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Algorithms for Computing the Magnetic Field, Vector Potential, and Field Derivatives for a Thin Solenoid with Uniform Current Density

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1 Abstract

A numerical algorithm for computing the field components B_r and B_z and their r and z derivatives with open boundaries in cylindrical coordinates for radially thin solenoids with uniform current density is described in this note. An algorithm for computing the vector potential A_θ is also described. For the convenience of the reader, derivations of the final expressions from their defining integrals are given in detail, since their derivations are not all easily found in textbooks. Numerical calculations are based on evaluation of complete elliptic integrals using the Bulirsch algorithm `cel`. The (apparently) new feature of the algorithms described in this note applies to cases where the field point is outside of the bore of the solenoid and the field-point radius approaches the solenoid radius. Since the elliptic integrals of the third kind normally used in computing B_z and A_θ become infinite in this region of parameter space, fields for points with the axial coordinate z outside of the ends of the solenoid and near the solenoid radius are treated by use of elliptic integrals of the third kind of modified argument, derived by use of an addition theorem. Also, the algorithms also avoid the numerical difficulties the textbook solutions have for points near the axis arising from explicit factors of $1/r$ or $1/r^2$ in the some of the expressions.

2 Integrals for the field components B_r and B_z.

All units used in this note are MKS. Fields and field derivatives are computed in a cylindrical coordinate system. The thin solenoid has radius a and extends from $-b$ to b in z. Solenoids with centers offset from $z = 0$ are treated by a shift in z of the field point. The current density is denoted by J_0, and has units of A/m. The field is given by the Biot-Savart law, which in this case takes the form

$$\vec{B} = \frac{\mu_0 J_0 a}{4\pi} \int_{-b}^{+b} \int_0^{2\pi} \frac{\hat{\phi} \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} dz' d\phi \tag{1}$$

In Eq. 1, $\vec{r}$ is the field point and $\vec{r}'$ is the source point. The current density vector is $J_0\hat{\phi}$. The field point has coordinates r, θ, z and the source point coordinates a, ϕ, z'. Since the field has axisymmetry, $B_\theta = 0$ and with no loss of generality the field point can be taken to be $r, 0, z$. Then Eq. 1 gives for the field components

$$B_r(r,z) = \frac{\mu_0 J_0 a}{2\pi} \int_0^\pi \int_{-b}^{+b} \frac{(z-z')\cos\phi\, dz' d\phi}{\left[r^2 + a^2 - 2ar\cos\phi + (z-z')^2\right]^{3/2}} \tag{2}$$

$$B_\theta(r,z) = 0 \tag{3}$$

$$B_z(r,z) = \frac{\mu_0 J_0 a}{2\pi} \int_0^\pi \int_{-b}^{+b} \frac{(a - r\cos\phi)\, dz' d\phi}{\left[r^2 + a^2 - 2ar\cos\phi + (z-z')^2\right]^{3/2}} \tag{4}$$

In terms of the variable $u = z - z'$, the integrals for B_r and B_z become

$$B_r(r,z) = \frac{\mu_0 J_0 a}{2\pi} \int_0^\pi \int_{u_1}^{u_2} \frac{u\cos\phi\, du\, d\phi}{(A+u^2)^{3/2}} \tag{5}$$

$$B_z(r,z) = \frac{\mu_0 J_0 a}{2\pi} \int_0^\pi \int_{u_1}^{u_2} \frac{(a - r\cos\phi)\, du\, d\phi}{(A+u^2)^{3/2}} \tag{6}$$

with $u_1 = z - b$, $u_2 = z + b$, and $A = r^2 + a^2 - 2ar\cos\phi$. The usual approach to evaluating the double integrals in Eqs. 5 and 6 is to first integrate over u. This gives

$$B_r(r,z) = -\frac{\mu_0 J_0 a}{2\pi} \left[\int_0^\pi \frac{\cos\phi\, d\phi}{\left(A + u_2^2\right)^{1/2}} - \int_0^\pi \frac{\cos\phi\, d\phi}{\left(A + u_1^2\right)^{1/2}} \right] \tag{7}$$

$$B_z(r,z) = \frac{\mu_0 J_0 a}{2\pi} \left[u_2 \int_0^\pi \frac{(a - r\cos\phi)\, d\phi}{A\left(A + u_2^2\right)^{1/2}} - u_1 \int_0^\pi \frac{(a - r\cos\phi)\, d\phi}{A\left(A + u_1^2\right)^{1/2}} \right] \tag{8}$$

