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Title: Algorithms for Computing the Magnetic Field, Vector Potential, and Field Derivatives for Circular Current Loops in Cylindrical Coordinates

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Algorithms for Computing the Magnetic Field, Vector Potential, and Field Derivatives for Circular Current Loops in Cylindrical Coordinates

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1 Abstract

A numerical algorithm for computing the field components B_r and B_z and their r and z derivatives with open boundaries in cylindrical coordinates for circular current loops is described. An algorithm for computing the vector potential is also described. For the convenience of the reader, derivations of the final expressions from their defining integrals are given in detail, since their derivations (especially for the field derivatives) are not all easily found in textbooks. Numerical calculations are based on evaluation of complete elliptic integrals using the Bulirsch algorithm `cel`. Since `cel` can evaluate complete elliptic integrals of a fairly general type, in some cases the elliptic integrals can be evaluated without first reducing them to forms containing standard Legendre forms. The algorithms avoid the numerical difficulties that many of the textbook solutions have for points near the axis because of explicit factors of $1/r$ or $1/r^2$ in the some of the expressions.

2 Integrals for the field components B_r and B_z.

All units used in this note are MKS. The current loop has radius a and is positioned at $z = 0$. Loops with centers offset from $z = 0$ are treated by a shift in z of the field point. The current is denoted by I, and has units of amperes. The field is given by the Biot-Savart law, which in this case takes the form

$$\vec{B} = \frac{\mu_0 I a}{4\pi} \int_0^{2\pi} \frac{\hat{\phi} \times (\vec{r} - \vec{r}\,')}{|\vec{r} - \vec{r}\,'|^3} d\phi \tag{1}$$

In Eq. 1, $\vec{r}$ is the field point and $\vec{r}\,'$ is the source point. The current vector is $I\hat{\phi}$. The field point has coordinates r, θ, z and the source point coordinates $a, \phi, 0$ in the cylindrical coordinate system. Since the field has axisymmetry, $B_\theta = 0$ and with no loss of generality the field point can be taken to be $r, 0, z$. With this notation, Eq. 1 gives for the field components

$$B_r(r,z) = \frac{\mu_0 I a}{2\pi} z \int_0^\pi \frac{\cos\phi d\phi}{(r^2 + a^2 - 2ar\cos\phi + z^2)^{3/2}} \tag{2}$$

$$B_\theta(r,z) = 0 \tag{3}$$

$$B_z(r,z) = \frac{\mu_0 I a}{2\pi} \int_0^\pi \frac{(a - r\cos\phi)\, d\phi}{(r^2 + a^2 - 2ar\cos\phi + z^2)^{3/2}} \tag{4}$$

3 Elliptic-integral expressions for B_r and B_z

The right-hand sides of Eqs. 2 and 4 are complete elliptic integrals. To put them in Legendre form, we make the change of variables $\psi = \pi/2 - \phi/2$. Then $d\phi = -2d\psi$ and $\cos\phi = 2\sin^2\psi - 1$. The expression for B_r becomes

$$B_r(r,z) = \frac{\mu_0 I a}{\pi} \frac{z}{[(a+r)^2 + z^2]^{3/2}} \int_0^{\pi/2} \frac{(\sin^2\psi - \cos^2\psi)\, d\psi}{\left(1 - k^2\sin^2\psi\right)^{3/2}} \tag{5}$$

with $k^2 = 4ar/[(a+r)^2 + z^2]$. The expression for B_z becomes

$$B_z(r,z) = \frac{\mu_0 I a}{\pi} \frac{1}{[(a+r)^2 + z^2]^{3/2}} \int_0^{\pi/2} \frac{[(a-r)\sin^2\psi + (a+r)\cos^2\psi]\, d\psi}{\left(1 - k^2\sin^2\psi\right)^{3/2}} \tag{6}$$

The elliptic integrals in Eqs. 5 and 6 can be expressed as linear combinations of the complete elliptic integrals of first and second kinds E and K, and unlike the case of the field from thin solenoids, do not involve elliptic integrals of the third kind. However, sometimes, for numerical reasons, elliptic integrals other than E and K will be used.

