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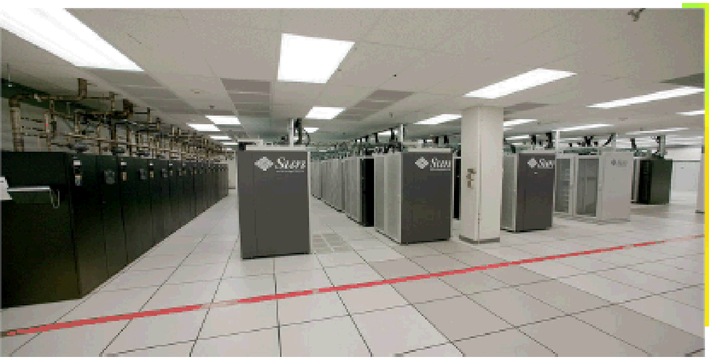
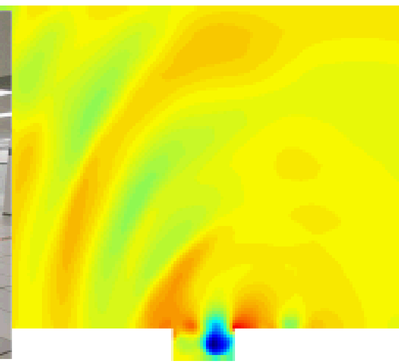
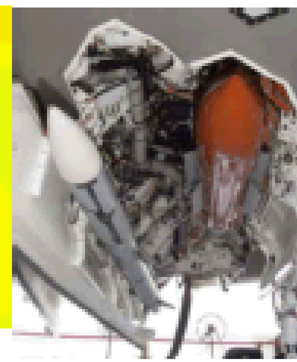
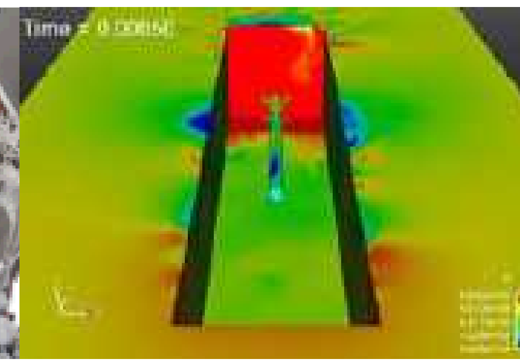

A minimal subspace rotation approach for obtaining stable & accurate low-order projection-based reduced order models for nonlinear compressible flow

I. Tezaur[1], M. Balajewicz[2]

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U.S. DEPARTMENT OF ENERGY

NNSA National Nuclear Security Administration

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Outline

1. Motivation
2. Projection-based model order reduction
3. Accounting for modal truncation
   - Traditional linear eddy-viscosity approach
   - New proposed approach via subspace rotation
4. Applications
   - High angle of attack laminar airfoil
   - Low Reynolds number channel driven cavity
   - Moderate Reynolds number channel driven cavity
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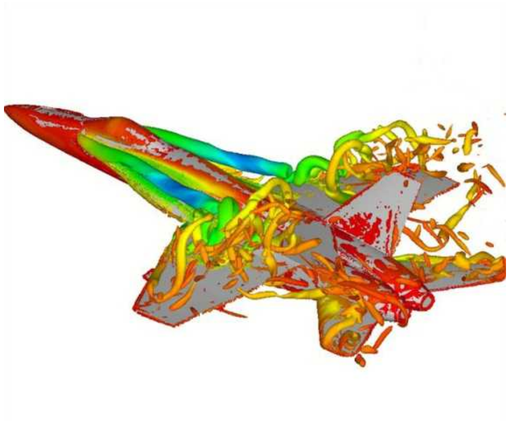
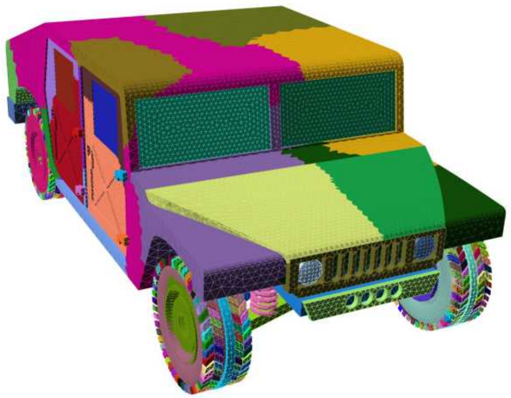
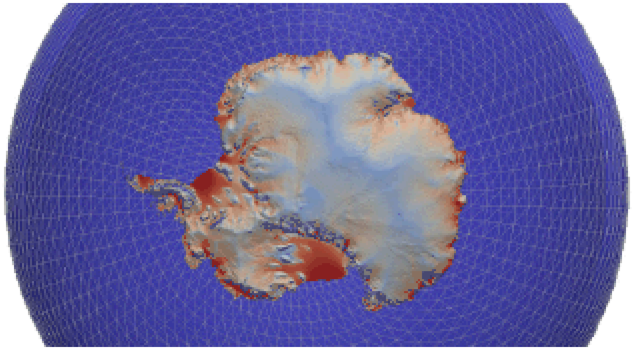
Motivation

Computational models of high-dimensional systems arise in a rich variety of engineering and scientific applications



Computational Fluid Dynamics (left)
Finite Element Analysis (right)
Climate Modeling (bottom)





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There are significant <u>scientific</u> and <u>engineering</u> benefits in developing and studying <u>low-dimensional</u> representations of high-dimensional systems and retain <u>physical fidelity</u> while substantially reducing the <u>size</u> and <u>cost</u> of the computational model.

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"Purpose of computing is insight, not numbers."

\- Richard Hamming, 1962

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Projection-based model order reduction

Data-driven model order reduction (MOR)[1]

[1] *Not to be confused with resolution decrease, model simplification or physics simplification.*

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1. **Compression:** solution of governing parametric partial differential equations (PDEs) lies in a subspace of significantly lower dimension.

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1. **Compression:** solution of governing parametric partial differential equations (PDEs) lies in a subspace of significantly lower dimension.

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2. **Offline training:** the low-dimensional subspace can be identified/learned offline via <u>training simulations</u> and high-fidelity model can be <u>reformulated</u> with respect to this subspace.

3. **Online prediction:** identified parametric reduced order models (ROMs) capable of providing new solutions at a <u>fraction</u> of the computational cost.

