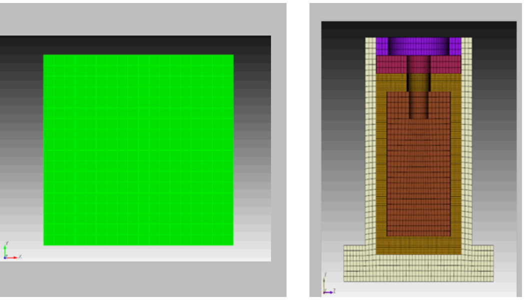
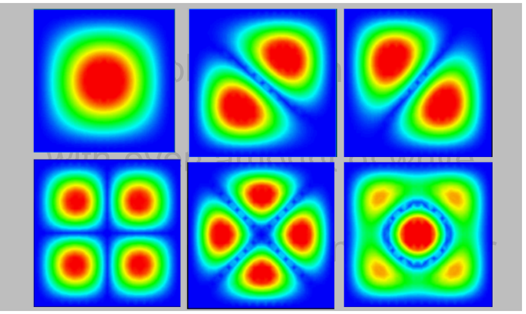
SAND2016-5153C





Time Domain Model Reduction of Linear Viscoelastic Finite Element Models

R.J. Kuether[1] & K.L. Troyer[1]

[1]Sandia National Laboratories

Presented at PROM workshop on June 2, 2016.

Exceptional service in the national interest

Sandia National Laboratories is a multi-program laboratory managed and operated by Sandia Corporation, a wholly owned subsidiary of Lockheed Martin Corporation, for the U.S. Department of Energy's National Nuclear Security Administration under contract DE-AC04-94AL85000.

What is our goal?

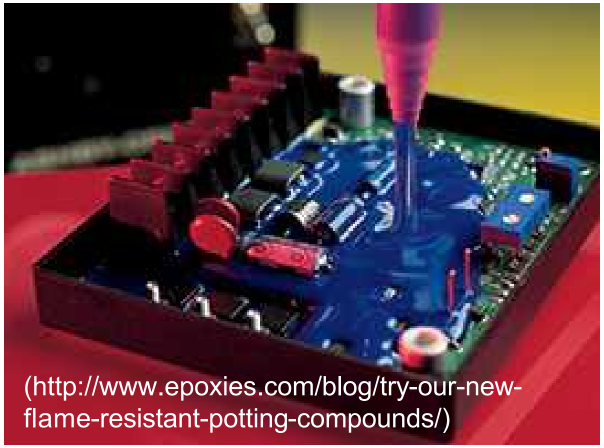
- Develop reduced order models (ROMs) of finite element models with linear viscoelastic material behavior for **time domain** structural dynamic simulations

- Reduce computational burden of repetitive numerical solutions while preserving the accuracy of the full order model

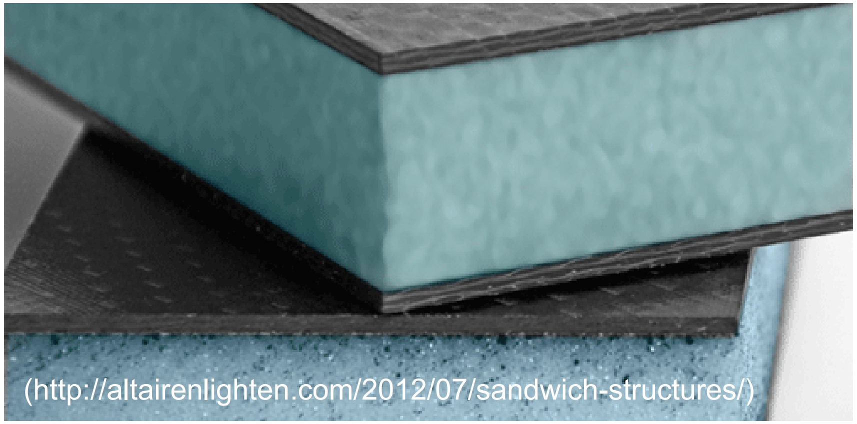
- Incorporate non-viscous damping into ROMs via material property data

Applications with Linear Viscoelastic Behavior

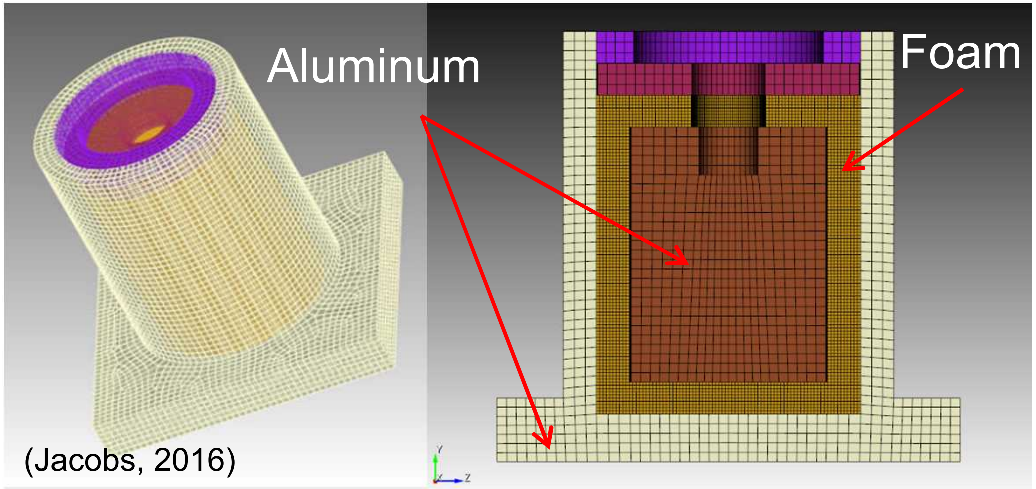
Encapsulation



Sandwich Structured Composites



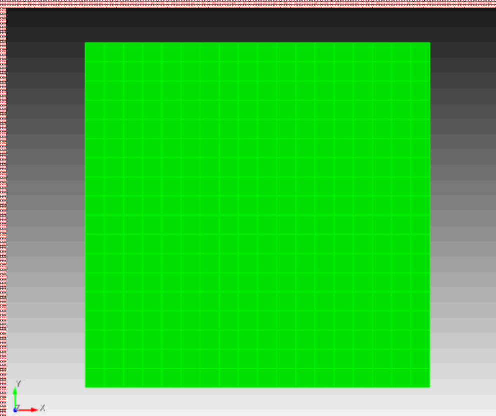
Ministack



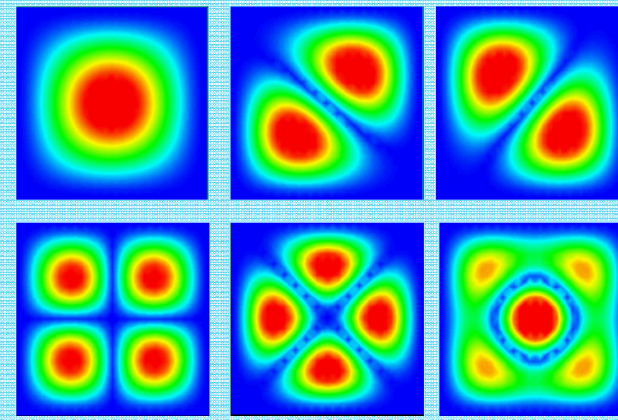
Reduced Order Modeling Approach

FE mesh in physical coordinates

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} + \mathbf{f}_{matl}(\mathbf{x},\dot{\mathbf{x}}) = \mathbf{f}(t)$$



Determine appropriate basis, or shape vectors based on the physical equations of motion



Project full equations of motion onto a small set of basis vectors

Related by transformation:

$$\mathbf{x}(\mathrm{t}) = \mathbf{T}\mathbf{q}(t)$$

$$^{*}\mathbf{q}(\mathrm{t}) \ll \mathbf{x}(t)$$

Solve reduced equations

$$\hat{\mathbf{M}}\ddot{\mathbf{q}} + \hat{\mathbf{K}}\mathbf{q} + \mathbf{f}_{matl}(\mathbf{q},\dot{\mathbf{q}}) = \hat{\mathbf{f}}(t)$$

Linear Viscoelasticity with Prony Series

- **Stress dependent upon time**

$$\boldsymbol{\sigma}(t) = \int_0^t E(t-\tau)\frac{d\boldsymbol{\varepsilon}}{d\tau}d\tau$$

- **Prony series**

$$E(t) = E_\infty + (E_g - E_\infty)\zeta(t)$$

$$\zeta(t) = \sum_{i=1}^{N} E_i e^{-t/\tau_i}$$

Relaxation modulus: describes time- and history-dependent behavior!

