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An optimization-based approach to mesh tying and transmission problems

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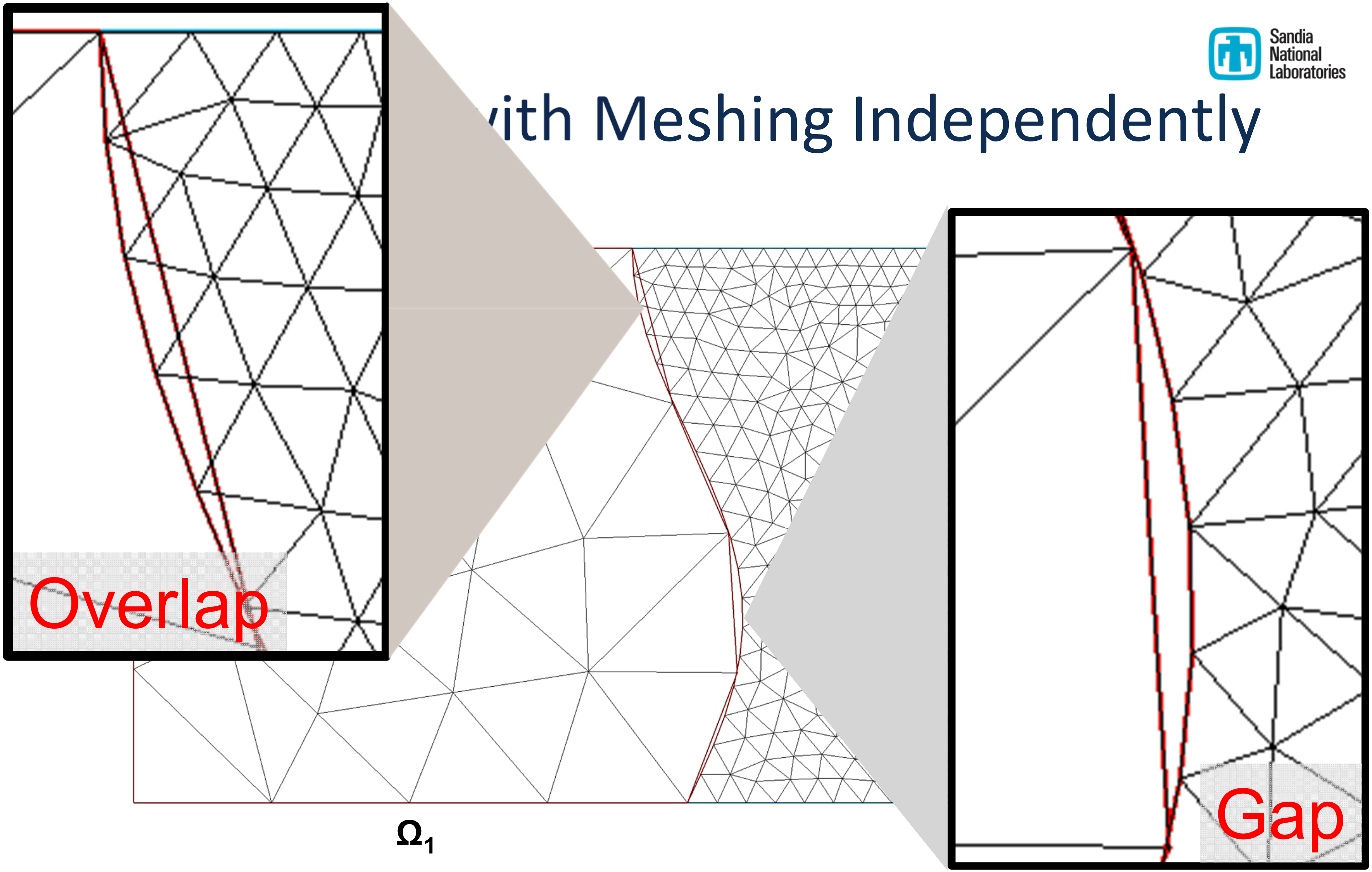
Outline

- Motivation
- Problem formulation
- Reformulation as an optimization problem
- Explanation of extension operators
- Numerical results

...with Meshing Independently



When meshing two domains sharing a smooth interface, it is difficult to avoid introducing overlaps and gaps.

Complications

With the introduction of gaps and overlaps, for any coupling technique it becomes difficult to:

- Compare solutions variables between the two domains
- Determine boundary conditions on the interface
- Preserve physical properties such as linear momentum, energy conservation, etc…

Common approaches are based on Lagrange multipliers

- FETI [Farhat & Roux]
- domain decomposition [Toselli & Widlund]
- solve the saddle point problem [Bochev, Parks & Romero]
- mortar methods [Anagnostou, Mavriplis & Patera; Bernardi, Maday & Patera; Flemisch, Melenk & Wohlmuth]

Traditional Formulation for Interfaces with Matching Nodes

Original weak form of an elliptic problem with matching conditions

$$\begin{aligned} \kappa(\nabla \mathbf{u}_i, \nabla \boldsymbol{v}_i)_{\Omega_i} &= (\mathbf{f_i}, \boldsymbol{v}_i)_{\Omega_i} + \langle \partial_{\mathbf{n}} \mathbf{u}_i, \boldsymbol{v}_i \rangle_\sigma & \forall \boldsymbol{v}_i \in \mathbf{H}_{\Gamma_i}(\Omega_i),\ i = 1,2 \\ \partial_{\mathbf{n}} \mathbf{u}_1 &= -\partial_{\mathbf{n}} \mathbf{u}_2 & \text{on } \sigma \\ \boxed{\mathbf{u}_1 = \mathbf{u}_2} & & \text{on } \sigma. \end{aligned}$$

