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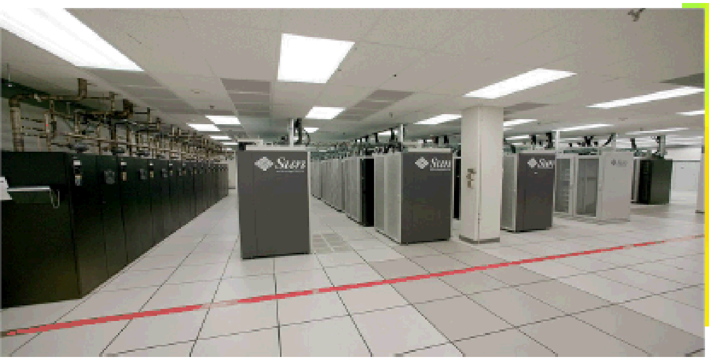
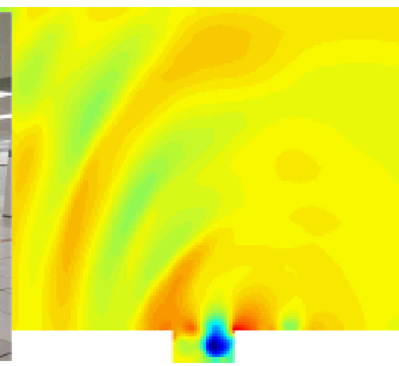
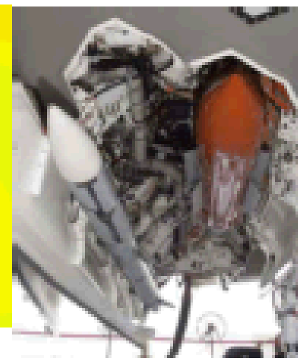
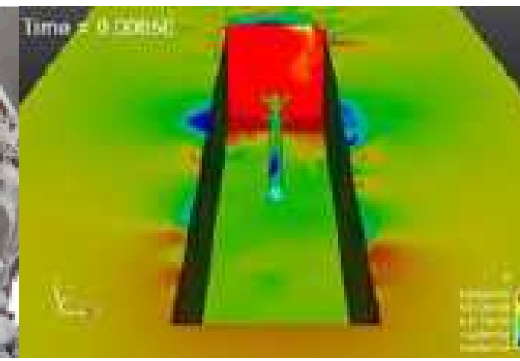

# A minimal subspace rotation approach for obtaining stable & accurate low-order projection-based reduced order models for nonlinear compressible flow


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1. Motivation
2. Projection-based model order reduction
3. Accounting for modal truncation
   - Traditional linear eddy-viscosity approach
   - New proposed approach via subspace rotation
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   - Low Reynolds number channel driven cavity
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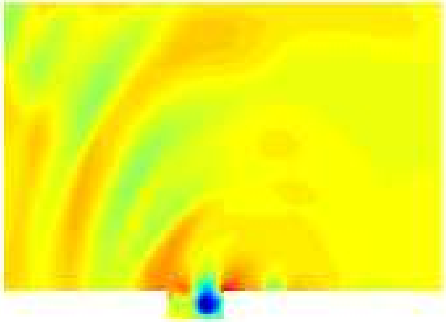

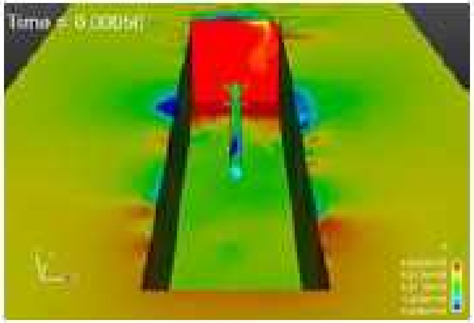

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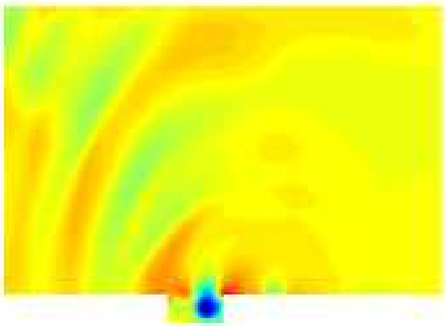