(skipping detailed mathematics)

- **FEA equations of motion**

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}_{v,K}\int_0^t \zeta_K(t-\tau)\dot{\mathbf{x}}(\tau)d\tau + \mathbf{K}_{v,G}\int_0^t \zeta_G(t-\tau)\dot{\mathbf{x}}(\tau)d\tau + \mathbf{K}_e\mathbf{x} = \mathbf{f}(t)$$

*Typically have many degrees-of-freedom!

Real Eigenmode Basis

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}_{v,K}\int_0^t \zeta_K(t-\tau)\dot{\mathbf{x}}(\tau)d\tau + \mathbf{K}_{v,G}\int_0^t \zeta_G(t-\tau)\dot{\mathbf{x}}(\tau)d\tau + \mathbf{K}_e\mathbf{x} = \mathbf{f}(t)$$

- **Real Eigenmodes with Static Correction**

$$\left(\mathbf{K}_e - \omega_r^2\mathbf{M}\right)\boldsymbol{\varphi}_r = \mathbf{0}$$

$$\mathbf{T} = [\boldsymbol{\varphi}_1 \quad \boldsymbol{\varphi}_2 \quad \ldots \quad \boldsymbol{\varphi}_{N_R} \quad \mathbf{R}_s]$$

$$\mathbf{M}\ddot{\mathbf{x}} + \alpha\mathbf{K}_e\int_0^t \zeta(t-\tau)\dot{\mathbf{x}}(\tau)d\tau + \mathbf{K}_e\mathbf{x} = \mathbf{f}(t)$$

- *Must satisfy three conditions to work
  - All elements in the FEA model made of same viscoelastic material
  - Kernel functions equal for shear and bulk relaxation moduli $\boldsymbol{\zeta}_K(t) = \boldsymbol{\zeta}_G(t)$
  - It must hold that $\frac{G_g - G_\infty}{K_g - K_\infty} = \frac{G_\infty}{K_\infty}$

$$\therefore \quad \mathbf{M}\ddot{\mathbf{x}} + \eta\mathbf{K}_e\int_0^t \zeta(t-\tau)\dot{\mathbf{x}}(\tau)d\tau + \mathbf{K}_e\mathbf{x} = \mathbf{f}(t)$$

Linearized Complex Eigenmode Basis

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}_{v,K}\int_0^t \zeta_K(t-\tau)\dot{\mathbf{x}}(\tau)d\tau + \mathbf{K}_{v,G}\int_0^t \zeta_G(t-\tau)\dot{\mathbf{x}}(\tau)d\tau + \mathbf{K}_e\mathbf{x} = \mathbf{f}(t)$$

- **Linearized Complex Eigenmodes**
  - Iterative approach that uses linearized quadratic eigensolver in Sierra/SD

Linearized quadratic eigenvalue problem: for each mode, iterate until $\mathbf{Im}(\boldsymbol{\lambda_r}) = \boldsymbol{\omega_0}$

$$\left(\lambda_r^2\mathbf{M} + \lambda_r\mathbf{K}_{v,K}\sum_{i=i}^{N_K}\frac{K_{coeff,i}}{\lambda_0 + 1/\tau_{K,i}} + \lambda_r\mathbf{K}_{v,G}\sum_{i=1}^{N_G}\frac{G_{coeff,i}}{\lambda_0 + 1/\tau_{G,i}} + \mathbf{K}_e\right)\mathbf{u}_r(\lambda_0) = \mathbf{0} \quad \textbf{with} \quad \lambda_0 = i\omega_0$$

Quasi-static Correction

$$\mathbf{R}_{qs} = \left(i\omega\mathbf{K}_{v,K}\sum_{i=1}^{N_K}\frac{K_{coeff,i}}{i\omega + 1/\tau_{K,i}} + i\omega\mathbf{K}_{v,G}\sum_{i=1}^{N_G}\frac{G_{coeff,i}}{i\omega + 1/\tau_{G,i}} + \mathbf{K}_e\right)^{-1}\mathbf{b} \quad \textbf{with} \quad \omega >> 0$$

$$\Rightarrow \quad \mathbf{T} = [\mathrm{Re}(\mathbf{u}_1 \quad \mathbf{u}_2 \quad \dots \quad \mathbf{u}_{N_R} \quad \mathbf{R}_{qs}) \quad \mathrm{Im}(\mathbf{u}_1 \quad \mathbf{u}_2 \quad \dots \quad \mathbf{u}_{N_R} \quad \mathbf{R}_{qs})]$$

Numerical Time Integration

- **Conditionally stable, second order accurate HHT method [1]**

$$\mathbf{q}_{n+1} = \mathbf{q}_n + \Delta t_n \dot{\mathbf{q}}_n + \frac{\Delta t_n^2}{2} \ddot{\mathbf{q}}_n$$

$$\ddot{\mathbf{q}}_{n+1} = -\hat{\mathbf{M}}^{-1}[\mathbf{f}_{visco}(\mathbf{q}_{n+1}) + \mathbf{f}_e(\mathbf{q}_{n+1}) - \mathbf{f}_{ext}(t_{n+1})]$$

$$\dot{\mathbf{q}}_{n+1} = \dot{\mathbf{q}}_n + \frac{\Delta t_n}{2}(\ddot{\mathbf{q}}_n + \ddot{\mathbf{q}}_{n+1})$$

- **Viscoelastic force recursively updated through incremental steps in time [2]**

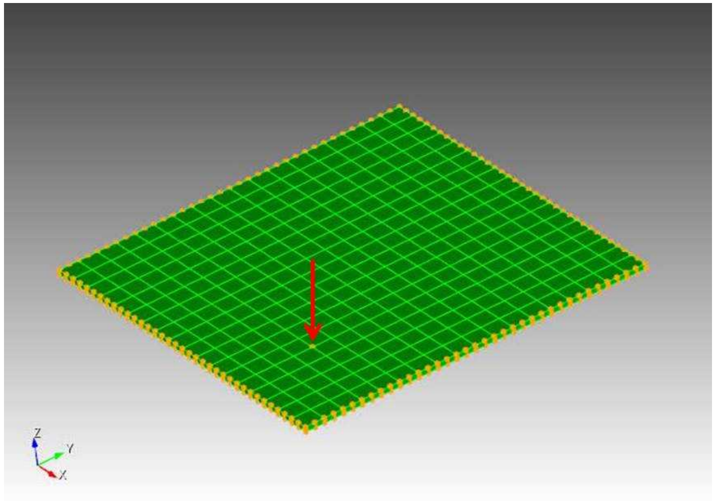
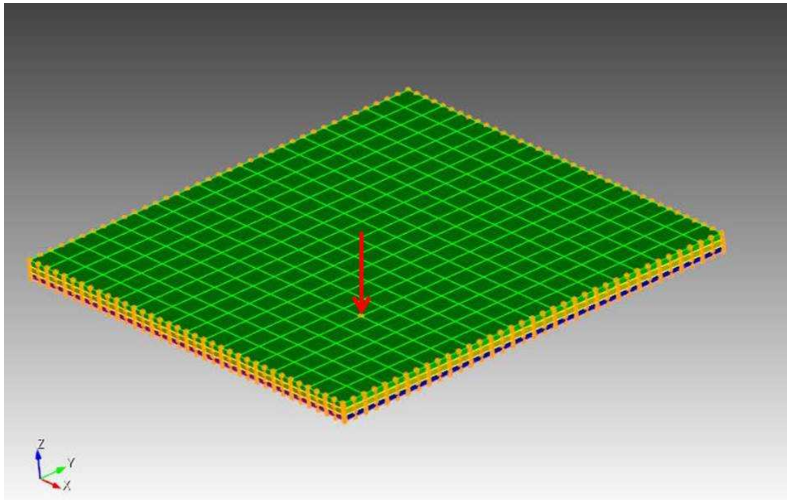
$$\mathbf{f}_{visco}(\mathbf{q}_{n+1}) = \sum_{i=1}^{N_K} [e^{-\frac{\Delta t_n}{\tau_{K,i}}} \mathbf{h}_i(t_n) + \frac{\tau_{K,i} K_{coeff,i}(1 - e^{-\frac{\Delta t_n}{\tau_{K,i}}})}{\Delta t_n}(\mathbf{q}_{n+1} - \mathbf{q}_n)]$$

$$\mathbf{h}_i(t_{n+1}) = e^{-\frac{\Delta t}{\tau_{K,i}}} \mathbf{h}_i(t_n) + \frac{\tau_{K,i} K_{coeff,i}(1 - e^{-\frac{\Delta t_n}{\tau_{K,i}}})}{\Delta t}(\mathbf{q}_{n+1} - \mathbf{q}_n)$$

History state variable!