Traditional Lagrange multiplier approach:

$$\begin{aligned} \kappa(\nabla \mathbf{u}_i, \nabla \boldsymbol{v}_i)_{\Omega_i} &= (\mathbf{f_i}, \boldsymbol{v}_i)_{\Omega_i} + (-1)^{i+1} \langle \boldsymbol{\lambda}, \boldsymbol{v}_i \rangle_\sigma & \forall \boldsymbol{v}_i \in \mathbf{H}_{\Gamma_i}(\Omega_i),\ i = 1,2 \\ \boxed{\langle \mathbf{u}_1 - \mathbf{u}_2, \boldsymbol{\mu} \rangle_\sigma} &= 0 & \forall \boldsymbol{\mu} \in \mathbf{H}^{-\mathbf{1/2}}(\sigma). \end{aligned}$$

Alternate Formulation as a Minimization Problem

Reformulation as an optimization problem [Gunzburger & Lee]

$$
\begin{aligned}
\kappa(\nabla \mathbf{u}_i, \nabla \boldsymbol{v}_i)_{\Omega_i} &= (\mathbf{f_i}, \boldsymbol{v}_i)_{\Omega_i} + \langle \partial_{\mathbf{n}} \mathbf{u}_i, \boldsymbol{v}_i \rangle_\sigma && \forall \boldsymbol{v}_i \in \mathbf{H}_{\Gamma_i}(\Omega_i),\ i = 1, 2 \\
\partial_{\mathbf{n}} \mathbf{u}_1 &= -\partial_{\mathbf{n}} \mathbf{u}_2 && \text{on } \sigma \\
\mathbf{u}_1 &= \mathbf{u}_2 && \text{on } \sigma.
\end{aligned}
$$

becomes

$$
\min_{\mathbf{g}} \quad \frac{1}{2} \|\mathbf{u}_1 - \mathbf{u}_2\|_\sigma^2 + \frac{\delta}{2} \|\mathbf{g}\|_\sigma^2
$$

$s.t.$

$$
\kappa(\nabla \mathbf{u}_i, \nabla \boldsymbol{v}_i)_{\Omega_i} = (\mathbf{f_i}, \boldsymbol{v}_i)_{\Omega_i} + (-1)^{i+1} \langle \mathbf{g}, \boldsymbol{v}_i \rangle_\sigma \quad \forall \boldsymbol{v}_i \in \mathbf{H}_{\Gamma_i}(\Omega_i),\ i = 1, 2
$$

However, this was designed with the matched node case in mind. We wish to extend this approach to noncoincident interfaces.

Extension to Noncoincident Meshes

$$\min_{\mathbf{g}} \quad \frac{1}{2}\left\|\mathbf{u}_1 - \mathbf{u}_2\right\|_\sigma^2 + \frac{\delta}{2}\left\|\mathbf{g}\right\|_\sigma^2$$

$$s.t.$$

$$\kappa(\nabla \mathbf{u}_i, \nabla \boldsymbol{v}_i)_{\Omega_i} = (\mathbf{f_i}, \boldsymbol{v}_i)_{\Omega_i} + (-1)^{i+1}\langle \mathbf{g}, \boldsymbol{v}_i\rangle_\sigma \quad \forall \boldsymbol{v}_i \in \mathbf{H}_{\Gamma_i}(\Omega_i),\ i = 1, 2$$

We introduce an extension operator, $E_{2\to 1}(\mathbf{u}_2)$, which in some way takes $\mathbf{u}_2$ from domain 2, and maps its values onto domain 1.

This can be done similarly for $\mathbf{u}_1$.

We now replace the single term in the objective with two terms,

$$\min_{\mathbf{g}} \quad \frac{\beta_C}{2}\left\|E_{2\to C}\mathbf{u}_1 - E_{2\to C}\mathbf{u}_2\right\|_{\sigma_C}^2 + \frac{\beta_1}{2}\left\|\mathbf{u}_1 - E_{2\to 1}\mathbf{u}_2\right\|_{\sigma_1}^2 + \frac{\beta_2}{2}\left\|\mathbf{u}_2 - E_{1\to 2}\mathbf{u}_1\right\|_{\sigma_2}^2 + \frac{\delta}{2}\left\|\mathbf{g}\right\|_\sigma$$

$$s.t.$$

$$\kappa(\nabla \mathbf{u}_i, \nabla \boldsymbol{v}_i)_{\Omega_i} = (\mathbf{f_i}, \boldsymbol{v}_i)_{\Omega_i} + (-1)^{i+1}\langle \mathbf{g}, \boldsymbol{v}_i\rangle_\sigma \qquad \forall \boldsymbol{v}_i \in \mathbf{H}_{\Gamma_i}(\Omega_i),\ i = 1, 2$$

Extension to Noncoincident Meshes

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Difficulty

$$\min_{\mathbf{g}} \quad \frac{\beta_C}{2}\|E_{2\to C}\mathbf{u}_1 - E_{2\to C}\mathbf{u}_2\|^2_{\sigma_C} + \frac{\beta_1}{2}\|\mathbf{u}_1 - E_{2\to 1}\mathbf{u}_2\|^2_{\sigma_1} + \frac{\beta_2}{2}\|\mathbf{u}_2 - E_{1\to 2}\mathbf{u}_1\|^2_{\sigma_2} + \frac{\delta}{2}\|\mathbf{g}\|_{\sigma}$$

$$s.t. \quad \kappa(\nabla\mathbf{u}_i, \nabla\boldsymbol{v}_i)_{\Omega_i} = (\mathbf{f_i}, \boldsymbol{v}_i)_{\Omega_i} + (-1)^{i+1}\langle\mathbf{g}, \boldsymbol{v}_i\rangle_{\sigma} \qquad \forall\boldsymbol{v}_i \in \mathbf{H}$$

We can no longer rely on $\partial_{\mathbf{n}}\mathbf{u}_1 = \mathbf{g} = -(-\mathbf{g}) = -\partial_{\mathbf{n}}\mathbf{u}_2$ to enforce the Neumann matching condition is satisfied.

