^{m+1} = \mathcal{G}(\boldsymbol{x}_0^m; T_m, T_{m+1})$$

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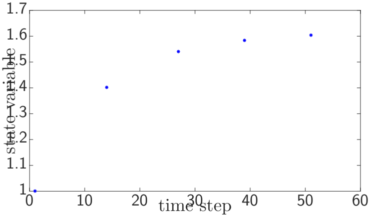

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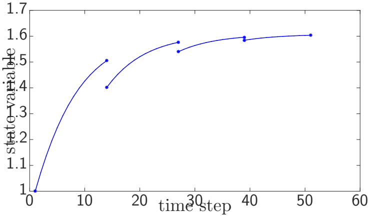

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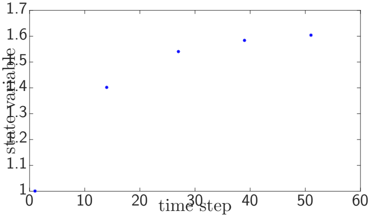
Parareal: sequential and parallel steps [Lions et al., 2001a]



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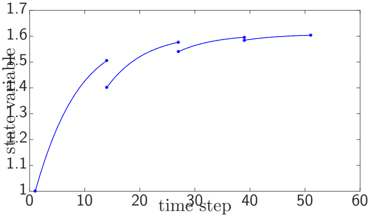

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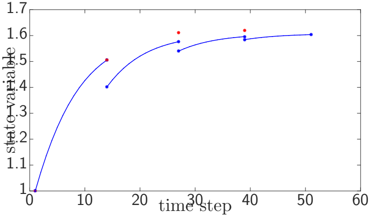

$$\boldsymbol{x}_1^{m+1} = \mathcal{F}(\boldsymbol{x}_0^m; T_m, T_{m+1}) + \mathcal{G}(\boldsymbol{x}_1^m; T_m, T_{m+1}) - \mathcal{G}(\boldsymbol{x}_0^m; T_m, T_{m+1})$$

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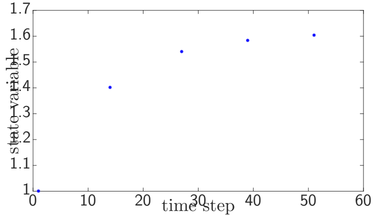

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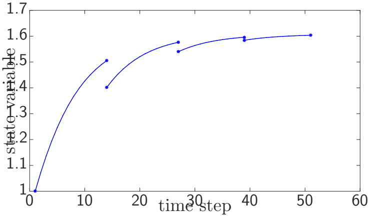

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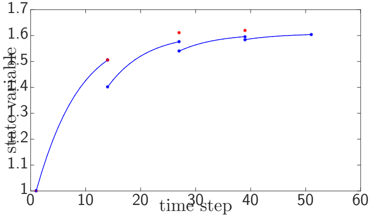

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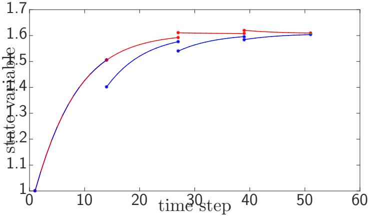

$$\mathcal{F}(\mathbf{x}_1^m; T_m, T_{m+1})$$

Coarse propagator

Critical: coarse propagator should be fast, accurate, stable

- Existing coarse propagators
  - Same integrator [Lions et al., 2001b, Bal and Maday, 2002]
  - Coarse spatial discretization
    [Fischer et al., 2005, Farhat et al., 2006, Cortial and Farhat, 2009]
  - Simplified physics model [Baffico et al., 2002, Maday and Turinici, 2003, Blouza et al., 2011, Engblom, 2009, Maday, 2007]
  - Relaxed solver tolerance [Guibert and Tromeur-Dervout, 2007]
  - Reduced-order model (on the fly) [Farhat et al., 2006, Cortial and Farhat, 2009, Ruprecht and Krause, 2012, Chen et al., 2014]

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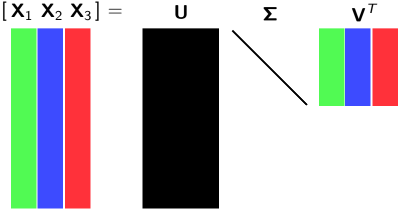
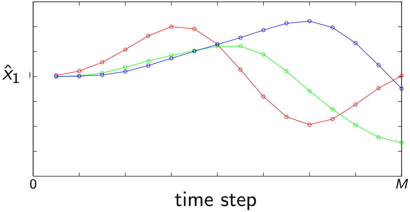
ROM context: can we leverage offline data to improve the coarse propagator?