[1] H. M. Hilber, T. J. R. Hughes, and R. L. Taylor, "Improved Numerical Dissipation for Time Integration Algorithms in Structural Dynamics," *Earthquake Engineering & Structural Dynamics,* vol. 5, pp. 283-292, 1977.
[2] J. C. Simo and T. J. R. Hughes, *Computational inelasticity*. New York: Springer, 1998.

Application to Plate Model in Sierra/SD

Case 1: Viscoelastic Plate	Case 2: Viscoelastic Sandwich Plate
	

Case 1: Viscoelastic Plate

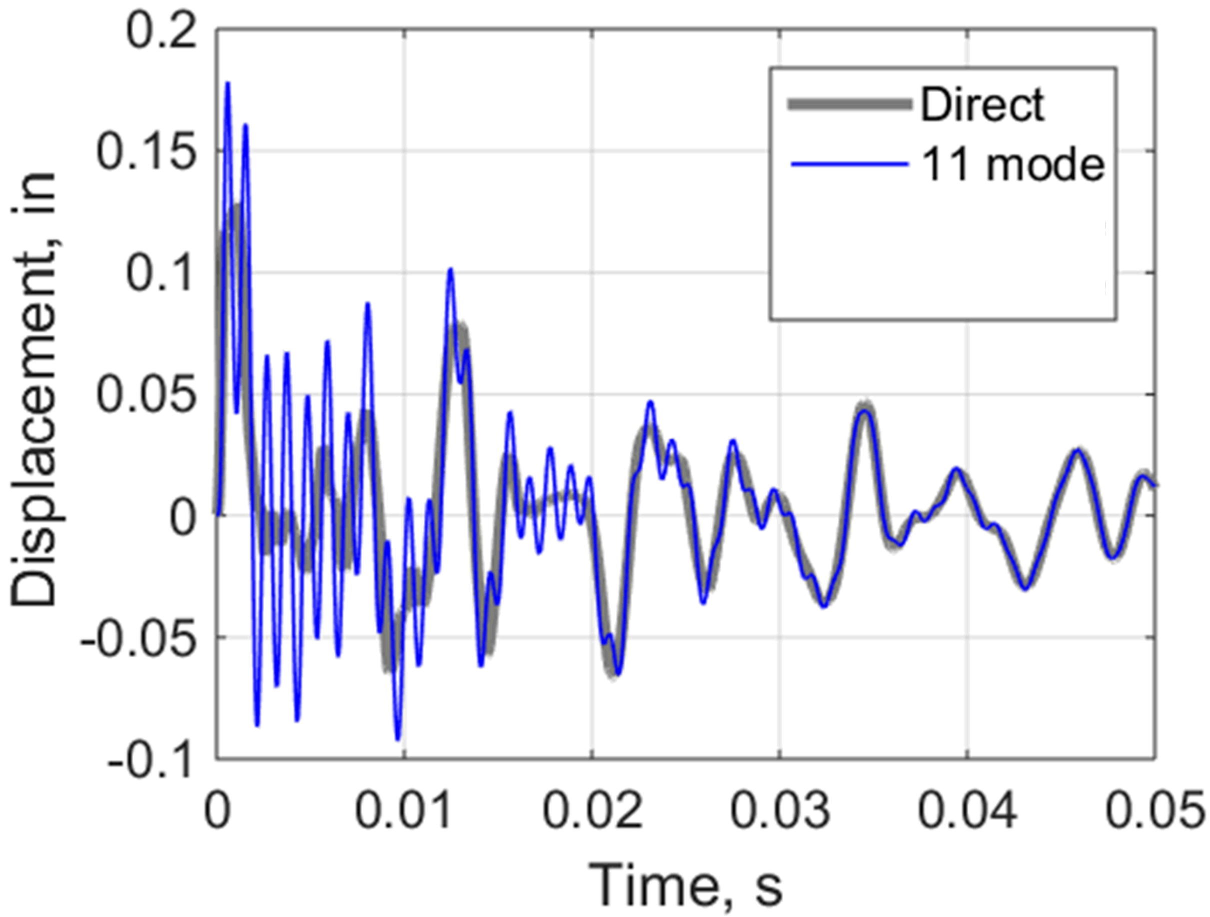
- PMDI 22 foam with 20 exp. terms
- Shear, bulk relaxation have same kernel function
- Rubbery & glassy moduli hold specific ratio:

$$\frac{G_g - G_\infty}{K_g - K_\infty} = \frac{G_\infty}{K_\infty}$$

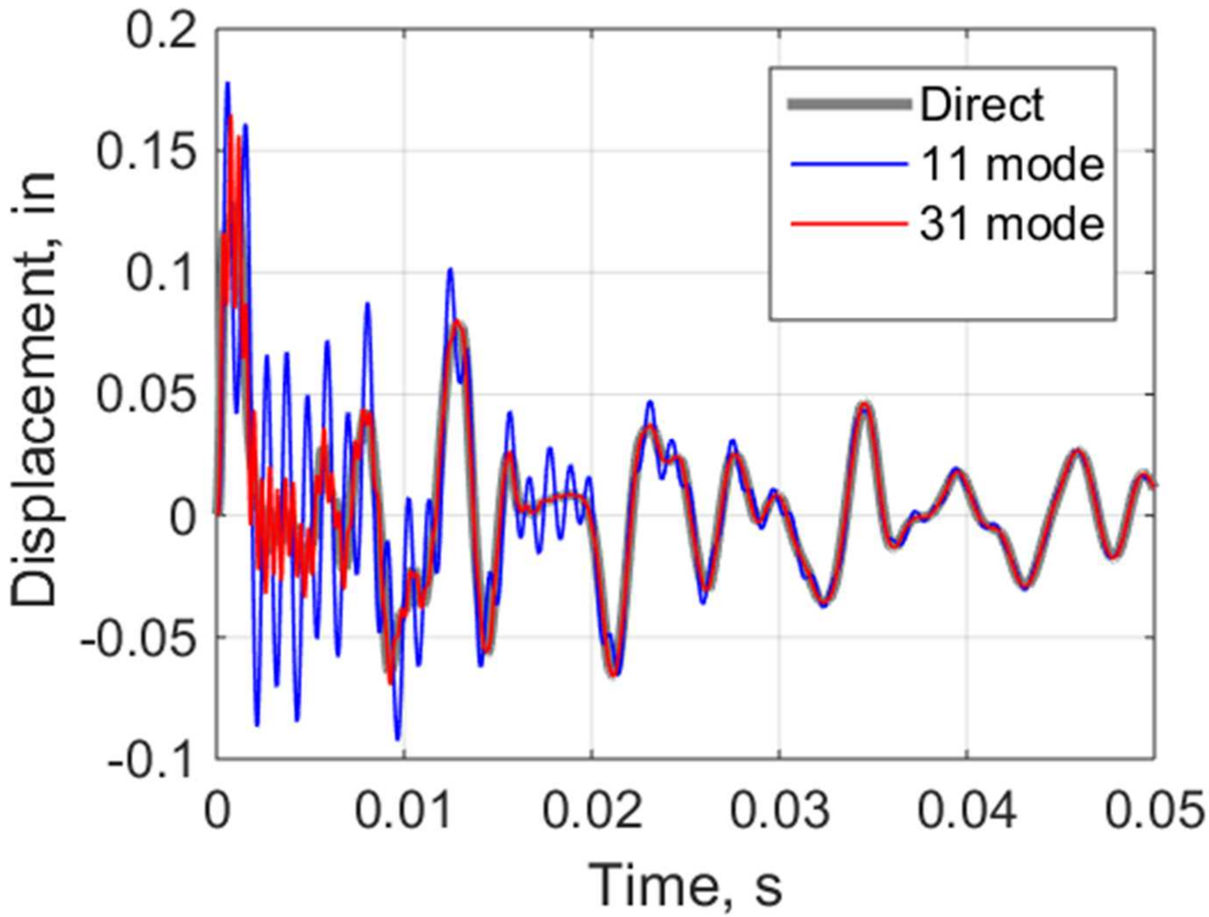
Case 2: Viscoelastic Sandwich Plate

- PMDI 22 foam sandwiched between two AL 6061-T6 plates
- **Criterion for real modes not met**