Solution

Instead, we introduce the operators B_1 and B_2 , described later, which exactly satisfy the condition $\int_{\sigma_1} B_1\mathbf{g}\, d\sigma_2 = -\int_{\sigma_2} B_2\mathbf{g}\, d\sigma_2$, and approximately satisfy the condition $B_1\mathbf{g} \approx -B_2\mathbf{g}$.

$$\min_{\mathbf{g}} \quad \frac{\beta_C}{2}\|E_{2\to C}\mathbf{u}_1 - E_{2\to C}\mathbf{u}_2\|^2_{\sigma_C} + \frac{\beta_1}{2}\|\mathbf{u}_1 - E_{2\to 1}\mathbf{u}_2\|^2_{\sigma_1} + \frac{\beta_2}{2}\|\mathbf{u}_2 - E_{1\to 2}\mathbf{u}_1\|^2_{\sigma_2} + \frac{\delta}{2}\|\mathbf{g}\|_{\sigma_C}$$

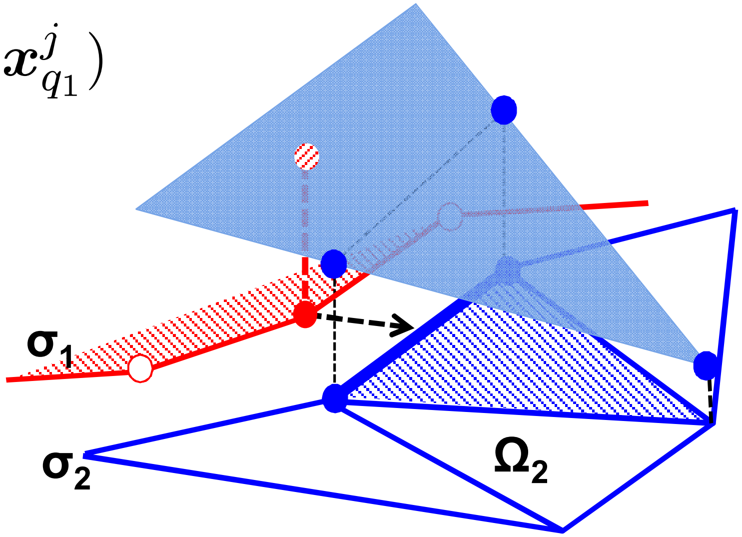
$$s.t. \quad \kappa(\nabla\mathbf{u}_i, \nabla\boldsymbol{v}_i)_{\Omega_i} = (\mathbf{f_i}, \boldsymbol{v}_i)_{\Omega_i} + \langle B_i\mathbf{g}, \boldsymbol{v}_i\rangle_{\sigma_i} \qquad \forall\boldsymbol{v}_i \in \mathbf{H}_{\Gamma_i}(\Omega_i),\ i = 1, 2$$

Extension Operator

$$\sum_{k=0}^{|N_1|}(E_{2\to 1}\vec{\mathbf{u}}_2^h)^k N_1(\boldsymbol{x}_{q_1}^j)$$

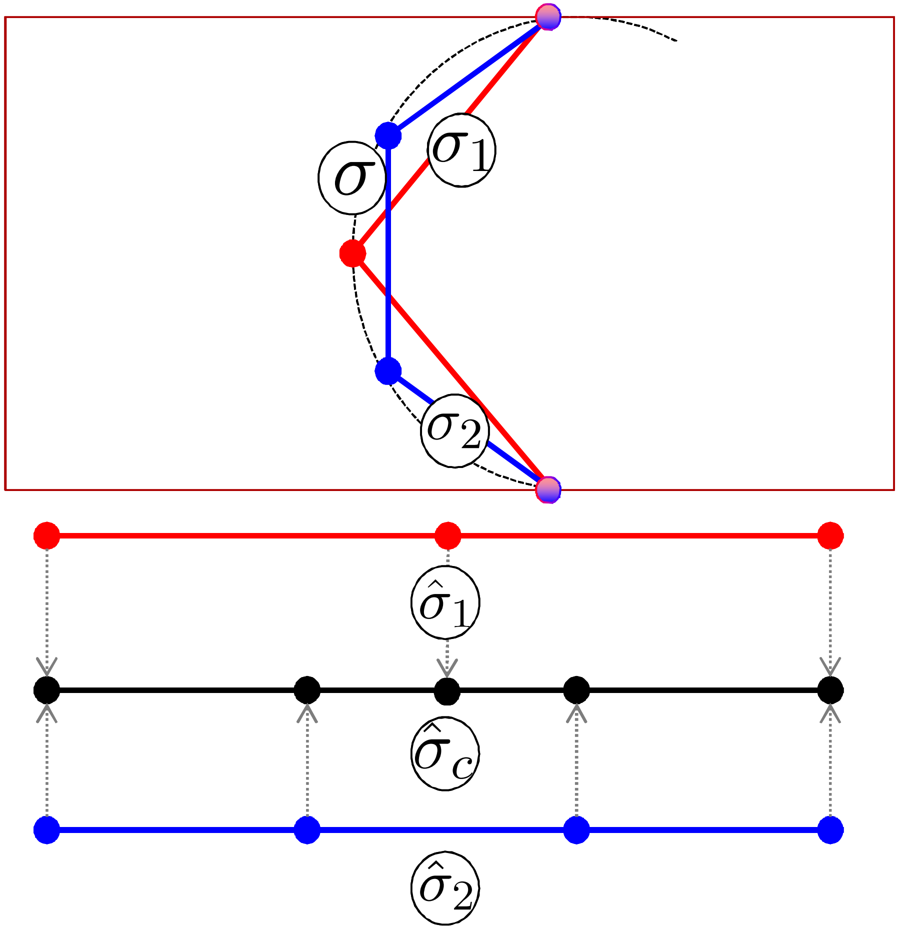
$$= \sum_{k=0}^{|N_1|}(\mathbf{u}_2^h + \nabla\mathbf{u}_2^h \Delta\boldsymbol{x})(\boldsymbol{x}_{q_1})N_1(\boldsymbol{x}_{q_1}^j)$$