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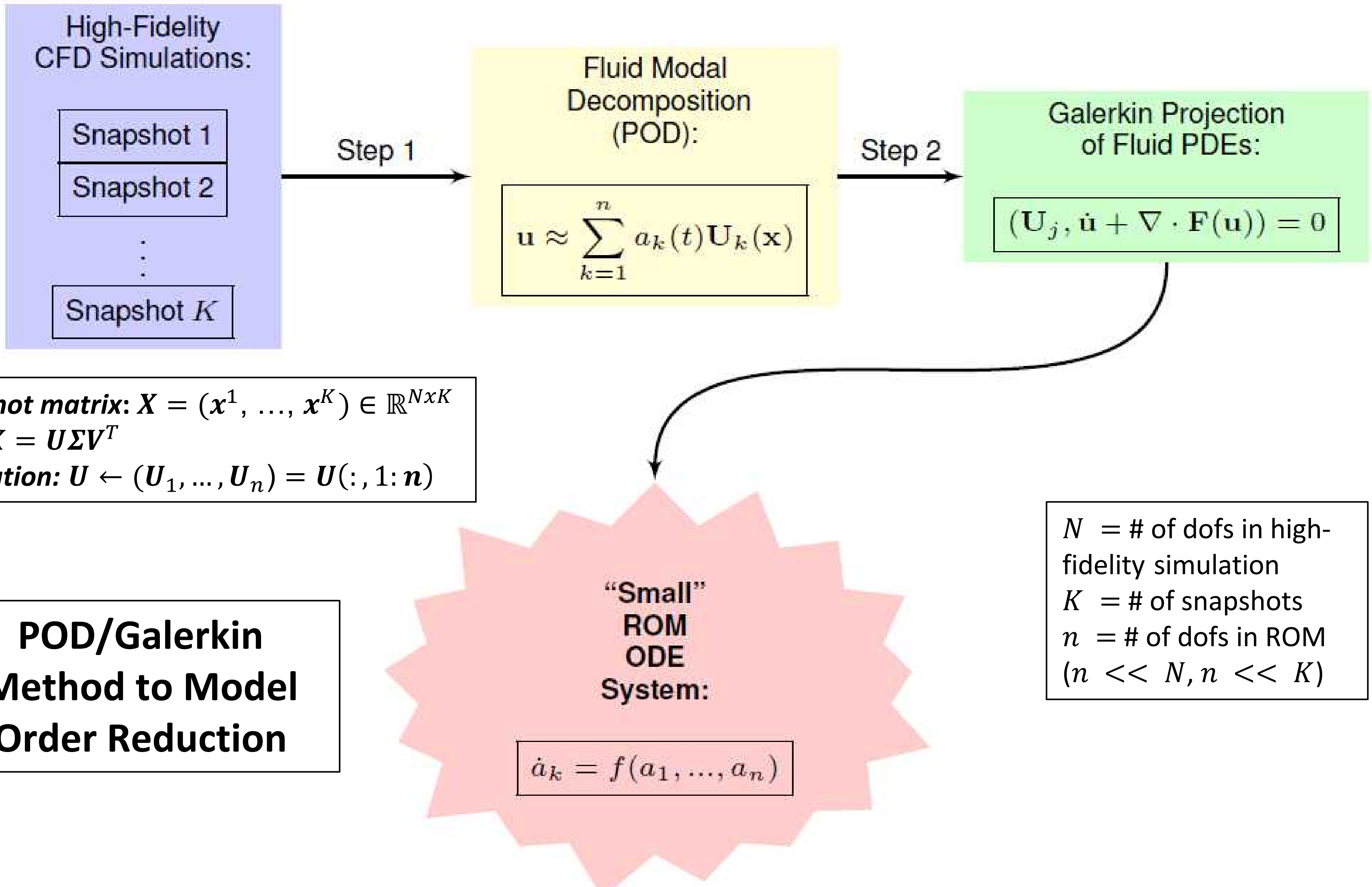
Projection-based model order reduction



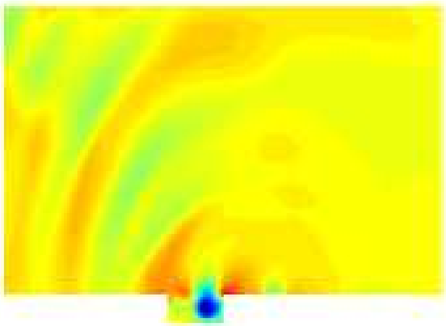
Snapshot matrix: $\boldsymbol{X} = (\boldsymbol{x}^1, \ldots, \boldsymbol{x}^K) \in \mathbb{R}^{NxK}$
SVD: $\boldsymbol{X} = \boldsymbol{U\Sigma V}^T$
Truncation: $\boldsymbol{U} \leftarrow (\boldsymbol{U}_1, \ldots, \boldsymbol{U}_n) = \boldsymbol{U}(:, 1{:}\,\boldsymbol{n})$

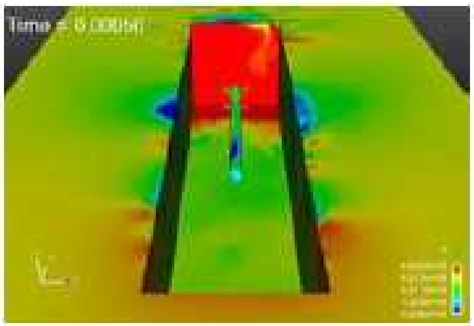
N = # of dofs in high-fidelity simulation
K = # of snapshots
n = # of dofs in ROM
$(n << N, n << K)$

POD/Galerkin Method to Model Order Reduction

Projection-based model order reduction

Targeted application: compressible fluid flow (e.g., captive-carry)





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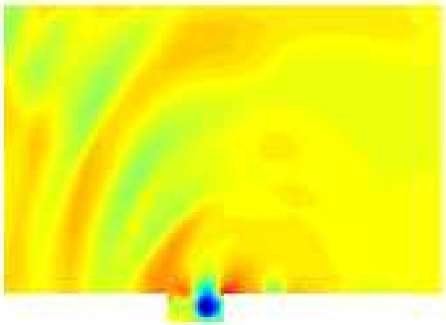

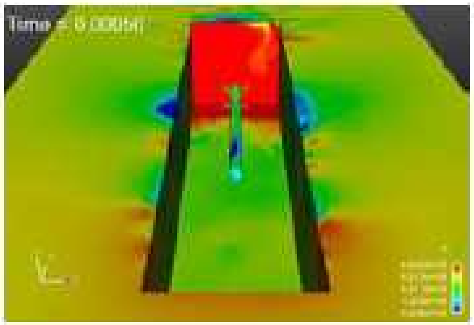

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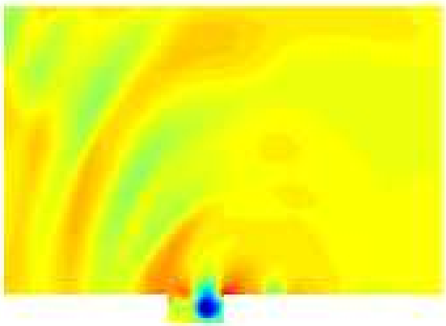

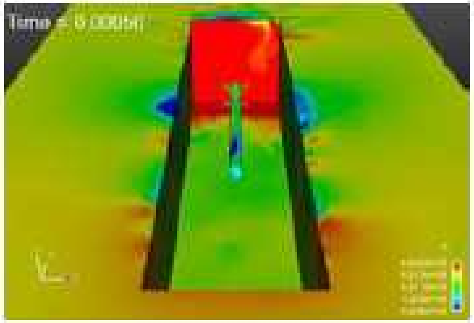

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- There has been some on MOR for compressible flows.

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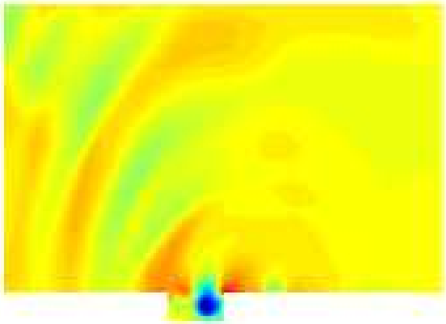

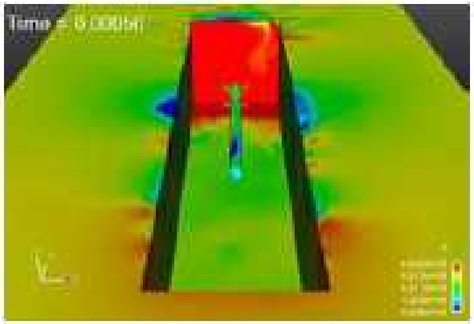

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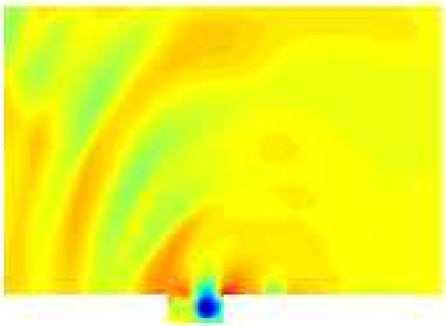
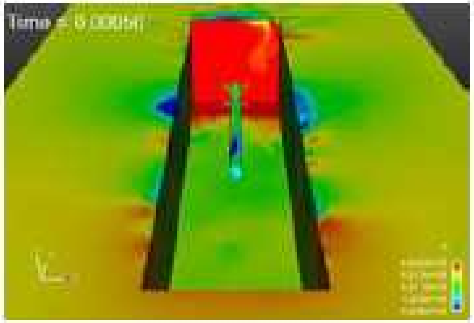