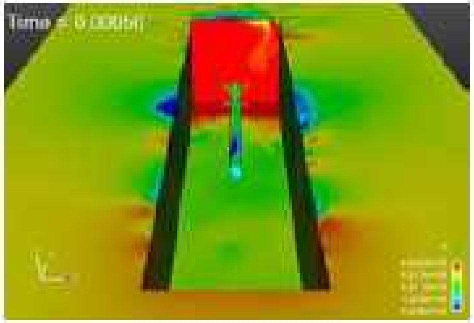

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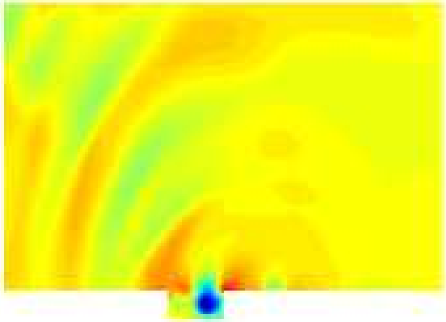

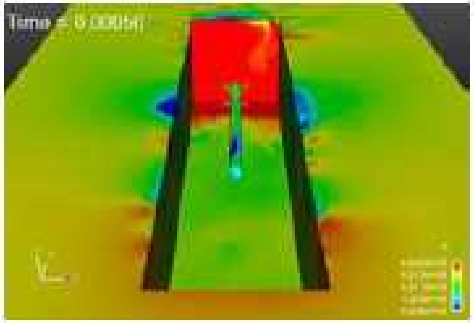

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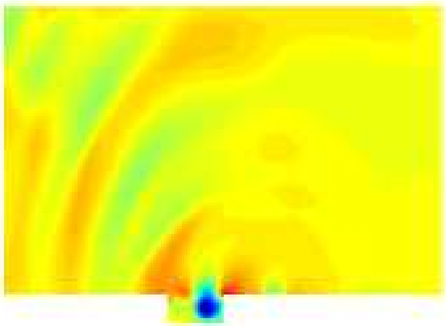

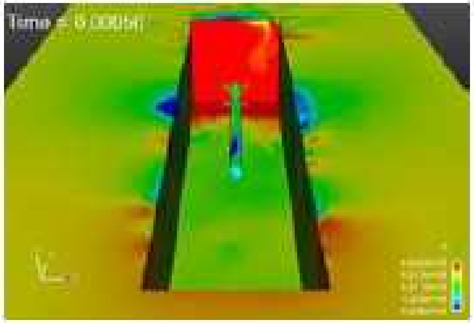

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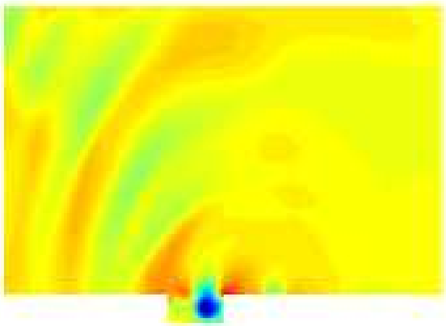

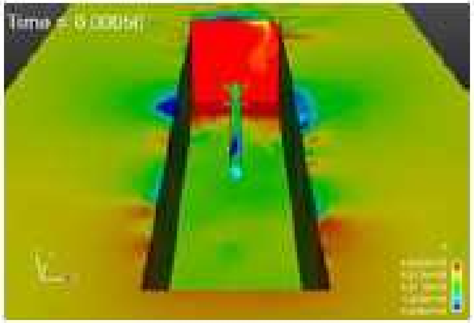

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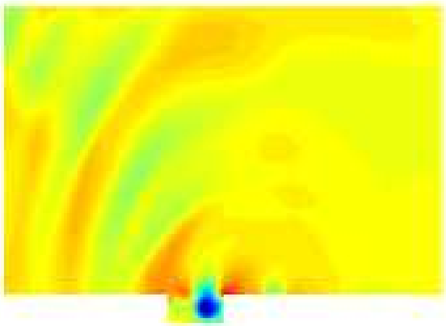

