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On-Off Minimum-Time Control With Limited Fuel Usage: Global Optima Via Linear Programming#

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Abstract: A method for finding a global optimum to the on-off minimum-time control problem with limited fuel usage is presented. Each control can take on only three possible values: maximum, zero, or minimum. The simplex method for linear systems naturally yields such a solution for the re-formulation presented herein because it always produces an extreme point solution to the linear program. Numerical examples for the benchmark linear flexible system are presented.

1. Introduction

The problem of generating minimum-time control for linear dynamic systems has been studied fairly extensively. The work in [Bashein, G.], [Keerthi, S. and Gilbert, E.], [Kim, M.], [Rasmy, M. and Hamza, M.], [Ryan, E.], [Torng, H.], [De Vlieger, J., et al], [Zadegh, L. and Whalen, B.], and [Chia-Ju, W.] used a fixed-size time step. Starting with one time-step and increasing to 2, 3, 4, etc. time steps until a phase I linear programming algorithm detected that the resulting linear program was feasible, they thereby obtained the minimum-time to within roughly the size of the time step Δt. In [Kim, M. and Engell, S.], however, a binary search on the final time was used to allow the algorithm to bisect or zero in on the minimum time more efficiently.

The on-off control problem (see [Singhose, et al, 1999]) presents a twist to solving the minimum-time problem, as does the limited fuel constraint. Instead of the usual point-to-point minimum-time problem with input bounds, we have an input that can take on only one of three discrete values: its maximum value, zero, and its minimum value. The limited fuel usage constraint (see [Singhose, et al, 1999]) involves the sum of the absolute values of inputs, which is itself a non-differentiable constraint. However, by making a change of variables, one can bring the problem to a linear program (a set of linear equations and inequalities) whose extreme points are essentially an on-off input history. The simplex

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method of linear programming, by its very nature, always produces such extreme points and is therefore very well-suited to this on-off control problem. From the resulting solution, the switches and switching times can be easily obtained.

2. Problem Statement

We are given a linear time-invariant dynamic system:

$$\dot{x} = \overline{A}x + \overline{B}u \tag{1}$$

For simplicity, we will assume a single input, i.e., $u \in R^1$, although all of the theory and methodology herein is trivially extendible to multi-input problems. There is a given initial state:

$$x(0) = x_0 \tag{2}$$

and a specified terminal state that must be reached at the unknown final time t_f:

$$x(t_f) = x_{f,des} \tag{3}$$

The input is constrained to be one of the following three values:

$$u \in \{-u_{\max}, 0, u_{\max}\} \tag{4}$$

In other words, the input history is an "on-off" one, with a finite number of switches. An additional constraint is that the total fuel used is limited, as given by the following equation (see [Singhose, et al, 1999])

$$\int_0^{t_f} |u| dt \leq U \tag{5}$$

The objective is to find u(t) that satisfies (1)-(5) and minimizes the total trajectory execution time t_f.

3. Method of Solution

We will first bring the problem to a discrete-time form:

$$x_{k+1} = Ax_k + Bu_k, \text{ (k=1,...,N)} \tag{6}$$

where N is the number of time-steps and where u(t) has been discretized with a stair-step time history and where the A and B matrices depend upon the sampling period $h = t_f / N$. We still have the given initial state:

$$x_1 = x_0 \tag{7}$$

and the required final state:

$$x_{N+1} = x_{f,des} \quad (8)$$

and the input constraint (4) becomes

$$u_k \in \{-u_{\max}, 0, u_{\max}\} \ \forall \text{ k} \quad (9)$$

and the fuel-usage constraint (5) becomes:

$$\sum_{k=1}^{N} h|u_k| \leq U \quad (10)$$

The problem can be re-formulated for linear programming as follows. Let

$$u_k = v_k + w_k \quad (11)$$

where

$$-u_{\max} \leq v_k \leq 0 \quad (12)$$

and

$$0 \leq w_k \leq u_{\max} \quad (13)$$

so that the fuel usage constraint can be written in a differentiable form as:

$$-\sum_{k=1}^{N} hv_k + \sum_{k=1}^{N} hw_k \leq U \quad (14)$$

The state equation (6) becomes

$$x_{k+1} = Ax_k + [B,B]\begin{pmatrix} v_k \\ w_k \end{pmatrix} \quad (15)$$

For a fixed final time t_f, determination of the feasibility of (11)-(15)/(7)-(8) is a phase I linear programming problem (see [Chvatal]). Moreover, since the simplex method of linear programming always finds an extreme-point solution, we are guaranteed that at least $N-n-1$, where n is the number of states or the length of $x(t_f)$, of the v_k and w_k will be at one of its bounds. Therefore, at least $N-n-1$ of the $u_k = v_k + w_k$ will be either $u_{\max}$, 0, or $-u_{\max}$, which is exactly what constraint (9) specifies. Finally, a simple binary search on t_f (i.e., a bisection algorithm with repeated calls to the simplex method) can be used to test feasibility/infeasibility of a given final time t_f. From a practical point of view, since $N >> n$, the u_k sequence found by the simplex method will be essentially bang-coast-bang and the switching times will be easy to identify.