4 Numerical evaluation of the elliptic integrals for B_r and B_z

The complete elliptic integrals for B_r and B_z can be efficiently evaluated by use of the Bulirsch algorithm `cel`[1], which evaluates a generalized complete elliptic integral of the form

$$\texttt{cel}(k_c, p, a, b) = \int_0^{\pi/2} \frac{\left(a\cos^2\psi + b\sin^2\psi\right) d\psi}{\left(\cos^2\psi + p\sin^2\psi\right)\left(\cos^2\psi + k_c^2\sin^2\psi\right)^{1/2}} \tag{7}$$

The quantity k_c in Eq. 7 is sometimes called the complementary modulus and is defined to be $k_c = (1 - k^2)^{1/2}$. We rewrite the expression for the square of the complementary modulus in order to reduce roundoff error:

$$k_c^2 = \frac{(a-r)^2 + z^2}{(a+r)^2 + z^2} \tag{8}$$

For evaluation of field derivatives we will need the canonical complete elliptic integrals of the first and second kind $K(k)$ and $E(k)$. They are evaluated by calls to `cel` as follows:

$$K(k) = \int_0^{\pi/2} \frac{d\psi}{\left(1 - k^2 \sin^2 \psi\right)^{1/2}} = \texttt{cel}(k_c, 1, 1, 1) \tag{9}$$

and

$$E(k) = \int_0^{\pi/2} \left(1 - k^2 \sin^2 \psi\right)^{1/2} d\psi = \texttt{cel}(k_c, 1, 1, k_c^2) \tag{10}$$

We will also need the elliptic integrals $D(k)$, defined by

$$D(k) = \frac{K(k) - E(k)}{k^2} = \int_0^{\pi/2} \frac{\sin^2 \psi d\psi}{\left(1 - k^2 \sin^2 \psi\right)^{1/2}} = \texttt{cel}(k_c, 1, 0, 1) \tag{11}$$

and $C(k)$, defined by

$$C(k) = \frac{K(k) - 2D(k)}{k^2} = \int_0^{\pi/2} \frac{\sin^2 \psi \cos^2 \psi d\psi}{\left(1 - k^2 \sin^2 \psi\right)^{3/2}} \tag{12}$$

Both $D(k)$ and $C(k)$ are finite for $k = 0$, with $D(0) = \pi/4$ and $C(0) = \pi/16$. It will be shown in a later section that even though $C(k)$ is not of the form of Eq. 7, it can nevertheless be evaluated with `cel`.

We see that `cel` is general enough to evaluate the integral in Eq. 5 with one call and we get for B_r

$$B_r(r, z) = \frac{\mu_0 I a}{\pi} \frac{z}{\left[(a + r)^2 + z^2\right]^{3/2}} \texttt{cel}(k_c, k_c^2, -1, 1) \tag{13}$$

Turning now to the equation for B_z (Eq. 6), we see that `cel` is again general enough to evaluate the elliptic integral in one call:

$$B_z(r, z) = \frac{\mu_0 I a}{\pi} \frac{1}{\left[(a + r)^2 + z^2\right]^{3/2}} \texttt{cel}(k_c, k_c^2, a + r, a - r) \tag{14}$$

If only B_r and B_z are to be computed, Eqs. 13 and 14 should be used. However, if field derivatives are also to be calculated, additional calls to `cel` will be required for evaluation of $K(k)$, $E(k)$, $D(k)$, and $C(k)$. Therefore, in order to minimize the total number of calls to `cel`, Eqs. 5 and 6 can be rewritten in such a way that the integrals for B_r and B_z are expressed as linear combinations of $K(k)$, $E(k)$, and $D(k)$. For B_r, we use the integrals **2.584** 40. and 42. of Gradshtein and Ryzhik [2], with the result

$$B_r(r, z) = \frac{\mu_0 I a}{\pi} \frac{z}{\left[(a + r)^2 + z^2\right]^{3/2}} \frac{1}{k_c^2} \left[2K(k) - E(k) - 2D(k)\right] \tag{15}$$