We now note that the two integrands in Eq. 7 are well-behaved when $r = a$ and $\phi = 0$, provided u_1 and u_2 are non-zero. However, due to the $1/A$ factor, the integrands in Eq. 8 become infinite when $r = a$ and $\phi = 0$, even when u_1 and u_2 are non-zero. This is the source of the numerical problems that occur in the usual algorithms for computing B_z when $|z| > b$ and r is near a.

Nevertheless, we know on physical grounds that B_z is finite when $r = a$, provided that $|z| > b$ (i.e. $u_1 < 0$ and $u_2 < 0$ or $u_1 > 0$ and $u_2 > 0$). Indeed, we can combine the two integrands in Eq. 8 into a single integrand that is finite when u_1 and u_2 are both non-zero and have the same sign. Setting $R_1^2 = A + u_1^2$ and $R_2^2 = A + u_2^2$, it is easy to prove the identity

$$\frac{1}{A}\left(\frac{u_2}{R_2} - \frac{u_1}{R_1}\right) = \frac{u_2^2 - u_1^2}{R_1 R_2 (u_2 R_1 + u_1 R_2)} \tag{9}$$

Inspection of Eq. 9 shows that indeed the right-hand side is finite when $r = a$ and $\phi = 0$, provided u_1 and u_2 are both non-zero and have the same sign. That is, the apparent singularity in the expression for B_z for $r = a$, $|z| > b$ is a *fictitious* singularity.

3 Legendre-form elliptic-integral expressions for B_r and B_z

The right-hand sides of Eqs. 7 and 8 are complete elliptic integrals. To put them in Legendre form, we make the change of variables $\psi = \pi/2 - \phi/2$. Then $d\phi = -2d\psi$ and $\cos\phi = \sin^2\psi - \cos^2\psi$. The expression for B_r becomes

$$B_r(r,z) = -\frac{\mu_0 J_0 a}{\pi}\left\{\frac{1}{D_2}\int_0^{\pi/2}\frac{(\sin^2\psi - \cos^2\psi)\,d\psi}{\left(1-k_2^2\sin^2\psi\right)^{1/2}} - \frac{1}{D_1}\int_0^{\pi/2}\frac{(\sin^2\psi - \cos^2\psi)\,d\psi}{\left(1-k_1^2\sin^2\psi\right)^{1/2}}\right\}, \quad (10)$$

with $D_1^2 = (a+r)^2 + u_1^2$, $D_2^2 = (a+r)^2 + u_2^2$, $k_1^2 = 4ar/D_1^2$ and $k_2^2 = 4ar/D_2^2$. The expression for B_z becomes

$$B_z(r,z) = \frac{\mu_0 J_0 a}{\pi}\left\{\frac{u_2}{(a+r)^2 D_2}\int_0^{\pi/2}\frac{(a+r-2r\sin^2\psi)\,d\psi}{(1-\alpha^2\sin^2\psi)\left(1-k_2^2\sin^2\psi\right)^{1/2}} - \frac{u_1}{(a+r)^2 D_1}\int_0^{\pi/2}\frac{(a+r-2r\sin^2\psi)\,d\psi}{(1-\alpha^2\sin^2\psi)\left(1-k_1^2\sin^2\psi\right)^{1/2}}\right\}, \quad (11)$$

with $\alpha^2 = 4ar/(a+r)^2$. In the following, we will refer to the complete elliptic integral of the first kind $K(k)$, the complete elliptic integral of the second kind $E(k)$, and the complete elliptic integral of the third kind $\Pi(\alpha^2, k)$. As usual, they are defined to be

$$K(k) = \int_0^{\pi/2}\frac{d\psi}{(1-k^2\sin^2\psi)^{1/2}}, \quad (12)$$

$$E(k) = \int_0^{\pi/2}(1-k^2\sin^2\psi)^{1/2}d\psi, \quad (13)$$

and

$$\Pi(\alpha^2,k) = \int_0^{\pi/2}\frac{d\psi}{(1-\alpha^2\sin^2\psi)(1-k^2\sin^2\psi)^{1/2}}. \quad (14)$$