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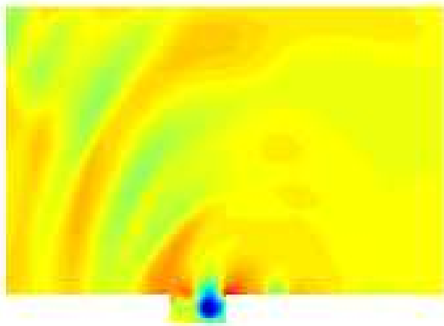

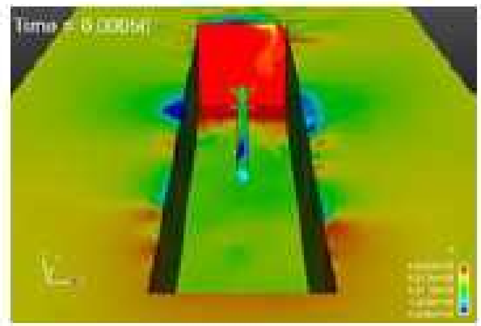

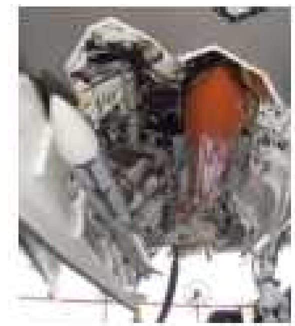

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MOR for <u>nonlinear</u>, <u>compressible</u> fluid flows is still in its infancy!

Projection-based model order reduction

Governing equations

- 3D compressible Navier-Stokes equations in ***primitive specific volume form:***

[PDEs]

$$\zeta_{,t} + \zeta_{,j} u_j - \zeta u_{j,j} = 0$$
$$u_{i,t} + u_{i,j} u_j + \zeta p_{,i} - \frac{1}{Re} \zeta \tau_{ij,j} = 0$$
$$p_{,t} + u_j p_{,j} + \gamma u_{j,j} p - \left(\frac{\gamma}{PrRe}\right) \left(\kappa (p\zeta)_{,j}\right)_{,j} - \left(\frac{\gamma - 1}{Re}\right) u_{i,j} \tau_{ij} = 0 \tag{1}$$

Projection-based model order reduction

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- Spectral discretization ($\boldsymbol{q}(\boldsymbol{x},t) \approx \sum_{i=1}^{n} a_i(t) \boldsymbol{U}_i(\boldsymbol{x})$) + Galerkin projection applied to (1) yields a system of *n coupled quadratic ODEs*:

[ROM]

$$\frac{d\boldsymbol{a}}{dt} = \boldsymbol{C} + \boldsymbol{L}\boldsymbol{a} + \left[\boldsymbol{a}^T \boldsymbol{Q}^{(1)} \boldsymbol{a} + \boldsymbol{a}^T \boldsymbol{Q}^{(2)} \boldsymbol{a} + \cdots + \boldsymbol{a}^T \boldsymbol{Q}^{(n)} \boldsymbol{a}\right]^T \qquad (2)$$

where $\boldsymbol{C} \in \mathbb{R}^n, \boldsymbol{L} \in \mathbb{R}^{n \times n}$ and $\boldsymbol{Q}^{(i)} \in \mathbb{R}^{n \times n}$ for all $i = 1, \dots, n$.

Projection-based model order reduction

Summary of technical challenges

Projection-based MOR necessitates ***truncation***

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↙ ***Turbulence Modeling*** (traditional approach)

↘ ***Subspace Rotation*** (our approach)

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Traditional linear eddy-viscosity approach

- Dissipative dynamics of truncated higher-order modes are modeled using an additional linear term:

$$\frac{d\boldsymbol{a}}{dt} = \boldsymbol{C} + \boldsymbol{L}\boldsymbol{a} + \left[\boldsymbol{a}^T\boldsymbol{Q}^{(1)}\boldsymbol{a} + \boldsymbol{a}^T\boldsymbol{Q}^{(2)}\boldsymbol{a} + \cdots + \boldsymbol{a}^T\boldsymbol{Q}^{(n)}\boldsymbol{a}\right]^T$$

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  1. Additional term destroys <u>*consistency*</u> between ROM and Navier-Stokes equations.
  2. Calibration is necessary to derive optimal $\boldsymbol{L}_\nu$ and optimal value is <u>*flow dependent.*</u>
  3. Inherently a <u>*linear model*</u> $\rightarrow$ cannot be expected to perform well for all classes of problems (e.g., nonlinear).

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Illustrative example

- *Standard approach:* retain only the most energetic POD modes, i.e., $\boldsymbol{U}_1, \boldsymbol{U}_2$, $\boldsymbol{U}_3$, $\boldsymbol{U}_4$, …
- *Proposed approach:* choose some higher order basis modes to increase dissipation, i.e., $\boldsymbol{U}_1, \boldsymbol{U}_2$, $\boldsymbol{U}_6$, $\boldsymbol{U}_8$, …

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- *More generally:* approximate the solution using a linear superposition of $n+p$ (with $p > 0$) most energetic modes:

$$\widetilde{\boldsymbol{U}}_i = \sum_{j=1}^{n+p} X_{ij}\, \boldsymbol{U}_j, \quad i = 1, \ldots, n, \tag{3}$$

where $\boldsymbol{X} \in \mathbb{R}^{(n+p)\times n}$ is an orthonormal ($\boldsymbol{X}^T\boldsymbol{X} = \boldsymbol{I}_{n\times n}$) "rotation" matrix.

Accounting for modal truncation

Goals of proposed new approach

Find $\boldsymbol{X}$ such that:

1. New modes $\widetilde{\boldsymbol{U}}$ remain *good approximations* of the flow.
2. New modes produce *stable* and *accurate* ROMs.

Accounting for modal truncation

Goals of proposed new approach

Find $\boldsymbol{X}$ such that:

1. New modes $\widetilde{\boldsymbol{U}}$ remain *good approximations* of the flow.
2. New modes produce *stable* and *accurate* ROMs.

- We formulate and solve a *constrained optimization problem* for $\boldsymbol{X}$:

$$\text{minimize}_{\boldsymbol{X}\in\mathcal{V}_{(n+p),n}} \; f(\boldsymbol{X})$$
$$\text{subject to} \quad g(\boldsymbol{X},\boldsymbol{L}) = 0$$

where $\mathcal{V}_{(n+p),n} \in \left\{\boldsymbol{X} \in \mathbb{R}^{(n+p)\times n} \colon \boldsymbol{X}^T\boldsymbol{X} = \boldsymbol{I}_n, p > 0\right\}$ is the *Stiefel manifold*.

Accounting for modal truncation

Sandia National Laboratories

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- Once $\boldsymbol{X}$ is found, the result is a system of the form (2) with:

$$Q^{(i)}{}_{jk} \leftarrow \sum_{s,q,r=1}^{n+p} X_{si} Q^{(s)}{}_{qr} X_{qr} X_{rk}, \quad \boldsymbol{L} \leftarrow \boldsymbol{X}^T\boldsymbol{L}\boldsymbol{X}, \quad \boldsymbol{C} \leftarrow \boldsymbol{X}^T\boldsymbol{C}^*$$

Accounting for modal truncation

Objective Function

$$\begin{aligned} &\text{minimize}_{\boldsymbol{X}\in\mathcal{V}_{(n+p),n}} \quad f(\boldsymbol{X}) \\ &\text{subject to} \quad g(\boldsymbol{X},\boldsymbol{L}) = 0 \end{aligned} \tag{5}$$

- We have considered two objectives $f(\boldsymbol{X})$ in (5):

Accounting for modal truncation

Objective Function

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- We have considered two objectives $f(\boldsymbol{X})$ in (5):
  - Minimize *subspace rotation*

$$f(\boldsymbol{X}) = \left\|\boldsymbol{X} - \boldsymbol{I}_{(n+p),n}\right\|_F = -\mathrm{tr}\left(\boldsymbol{X}^T\boldsymbol{I}_{(n+p)\times n}\right) \tag{6}$$

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  - Maximize resolved *turbulent kinetic energy (TKE)*

$$f(\boldsymbol{X}) = -||\boldsymbol{\Sigma} - \boldsymbol{X}\boldsymbol{X}^T\boldsymbol{\Sigma}||_F \tag{7}$$

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- TKE objective (7) comes from earlier work (Balajewicz *et al.,* 2013) involving stabilization of incompressible flow ROMs.