To do this for B_z, we rewrite Eq. 6 in the form

$$B_z(r,z) = \frac{\mu_0 I a}{\pi} \frac{1}{\left[(a+r)^2+z^2\right]^{3/2}} \int_0^{\pi/2} \frac{(a+r-2r\sin^2\psi)\,d\psi}{\left(1-k^2\sin^2\psi\right)^{3/2}} \tag{16}$$

Using the integrals **2.584** 37. and **2.584** 40. of Ref. [2], we get

$$B_z(r,z) = \frac{\mu_0 I a}{\pi} \frac{1}{\left[(a+r)^2+z^2\right]^{3/2}} \frac{1}{k_c^2} \left[(a+r)E(k) - 2rK(k) + 2rD(k)\right] \tag{17}$$

Expressions for B_r and B_z have been published in various places, e.g. Refs. [3], [4] and [5]. Reference [4] also has expressions for the four field derivatives. The expressions in Refs. [3] and [4] contain only the elliptic integrals E and K. As a result, the expressions for B_r and $\partial B_r/\partial z$ contain factors of $1/r$ (or equivalently, factors of $1/k^2$) and the expression for $\partial B_r/\partial r$ in Ref. [4] has a factor of $1/r^2$. Therefore, if the expressions in [3] and [4] are to be used, separate expansions in r must be used to avoid roundoff error and division by zero when r is small or zero. The expression for B_r in Ref. [5] uses the elliptic integral $B(k)$ and thereby avoids an explicit factor of $1/r$, but [5] does not have expressions for field derivatives.

In this memo, expressions containing $D(k)$ are used to eliminate explicit factors of $1/r$ in B_r and $\partial B_r/\partial z$, and an expression containing both $D(k)$ and $C(k)$ is used to eliminate a factor of $1/r^2$ in the evaluation of $\partial B_r/\partial r$. This eliminates the need for separate small-k expansions to evaluate fields and field derivatives for small r.

5 Loop Vector Potential

Having the capability of computing numerical values of the vector potential for an axisymmetric magnetic field is useful in fluxline plotting and in computing the canonical momentum in numerical charged-particle trajectory integration with Hamiltonian dynamics. For fluxline plotting, we note that as a consequence of Stoke's theorem, the magnetic flux Φ passing through a circular disk of radius r centered on and perpendicular to the axis of symmetry at axial position z is given by $\Phi(r,z) = 2\pi r A_\theta(r,z)$. Fluxlines in the r, z plane are contours of constant $\Phi(r,z)$.

In the usual gauge, the vector potential for a circular current loop in cylindrical coordinates r, θ, z is given by

$$\vec{A} = \frac{\mu_0 I a}{4\pi} \frac{\hat{\phi}}{|\vec{r}-\vec{r}'|} d\phi \tag{18}$$

As before in the derivation of the field components, $\vec{r}$ is the field point, $\vec{r}'$ the source point, and the current vector is $I\hat{\phi}$. The vector potential from Equation 18 has only a θ component, given by

$$A_\theta(r,z) = \frac{\mu_0 I a}{2\pi} \int_0^{\pi} \frac{\cos\phi\, d\phi}{\left[r^2+a^2-2ar\cos\phi+z^2\right]^{1/2}} \tag{19}$$

As before, the change of variables $\psi = \pi/2 - \phi/2$ is applied to the integral of Eq. 19 and we get

$$A_\theta(r,z) = \frac{\mu_0 I a}{\pi} \frac{1}{[(a+r)^2 + z^2]^{1/2}} \int_0^{\pi/2} \frac{(\sin^2\psi - \cos^2\psi)\, d\psi}{(1 - k_2^2 \sin^2\psi)^{1/2}} \tag{20}$$

where k^2 has the same definition as in Section 3. The elliptic integral in Eq. 20 can be directly evaluated with a single call to `cel`, giving:

$$A_\theta(r,z) = \frac{\mu_0 I a}{\pi} = \frac{1}{[(a+r)^2 + z^2]^{1/2}} \texttt{cel}(k_c, 1, -1, 1) \tag{21}$$