Revisit the SVD

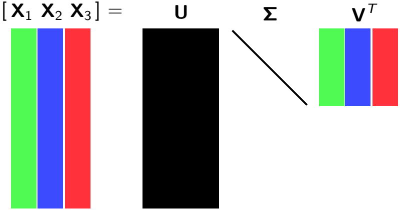


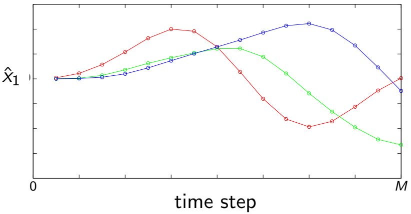


First row of $\boldsymbol{V}^T$

jth row of $\boldsymbol{V}^T$ contains a basis for time evolution of $\hat{x}_j$

Revisit the SVD





First row of $\boldsymbol{V}^T$

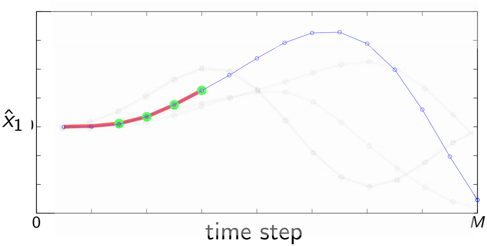
jth row of $\boldsymbol{V}^T$ contains a basis for time evolution of $\hat{x}_j$

- Construct $\boldsymbol{\Xi}_j$: **global time-evolution basis** for $\hat{x}_j$

$$\boldsymbol{\Xi}_j := \left[\xi_j^1 \ \cdots \ \xi_j^{n_{\text{train}}}\right], \quad \xi_j^i := \left[v_{M(i-1)+1,j} \ \cdots \ v_{Mi,j}\right]^T$$

First attempt [C. et al., 2015b]

1. compute **global forecast** by gappy POD in time domain:



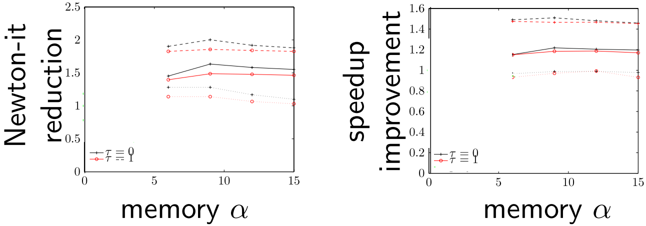
$\hat{\boldsymbol{x}}_1$ so far; memory $\alpha = 4$; forecast; temporal basis

$$\boldsymbol{z}_j = \arg\min_{\boldsymbol{z} \in \boldsymbol{R}^{a_j}} \|\boldsymbol{Z}(m-1,\alpha)\boldsymbol{\Xi}_j\boldsymbol{z} - \boldsymbol{Z}(m-1,\alpha)g(\hat{x}_j)\|_2$$

- Time sampling: $\boldsymbol{Z}(k,\beta) := \begin{bmatrix} \boldsymbol{e}_{k-\beta} & \cdots & \boldsymbol{e}_k \end{bmatrix}^T$
- Time unrolling: $g(\hat{x}_j) : \hat{x}_j \mapsto [\hat{x}_j(t_0) \;\; \cdots \;\; \hat{x}_j(t_M)]^T$

2. use $\boldsymbol{e}_m^T \boldsymbol{\Xi}_j \boldsymbol{z}_j$ as initial guess for $\hat{x}_j(t_m)$ in Newton solver

First attempt: structural dynamics [C. et al., 2015b]



+ Newton iterations reduced by up to ∼2x
+ Speedup improved by up to ∼1.5x
+ No accuracy loss
+ Applicable to any nonlinear ROM
- Insufficient for real-time computation

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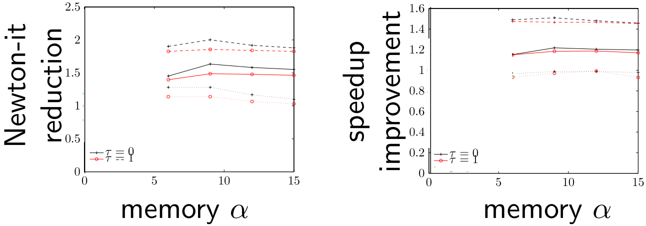
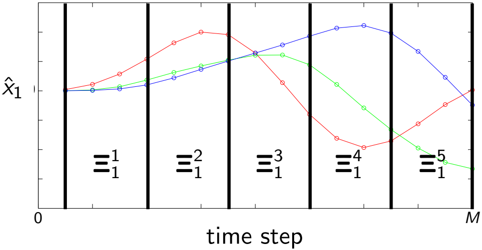