We will show that B_r can be expressed as a linear combination of the elliptic integrals of the first and second kinds only, while B_z is a linear combination of the elliptic integrals of the first and third kinds. We note that when r approaches a, $K(k)$ and $E(k)$ are finite, but α^2 approaches 1 and $\Pi(\alpha^2,k)$ approaches infinity, independently of k. That is, the fictitious singularity in B_z for $r = a$ can be isolated to the $\Pi(\alpha^2,k)$ terms.

We will also refer to the complete elliptic integrals $B(k)$, $D(k)$, and $C(k)$, defined by

$$B(k) = \int_0^{\pi/2}\frac{\cos^2\psi\,d\psi}{(1-k^2\sin^2\psi)^{1/2}}, \quad (15)$$

$$D(k) = \int_0^{\pi/2}\frac{\sin^2\psi\,d\psi}{(1-k^2\sin^2\psi)^{1/2}}, \quad (16)$$

and

$$C(k) = \int_0^{\pi/2} \frac{\sin^2\psi\cos^2\psi\,d\psi}{(1-k^2\sin^2\psi)^{3/2}}, \tag{17}$$

The elliptic integrals $B(k)$, $D(k)$, and $C(k)$ can all be expressed as linear combinations of $K(k)$ and $E(k)$, but the expressions contain explicit factors of $1/k^2$ or $1/k^4$, which makes their numerical evalaution problematic when r is near zero.

We will also use the integrals

$$\int_0^{\pi/2} \frac{\cos^2\psi\,d\psi}{(1-k^2\sin^2\psi)^{3/2}} = D(k) \tag{18}$$

and

$$\int_0^{\pi/2} \frac{d\psi}{(1-k^2\sin^2\psi)^{3/2}} = \frac{E(k)}{1-k^2} \tag{19}$$

We now use the identity $D(k) - B(k) = k^2C(k)$[1] for evaluating B_r (see Eq. 10). In terms of $C(k)$, the expression for B_r becomes

$$B_r(r,z) = -\frac{\mu_0 J_0 a}{\pi}\left[\frac{k_2^2}{D_2}C(k_2) - \frac{k_1^2}{D_1}C(k_1)\right] \tag{20}$$

The above expression for B_r avoids the explicit factor of $1/r$ appearing in equivalent expressions for B_r in many papers and textbooks (see, for example Ref. [2]) and, as shown in a following section, also allows evaluation of B_r/r without an explicit factor of $1/r^2$.

The integral for B_z (Equation 11) can be rewritten in such a way that the elliptic integrals are split into separate terms containing elliptic integrals of the first and third kinds:

$$\begin{aligned} B_z(r,z) = \frac{\mu_0 J_0 a}{\pi}\frac{1}{2a}\Bigg\{ &\frac{u_2}{D_2}\int_0^{\pi/2}\frac{d\psi}{\left(1-k_2^2\sin^2\psi\right)^{1/2}} - \frac{u_1}{D_1}\int_0^{\pi/2}\frac{d\psi}{\left(1-k_1^2\sin^2\psi\right)^{1/2}} + \\ &\left(\frac{a-r}{a+r}\right)\frac{u_2}{D_2}\int_0^{\pi/2}\frac{d\psi}{\left(1-\alpha^2\sin^2\psi\right)\left(1-k_2^2\sin^2\psi\right)^{1/2}} - \\ &\left(\frac{a-r}{a+r}\right)\frac{u_1}{D_1}\int_0^{\pi/2}\frac{d\psi}{\left(1-\alpha^2\sin^2\psi\right)\left(1-k_1^2\sin^2\psi\right)^{1/2}}\Bigg\}, \end{aligned} \tag{21}$$