Accounting for modal truncation

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$$\text{minimize}_{\boldsymbol{X}\in\mathcal{V}_{(n+p),n}} \; f(\boldsymbol{X}) \quad \text{subject to} \quad g(\boldsymbol{X},\boldsymbol{L}) = 0 \qquad (5)$$

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$$f(\boldsymbol{X}) = -||\boldsymbol{\Sigma} - \boldsymbol{X}\boldsymbol{X}^T\boldsymbol{\Sigma}||_F \qquad (7)$$

- TKE objective (7) comes from earlier work (Balajewicz *et al.,* 2013) involving stabilization of incompressible flow ROMs.
- Numerical experiments reveal objective (6) produces better results than objective (7) for compressible flow.

Accounting for modal truncation

Constraint

$$\begin{aligned} &\text{minimize}_{\boldsymbol{X}\in\mathcal{V}_{(n+p),n}} \quad f(\boldsymbol{X}) \\ &\text{subject to} \quad g(\boldsymbol{X},\boldsymbol{L}) = 0 \end{aligned} \qquad (5)$$

- We use the traditional *linear eddy-viscosity closure model ansatz* for the constraint $g(\boldsymbol{X},\boldsymbol{L}) = 0$ in (5):

$$g(\boldsymbol{X},\boldsymbol{L}) = \mathrm{tr}(\boldsymbol{X}^T\boldsymbol{L}\boldsymbol{X}) - \eta \qquad (8)$$

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- Specifically, constraint (8) involves overall balance between *linear energy production* and *dissipation*.
    - η = proxy for the balance between linear energy production and energy dissipation.
- Constraint comes from property that *averaged total power* (= $\mathrm{tr}(\boldsymbol{X}^T\boldsymbol{L}\boldsymbol{X})$ + energy transfer) has to vanish.

Accounting for modal truncation

Minimal subspace rotation: trace minimization on Stiefel manifold

$$\begin{array}{cc} \text{minimize}_{\boldsymbol{X} \in \mathcal{V}_{(n+p),n}} & -\operatorname{tr}\left(\boldsymbol{X}^T \boldsymbol{I}_{(n+p)\times n}\right) \\ \text{subject to} & \operatorname{tr}(\boldsymbol{X}^T \boldsymbol{L} \boldsymbol{X}) = \eta \end{array} \quad (9)$$

- $\eta \in \mathbb{R}$: proxy for the balance between linear energy production and energy dissipation (calculated iteratively using modal energy).
- $\mathcal{V}_{(n+p),n} \in \left\{\boldsymbol{X} \in \mathbb{R}^{(n+p)\times n} : \boldsymbol{X}^T \boldsymbol{X} = \boldsymbol{I}_n, p > 0\right\}$ is the *Stiefel manifold*.
- Equation (9) is solved efficiently offline using the method of Lagrange multipliers (`Manopt MATLAB toolbox`).
- See (Balajewicz *et al.,* 2016) and Appendix slide for Algorithm.

Accounting for modal truncation

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Remarks

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- <u>Disadvantages of proposed approach:</u>
  1. Off-line calibration of free parameter η is required.
  2. Stability cannot be proven like for incompressible case.
  3. Uniqueness of solution to (5) is <u>*not*</u> guaranteed.

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1. Motivation
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   - New proposed approach via subspace rotation
4. **Applications**
   - High angle of attack laminar airfoil
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Applications

High angle of attack laminar airfoil

2D flow around an inclined NACA0012 airfoil at Mach 0.7,
Re = 500, Pr = 0.72, AOA = 20° $\Longrightarrow n = 4$ ROM (86% snapshot energy).

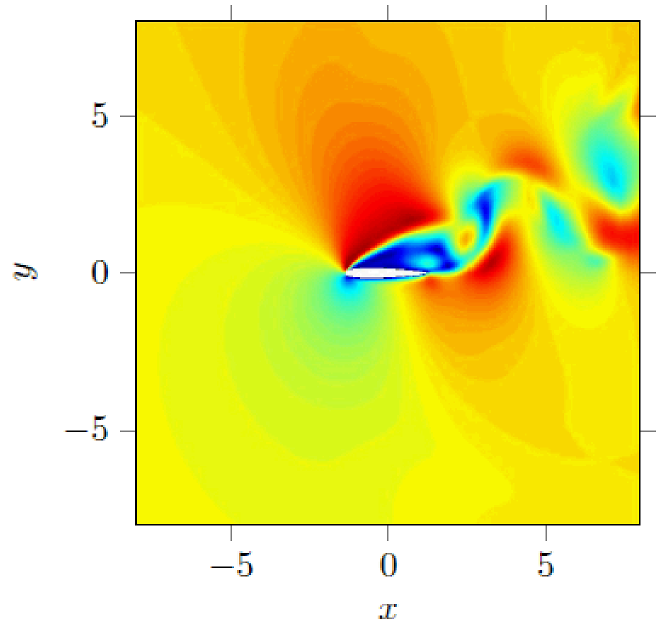


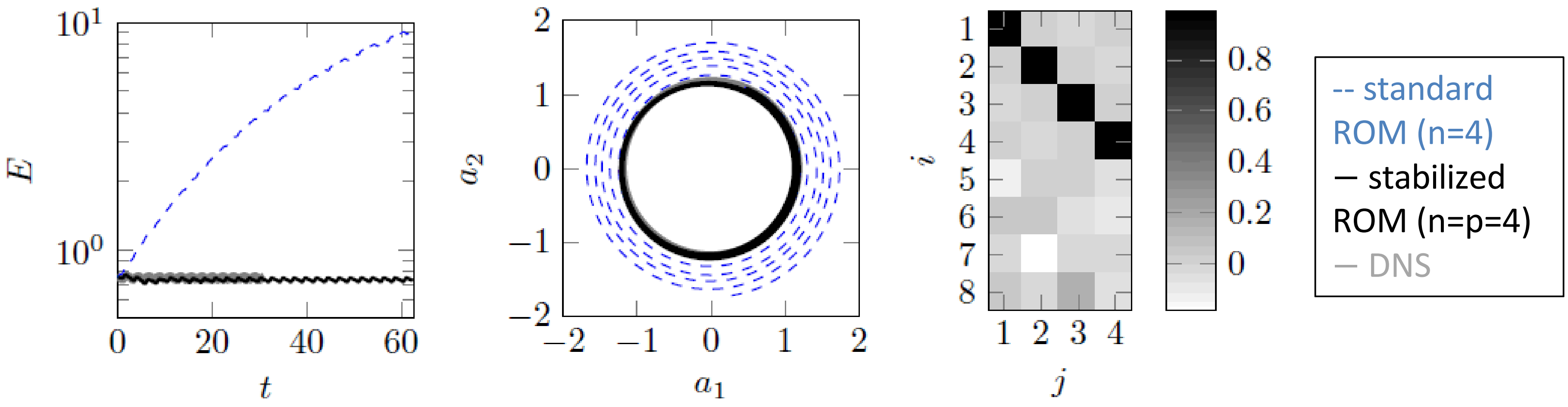
Figure 1: Contours of velocity magnitude at time of final snapshot.