- For each (quadrature) point
  - Find the triangle closest to the point of interest
  - Construct a gradient from the solution on the cell
  - Evaluate the plane at the point



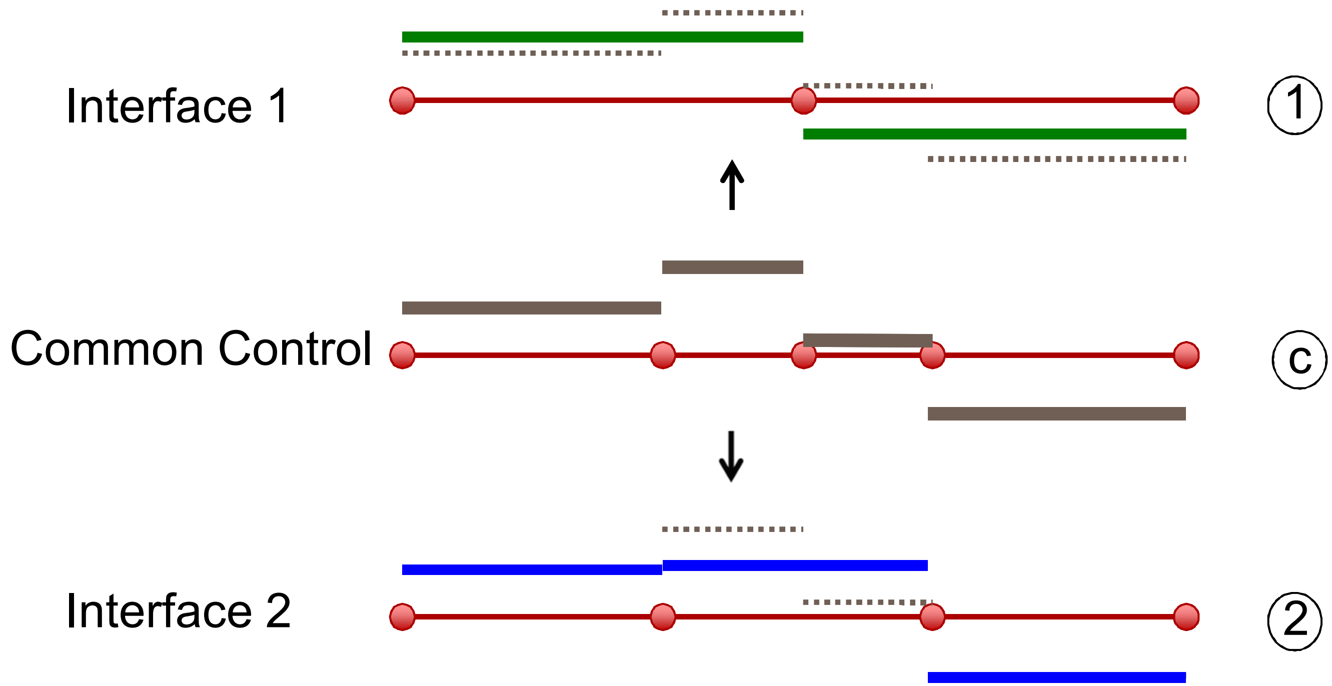
Neumann Operator

- We need to construct B_1 and B_2 that map a common control to both domains and:
  - Preserve global flux exactly
  - Approximately enforce that flux is equal but opposite pointwise



Parameterized interfaces $\hat{\sigma}_1$, $\hat{\sigma}_2$, and $\hat{\sigma}_c$.