This gives

$$B_z(r,z) = \frac{\mu_0 J_0 a}{\pi}\frac{1}{2a}\left\{\frac{u_2}{D_2}K(k_2) - \frac{u_1}{D_1}K(k_1) + \left(\frac{a-r}{a+r}\right)\left[\frac{u_2}{D_2}\Pi(\alpha^2,k_2) - \frac{u_1}{D_1}\Pi(\alpha^2,k_1)\right]\right\}, \tag{22}$$

When the elliptic integrals of the third kind in Eq. 22 are evaluated numerically for the two endpoint u values, they become increasingly large as r approaches a, but $u_2\Pi(\alpha^2,k_2)/D_2 -$

$u_1\Pi(\alpha^2, k_1)/D_1$ must approach a finite value. Indeed, the algorithm for B_z in Garrett's 1963 paper [2], which is equivalent to the algorithms described in a following section that use the Bulirsch algorithm `cel`[3], fails for $r = a$, independently of z. When r is close to a, many iterations are required for convergence and the finite value computed for B_z is the difference of two increasingly large numbers, which results in loss of precision and/or increased computation time. This feature is mentioned in Garrett's paper, but no explicit method for dealing with it is given. A remedy for this problem is described in the following section.

4 Use of an addition theorem for $\Pi(\alpha^2, k)$ to eliminate the fictitious singularity at $r = a$ in computation of B_z

This section describes an expression for B_z that eliminates the fictitious singularity that appears when $r = a$ and $|z| > b$. As far as the author knows, the expression has not been previously published.

The new expression for B_z is based on an addition theorem for complete elliptic integrals of the third kind [4]:

$$\Pi(\alpha^2, k) + \Pi(k^2/\alpha^2, k) = K(k) + \frac{\pi}{2}\left[\frac{\alpha^2}{(1-\alpha^2)(\alpha^2 - k^2)}\right]^{1/2} \tag{23}$$

Equation 23 is valid when $0 < k^2 < \alpha^2 < 1$, which is true here. When we multiply the addition formulas for $\Pi(\alpha^2, k_2)$ and $\Pi(\alpha^2, k_1)$ by factors $(\alpha^2 - k_2^2)^{1/2}$ and $(\alpha^2 - k_1^2)^{1/2}$, respectively, and subtract, the result is

$$(\alpha^2 - k_2^2)^{1/2}\Pi(\alpha^2, k_2) - (\alpha^2 - k_1^2)^{1/2}\Pi(\alpha^2, k_1) = (\alpha^2 - k_2^2)^{1/2}\left[K(k_2) - \Pi(k_2^2/\alpha^2, k_2)\right] - (\alpha^2 - k_1^2)^{1/2}\left[K(k_1) - \Pi(k_1^2/\alpha^2, k_1)\right] \tag{24}$$

We see that the elliptic integrals of the third kind on the right-hand side of Eq. 24 depend on the new moduli k_1^2/α^2 and k_2^2/α^2, neither of which approaches unity as r approaches a, provided that u_1 and u_2 are non-zero. Therefore all of the elliptic integrals on the right-hand side are well-behaved as r approaches a. Using the definitions of k_1^2, k_2^2, and α^2, we can write

$$(\alpha^2 - k_1^2)^{1/2} = \frac{2(ar)^{1/2}|u_1|}{(a+r)D_1}, \tag{25}$$

and

$$(\alpha^2 - k_2^2)^{1/2} = \frac{2(ar)^{1/2}|u_2|}{(a+r)D_2}, \tag{26}$$

We now apply Eqs. 23-25 to the last two terms in Eq. 22. The resulting expression for B_z, valid for any r, including $r = a$, but only for $|z| > b$ (i.e. $u_1 < 0$ and $u_2 < 0$ or $u_1 > 0$

and $u_2 > 0$) is

$$B_z(r,z) = \frac{\mu_0 J_0 a}{\pi}\frac{1}{2a}\left\{\frac{u_2}{D_2}K(k_2) - \frac{u_1}{D_1}K(k_1) + \right.$$

$$\left.\left(\frac{a-r}{a+r}\right)\left[\frac{u_2}{D_2}\left[K(k_2) - \Pi(k_2^2/\alpha^2, k_2)\right] - \frac{u_1}{D_1}\left[K(k_1) - \Pi(k_1^2/\alpha^2, k_1)\right]\right]\right\} \tag{27}$$

In practice, Eq. 27 is used to compute B_z only when $|z| > b$ and $0.8 < r/a < 1.2$, and Eq. 22 is used everywhere else.