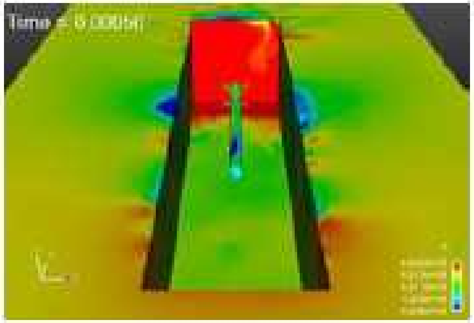

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**MOR for <u>nonlinear</u>, <u>compressible</u> fluid flows is still in its infancy!**

# Projection-based model order reduction

**Governing equations**

- 3D compressible Navier-Stokes equations in *primitive specific volume form*:

[PDEs]

$$\zeta_{,t} + \zeta_{,j} u_j - \zeta u_{j,j} = 0$$
$$u_{i,t} + u_{i,j} u_j + \zeta p_{,i} - \frac{1}{Re} \zeta \tau_{ij,j} = 0 \qquad (1)$$
$$p_{,t} + u_j p_{,j} + \gamma u_{j,j} p - \left(\frac{\gamma}{PrRe}\right)\left(\kappa (p\zeta)_{,j}\right)_{,j} - \left(\frac{\gamma - 1}{Re}\right) u_{i,j} \tau_{ij} = 0$$

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- Spectral discretization* ($\boldsymbol{q}(\boldsymbol{x},t) \approx \sum_{i=1}^{n} a_i(t) \boldsymbol{U}_i(\boldsymbol{x})$) + Galerkin projection applied to (1) yields a system of *$n$ coupled quadratic ODEs*:

[ROM]

$$\frac{d\boldsymbol{a}}{dt} = \boldsymbol{C} + \boldsymbol{L}\boldsymbol{a} + \left[\boldsymbol{a}^T \boldsymbol{Q}^{(1)} \boldsymbol{a} + \boldsymbol{a}^T \boldsymbol{Q}^{(2)} \boldsymbol{a} + \cdots + \boldsymbol{a}^T \boldsymbol{Q}^{(n)} \boldsymbol{a}\right]^T \qquad (2)$$

where $\boldsymbol{C} \in \mathbb{R}^n, \boldsymbol{L} \in \mathbb{R}^{n \times n}$ and $\boldsymbol{Q}^{(i)} \in \mathbb{R}^{n \times n}$ for all $i = 1, \ldots, n$.

* Here we use a *Proper Orthogonal Decomposition (POD)* basis $\boldsymbol{U}_i(\boldsymbol{x})$.

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***Turbulence Modeling*** (traditional approach)

***Subspace Rotation*** (our approach)

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  3. Inherently a *linear model* → cannot be expected to perform well for all classes of problems (e.g., nonlinear).

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**Illustrative example**

- *Standard approach:* retain only the most energetic POD modes, i.e., $\boldsymbol{U}_1, \boldsymbol{U}_2$, $\boldsymbol{U}_3$, $\boldsymbol{U}_4$, …
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- *More generally:* approximate the solution using a linear superposition of $n+p$ (with $p>0$) most energetic modes:

$$\widetilde{\boldsymbol{U}}_i = \sum_{j=1}^{n+p} X_{ij}\, \boldsymbol{U}_j, \quad i = 1, \ldots, n, \tag{3}$$

where $\boldsymbol{X} \in \mathbb{R}^{(n+p)\times n}$ is an orthonormal ($\boldsymbol{X}^T\boldsymbol{X} = \boldsymbol{I}_{n\times n}$) "rotation" matrix.

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- Once $\boldsymbol{X}$ is found, the result is a system of the form (2) with:

$$Q^{(i)}{}_{jk} \leftarrow \sum_{s,q,r=1}^{n+p} X_{si} Q^{(s)}{}_{qr} X_{qr} X_{rk}, \quad \boldsymbol{L} \leftarrow \boldsymbol{X}^T \boldsymbol{L} \boldsymbol{X}, \quad \boldsymbol{C} \leftarrow \boldsymbol{X}^T \boldsymbol{C}^*$$

# Accounting for modal truncation

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**Minimal subspace rotation:** trace minimization on Stiefel manifold

$$\begin{array}{ll} \text{minimize}_{\boldsymbol{X} \in \mathcal{V}_{(n+p),n}} & -\operatorname{tr}\left(\boldsymbol{X}^T \boldsymbol{I}_{(n+p)\times n}\right) \\ \text{subject to} & \operatorname{tr}(\boldsymbol{X}^T \boldsymbol{L} \boldsymbol{X}) = \eta \end{array} \quad (9)$$