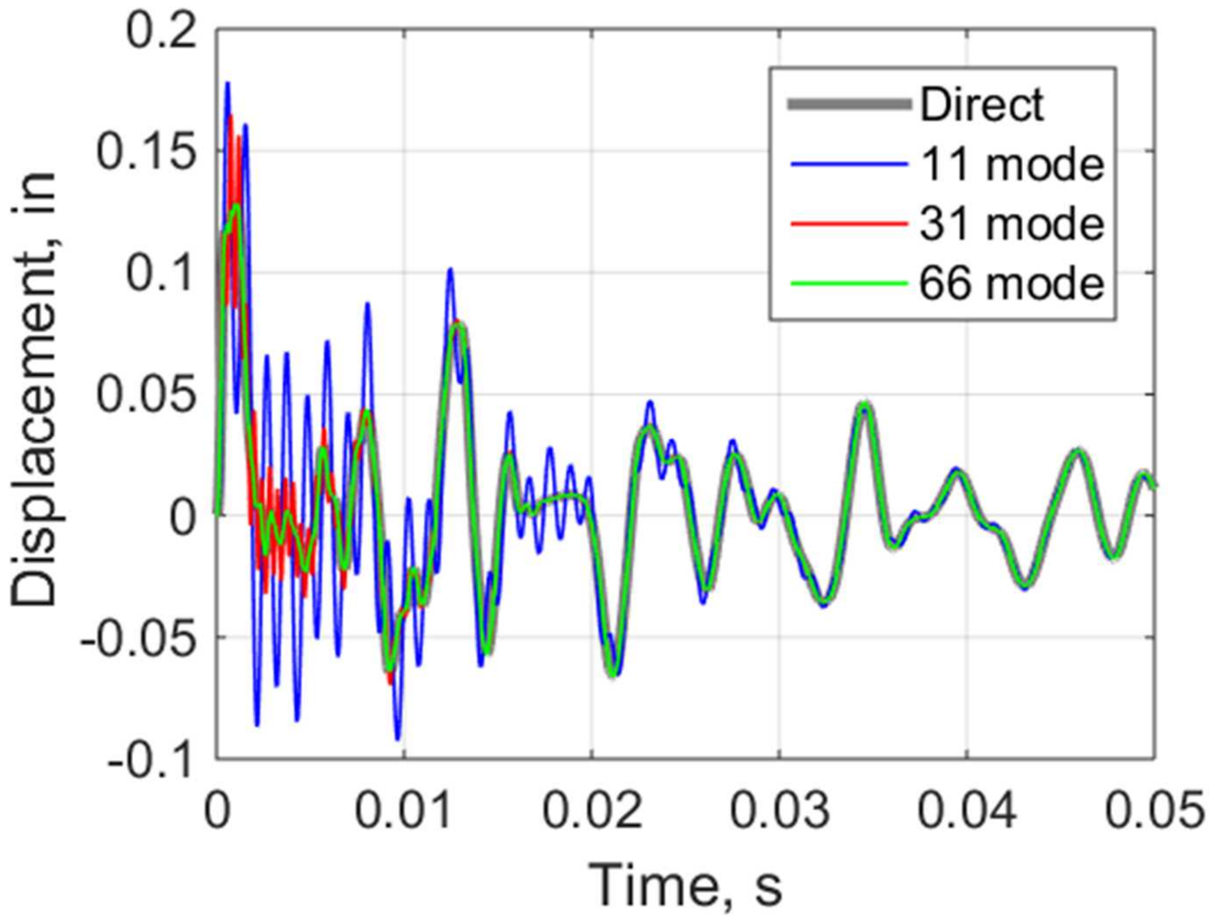
Case 1: Viscoelastic Plate with Real Modes



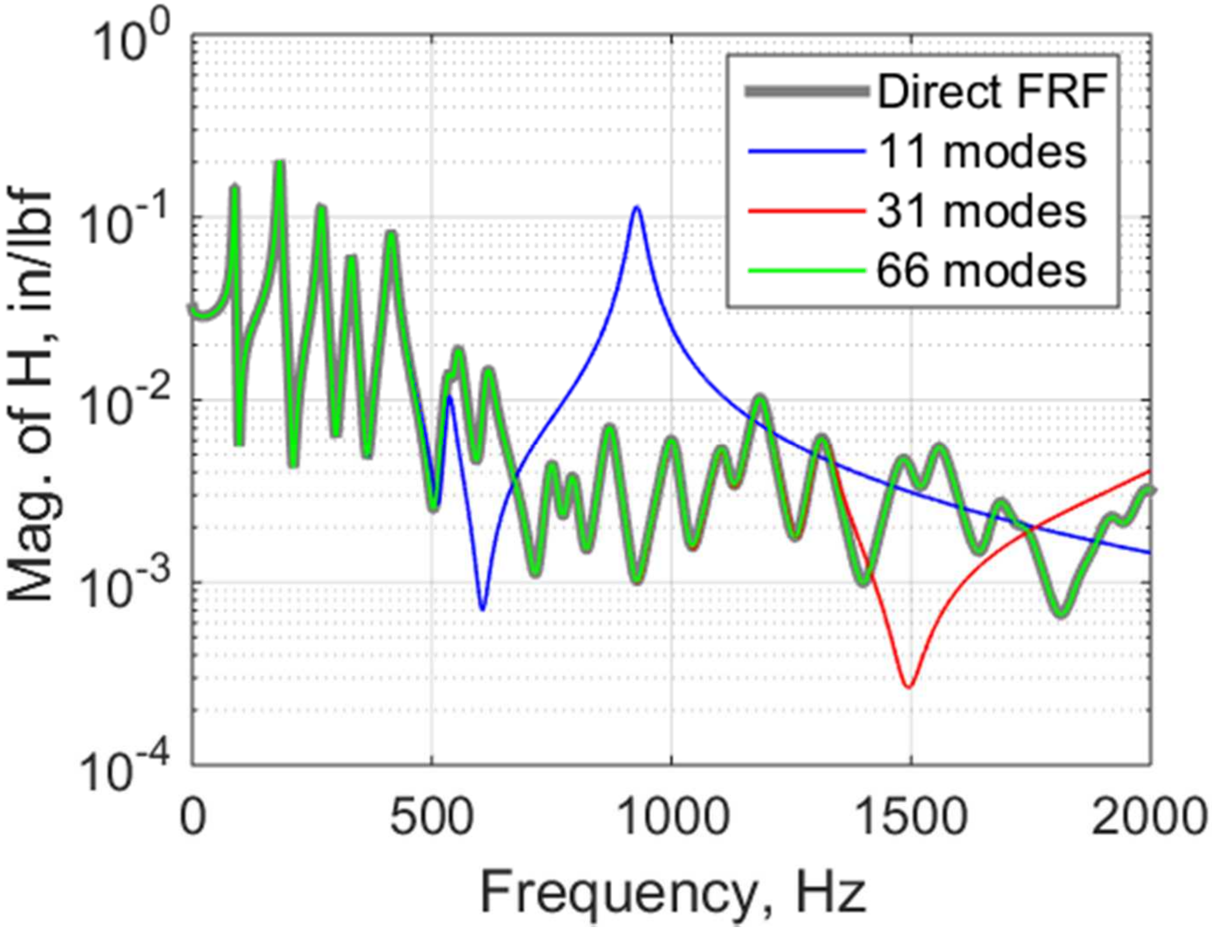
Case 1: Viscoelastic Plate with Real Modes