5 Elliptic-integral expressions for the vector potential

Having the capability of computing numerical values of the vector potential for an axisymmetric magnetic field is useful in fluxline plotting and in computing the canonical momentum in numerical trajectory integration with Hamiltonian dynamics of charged particles. For fluxline plotting, we note that as a consequence of Stoke's theorem, the magnetic flux Φ passing through a circular disk of radius r centered on and perpendicular to the axis of symmetry at axial position z is given by $\Phi(r,z) = 2\pi r A_\theta(r,z)$. Fluxlines in the r,z plane are contours of constant $\Phi(r,z)$.

In the usual gauge, the vector potential for a solenoid in cylindrical coordinates r,θ,z is given by

$$\vec{A} = \frac{\mu_0 J_0 a}{4\pi}\int_{-b}^{+b}\int_0^{2\pi}\frac{\hat{\phi}}{|\vec{r} - \vec{r}'|}dz'd\phi \tag{28}$$

As before in the derivation of the field components, $\vec{r}$ is the field point, $\vec{r}'$ the source point, and the current density vector is $J_0\hat{\phi}$. The vector potential from Equation 28 has only a θ component, given by

$$A_\theta(r,z) = \frac{\mu_0 J_0 a}{2\pi}\int_0^{\pi}\int_{u_1}^{u_2}\frac{\cos\phi\, du d\phi}{\left[r^2 + a^2 - 2ar\cos\phi + u^2\right]^{1/2}}, \tag{29}$$

where again $u_1 = z - b$ and $u_2 = z + b$. Performing the integration over u gives

$$A_\theta(r,z) = \frac{\mu_0 J_0 a}{2\pi}\int_0^{\pi}\cos\phi\log\left[\frac{u_2 + \left(u_2^2 + A\right)^{1/2}}{u_1 + \left(u_1^2 + A\right)^{1/2}}\right]d\phi, \tag{30}$$

where again $A = a^2 + r^2 - 2ar\cos\phi$. Although at first glance the right-hand side of Eq. 30 does not appear to be an elliptic integral, it can be transformed by means of integration by parts into an integral that is manifestly an elliptic integral, plus terms that vanish at the endpoints 0 and π of the angular integration. The result is

$$A_\theta(r,z) = \frac{\mu_0 J_0 a^2 r}{2\pi}\int_0^{\pi}\left[\frac{u_2\sin^2\phi}{A\left(u_2^2 + A\right)^{1/2}} - \frac{u_1\sin^2\phi}{A\left(u_1^2 + A\right)^{1/2}}\right]d\phi, \tag{31}$$

Equation 31 is derived in Ref. [5], but later in that note A_θ is evaluated by use of the Heumann lambda function. Here we derive algorithms based on `cel`. As before, the change of variables $\psi = \pi/2 - \phi/2$ is used in the integrals of Eq. 31 and we get

$$A_\theta(r,z) = \frac{\mu_0 J_0 a}{\pi}\alpha^2 \left[\frac{u_2}{D_2}\int_0^{\pi/2} \frac{\sin^2\psi\cos^2\psi\, d\psi}{(1-\alpha^2\sin^2\psi)(1-k_2^2\sin^2\psi)^{1/2}} - \frac{u_1}{D_1}\int_0^{\pi/2} \frac{\sin^2\psi\cos^2\psi\, d\psi}{(1-\alpha^2\sin^2\psi)(1-k_1^2\sin^2\psi)^{1/2}}\right], \tag{32}$$

where α^2, D_1, D_2, k_1^2, and k_2^2 have the same definitions as in Sections 3 and 4. We see that due to the presence of the factor of $1/(1-\alpha^2\sin^2\psi)$, the integrand for A_θ has the same singularity as that for B_z, and the addition theorem for Π will be used in a similar way to eliminate the fictitious singularity for $r = a$, $|z| > b$.

Except for an overall multiplicative factor, Eq. 32 is the same as the equation for the mutual inductance between a circular loop and a solenoid given in Ref. [6]. This correspondence is pointed out in Garrett's paper [2].