- $+$ Newton iterations reduced by up to ∼2x
- $+$ Speedup improved by up to ∼1.5x
- $+$ No accuracy loss
- $+$ Applicable to any nonlinear ROM
- \- Insufficient for real-time computation

Can we apply the same idea for the coarse propagator?

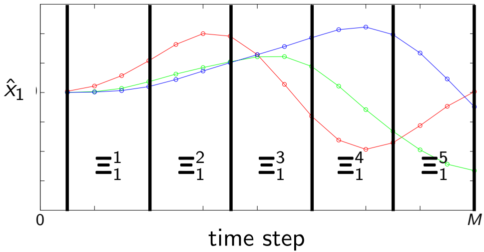
Coarse propagator via local forecasting

- **Offline**: Construct **local time-evolution basis** $\boldsymbol{\Xi}_j^m$



Coarse propagator via local forecasting

- **Offline**: Construct **local time-evolution basis $\boldsymbol{\Xi}_j^m$**

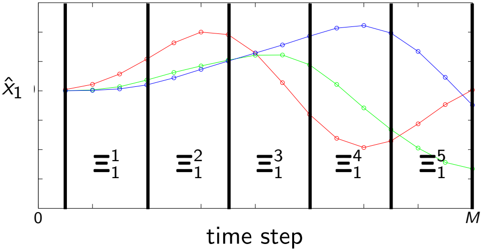


- **Online**: Coarse propagator $\mathcal{G}_j^m$ defined via forecasting:
  1. Compute α time steps with fine propagator

$$\begin{bmatrix} \mathcal{F}(\hat{\mathbf{x}}_j; T_m, T_m + \Delta t) \\ \vdots \\ \mathcal{F}(\hat{\mathbf{x}}_j; T_m, T_m + \Delta t\alpha) \end{bmatrix}$$

Coarse propagator via local forecasting

- **Offline**: Construct **local time-evolution basis** $\boldsymbol{\Xi}_j^m$

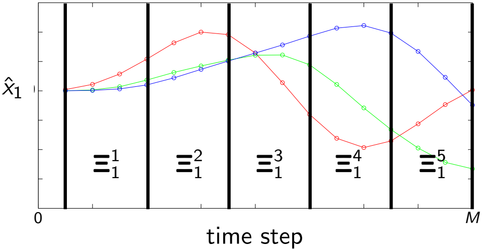


- **Online**: Coarse propagator $\mathcal{G}_j^m$ defined via forecasting:
  1. Compute α time steps with fine propagator
  2. Compute **local forecast** via gappy POD

$$\boldsymbol{\Xi}_j^m \left[\boldsymbol{Z}(\alpha+1,\alpha)\boldsymbol{\Xi}_j^m\right]^+ \begin{bmatrix} \mathcal{F}(\hat{\mathbf{x}}_j; T_m, T_m+\Delta t) \\ \vdots \\ \mathcal{F}(\hat{\mathbf{x}}_j; T_m, T_m+\Delta t\alpha) \end{bmatrix}$$

Coarse propagator via local forecasting

- **Offline**: Construct **local time-evolution basis $\boldsymbol{\Xi}_j^m$**



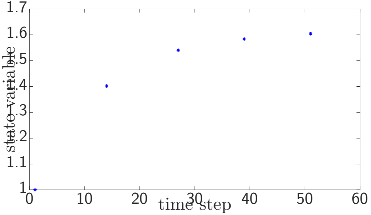
- **Online**: Coarse propagator $\mathcal{G}_j^m$ defined via forecasting:
  1. Compute α time steps with fine propagator
  2. Compute **local forecast** via gappy POD
  3. Select last timestep of **local forecast**

$$\mathcal{G}_j^m : (\hat{\boldsymbol{x}}_j; T_m, T_{m+1}) \mapsto \boldsymbol{e}_{\Delta T/\Delta t}^T \boldsymbol{\Xi}_j^m \left[\boldsymbol{Z}(\alpha+1,\alpha)\boldsymbol{\Xi}_j^m\right]^+ \begin{bmatrix} \mathcal{F}(\hat{\boldsymbol{x}}_j; T_m, T_m + \Delta t) \\ \vdots \\ \mathcal{F}(\hat{\boldsymbol{x}}_j; T_m, T_m + \Delta t\alpha) \end{bmatrix}$$