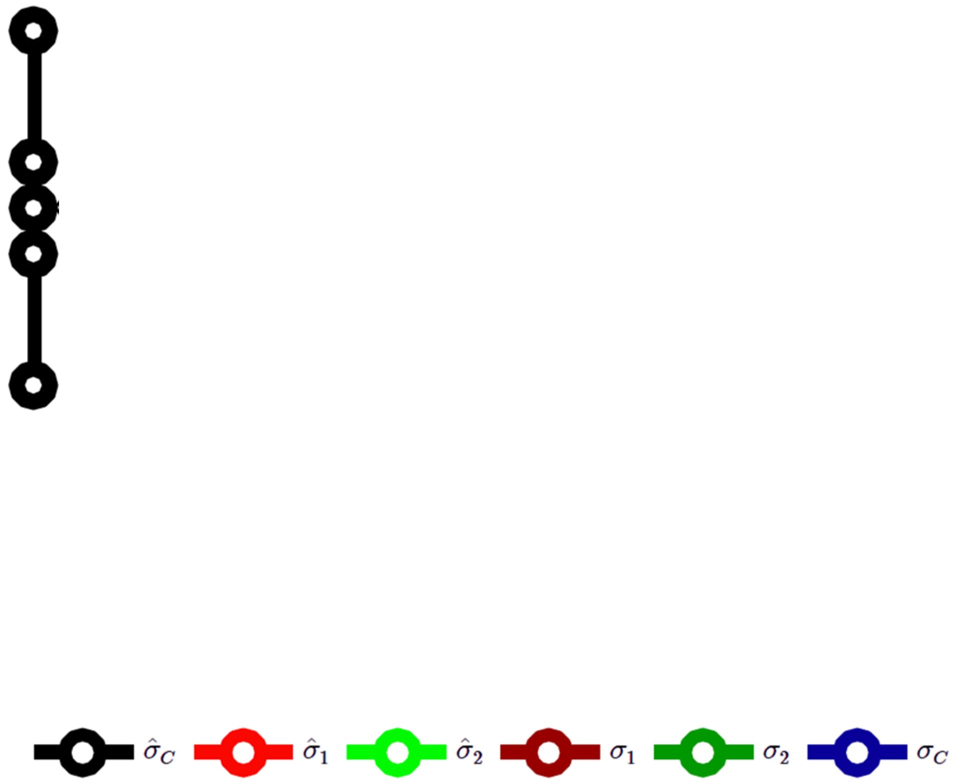
Neumann Operator



By parameterizing and then weighting values according to support of piecewise discontinuous basis functions (transformed onto the common interface), normal directions are ignored by assuming they are the same.

This will cause problems in passing a patch test, but can ensure global flux conservation.

Virtual Interface σ_c



Numerical Studies

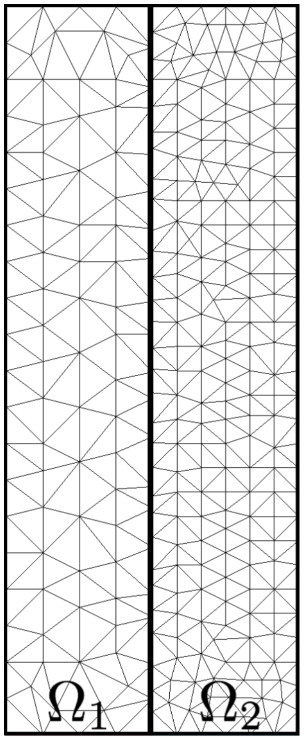
- Minimization problem

$$
\begin{aligned}
\min_{\mathbf{g}} \quad & \frac{\beta_C}{2}\left\|E_{2\to C}\mathbf{u}_1 - E_{2\to C}\mathbf{u}_2\right\|^2_{\sigma_C} + \frac{\beta_1}{2}\left\|\mathbf{u}_1 - E_{2\to 1}\mathbf{u}_2\right\|^2_{\sigma_1} + \frac{\beta_2}{2}\left\|\mathbf{u}_2 - E_{1\to 2}\mathbf{u}_1\right\|^2_{\sigma_2} + \frac{\delta}{2}\left\|\mathbf{g}\right\|_{\sigma_C} \\
s.t. \quad & \\
& \kappa(\nabla\mathbf{u}_i, \nabla\boldsymbol{v}_i)_{\Omega_i} = (\mathbf{f_i}, \boldsymbol{v}_i)_{\Omega_i} + \langle B_i\mathbf{g}, \boldsymbol{v}_i\rangle_{\sigma_i} \qquad \forall \boldsymbol{v}_i \in \mathbf{H}_{\Gamma_i}(\Omega_i),\ i = 1,2
\end{aligned}
$$

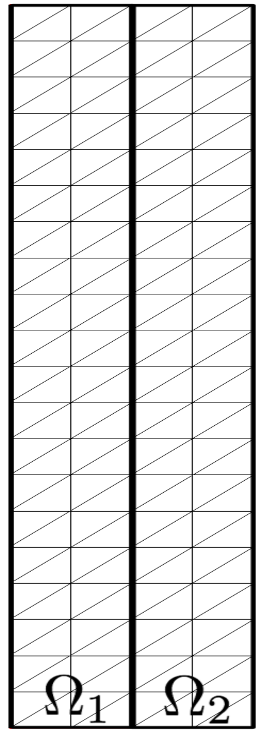
is solved by eliminating the states and solving the Hessian against the Jacobian of the objective

- Only one Hessian solve needed
  (linear constraints with quadratic objective results in a quadratic objective after reducing the states)
- Many other choices for optimization algorithm