A first step in reducing Eq. 32 to standard form is to elimininate the $\sin^2\psi$ factors in the numerators, with the result

$$A_\theta(r,z) = \frac{\mu_0 J_0 a}{\pi}\left\{\frac{u_2}{D_2}\left[\int_0^{\pi/2} \frac{\cos^2\psi\, d\psi}{(1-\alpha^2\sin^2\psi)(1-k_2^2\sin^2\psi)^{1/2}} - \int_0^{\pi/2} \frac{\cos^2\psi\, d\psi}{(1-k_2^2\sin^2\psi)^{1/2}}\right] - \frac{u_1}{D_1}\left[\int_0^{\pi/2} \frac{\cos^2\psi\, d\psi}{(1-\alpha^2\sin^2\psi)(1-k_1^2\sin^2\psi)^{1/2}} - \int_0^{\pi/2} \frac{\cos^2\psi\, d\psi}{(1-k_1^2\sin^2\psi)^{1/2}}\right]\right\} \tag{33}$$

The integrals in Eq. 33 can be evaluated in terms of D, Π, and K as follows:

$$A_\theta(r,z) = \frac{\mu_0 J_0 a}{\pi}\left\{\frac{u_2}{D_2}\left[D(k_2) - \left(\frac{1-\alpha^2}{\alpha^2}\right)\left[\Pi(\alpha^2,k_2) - K(k_2)\right]\right] - \frac{u_1}{D_1}\left[D(k_1) - \left(\frac{1-\alpha^2}{\alpha^2}\right)\left[\Pi(\alpha^2,k_1) - K(k_1)\right]\right]\right\} \tag{34}$$

Equation 34 could be used for numerical evaluation of A_θ except for the cases $r \sim a$, $|z| > b$ where there is the fictitious singularity, and $r \sim 0$, where the $1/\alpha^2$ factor becomes large. However, it will not be used directly for computing A_θ in the algorithms described in this note, but rather as a starting point for applying the addition theorem.

Dealing first with the case $r \sim a$, $|z| > b$, we use the addition theorem for Π (see Eq. 24) as before in evaluation of B_z. This gives

$$A_\theta(r,z) = \frac{\mu_0 J_0 a}{\pi}\left\{\frac{u_2}{D_2}\left[\frac{(a-r)^2}{4ar}\Pi(k_2^2/\alpha^2, k_2) + D(k_2)\right] - \frac{u_1}{D_1}\left[\frac{(a-r)^2}{4ar}\Pi(k_1^2/\alpha^2, k_1) + D(k_1)\right]\right\} \tag{35}$$

In order to reduce roundoff error, the ratio $(1-\alpha^2)/\alpha^2$ in Eq. 34 has been replaced by the equivalent ratio $(a-r)^2/(4ar)$.

In practice, Eq. 35 is used to compute A_θ only when $|z| > b$ and $0.8 < r/a < 1.2$, and in order to avoid the factor of $1/\alpha^2$ in Eq. 34, Eq. 33 (with the integrals evaluated directly by `cel`) is used everywhere else.

6 Field-component derivatives

Field-component derivatives are needed, for example, in tracking neutral particles that are subject to spin-field gradient forces. Since the geometry of the problem is axisymmetric, the θ derivatives are zero and we are left with the four derivatives $\partial B_r/\partial r$, $\partial B_r/\partial z$, $\partial B_z/\partial r$, and $\partial B_z/\partial z$. If the field point does not lie on the solenoid itself, both the curl and divergence of the field are zero. The zero curl condition gives $\partial B_r/\partial z = \partial B_z/\partial r$. The zero divergence condition gives $\partial B_r/\partial r = -\partial B_z/\partial z - B_r/r$. This leaves only two independent derivatives. It is most convenient to compute the two z derivatives $\partial B_r/\partial z$ and $\partial B_z/\partial z$ and use the zero-curl and zero-divergence conditions to compute the remaining two derivatives. Then, for computation of $\partial B_r/\partial r$, the ratio B_r/r is needed. For this, we go back to Eq. 20. Writing

$$\frac{1}{r} = \frac{1}{k^2}\frac{4a}{(a+r)^2+u^2} \tag{36}$$

we get for B_r/r

$$\frac{B_r(r,z)}{r} = -\frac{\mu_0 J_0 a}{\pi}\left[\frac{4a}{D_2^3}C(k_2) - \frac{4a}{D_1^3}C(k_1)\right] \tag{37}$$

We see that, since $C(k)$ approaches a finite limit as k approaches 0, B_r/r approaches a finite limit as r approaches zero.