If, in addition to B_r, B_z and A_θ, field derivatives are also to be calculated, an equivalent expression derived from **2.584** 4. and 6. of Ref. [2] should be used:

$$A_\theta(r,z) = \frac{\mu_0 I a}{\pi} = \frac{1}{[(a+r)^2 + z^2]^{1/2}} [2D(k) - K(k)] \tag{22}$$

6 Loop Field-Component Derivatives

Field-component derivatives are needed, for example, in tracking neutral particles that are subject to spin-field gradient forces. Since the geometry of the problem is axisymmetric, the θ derivatives are zero and we are left with the four derivatives $\partial B_r/\partial r$, $\partial B_r/\partial z$, $\partial B_z/\partial r$, and $\partial B_z/\partial z$. The zero-curl condition gives $\partial B_r/\partial z = \partial B_z/\partial r$. The zero-divergence condition gives $\partial B_r/\partial r = -\partial B_z/\partial z - B_r/r$. This leaves only two independent derivatives. It is most convenient to compute the two z derivatives $\partial B_r/\partial z$ and $\partial B_z/\partial z$ and use the zero-curl and zero-divergence conditions to compute the remaining two derivatives. Then, for computation of $\partial B_r/\partial r$, the ratio B_r/r is needed. For r that is not small in comparison to the loop radius, B_r/r can be computed by dividing B_r by r. For small r, a series expansion in r can be used to avoid numerical difficulties. As r approaches zero, $B_r(r,z)/r$ approaches a finite function of z only. This can be shown by Taylor-expanding the integrand in Eq. 2 in the small parameter $\epsilon = r/(a^2+z^2)^{1/2}$ and integrating over ϕ. This gives

$$\frac{B_r(r,z)}{r} \approx \frac{\mu_0 I a}{\pi} \frac{3\pi}{4} \frac{az}{(a^2+z^2)^{5/2}} \left[1 - \frac{5}{2}\frac{r^2}{a^2+z^2} + \frac{35}{8}\frac{a^2 r^2}{(a^2+z^2)^2}\right] \tag{23}$$

However, if we have a numerical algorithm for the elliptic integral $C(k)$ (see Eq. 12), we can go back to Eq. 15 and write $2K(k) - E(k) - 2D(k) = k^2[D(k) - C(k)]$, after which we have

$$\frac{B_r}{r}(r,z) = \frac{\mu_0 I a}{\pi} \frac{4az}{[(a+r)^2+z^2]^{5/2}} \frac{1}{k_c^2} [D(k) - C(k)], \tag{24}$$

thereby eliminating the explicit factor of $1/r$ (i.e., $1/k^2$) in the expression for B_r/r. It is shown in Ref. [6] that $C(k)$ can be computed with the older Bulirsch algorithm `cel2`, as follows:

$$C(k) = \texttt{cel2}\left(\frac{2k_c^{1/2}}{1+k_c}, 0, \frac{2}{(1+k_c)^3}\right) \tag{25}$$

It turns out that the newer Bulirsch algorithm `cel`, used everywhere else in this note, can also be used to compute $C(k)$:

$$C(k) = \texttt{cel}\left(\frac{2k_c^{1/2}}{1+k_c}, 1, 0, \frac{2}{(1+k_c)^3}\right) \tag{26}$$

To get the derivative $\partial B_r/\partial z$, we go back to Eq. 2. Differention of the right-hand side with respect to z gives

$$\frac{\partial B_r}{\partial z}(r,z) = \frac{\mu_0 I a}{2\pi}\left[\int_0^\pi \frac{\cos\phi d\phi}{[a^2+r^2-2ar\cos\phi+z^2]^{3/2}} - 3z^2\int_0^\pi \frac{\cos\phi d\phi}{[a^2+r^2-2ar\cos\phi+z^2]^{5/2}}\right] \tag{27}$$