- $\mathcal{V}_{(n+p),n} \in \left\{\boldsymbol{X} \in \mathbb{R}^{(n+p)\times n} : \boldsymbol{X}^T \boldsymbol{X} = \boldsymbol{I}_n, p > 0\right\}$ is the *Stiefel manifold*.
- Constraint is traditional *linear eddy-viscosity closure model ansatz* → involves overall balance between *linear energy production* and *dissipation* / vanishing of *averaged total power* ($= \operatorname{tr}(\boldsymbol{X}^T \boldsymbol{L} \boldsymbol{X}) +$ energy transfer).
  - $\eta \in \mathbb{R}$: proxy for the balance between linear energy production and energy dissipation (calculated iteratively using modal energy).
- Equation (9) is solved efficiently offline using the method of Lagrange multipliers (`Manopt MATLAB toolbox`).
- See (Balajewicz, Tezaur, Dowell, 2016) and Appendix slide for Algorithm.

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  3. Works with *any* basis and Petrov-Galerkin projection.

- Disadvantages of proposed approach:
  1. Off-line calibration of free parameter $\eta$ is required.
  2. Stability cannot be proven like for incompressible case.

# Outline

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# Applications

**Channel driven cavity: low Reynolds number case**

Flow over square cavity at Mach 0.6, Re = 1453.9, Pr = 0.72
$\Rightarrow n = 4$ ROM (91% snapshot energy).

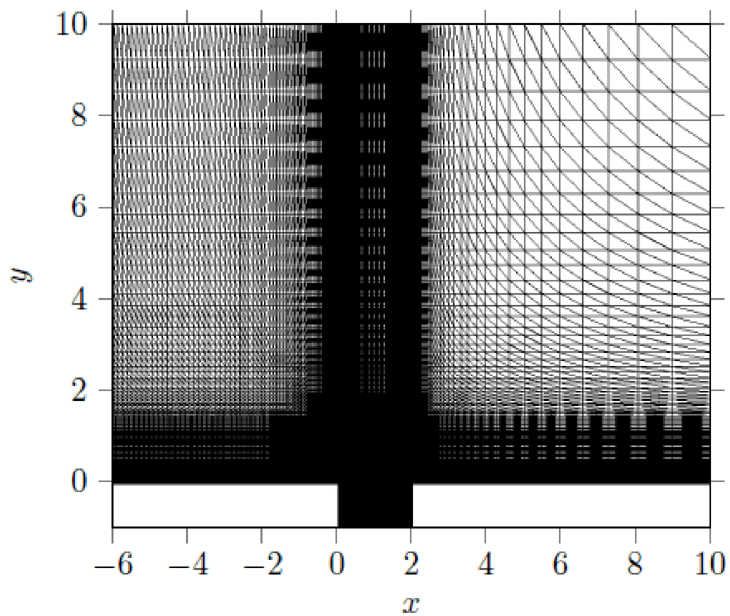


Figure 1: Domain and mesh for viscous channel driven cavity problem.