Applications

High angle of attack laminar airfoil

- *Minimizing subspace rotation:*

$$f(\boldsymbol{X}) = \left\|\boldsymbol{X} - \boldsymbol{I}_{(n+p),n}\right\|_F = -\mathrm{tr}\left(\boldsymbol{X}^T \boldsymbol{I}_{(n+p)\times n}\right)$$



(a) (b) (c)

Figure 2: (a) evolution of modal energy, (b) phase plot of first and second temporal basis $a_1(t)$ and $a_2(t)$, (c) illustration of stabilizing rotation showing that rotation is small: $\frac{\left\|\boldsymbol{X}-\boldsymbol{I}_{(n+p),n}\right\|_F}{n} = 0.083, \boldsymbol{X} \approx \boldsymbol{I}_{(n+p),n}$

Applications

High angle of attack laminar airfoil

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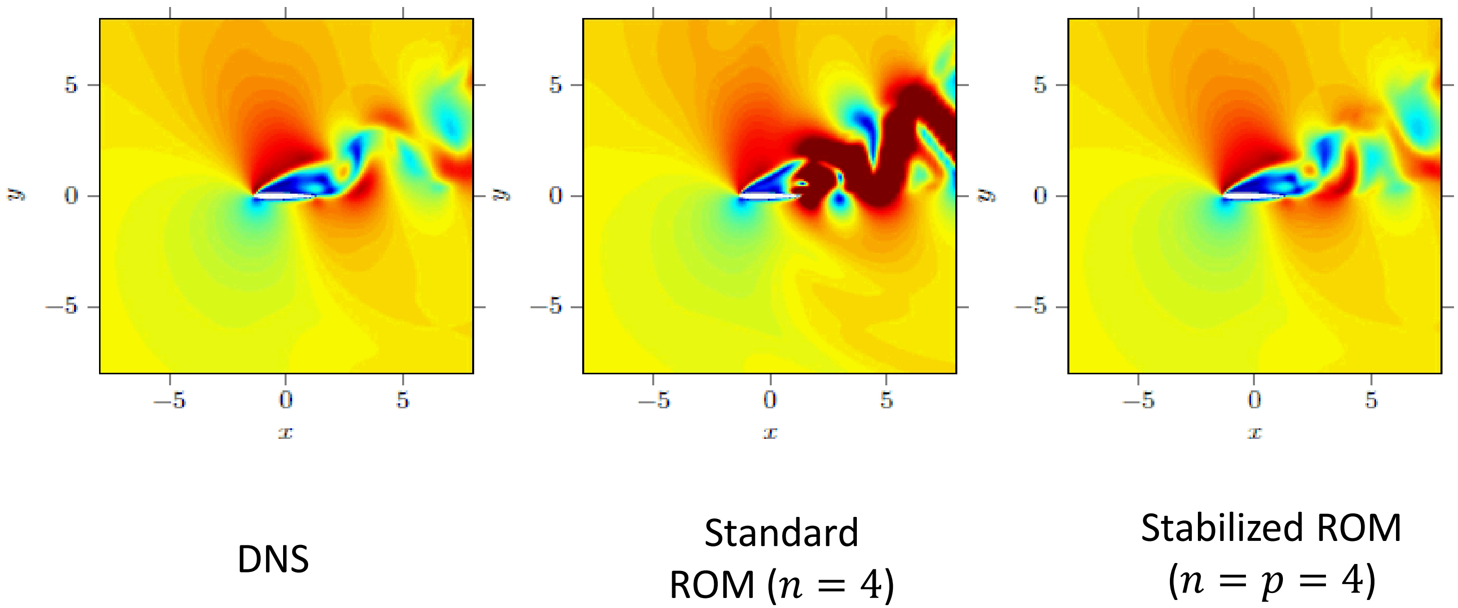

Figure 3: High angle of attack laminar airfoil contours of velocity magnitude at time of final snapshot.

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Applications

Channel driven cavity: low Reynolds number case

Flow over square cavity at Mach 0.6, Re = 1453.9, Pr = 0.72
$\Rightarrow n = 4$ ROM (91% snapshot energy).

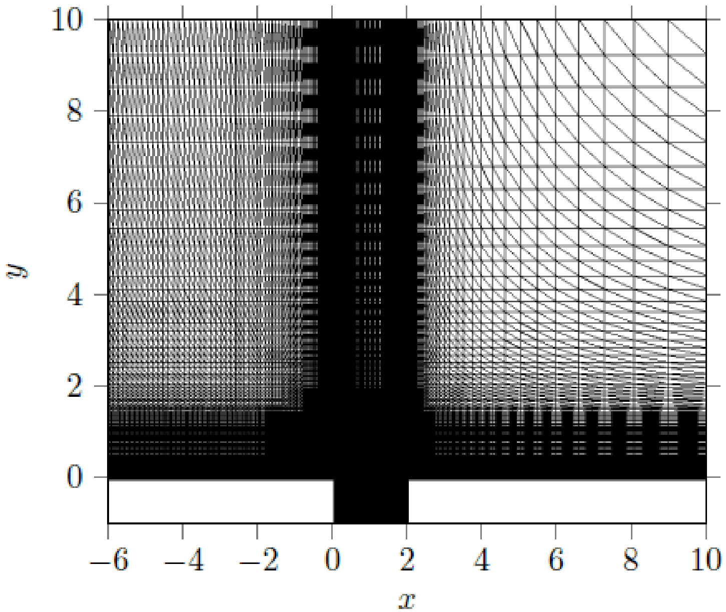

Figure 4: Domain and mesh for viscous channel driven cavity problem.

Applications

Channel driven cavity: low Reynolds number case

- *Minimizing subspace rotation:*

$$f(\boldsymbol{X}) = \left\|\boldsymbol{X} - \boldsymbol{I}_{(n+p),n}\right\|_F = -\mathrm{tr}\left(\boldsymbol{X}^T \boldsymbol{I}_{(n+p)\times n}\right)$$

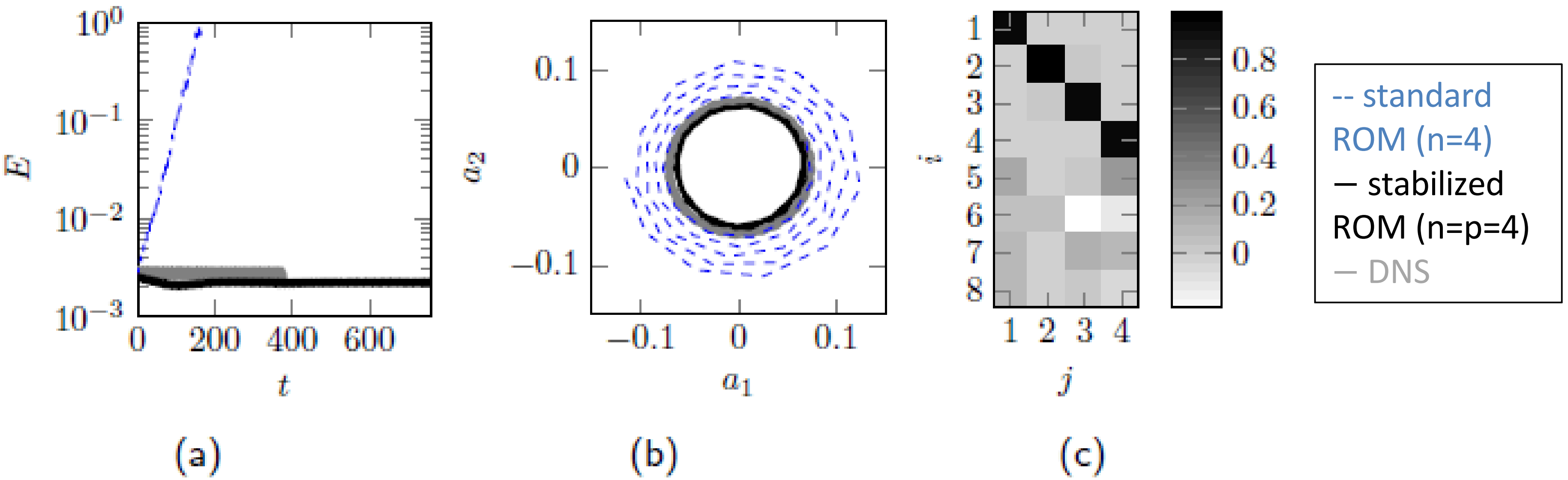

Figure 5: (a) evolution of modal energy, (b) phase plot of first and second temporal basis $a_1(t)$ and $a_2(t)$, (c) illustration of stabilizing rotation showing that rotation is small: $\frac{\left\|\boldsymbol{X}-\boldsymbol{I}_{(n+p),n}\right\|_F}{n} = 0.188, \boldsymbol{X} \approx \boldsymbol{I}_{(n+p),n}$