Initial seed

$$\mathbf{x}_{k+1}^{m+1} = \mathcal{G}(\mathbf{x}_{k+1}^{m}; T_m, T_{m+1}) + \mathcal{F}(\mathbf{x}_{k}^{m}; T_m, T_{m+1}) - \mathcal{G}(\mathbf{x}_{k}^{m}; T_m, T_{m+1})$$

How to compute initial seed $\mathbf{x}_0^m$, $m = 0, \dots, \bar{M}$?



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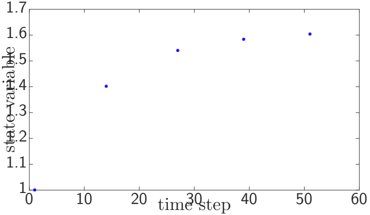


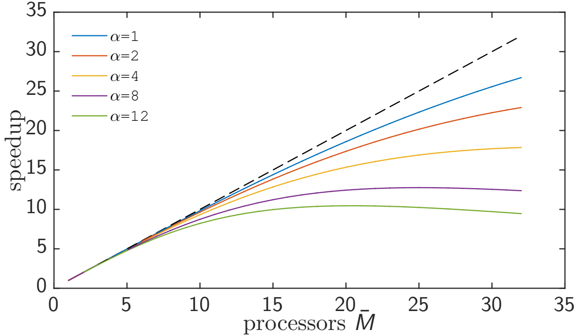
1. Typical time integrator
2. Local forecast
3. Global forecast

Ideal-conditions speedup

Theorem

If $g(\hat{x}_j) \in \mathrm{range}(\boldsymbol{\Xi}_j)$, $j = 1, \ldots, p$, then the proposed method converges in one parareal iteration and realizes a speedup of

$$\frac{\bar{M}}{\bar{M}(\bar{M}-1)\alpha/M + 1}.$$



Ideal-conditions speedup for $M = 5000$

Ideal-conditions speedup with initial guesses

Corollary

If $\boldsymbol{f}$ *is nonlinear,* $g(\hat{x}_j) \in \mathit{range}(\boldsymbol{\Xi}_j), \quad j = 1, \ldots, p$, *and the forecasting method also provides Newton-solver initial guesses, then*

1. *the method converges in* **one parareal iteration**, *and*
2. *only* α *nonlinear systems of algebraic equations are solved in each time interval.*

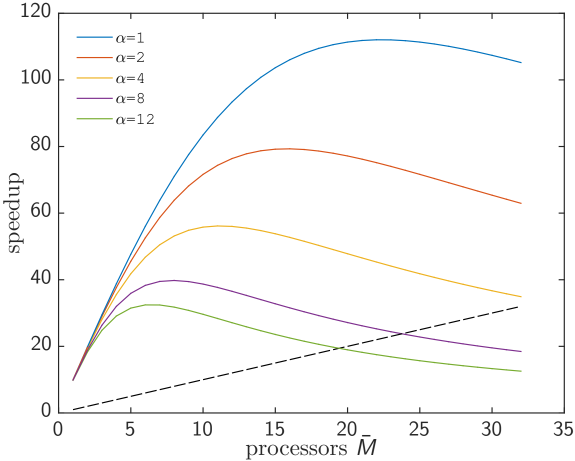
The method then realizes a theoretical speedup of

$$\frac{M}{(\bar{M}\alpha) + (M/\bar{M} - \alpha)\tau_r}$$

relative to the sequential algorithm without forecasting. Here,

$$\tau_r = \frac{\text{residual computation time}}{\text{nonlinear-system solution time}}.$$

Ideal-conditions speedup with initial-guesses



Ideal-condition speedup for $M = 5000$, $\tau_r = 1/10$

Significant speedups possible by leveraging time-domain data!