# Applications

**Channel driven cavity: low Reynolds number case**

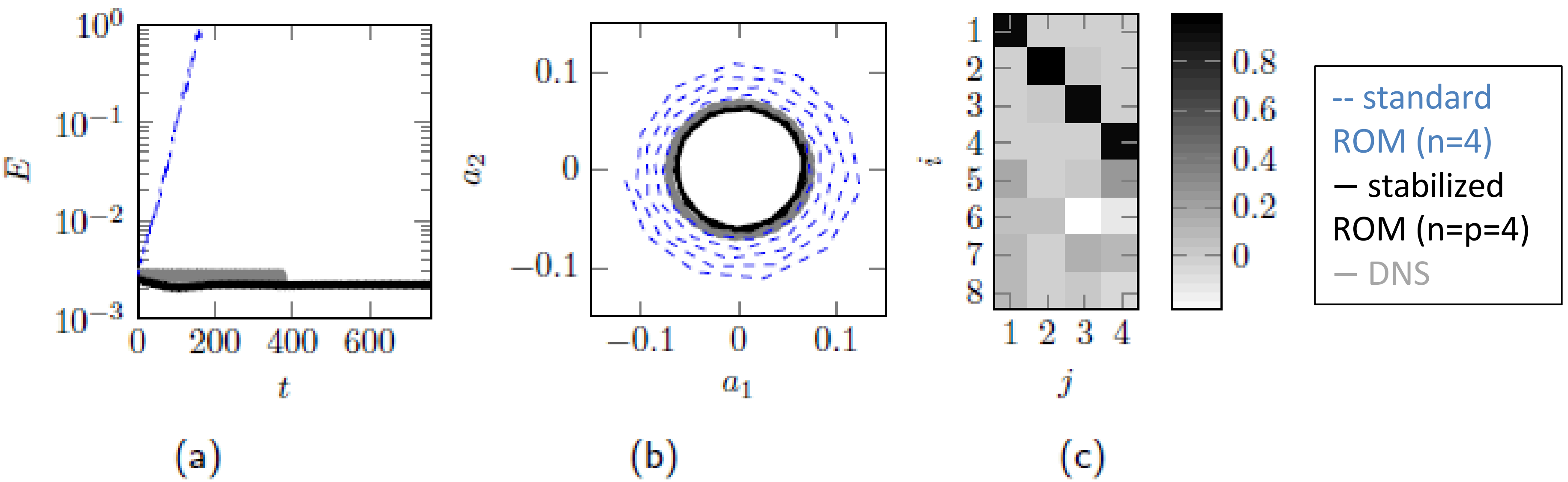


Figure 2: (a) evolution of modal energy, (b) phase plot of first and second temporal basis $a_1(t)$ and $a_2(t)$, (c) illustration of stabilizing rotation showing that rotation is small: $\frac{\|\boldsymbol{X}-\boldsymbol{I}_{(n+p),n}\|_F}{n} = 0.188, \boldsymbol{X} \approx \boldsymbol{I}_{(n+p),n}$

# Applications

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**Channel driven cavity: low Reynolds number case**

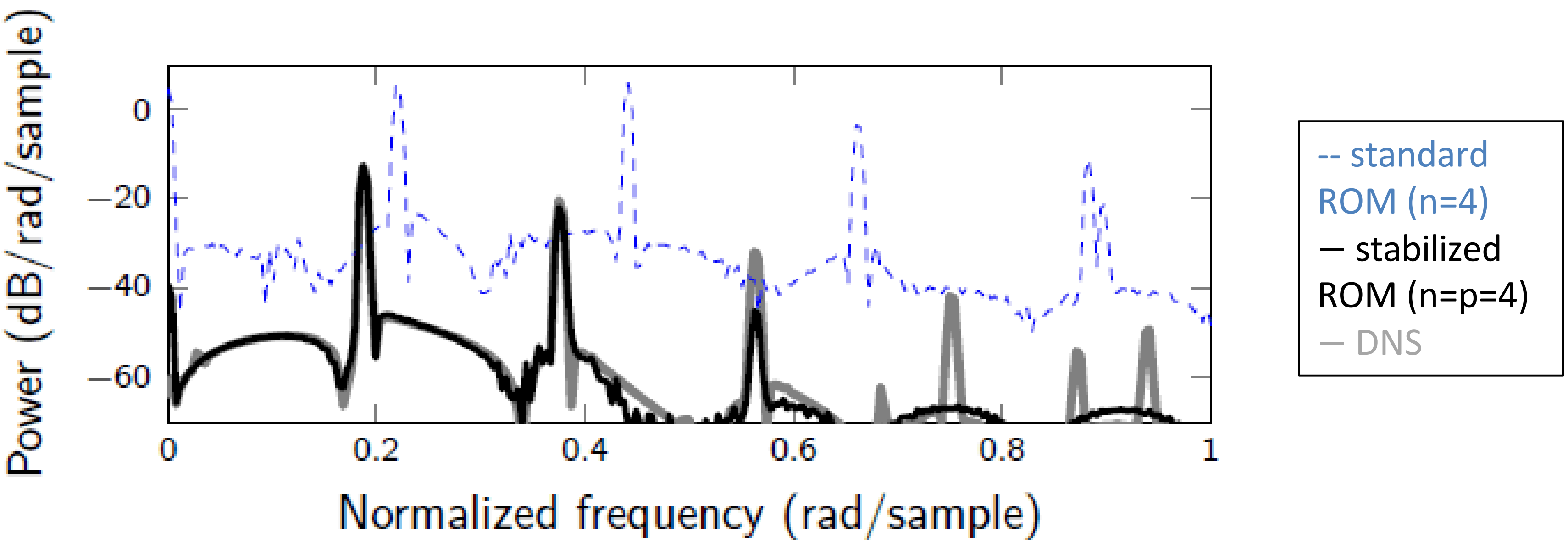
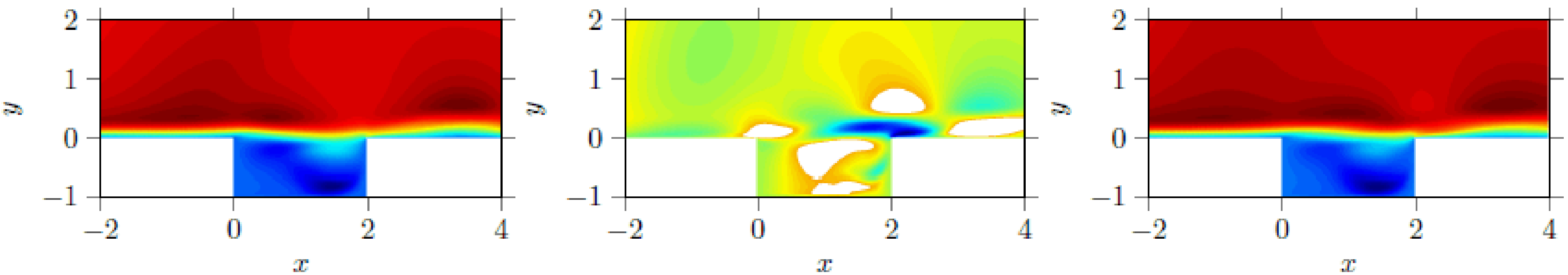



