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Abstract

We demonstrate that discrete Lyapunov exponents are isomorphic to numeric overflows of the capacity of an arbitrary noiseless and memoryless channel in a Shannon communication model with feedback. The isomorphism allows the understanding of Lyapunov exponents in terms of Information Theory, rather than the traditional definitions in chaos theory. The result also implies alternative approaches to the calculation of related quantities, such as the Kolmogorov Sinai entropy which has been linked to thermodynamic entropy. This work provides a bridge between fundamental physics and information theory. It suggests, among other things, that machine learning and other information theory methods can be employed at the core of physics simulations.

1 Introduction

In physics, the Lyapunov exponents are quantities that characterize the rate of separation of infinitesimally close time trajectories [Boe16, WSSV85, CCCP97] in the phase space of a physical system. Quantitatively, two trajectories in the phase space with initial separation $\delta\mathbf{Z}_0$ will diverge (provided that the divergence can be treated within the linearized approximation) at a rate given by

$$|\delta\mathbf{Z}(t)| \approx e^{\lambda t}|\delta\mathbf{Z}_0| \tag{1}$$

where λ is the Lyapunov exponent.

The rate of separation can be different for different orientations of the initial separation vector. Thus, there is a spectrum of Lyapunov exponents equal in number to the dimensionality of the phase space. It is common to refer to the largest one as the Maximal Lyapunov exponent (MLE), because it determines a notion of predictability for a dynamical system. Negative exponents indicate convergence, while a positive MLE is usually treated as an indication that the system is chaotic.

With the recent rising popularity of the topic [Wil17], an often-posed question is how to connect this metric of predictability in physics to Shannon's model of communication [Sha01] in information theory. Intuitively, information theory also provides a metric of predictability by measuring the distortion of signals. This article shows that this intuition is indeed correct. The distortion of the 'trajectory signal' sent over the channel is caused by a bit rate larger than the channel capacity. However, in a noiseless channel, the cause of such information loss is due to numerical overflow in the decoding of output message instead of an additive noise source.

2 Model of Communication in Information Theory

Figure 1 shows the communication model as assumed in this article. Sender and receiver "communicate" one iteration of a trajectory evolution $|\delta\mathbf{Z}_n|$ to $|\delta\mathbf{Z}_{n+1}|$ by encoding each vector canonically as an integer $\mathbf{N}_n$. The memoryless and noiseless channel transforms the integer with some arbitrary set of operations into a new integer $\mathbf{N}_{n+1}$ which can then be decoded canonically as well. The feedback is introduced to model the iteration, i.e. the feedback is assumed to be an identical copy operation. It is well known that feedback does not change the capacity of a discrete memoryless channel.

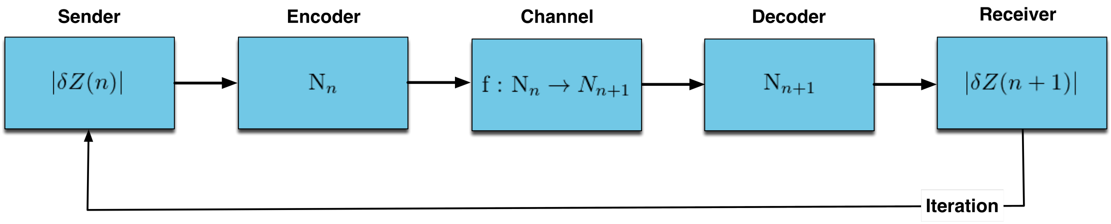

Figure 1: An isomorphism between Shannon's model of communication and Lyapunov exponents.

3 Isomorphism

Without loss of generality, let $|\delta \mathbf{Z}_0|$ be encoded as an integer $\mathbf{N_0}$. This will take $\log_2(\mathbf{N_0})$ bits. Since the channel is noiseless, each bit arrives at the decoder without loss if and only if the output of the channel transition function $\mathbf{f} : N_n \to N_{n+1}$ does not exceed $\log_2(\mathbf{N_0})$ bits. In other words, the channel capacity is $\log_2(\mathbf{N_0})$.

We can now define a function $f(N_n, N_{n+1}) \equiv \lambda_i$ that characterizes the channel transition:

$$\lambda_i = \log_2\left(\frac{N_{i+1}}{N_i}\right). \tag{2}$$

In words, λ_i measures the difference in bits between the received message N_{i+1} and sent message N_i.

Let t be the number of iterations to go from $|\delta Z_0|$ to $|\delta Z_t|$. It is self-evident that the channel is lossless only for when $\lambda_i \leq 0$. In fact, if $\lambda_i < 0$ for sufficiently small λ_i, and sufficiently large number of iterations, then $|\delta Z_t|$ will converge to 0. For $\lambda_i > 0$ the transformation itself exceeds the channel capacity, i.e. λ_i bits are lost. With regards to N_{i+1}, this means that the error is $2^{\lambda_i} N_i$. For example, a full one bit lost means an error of $2^{1.0} N_i$. Therefore, we can establish that:

$$N_{i+1} = 2^{\lambda_i} N_i. \tag{3}$$

The error induced over all iterations t is therefore $\sum_{i=0}^{t} \lambda_i$, with $\lambda_i > 0$. The maximum possible error in our model is therefore yielded when all λ_i are positive. The total error is then $\sum_{i=0}^{t} \lambda_i$. We can then define:

$$\lambda = \frac{1}{t} \sum_{i=0}^{t} l_i, l_i = \begin{cases} \lambda_i, & \text{for } \lambda_i > 0 \\ 0, & \text{for } \lambda_i \leq 0 \end{cases} \tag{4}$$

as the mean error over all λ_i. The deviation from $N_t - N_0 = N_t$ then becomes:

$$N_t \leq 2^{\lambda t} \mathbf{N}_0. \tag{5}$$

Since changing the base of a logarithm only involves a multiplication with a constant, which can be done in the encoder (and reversed in the decoder), we have shown that Equation 1 and Equation 5 are isomorphic. $\square$

4 Conclusion

Negative λ_i may contribute to canceling out errors, therefore our method yields an inequality. Having said that, in a practical setting, there may also be the quantization error. However, since the channel errors are the result of overflows they can be avoided by allocating enough bits and trading off compute time for arbitrary precision. This changes the predictability discussion into a discussion about computational complexity. Since we have shown our λ_i to be isomorphic to discrete Lyapunov exponents, we can assume that the sum of the positive λ_i also corresponds to the Kolmogorov Sinai entropy [Rue78], which was originally derived from the Shannon entropy.

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References

[Boe16] Geoff Boeing. Visual analysis of nonlinear dynamical systems: Chaos, fractals, self-similarity and the limits of prediction. *Systems*, 4(4):37, 2016.

[CCCP97] Lando Caiani, Lapo Casetti, Cecilia Clementi, and Marco Pettini. Geometry of dynamics, lyapunov exponents, and phase transitions. *Physical review letters*, 79(22):4361, 1997.

[Rue78] David Ruelle. An inequality for the entropy of differentiable maps. *Bulletin of the Brazilian Mathematical Society*, 9(1):83–87, 1978.

[Sha01] Claude Elwood Shannon. A mathematical theory of communication. *ACM SIGMOBILE Mobile Computing and Communications Review*, 5(1):3–55, 2001.

[Wil17] Amie Wilkinson. What are lyapunov exponents, and why are they interesting? *Bulletin of the American Mathematical Society*, 54(1):79–105, 2017.

[WSSV85] Alan Wolf, Jack B Swift, Harry L Swinney, and John A Vastano. Determining lyapunov exponents from a time series. *Physica D: Nonlinear Phenomena*, 16(3):285–317, 1985.