To get the derivative $\partial B_r/\partial z$, we go back to Eq. 5. Since u in the integrand is a dummy variable, the z derivative involves only the limits, for which $\partial u_1/\partial z = \partial u_2/\partial z = 1$, and we get

$$\frac{\partial B_r}{\partial z}(r,z) = \frac{\mu_0 J_0 a}{2\pi}\left\{u_2\int_0^\pi \frac{\cos\phi}{\left[u_2^2+A\right]^{3/2}}d\phi - u_1\int_0^\pi \frac{\cos\phi}{\left[u_1^2+A\right]^{3/2}}d\phi\right\} \tag{38}$$

Using the change of variables $\psi = \pi/2 - \phi/2$ now gives

$$\frac{\partial B_r}{\partial z}(r,z) = \frac{\mu_0 J_0 a}{\pi}\left\{\frac{u_2}{D_2^3}\int_0^{\pi/2}\frac{(\sin^2\psi - \cos^2\psi)\,d\psi}{\left(1-k_2^2\sin^2\psi\right)^{3/2}} - \frac{u_1}{D_1^3}\int_0^{\pi/2}\frac{(\sin^2\psi - \cos^2\psi)\,d\psi}{\left(1-k_1^2\sin^2\psi\right)^{3/2}}\right\}, \tag{39}$$

The integrals in Eq. 39 can be evaluated in terms of the elliptic integrals E and D:

$$\frac{\partial B_r}{\partial z}(r,z) = \frac{\mu_0 J_0 a}{\pi}\left\{\frac{u_2}{D_2^3}\left[\frac{E(k_2)}{1-k_2^2} - 2D(k_2)\right] - \frac{u_1}{D_1^3}\left[\frac{E(k_1)}{1-k_1^2} - 2D(k_1)\right]\right\} \tag{40}$$

For $\partial B_z/\partial z$ we differentiate the right-hand side of Eq. 6 with respect to z, and we get

$$\frac{\partial B_z}{\partial z}(r,z) = \frac{\mu_0 J_0 a}{2\pi}\left\{\int_0^{\pi}\frac{(a - r\cos\phi)d\phi}{\left[u_2^2 + A\right]^{3/2}} - \int_0^{\pi}\frac{(a - r\cos\phi)d\phi}{\left[u_1^2 + A\right]^{3/2}}\right\} \tag{41}$$

The change of variables $\psi = \pi/2 - \phi/2$ gives

$$\frac{\partial B_z}{\partial z}(r,z) = \frac{\mu_0 J_0 a}{\pi}\left\{\frac{1}{D_2^3}\int_0^{\pi/2}\frac{\left[a - r + 2r\cos^2\psi\right]d\psi}{\left(1-k_2^2\sin^2\psi\right)^{3/2}} - \frac{1}{D_1^3}\int_0^{\pi/2}\frac{\left[a - r + 2r\cos^2\psi\right]d\psi}{\left(1-k_1^2\sin^2\psi\right)^{3/2}}\right\}, \tag{42}$$

The two elliptic integrals in Eq. 42 can again be evaluated in terms of E and D:

$$\frac{\partial B_z}{\partial z}(r,z) = \frac{\mu_0 J_0 a}{\pi}\left\{\frac{1}{D_2^3}\left[\frac{(a-r)E(k_2)}{1-k_2^2} + 2rD(k_2)\right] - \frac{1}{D_1^3}\left[\frac{(a-r)E(k_1)}{1-k_2^2} + 2rD(k_1)\right]\right\} \tag{43}$$