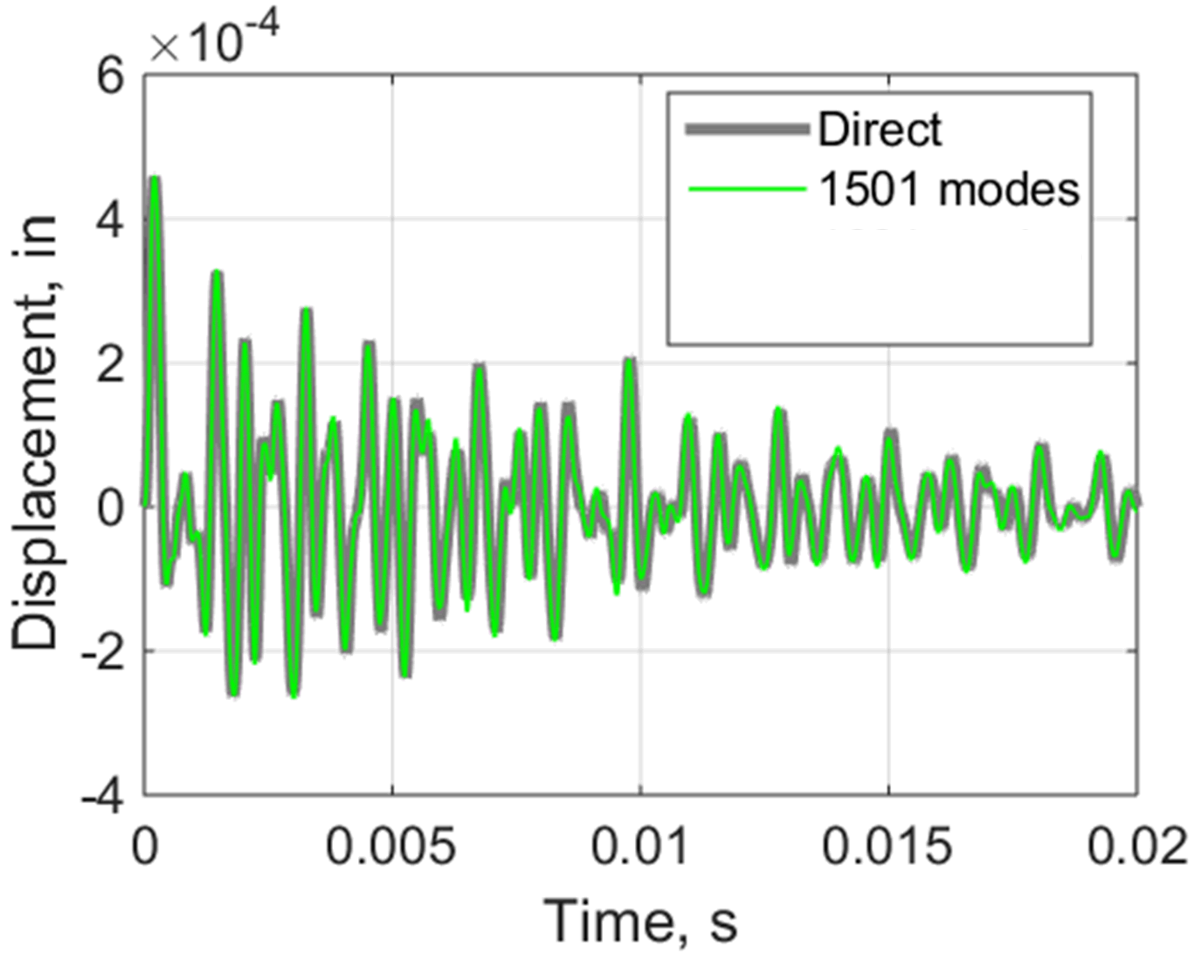
Case 1: Viscoelastic Plate with Real Modes



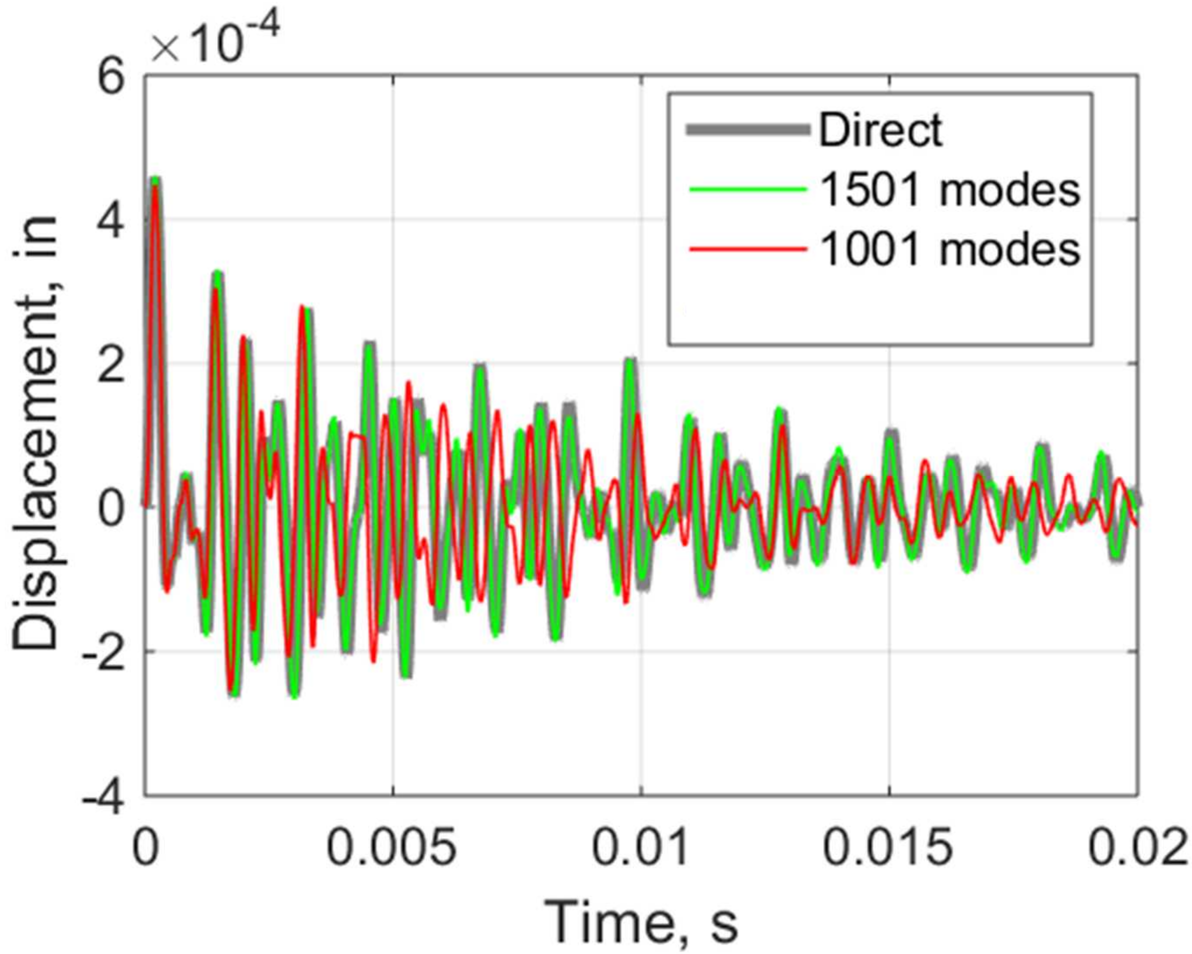
Case 1: Viscoelastic Plate with Real Modes