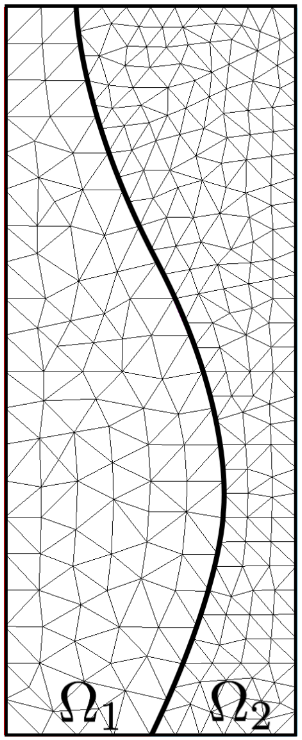
Domains



Flat-Perturbed

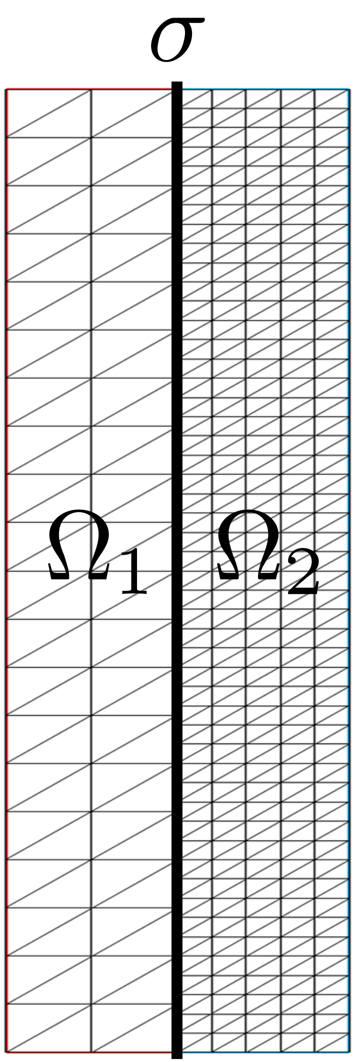


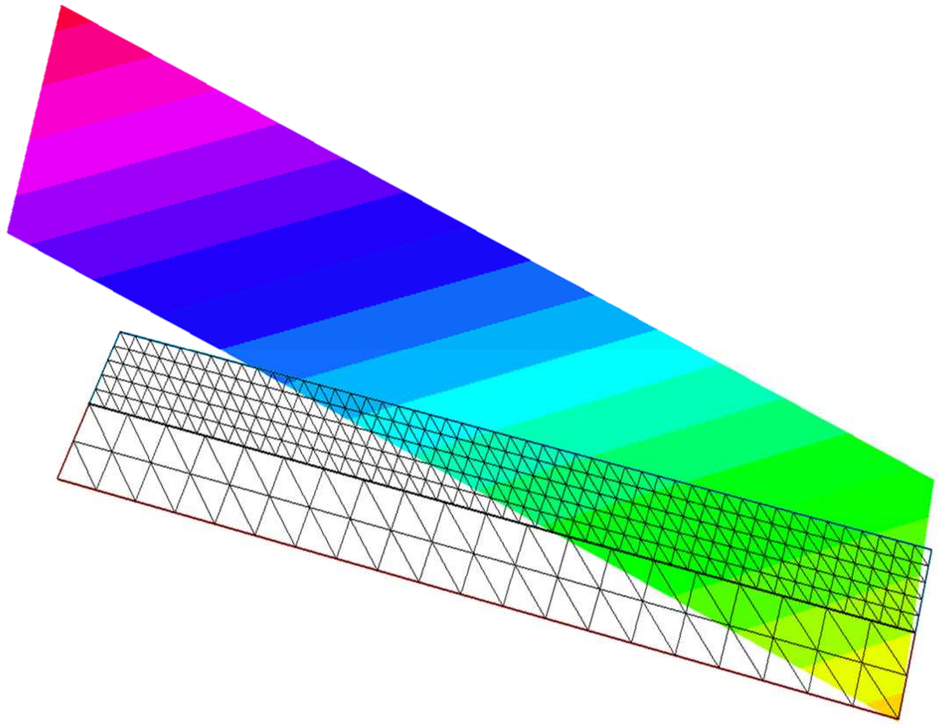
Flat-Uniform



S-Curve

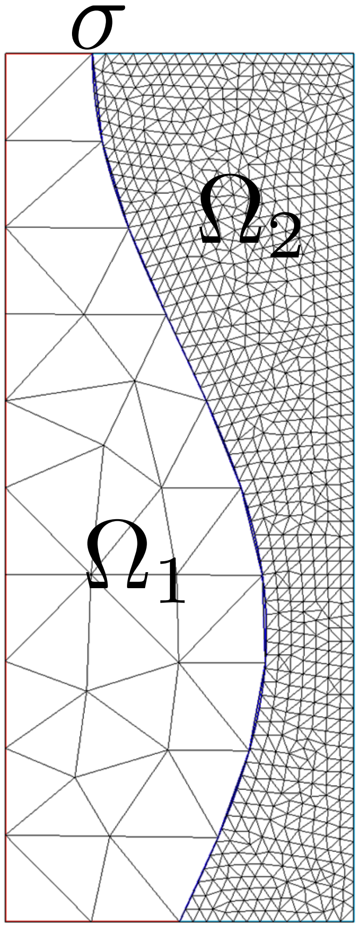
Numerical Results
[Flat Interface]

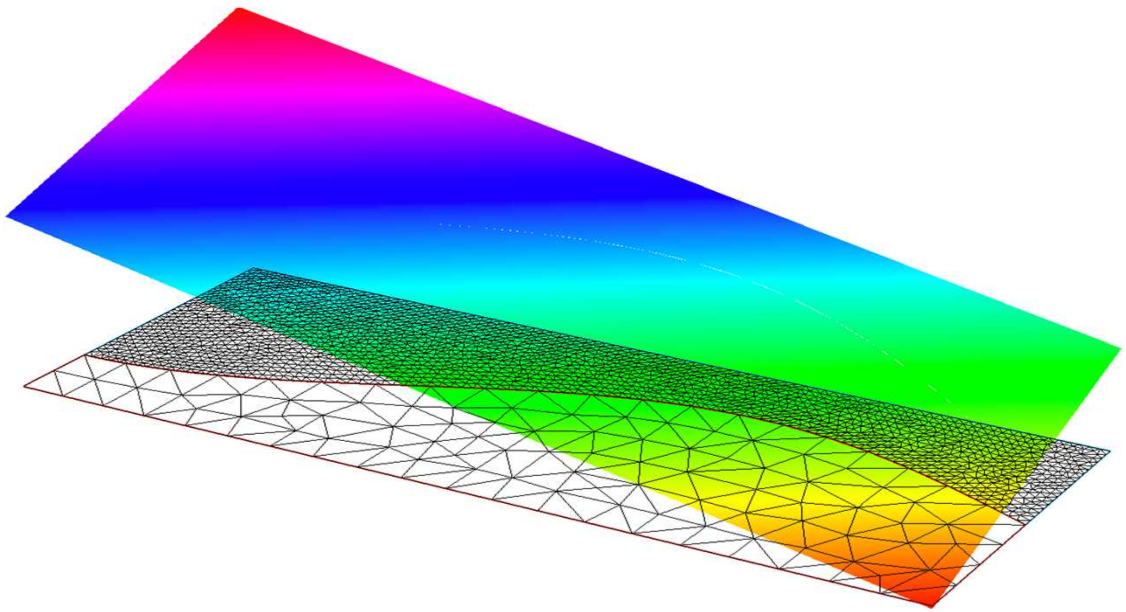




✔ **DOES** pass patch test
For **any** left to right ratio

Numerical Results
[S-Curve Interface]