Applications

Channel driven cavity: low Reynolds number case

- *Minimizing subspace rotation:*

$$f(\boldsymbol{X}) = \left\|\boldsymbol{X} - \boldsymbol{I}_{(n+p),n}\right\|_F = -\mathrm{tr}\left(\boldsymbol{X}^T \boldsymbol{I}_{(n+p)\times n}\right)$$

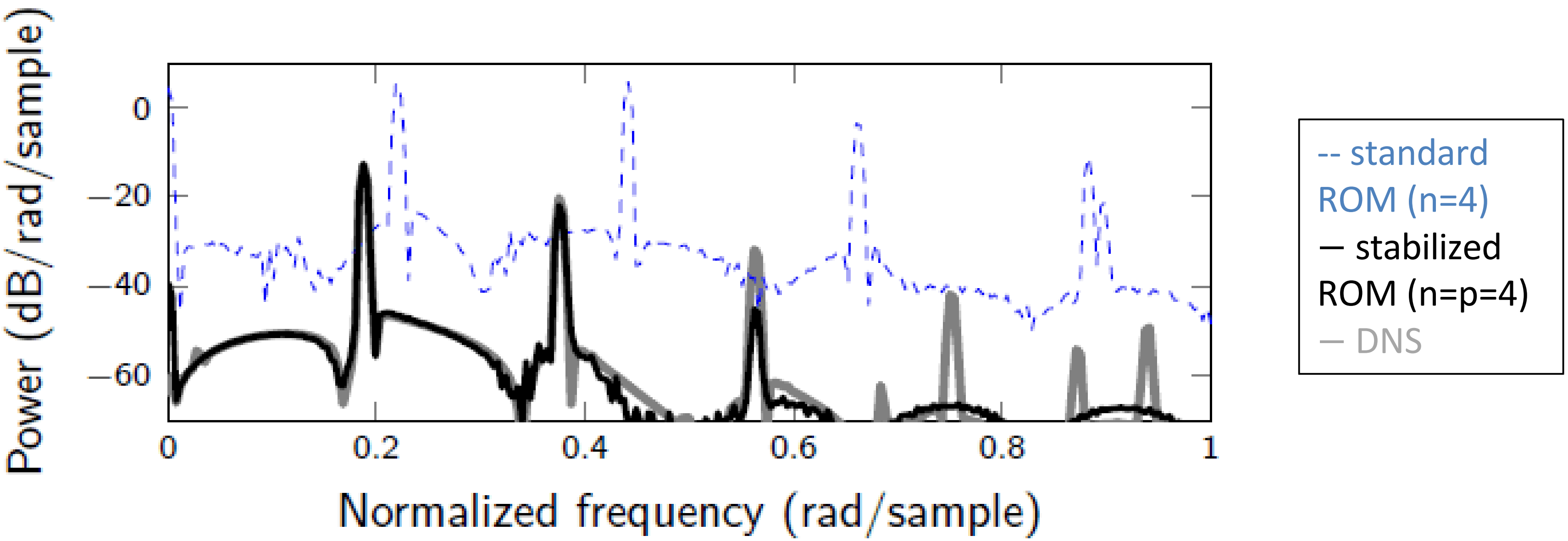

Figure 6: Pressure power spectral density (PSD) at location $\boldsymbol{x} = (2, -1)$; stabilized ROM minimizes subspace rotation.

Applications

Channel driven cavity: low Reynolds number case

- *Maximizing resolved TKE:*

$$f(\boldsymbol{X}) = -||\boldsymbol{\Sigma} - \boldsymbol{X}\boldsymbol{X}^T\boldsymbol{\Sigma}||_F$$

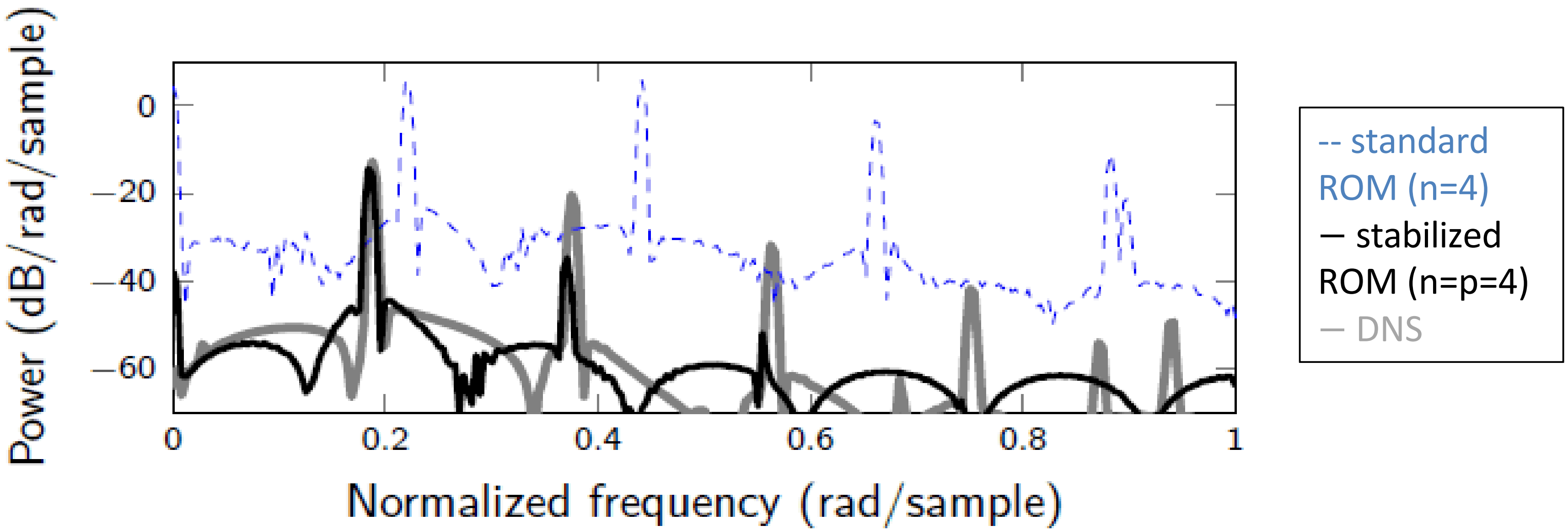

Figure 7: Pressure power spectral density (PSD) at location $\boldsymbol{x} = (2, -1)$; stabilized ROM maximizes resolved TKE.

Applications

Channel driven cavity: low Reynolds number case

- *Minimizing subspace rotation:*

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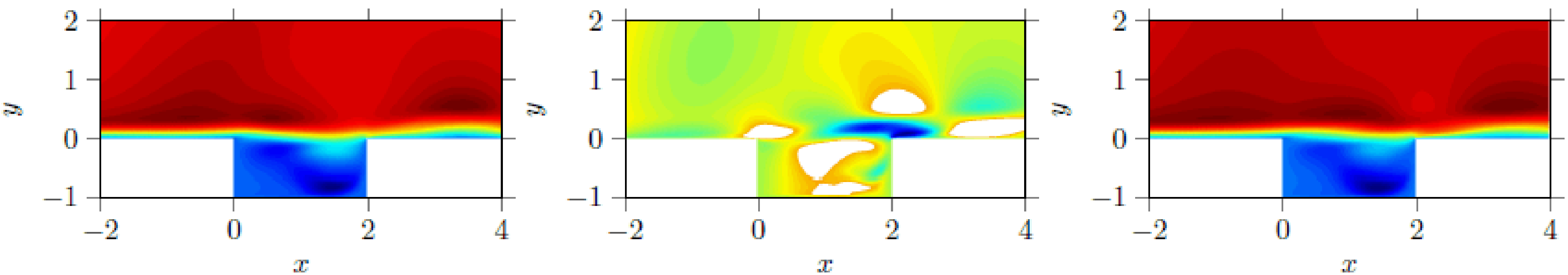

DNS

Standard ROM ($n = 4$)

Stabilized ROM ($n = p = 4$)

Figure 8: Channel driven cavity Re $\approx$ 1500 contours of u-velocity at time of final snapshot.