Figure 3: Pressure power spectral density (PSD) at location $\boldsymbol{x} = (2, -1)$; stabilized ROM minimizes subspace rotation.

# Applications

**Channel driven cavity: low Reynolds number case**




DNS

Standard ROM ($n = 4$)

Stabilized ROM ($n = p = 4$)

Figure 4: Channel driven cavity Re ≈ 1500 contours of $u$-velocity at time of final snapshot.

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**Channel driven cavity: moderate Reynolds number case**

Flow over square cavity at Mach 0.6, Re = 5452.1, Pr = 0.72
$\Rightarrow n = 20$ ROM (71.8% snapshot energy).

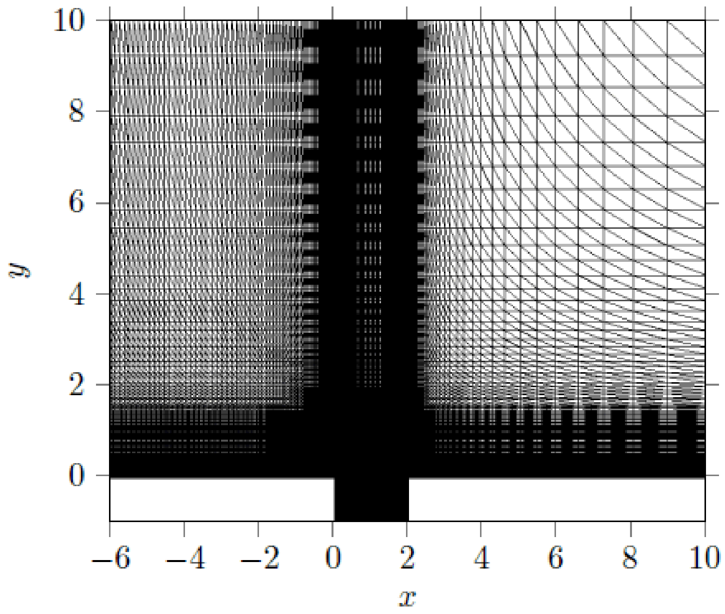
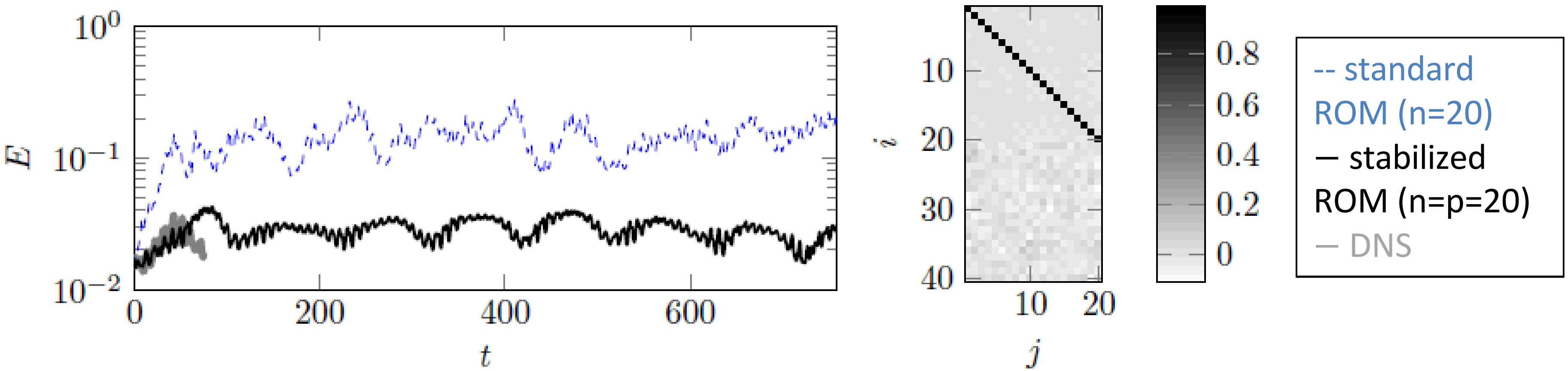