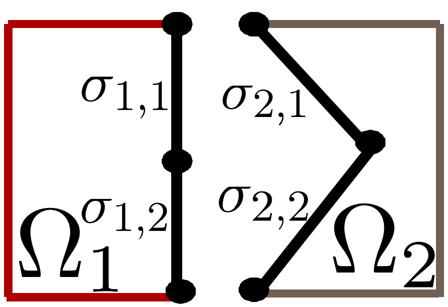


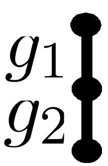
✔ Passes patch test for left = right ratio

✖ Does **NOT** pass patch test for left ≠ right ratio

Linear Patch Test

- Globally linear solution, $u = 3x + 2y$
- Expectation:
  - Will pass for coincident interfaces
    - Constant flux corresponding to manufactured solution lies in the solution space being optimized over
  - Will generally fail for noncoincident interfaces
    - Counter example:
      - No common control can be defined for the following case



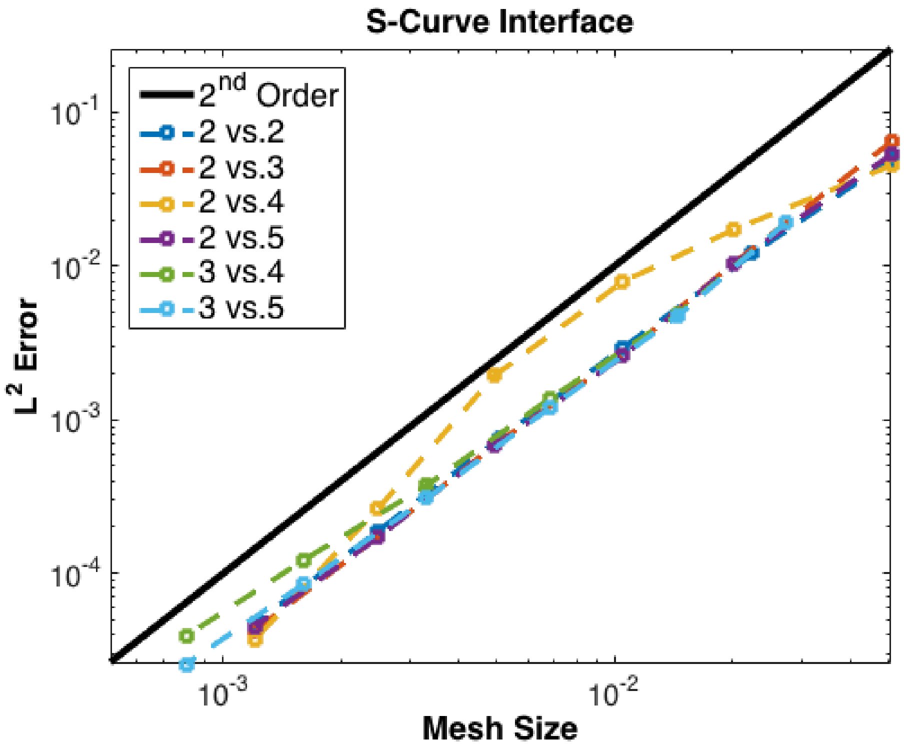


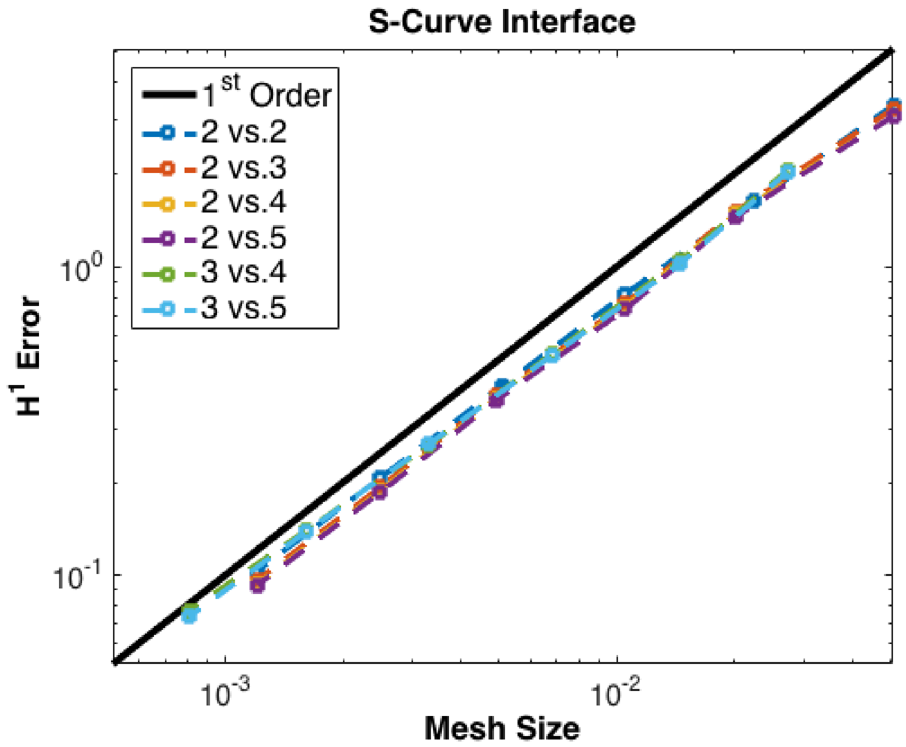
$$\vec{g} = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$$