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Applications

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Channel driven cavity: moderate Reynolds number case

Flow over square cavity at Mach 0.6, Re = 5452.1, Pr = 0.72
$\Rightarrow n = 20$ ROM (71.8% snapshot energy).

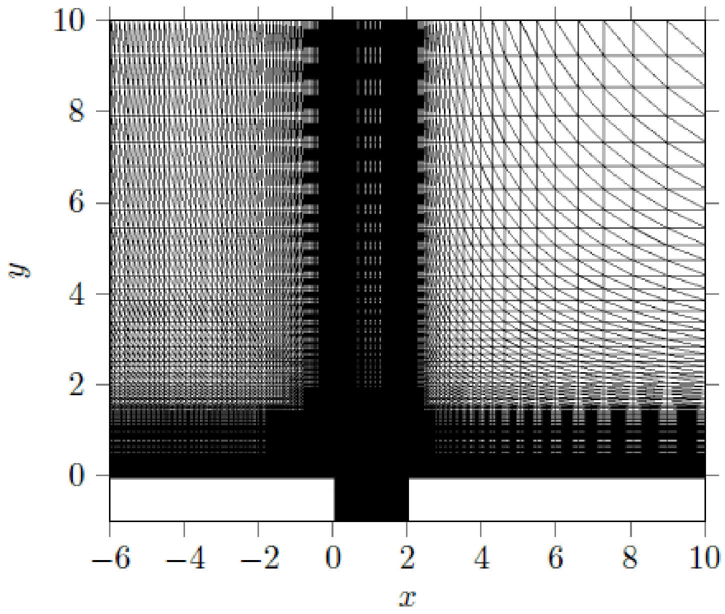

Figure 9: Domain and mesh for viscous channel driven cavity problem.

Applications

Channel driven cavity: moderate Reynolds number case

- *Minimizing subspace rotation:*

$$f(\boldsymbol{X}) = \left\|\boldsymbol{X} - \boldsymbol{I}_{(n+p),n}\right\|_F = -\mathrm{tr}\left(\boldsymbol{X}^T \boldsymbol{I}_{(n+p)\times n}\right)$$

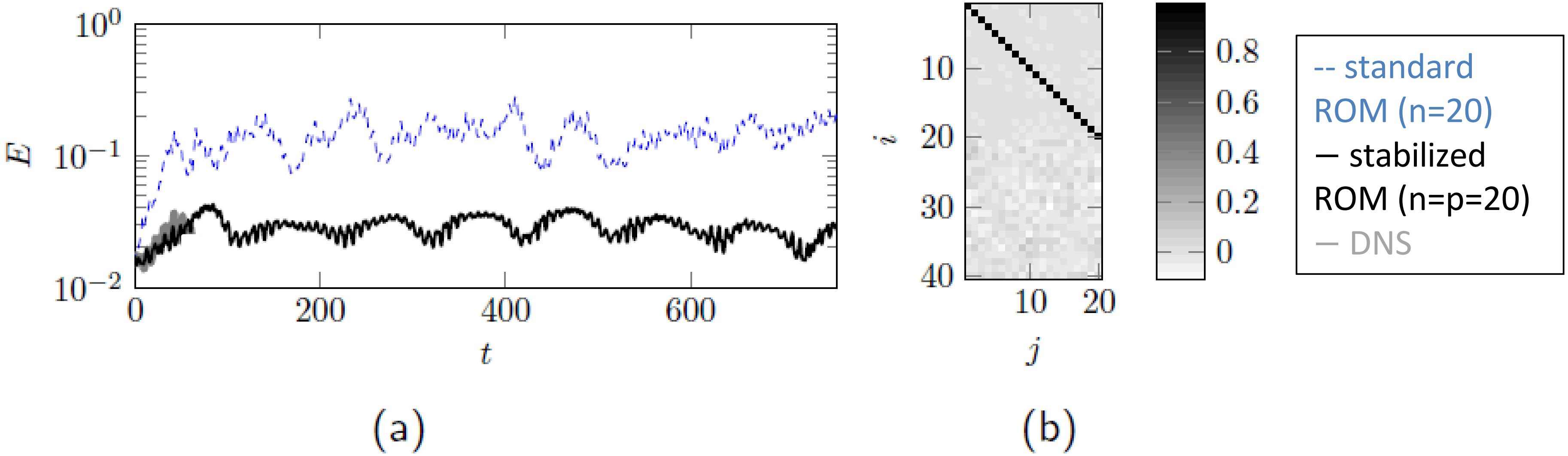

Figure 10: (a) evolution of modal energy, (b) illustration of stabilizing rotation showing that rotation is small: $\frac{\left\|\boldsymbol{X}-\boldsymbol{I}_{(n+p),n}\right\|_F}{n} = 0.038, \boldsymbol{X} \approx \boldsymbol{I}_{(n+p),n}$

Applications

Channel driven cavity: moderate Reynolds number case

- *Minimizing subspace rotation:*

$$f(\boldsymbol{X}) = \left\|\boldsymbol{X} - \boldsymbol{I}_{(n+p),n}\right\|_F = -\mathrm{tr}\left(\boldsymbol{X}^T \boldsymbol{I}_{(n+p)\times n}\right)$$

— stabilized ROM (n=p=20)
— DNS

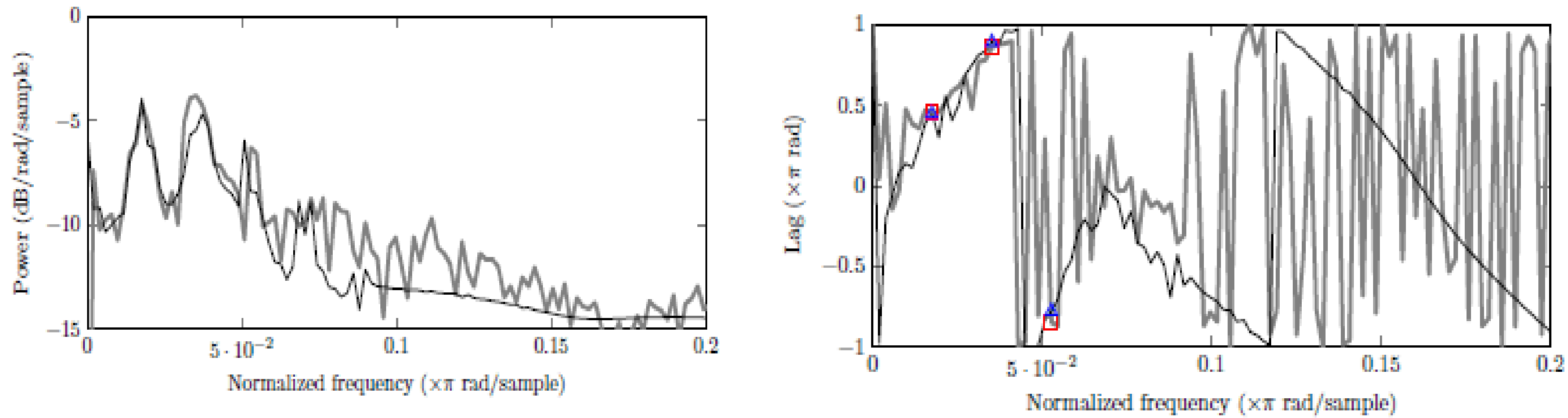

Figure 11: Pressure cross PSD of of $p(\boldsymbol{x}_1, t)$ and $p(\boldsymbol{x}_2, t)$ where $\boldsymbol{x}_1 = (2, -0.5)$, $\boldsymbol{x}_2 = (0, -0.5)$