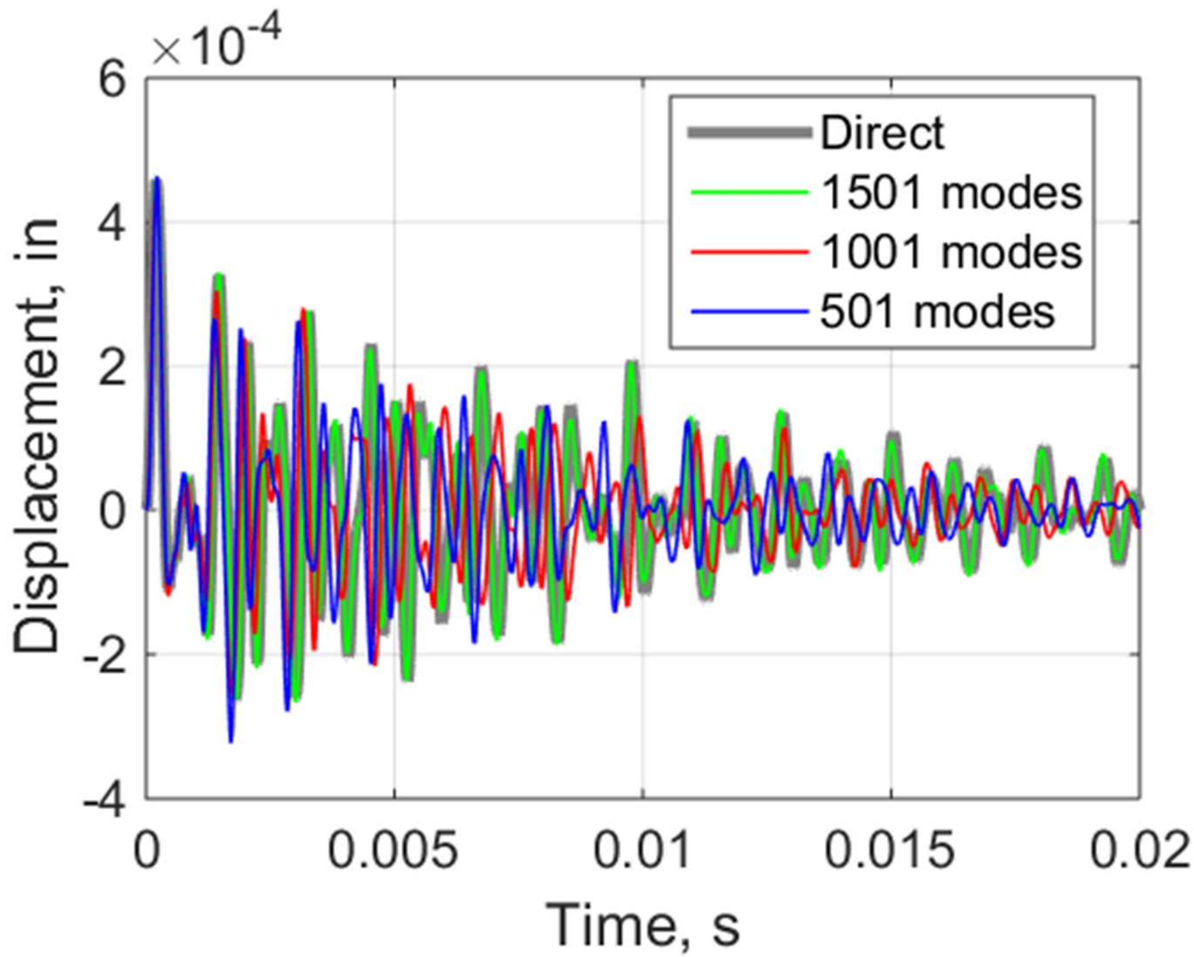
Case 2: Viscoelastic Sandwich Plate with Real Modes



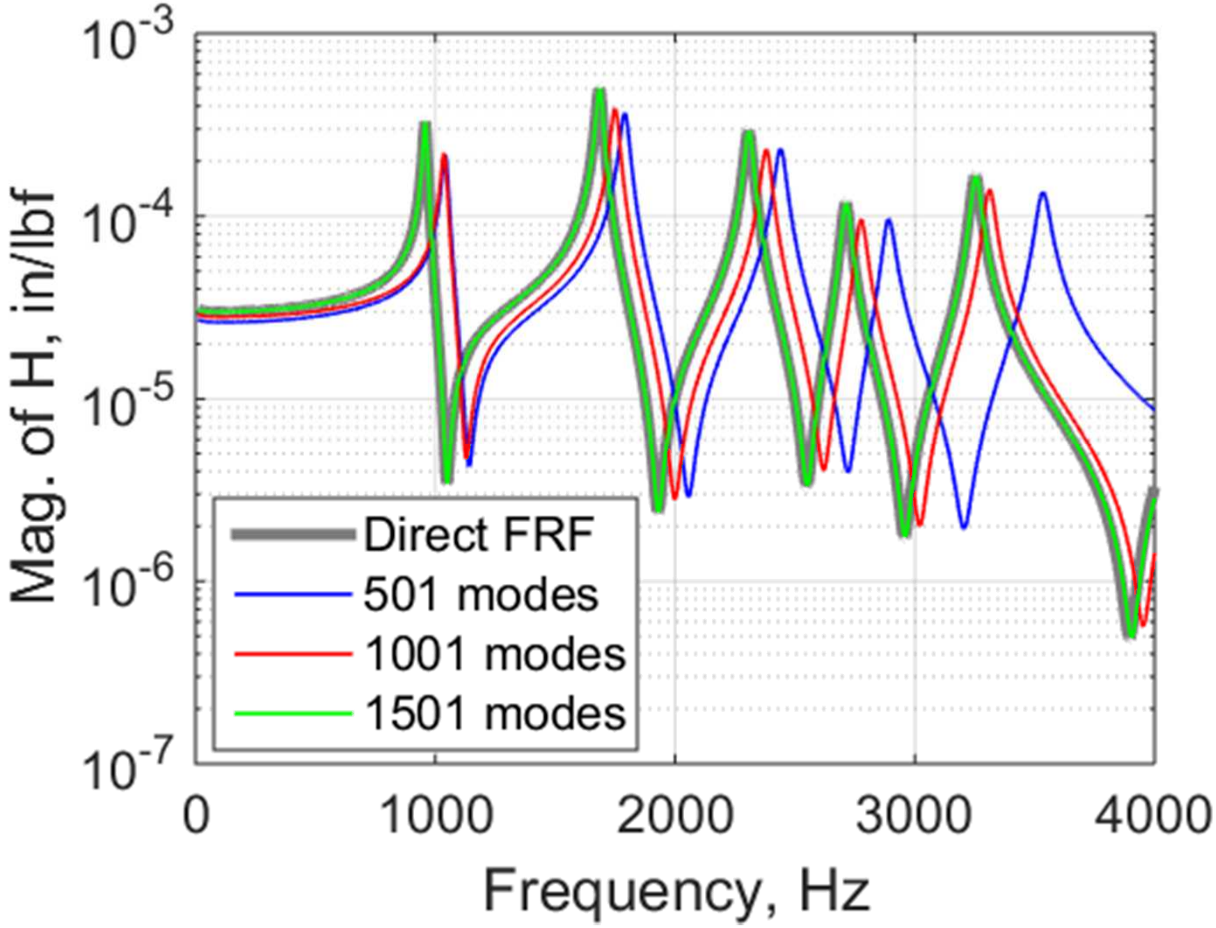
Case 2: Viscoelastic Sandwich Plate with Real Modes