Using the change of variables $\psi = \pi/2 - \phi/2$ now gives

$$\frac{\partial B_r}{\partial z}(r,z) = \frac{\mu_0 I a}{\pi}\left\{\frac{1}{[(a+r)^2+z^2]^{3/2}}\int_0^{\pi/2}\frac{\left(\sin^2\psi-\cos^2\psi\right)d\psi}{\left(1-k^2\sin^2\psi\right)^{3/2}} - \frac{3z^2}{[(a+r)^2+z^2]^{5/2}}\int_0^{\pi/2}\frac{\left(\sin^2\psi-\cos^2\psi\right)d\psi}{\left(1-k^2\sin^2\psi\right)^{5/2}}\right\} \tag{28}$$

The first integral in Eq. 28 was already evaluated in computation of B_r (see Eq. 13), but with a call to evaluate a non-standard elliptic integral. Therefore, in to reduce the total number of calls to `cel`, the alternative expression in Eq. 15 (without the factor of z) should be used for this term. The second integral can be evaluated by use of **2.584** 61. and **2.584** 63. of Gradshtein and Ryzhik [2]. We get

$$\int_0^{\pi/2}\frac{(\sin^2\psi-\cos^2\psi)\,d\psi}{\left(1-k^2\sin^2\psi\right)^{5/2}} = \frac{1}{3k_c^4}\left[(2+k_c^2)K(k) - 2k_c^2E(k) - 2D(k)\right] \tag{29}$$

The final expression for $\partial B_r/\partial z$ is then

$$\frac{\partial B_r}{\partial z}(r,z) = \frac{\mu_0 I a}{\pi}\left\{\frac{1}{[(a+r)^2+z^2]^{3/2}}\frac{1}{k_c^2}\left[2K(k)-E(k)-2D(k)\right] - \frac{z^2}{[(a+r)^2+z^2]^{5/2}}\frac{1}{k_c^4}\left[(2+k_c^2)K(k)-2k_c^2E(k)-2D(k)\right]\right\} \tag{30}$$

To get an expression for $\partial B_z/\partial z$, we go back to Eq. 4 and differentiate the right-hand side with respect to z:

$$\frac{\partial B_z}{\partial z}(r,z) = -3z\frac{\mu_0 I a}{2\pi}z\int_0^\pi \frac{(a - r\cos\phi)d\phi}{(r^2 + a^2 - 2ar\cos\phi + z^2)^{5/2}} \tag{31}$$

The change of variables $\psi = \pi/2 - \phi/2$ gives

$$\frac{\partial B_z}{\partial z}(r,z) = -3z\frac{\mu_0 I a}{\pi}\frac{1}{[(a+r)^2 + z^2]^{5/2}}\int_0^{\pi/2} \frac{\left(a + r - 2r\sin^2\psi\right)\,d\psi}{\left(1 - k^2\sin^2\psi\right)^{5/2}} \tag{32}$$

The integrals

$$\int \frac{d\psi}{\left(1 - k^2\sin^2\psi\right)^{5/2}} \tag{33}$$

and

$$\int \frac{\sin^2\psi\,d\psi}{\left(1 - k^2\sin^2\psi\right)^{5/2}} \tag{34}$$

are evaluated, respectively, in **2.584** 48. and **2.584** 61. of Gradshtein and Ryzhik [2]. Combining their expressions with the other factors in Eq. 32 yields the final expression

$$\frac{\partial B_z}{\partial z}(r,z) = -z\frac{\mu_0 I a}{\pi}\frac{1}{[(a+r)^2 + z^2]^{5/2}}\frac{1}{k_c^4}\left\{\left[2(k_c^2+1)E(k) - k_c^2 K(k)\right]a + \frac{(a+r)^2 + z^2}{4a}\left[k_c^2(1+k_c^2)K(k) - 2(k_c^4 - k_c^2 + 1)E(k)\right]\right\} \tag{35}$$

In deriving Eq. 35, a factor of $1/k^2$ in the second term was eliminated by substituting $4ar/[(a+r)^2 + z^2]$ for k^2.

7 References

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