Figure 5: Domain and mesh for viscous channel driven cavity problem.

# Applications

**Channel driven cavity: moderate Reynolds number case**



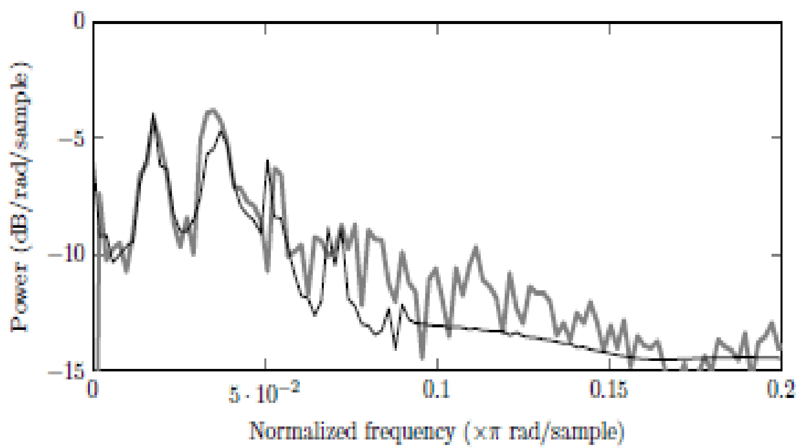

(a) (b)

Figure 6: (a) evolution of modal energy, (b) illustration of stabilizing rotation showing that rotation is small: $\frac{\|\boldsymbol{X}-\boldsymbol{I}_{(n+p),n}\|_F}{n} = 0.038, \boldsymbol{X} \approx \boldsymbol{I}_{(n+p),n}$