It *must* be emphasized that, so long as the terminal state $x_{f,des}$ is maintainable by the system, the above approach is *guaranteed* to produce a globally optimal solution to the minimum-time control problem, within the accuracy of the discrete-time approximation (9) and the tolerance set on t_f in the bisection outer loop of the method.

4. Numerical Examples

The numerical examples presented in this section will be based upon the benchmark problem given in [Liu and Wie, 1992], [Wie and Bernstein, 1992], [Wie and Liu, 1992], [Singh and Vadali, 1994], [Pao, 1996], [Singhose, et al, 1996b], and [Singhose, et al, 1999], which is illustrated below in Figure 1.

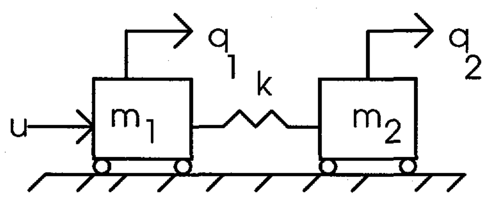

Figure 1. Schematic of Example Problem

The parameter values used are $m_1 = m_2 = 1$, $k = 1$, and $u_{max} = 1$. The maximum fuel usage U=1, 2, and 5. Letting the state vector $x \equiv (q_1, q_2, \dot{q}_1, \dot{q}_2)^T$, the initial state is $x(0) = (0,0,0,0)^T$ and the desired final state is $x_{f,des} = (5,5,0,0)^T$.

The input force histories for maximum fuel usages of U=1, U=2, and U=5, are shown in Figures 2, 3, and 4 below, respectively.

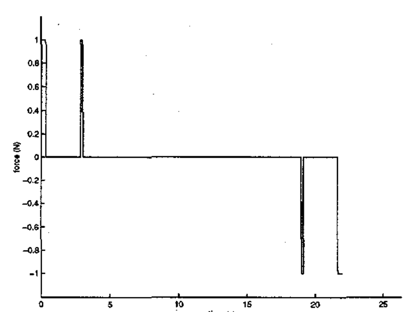

Figure 2. Input force (N) versus time (s), maximum fuel usage U=1

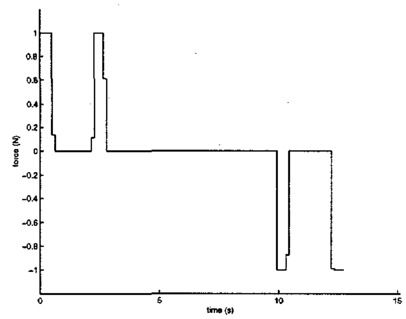

Figure 2. Input force (N) versus time (s), maximum fuel usage U=2

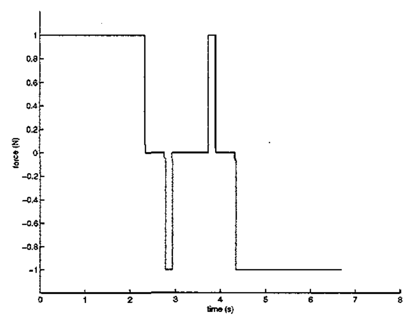

Figure 3. Input force (N) versus time (s), maximum fuel usage U=5

The respective t_f values are 21.99 seconds, 12.71 seconds, and 6.680 seconds. These t_f values are consistent with Figure 4 of [Singhose, et al, 1999]. The value of N used was 500 and the bisection tolerance on t_f was set to 5×10^{-4}. Although the method was coded in MATLAB, rather than a compilable language like C or Fortran, we report that the CPU run times were 40 seconds, 35 seconds, and 21 seconds, respectively, on a Sun Ultra-2 workstation. We can use Figures 2-4 to ascertain the switching times, which is facilitated using MATLAB's "zoom" feature for zooming or windowing in on the switches.

5. Conclusion

We presented and demonstrated a method for finding a global optimum to the problem of on-off minimum-time control with limited fuel usage. The method is guaranteed to find a global optimum and utilizes the simplex method of linear programming which naturally yields solutions that are "on-off," thus providing a unified and rigorous approach to the problem. The method was demonstrated on a benchmark flexible system problem.

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