Power and phase lag at fundamental frequency, and first two super harmonics are predicted accurately using the fine-tuned ROM (Δ = stabilized ROM, □ = DNS)

Applications

Channel driven cavity: moderate Reynolds number case

- *Minimizing subspace rotation:*

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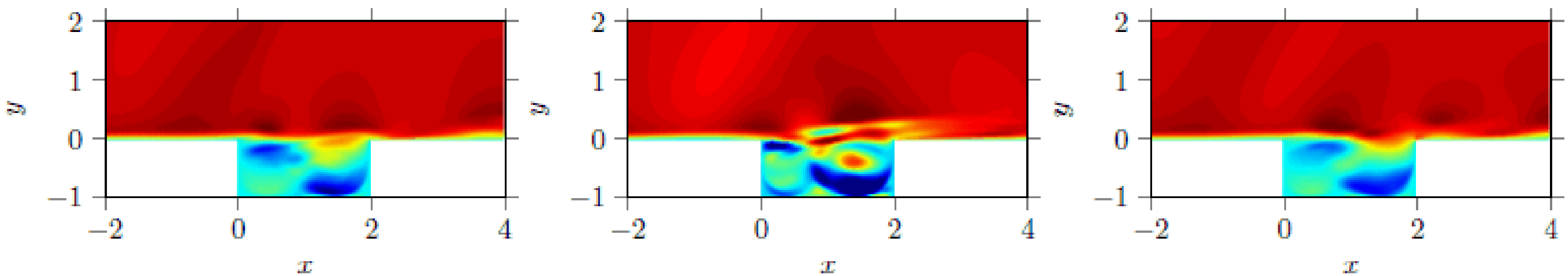

DNS

Standard ROM ($n = 20$)

Stabilized ROM ($n = p = 20$)

Figure 12: Channel driven cavity Re ≈ 5500 contours of u-velocity at time of final snapshot.

Applications

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CPU times (CPU-hours) for offline and online computations*

	Procedure	Airfoil	Low Re Cavity	Moderate Re Cavity
	FOM # of DOF	360,000	288,250	243,750
offline	Time-integration of FOM	7.8 hrs	72 hrs	179 hrs
offline	Basis construction (size $n+p$ ROM)	0.16 hrs	0.88 hrs	3.44 hrs
offline	Galerkin projection (size $n+p$ ROM)	0.74 hrs	5.44 hrs	14.8 hrs
offline	Stabilization	28 sec	14 sec	170 sec
	ROM # of DOF	4	4	20
online	Time-integration of ROM	0.31 sec	0.16 sec	0.83 sec
	Online computational speed-up	9.1e4	1.6e6	7.8e5

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- Stabilization is <u>*fast*</u> (O(sec) or O(min)).

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- Stabilization is *fast* (O(sec) or O(min)).
- Significant *online computational speed-up*!

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Summary

- We have developed a non-intrusive approach for *stabilizing* and *fine-tuning* projection-based ROMs for compressible flows.
- The standard POD modes are *"rotated"* into a more dissipative regime to account for the dynamics in the higher order modes truncated by the standard POD method.
- The new approach is *consistent* and does not require the addition of empirical turbulence model terms unlike traditional approaches.
- Mathematically, the approach is formulated as a *quadratic matrix program* on the Stiefel manifold.
- The constrained minimization problem is solved *offline* and *small* enough to be solved in MATLAB.
- The method is demonstrated on several compressible flow problems and shown to deliver *stable* and *accurate* ROMs.

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Future work

- Application to <u>*higher Reynolds number*</u> problems.

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- Extension of the proposed approach to problems with *generic nonlinearities*, where the ROM involves some form of hyper-reduction (e.g., DEIM, gappy POD).
- Extension of the method to *predictive applications*, e.g., problems with varying Reynolds number and/or geometry.

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- Application to *higher Reynolds number* problems.
- Extension of the proposed approach to problems with *generic nonlinearities*, where the ROM involves some form of hyper-reduction (e.g., DEIM, gappy POD).
- Extension of the method to *predictive applications*, e.g., problems with varying Reynolds number and/or geometry.
- Selecting different *goal-oriented* objectives and constraints in our optimization problem:

$$\begin{aligned} &\text{minimize}_{\boldsymbol{X} \in \mathcal{V}_{(n+p),n}} \quad f(\boldsymbol{X}) \\ &\text{subject to} \quad g(\boldsymbol{X}, \boldsymbol{L}) = 0 \end{aligned}$$

Future work

- Application to *higher Reynolds number* problems.
- Extension of the proposed approach to problems with *generic nonlinearities*, where the ROM involves some form of hyper-reduction (e.g., DEIM, gappy POD).
- Extension of the method to *predictive applications*, e.g., problems with varying Reynolds number and/or geometry.
- Selecting different *goal-oriented* objectives and constraints in our optimization problem:

$$\begin{array}{ll} \text{minimize}_{\boldsymbol{X}\in\mathcal{V}_{(n+p),n}} & f(\boldsymbol{X}) \\ \text{subject to} & g(\boldsymbol{X},\boldsymbol{L}) = 0 \end{array}$$

e.g.,

- Maximize parametric robustness: $f = \sum_{i=1}^{k} \beta_i \|\boldsymbol{U}^*(\mu_i)\boldsymbol{X} - \boldsymbol{U}^*(\mu_i)\|_F$.
- ODE constraints: $g = \|\boldsymbol{a}(t) - \boldsymbol{a}^*(t)\|$.

Outline

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3. Accounting for modal truncation
   - Traditional linear eddy-viscosity approach
   - New proposed approach via subspace rotation
4. Applications
   - High angle of attack laminar airfoil
   - Low Reynolds number channel driven cavity
   - Moderate Reynolds number channel driven cavity
5. Summary
6. Future work
7. **References**

References

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8. **Appendix**

Appendix: Accounting for modal truncation

Stabilization algorithm: returns stabilizing rotation matrix $\boldsymbol{X}$.

Inputs: Initial guess $\eta^{(0)} = \text{tr}(L(1:n, 1:n))$ $(\boldsymbol{X} = \boldsymbol{I}_{(n+p)\times n})$, ROM size n and $p \geq 1$, ROM matrices associated with the first $n+p$ most energetic POD modes, convergence tolerance TOL, maximum number of iterations k_{max}.

for $k = 0, \cdots, k_{max}$

Solve constrained optimization problem on Stiefel manifold:

$$\underset{\boldsymbol{X}^{(k)} \in \mathcal{V}_{(n+p),n}}{\text{minimize}} \quad -\text{tr}\left(\boldsymbol{X}^{(k)\text{T}} \boldsymbol{I}_{(n+p)\times n}\right)$$

$$\text{subject to} \quad \text{tr}(\boldsymbol{X}^{(k)\text{T}} \boldsymbol{L} \boldsymbol{X}^{(k)}) = \eta^{(k)}.$$

Construct new Galerkin matrices using (4).
Integrate numerically new Galerkin system.
Calculate "modal energy" $E(t)^{(k)} = \sum_i^n (a(t)_i^{(k)})^2$.
Perform linear fit of temporal data $E(t)^{(k)} \approx c_1^{(k)} t + c_0^{(k)}$, where $c_1^{(k)}$ =energy growth.
Calculate ϵ such that $c_1^{(k)}(\epsilon) = 0$ (no energy growth) using root-finding algorithm.
Perform update $\eta^{(k+1)} = \eta^{(k)} + \epsilon$.
if $||c_1^{(k)}|| < TOL$
    $\boldsymbol{X} := \boldsymbol{X}^{(k)}$.
    terminate the algorithm.
end
end