Stability

Theorem

If the fine propagator is stable, i.e.,

$$\|\mathcal{F}(\boldsymbol{x};\tau,\tau+\Delta T)\| \leq (1+C_{\mathcal{F}}\Delta T)\|\boldsymbol{x}\|, \quad \forall 0\leq\tau\leq\tau+\Delta T$$

then the proposed method is also stable, i.e.,

$$\|\hat{\boldsymbol{x}}_{k+1}^{m}\| \leq C_m \exp(C_{\mathcal{F}} m\Delta T)\|\hat{\boldsymbol{x}}^{0}\|.$$

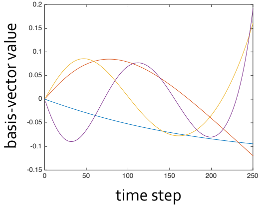
- $C_m := \sum_{k=1}^{m}\binom{k}{m}\beta_k\gamma^m\alpha^k(\Delta T/\Delta t)^{m-k}$
- $\beta_k := \exp(-C_{\mathcal{F}}k(\Delta T-\Delta t\alpha)) \leq 1$
- $\gamma := \max(\max_{m,j} 1/\|\boldsymbol{Z}(\alpha+1,\alpha)\boldsymbol{\Xi}_j^m\|, 1/\sigma_{\min}(\boldsymbol{Z}(\alpha+1,\alpha)\boldsymbol{\Xi}_j^m))$

Example: inviscid Burgers equation [Rewienski, 2003]

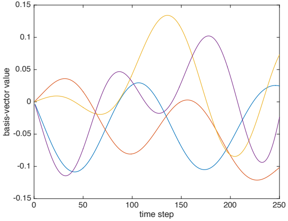
$$\frac{\partial u(x,\tau)}{\partial \tau} + \frac{1}{2}\frac{\partial\left(u^2\left(x,\tau\right)\right)}{\partial x} = 0.02e^{\mu_2 x}$$
$$u(0,\tau) = \mu_1,\ \forall \tau \in [0, 25]$$
$$u(x,0) = 1,\ \forall x \in [0, 100]\,,$$

- Discretization: Godunov's scheme
- $(\mu_1, \mu_2) \in [2.5, 3.5] \times [0.02, 0.03]$
- $\Delta t = 0.1$, $M = 250$ fine time steps
- FOM: $N = 500$ degrees of freedom
- ROM: LSPG [C. et al., 2011a], POD basis dimension $p = 100$
- $n_{\text{train}} = 4$ training points (LHS sampling); random online point
- **2 coarse propagators**: Backward Euler and local forecast
- **3 initial seeds**: Backward Euler, local forecast, global forecast

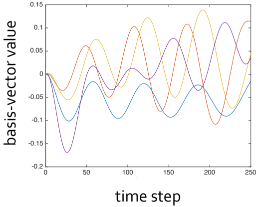
Global temporal bases



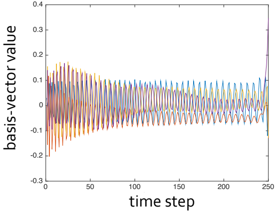
(a) coordinate 1



(b) coordinate 5



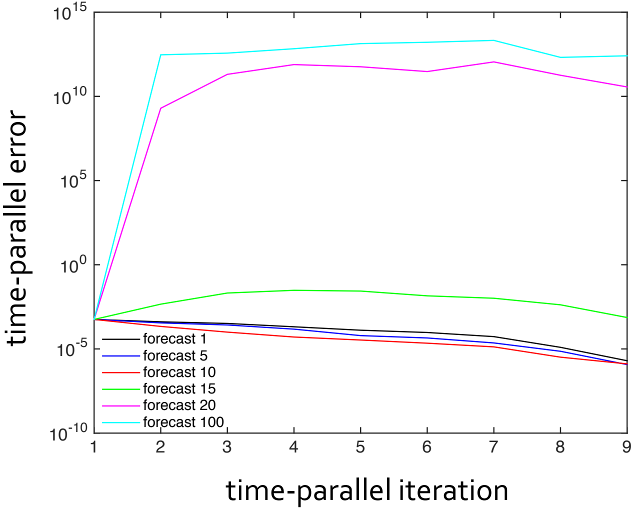
(c) coordinate 10



(d) coordinate 100

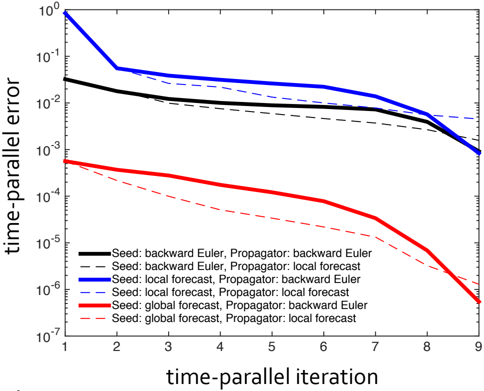
Higher-index generalized coordinates not 'forecastable'