7 Numerical evaluation of field components, vector potential, and field-component derivatives using Bulirsch's `cel`

The Bulirsch algorithm `cel` [3] evaluates a generalized complete elliptic integral of the form

$$\texttt{cel}(k_c, p, a, b) = \int_0^{\pi/2}\frac{\left(a\cos^2\psi + b\sin^2\psi\right)d\psi}{\left(\cos^2\psi + p\sin^2\psi\right)\left(\cos^2\psi + k_c^2\sin^2\psi\right)^{1/2}} \tag{44}$$

The quantities k_c and p in Eq. 44 are sometimes called the complementary moduli and are defined to be $k_c = (1-k^2)^{1/2}$ and $p = 1 - \alpha^2$. We rewrite the expressions for the complementary moduli in order to reduce roundoff error:

$$k_{c,1}^2 = \frac{(a-r)^2 + u_1^2}{(a+r)^2 + u_1^2} \tag{45}$$

$$k_{c,2}^2 = \frac{(a-r)^2 + u_2^2}{(a+r)^2 + u_2^2} \tag{46}$$

$$p = \frac{(a-r)^2}{(a+r)^2} \tag{47}$$

In order to compute $\Pi(k_1^2/\alpha^2, k_1)$ and $\Pi(k_2^2/\alpha^2, k_2)$ with `cel`, we will need $p_1 = 1 - k_1^2/\alpha^2$ and $p_2 = 1 - k_2^2/\alpha^2$. In order to reduce roundoff error, we use for them the expressions

$$p_1 = 1 - k_1^2/\alpha^2 = \frac{u_1^2}{(a+r)^2 + u_1^2} \tag{48}$$

and

$$p_2 = 1 - k_2^2/\alpha^2 = \frac{u_2^2}{(a+r)^2 + u_2^2}. \tag{49}$$

From Eq. 44 we see the standard elliptic integrals K, E, Π, B, and D are given by

$$K(k) = \texttt{cel}(k_c, 1, 1, 1), \tag{50}$$

$$E(k) = \texttt{cel}(k_c, 1, 1, k_c^2), \tag{51}$$

$$\Pi(\alpha^2, k) = \texttt{cel}(k_c, p, 1, 1), \tag{52}$$

$$B(k) = \texttt{cel}(k_c, 1, 1, 0), \tag{53}$$

and

$$D(k) = \texttt{cel}(k_c, 1, 0, 1). \tag{54}$$

When evaluating A_θ for points outside of the zone of the fictitious singularity, we use `cel` directly to evaluate the elliptic integral in Eq. 33, since this avoids the problem of the $1/\alpha^2$ factor in Eq. 34. This gives

$$A_\theta(r,z) = \frac{\mu_0 J_0 a}{\pi} \left\{ \frac{u_2}{D_2} \left[\texttt{cel}(k_{c,2}, p, 1, 0) - \texttt{cel}(k_{c,2}, 1, 1, 0)\right] - \frac{u_1}{D_1} \left[\texttt{cel}(k_{c,1}, p, 1, 0) - \texttt{cel}(k_{c,1}, 1, 1, 0)\right] \right\}. \tag{55}$$

Although $C(k)$ as defined by Eq. 17 is not of the form of Eq. 44, it is shown in Ref. [7] that $C(k)$ can be computed with the older Bulirsch algorithm `cel2`, as follows:

$$C(k) = \texttt{cel2}\left(\frac{2k_c^{1/2}}{1+k_c}, 0, \frac{2}{(1+k_c)^3}\right) \tag{56}$$

It turns out that the newer Bulirsch algorithm `cel`, used everywhere else in this note, can also be used to compute $C(k)$:

$$C(k) = \texttt{cel}\left(\frac{2k_c^{1/2}}{1+k_c}, 1, 0, \frac{2}{(1+k_c)^3}\right) \tag{57}$$

If only B_r, B_z, and A_θ are required, the number of calls to `cel` required is 8. If the field derivatives are also required, a total of 10 calls to `cel` are required when r of the field point is in the fictitious singularity region $r \sim a$, $|z| > b$. Since, as previously mentioned, the region outside of the zone of the fictitious singularity includes the region of small r (small α^2), Eq. 55 is used in that region. The total number of calls to `cel` is then 12.

8 References

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