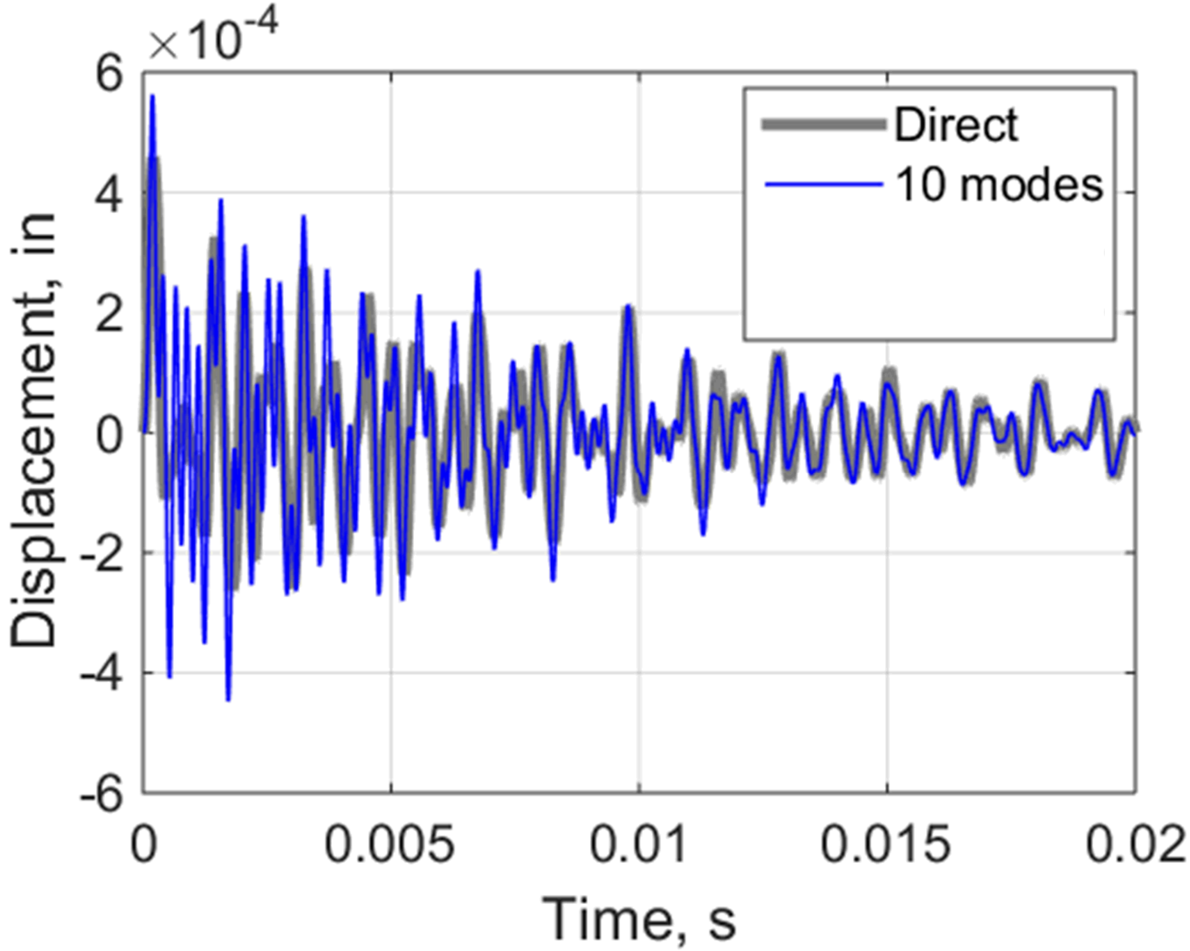
Case 2: Viscoelastic Sandwich Plate with Real Modes



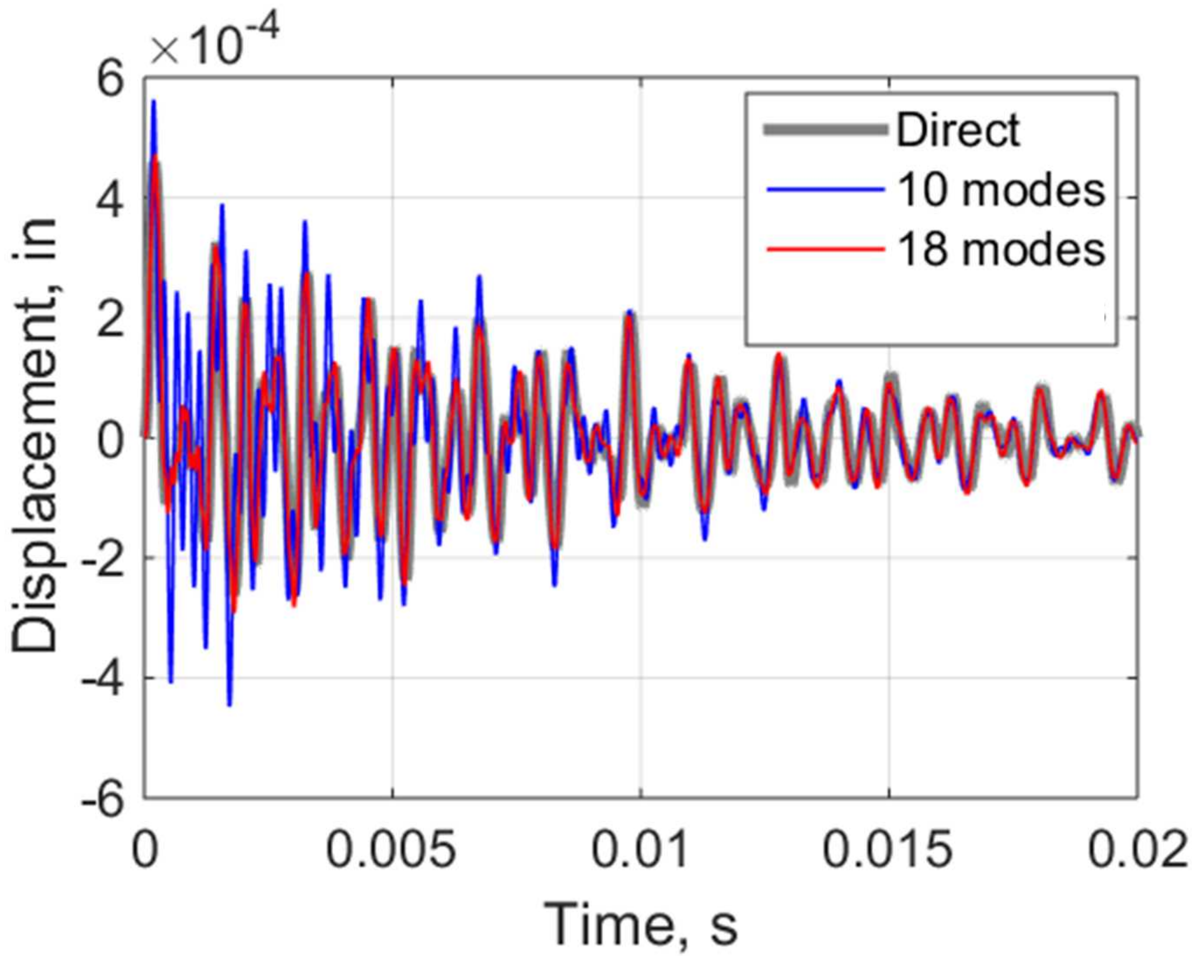
Case 2: Viscoelastic Sandwich Plate with Real Modes