$$\sigma_1 \cdot \mathbf{n}_1 = u_{1,x} = 3 \text{ on } \sigma_{1,1}$$
$$\sigma_1 \cdot \mathbf{n}_1 = u_{1,x} = 3 \text{ on } \sigma_{1,2}$$
$$\sigma_2 \cdot \mathbf{n}_2 = \frac{1}{\sqrt{2}}(-u_{2,x} - u_{2,y}) = \frac{-5}{\sqrt{2}} \text{ on } \sigma_{2,1}$$
$$\sigma_2 \cdot \mathbf{n}_2 = \frac{1}{\sqrt{2}}(-u_{2,x} + u_{2,y}) = \frac{-1}{\sqrt{2}} \text{ on } \sigma_{2,2}$$
$$\therefore \quad g_1 = 3 \text{ and } g_2 = 3 \text{ for } \sigma_1$$
$$\text{and } g_1 = \tfrac{5}{2} \text{ and } g_2 = \tfrac{1}{2} \text{ for } \sigma_2. \quad ⨳$$

Manufactured Solution [S-Curve Interface]

- Solution: $x^2(y-2)^3\sin(2\pi x) - (x-3)^3\cos(2\pi x - y)$
- $\delta = 1\text{e-}11,\ \kappa = 1,\ \beta_1 = 0, \beta_2 = 0, \beta_C = 1$





Manufactured Solution [Flat-Perturbed]

Parameter			δ=0		δ=1e-14		δ=1e-11	
β_1	β_2	β_C	L^2	H^1	L^2	H^1	L^2	H^1
1	h	0	1.19	0.81	1.76	0.93	1.72	0.92
1	h^2	0	1.71	0.93	1.73	0.92	1.72	0.90
1	h^3	0	1.11	0.81	1.74	0.91	1.71	0.90
h	1	0	1.36	0.89	1.75	0.93	1.71	0.92
h^2	1	0	1.58	0.93	1.72	0.92	1.70	0.91
h^3	1	0	0.87	0.67	1.72	0.91	1.69	0.91
1	0	0	1.41	0.76	1.71	0.91	1.71	0.91
0	1	0	1.47	0.88	1.70	0.91	1.70	0.91
1	1	0	1.13	0.87	1.77	0.93	1.71	0.93
0	0	1	1.30	0.89	1.78	0.93	1.70	0.93

Flat perturbed interface with 2:3 ratio between σ_1 and σ_2

Manufactured Solution [Flat-Uniform]

Parameter			δ=0		δ=1e-14		δ=1e-11	
β_1	β_2	β_C	L^2	H^1	L^2	H^1	L^2	H^1
1	h	0	2.00	1.00	2.00	1.00	2.00	1.00
1	h^2	0	2.00	1.00	2.00	1.00	2.00	1.00
1	h^3	0	2.00	1.00	2.00	1.00	2.00	1.00
h	1	0	2.00	1.00	2.00	1.00	2.00	1.00
h^2	1	0	2.00	1.00	2.00	1.00	2.00	1.00
h^3	1	0	2.00	1.00	2.00	1.00	2.00	1.00
1	0	0	2.00	1.00	2.00	1.00	2.00	1.00
0	1	0	2.00	1.00	2.00	1.00	2.00	1.00
1	1	0	2.00	1.00	2.00	1.00	2.00	1.00
0	0	1	2.00	1.00	2.00	1.00	2.00	1.00

Flat uniform interface with 1:1 ratio between σ_1 *and* σ_2

Manufactured Solution [S-Curve]

Parameter			δ=0		δ=1e-14		δ=1e-11	
β_1	β_2	β_C	L^2	H^1	L^2	H^1	L^2	H^1
1	h	0	2.30	1.21	1.86	0.94	2.09	0.95
1	h^2	0	2.30	1.13	2.09	0.95	1.84	0.93
1	h^3	0	2.35	1.07	1.97	0.94	1.73	0.92
h	1	0	2.39	1.21	1.87	0.94	2.08	0.95
h^2	1	0	2.34	1.14	2.10	0.95	1.96	0.94
h^3	1	0	2.54	1.15	2.06	0.94	1.86	0.93
1	0	0	1.19	0.56	1.75	0.92	1.75	0.92
0	1	0	1.02	0.33	1.86	0.93	1.86	0.93
1	1	0	1.80	1.08	1.60	0.95	1.97	0.94
0	0	1	2.21	1.20	1.78	0.94	2.01	0.94

S-Curve interface with 2:3 ratio between σ_1 and σ_2

Conclusions

- Observed optimal H^1 convergence rates (1.0) with various weightings of terms in the objective, and nearly optimal L^2 convergence rates (1.7-2.0)
- Developed an extension technique for evaluating jumps in solution variables from opposite subdomains using the gradient
- Used parameterizations of the 1d interface to generate a common refinement and then used this common refinement as a mesh on which to define a control space
- Reformulated the coupled problem posed over mismatched and noncoincident subdomains as a minimization problem by using the common interface and extension operators
- Passed a linear patch test in the case of coincident interfaces

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