Forecasting ‘high-frequency’ coordinates is dangerous



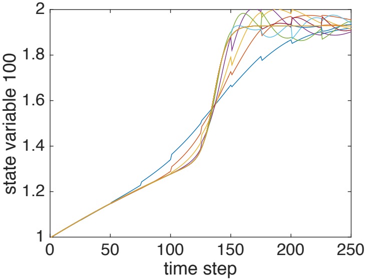
Proceed by forecasting the first 10 coordinates

Comparison: Initial seed and coarse propagator

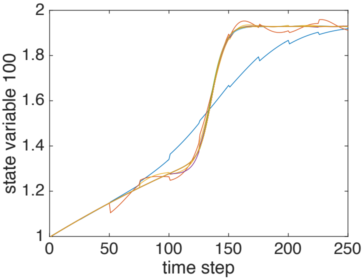


- Initial seed:
  + best performance: global forecast
  - worst performance: local forecast (error accumulation)
- Coarse propagator:
  + local forecast outperforms backward Euler

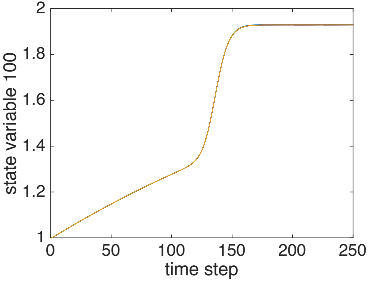
Forecasting improves improves initial seed and coarse propagator!



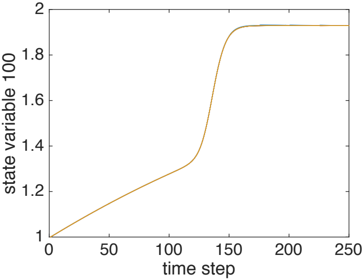
(e) Seed: Euler, Prop: Euler



(f) Seed: Euler, Prop: local forecast

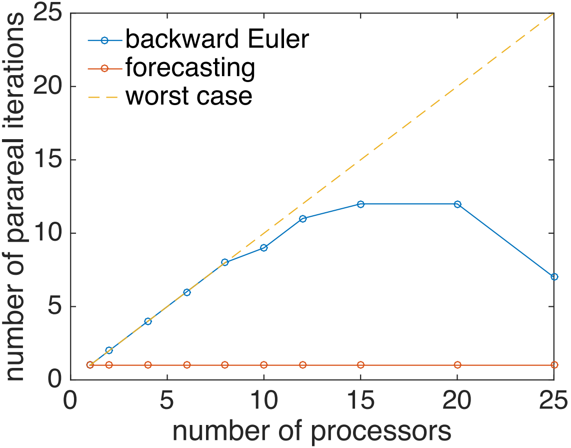


(g) Seed: glob forecast, Prop: Euler



(h) Seed: glob forecast, Prop: loc fore-

Parareal performance



+ *Forecasting*: minimum possible iterations
- *Backward Euler*: often close to worst-case performance

Conclusions

Use temporal data to reduce ROM simulation time

- **offline**: time-evolution bases from right singular vectors
- **online**:
  1. global forecast as initial seed
  2. local forecast as coarse propagator

+ theory: excellent speedup and stability
+ ideal parareal performance observed
+ significant improvement over Backward Euler
+ no additional error introduced

- **References**:

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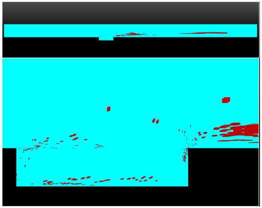
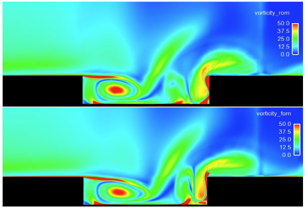
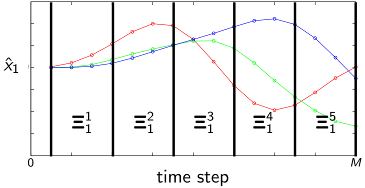
Questions?




vorticity_rom
50.0
37.5
25.0
12.5
0.0
vorticity_fom
50.0
37.5
25.0
12.5
0.0


$\hat{x}_1$
Ξ_1^1
Ξ_1^2
Ξ_1^3
Ξ_1^4
Ξ_1^5
0
M
time step

Acknowledgments

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