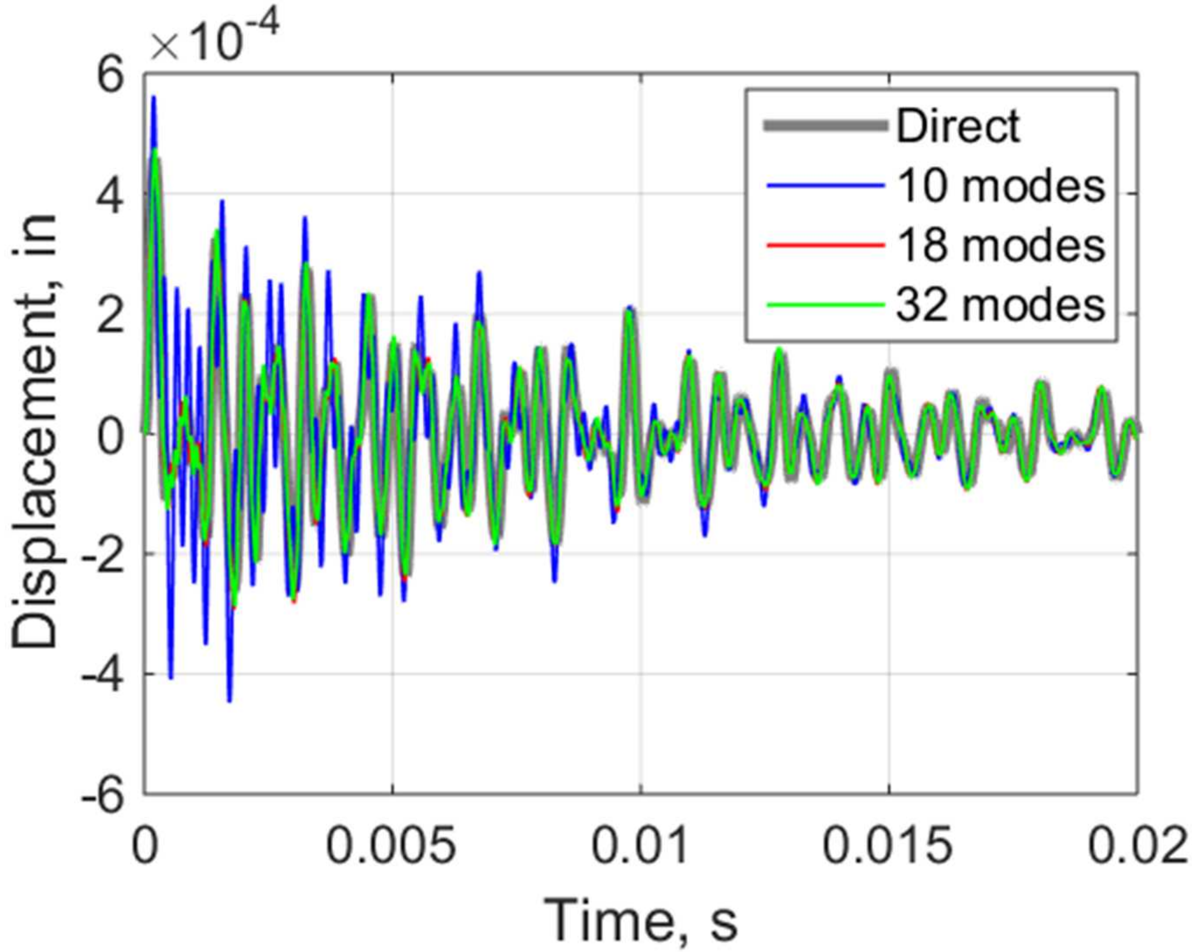
Case 2: Viscoelastic Sandwich Plate with Linearized Complex Modes



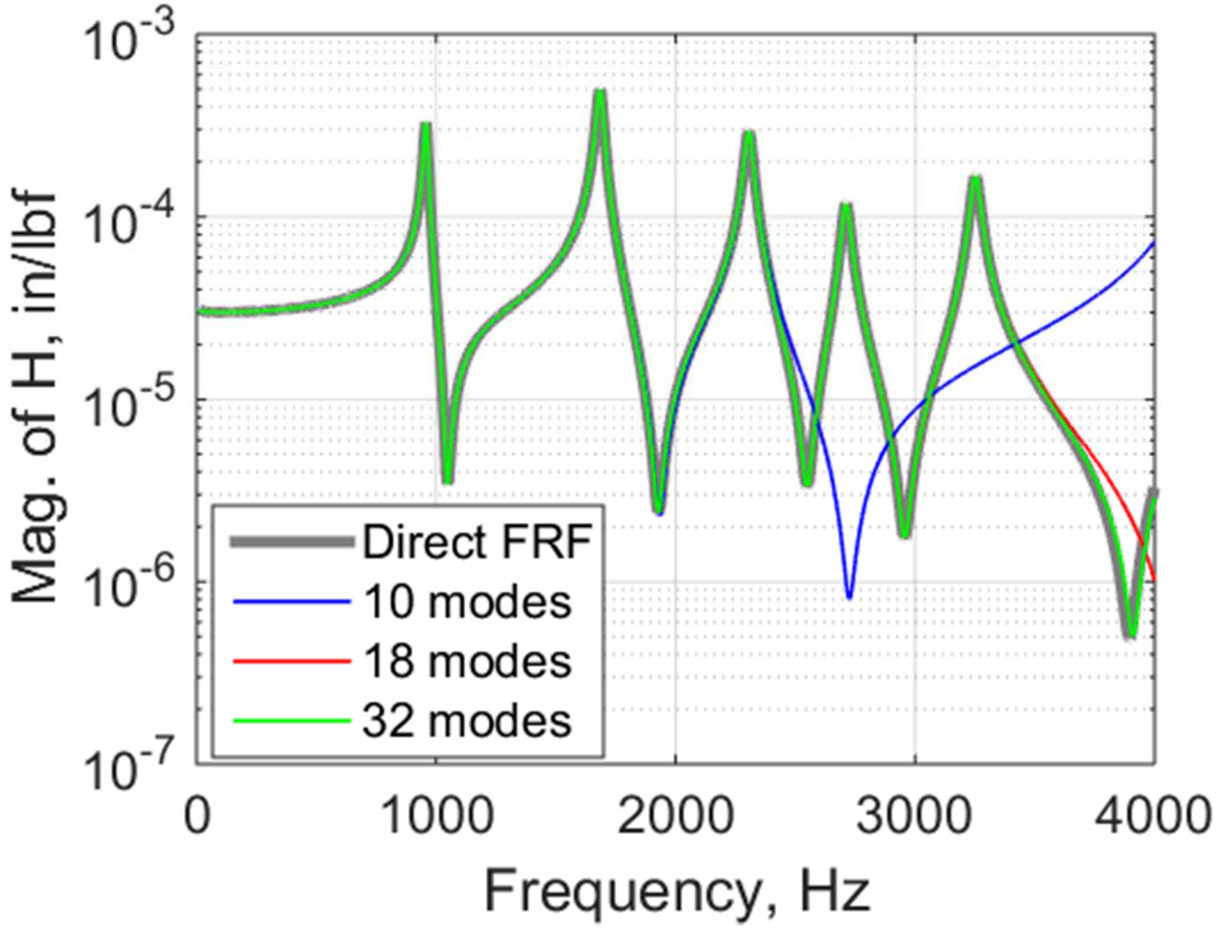
Case 2: Viscoelastic Sandwich Plate with Linearized Complex Modes



Case 2: Viscoelastic Sandwich Plate with Linearized Complex Modes



Case 2: Viscoelastic Sandwich Plate with Linearized Complex Modes



Case 2: Viscoelastic Sandwich Plate with Linearized Complex Modes

Transient Solutions of Sandwich Plate			
Model	**Eigensolution**	**Solution Time**	**Total Cost**
Full FEA model	n/a	~10 days* (dt = 1e-7)	~864,000 seconds (~14,400 min)
1501 mode ROM with real modes	229 seconds	2571 seconds (dt = 1e-6)	2800 seconds (~47 min)
32 mode ROM with linearized complex modes	4923 seconds	4.2 seconds (dt = 5e-6)	4927 seconds (~82 min)

*Estimate based on integration in Matlab on single processor

Conclusions

- Developed **two approaches to reduce the size** of a finite element model with linear viscoelastic material behavior
  - Real eigenmodes
  - Linearized complex eigenmodes

- Real eigenmodes perform well when the **three conditions are met**; linearized complex eigenmodes are efficient for the **general case**.

- Obtained reduced order models with **non-viscous damping** from Prony series representation of the viscoelastic material

Any Questions?

- This research was supported by the Laboratory Directed Research and Development program at Sandia National Laboratories, a multi-program laboratory managed and operated by Sandia Corporation, a wholly owned subsidiary of Lockheed Martin Corporation, for the U.S. Department of Energy's National Nuclear Security Administration under contract DE-AC04-94AL85000.

- Contact Information
  - Rob Kuether: rjkueth@sandia.gov
  - Kevin Troyer: kltroye@sandia.gov

Extra Slides

Sandia National Laboratories

Pro's/Con's

	Real Modes	Complex Modes
Pro	• Upfront cost of computing modes is significantly less	• Accounts for viscoelastic forces • Produces accurate model with efficient solve times
Con	• Inaccurate for models when special conditions not met • Requires many more modes to achieve acceptable accuracy	• Expensive to calculate each mode • Potential convergence issues solving quadratic eigenvalue problem