# Applications

**Channel driven cavity: moderate Reynolds number case**

— stabilized ROM (n=p=20)
— DNS



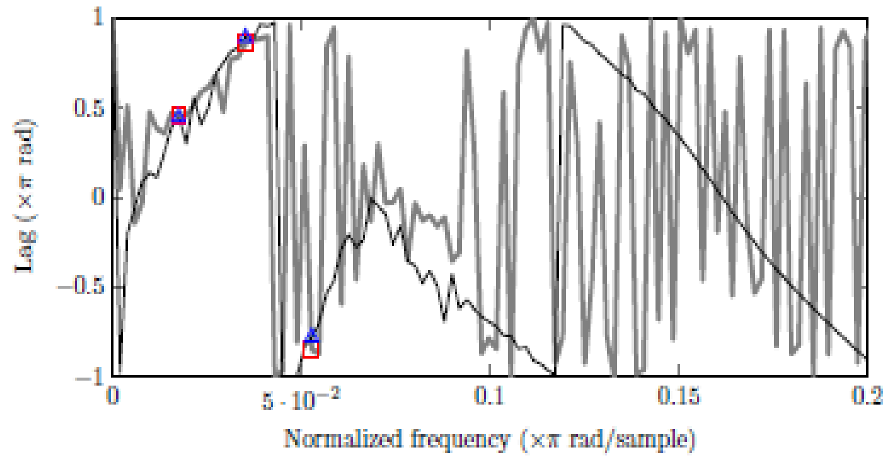
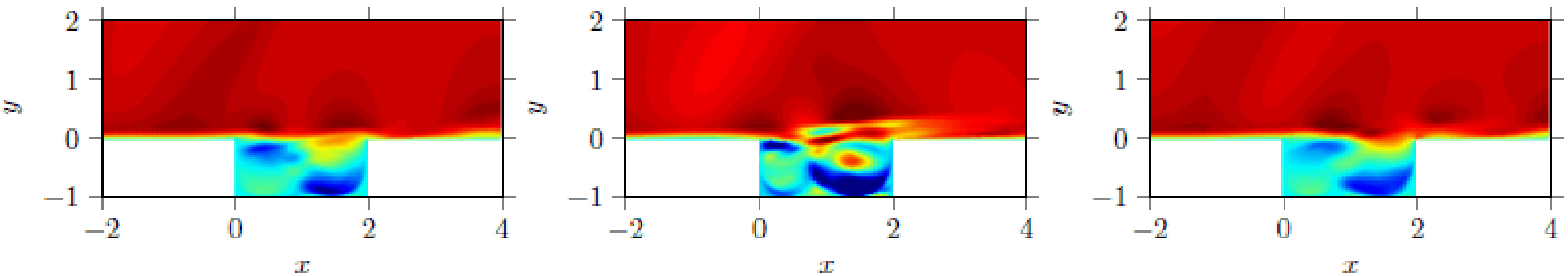




Figure 7: Pressure cross PSD of of $p(\boldsymbol{x}_1, t)$ and $p(\boldsymbol{x}_2, t)$ where $\boldsymbol{x}_1 = (2, -0.5)$, $\boldsymbol{x}_2 = (0, -0.5)$

Power and phase lag at fundamental frequency, and first two super harmonics are predicted accurately using the fine-tuned ROM ($\Delta$ = stabilized ROM, □ = DNS)

# Applications

## Channel driven cavity: moderate Reynolds number case




DNS

Standard ROM
$(n = 20)$

Stabilized ROM
$(n = p = 20)$

Figure 8: Channel driven cavity Re $\approx$ 5500 contours of $u$-velocity at time of final snapshot.

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# Summary

- We have developed a non-intrusive approach for *stabilizing* and *fine-tuning* projection-based ROMs for compressible flows.
- The standard POD modes are *"rotated"* into a more dissipative regime to account for the dynamics in the higher order modes truncated by the standard POD method.
- The new approach is *consistent* and does not require the addition of empirical turbulence model terms unlike traditional approaches.
- Mathematically, the approach is formulated as a *quadratic matrix program* on the Stiefel manifold.
- The constrained minimization problem is solved *offline* and *small* enough to be solved in MATLAB.
- The method is demonstrated on several compressible flow problems and shown to deliver *stable* and *accurate* ROMs.

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# Future work

- Application to *higher Reynolds number* problems.
- Extension of the proposed approach to problems with *generic nonlinearities*, where the ROM involves some form of hyper-reduction (e.g., DEIM, gappy POD).
- Extension of the method to *minimal-residual-based* nonlinear ROMs.
- Extension of the method to *predictive applications*, e.g., problems with varying Reynolds number and/or Mach number.
- Selecting different *goal-oriented* objectives and constraints in our optimization problem:

$$\begin{aligned} &\text{minimize}_{\boldsymbol{X}\in\mathcal{V}_{(n+p),n}} \quad f(\boldsymbol{X}) \\ &\text{subject to} \quad g(\boldsymbol{X},\boldsymbol{L}) = 0 \end{aligned}$$

  e.g.,
  - Maximize parametric robustness:
  $$f = \sum_{i=1}^{k} \beta_i \|\boldsymbol{U}^*(\mu_i)\boldsymbol{X} - \boldsymbol{U}^*(\mu_i)\|_F.$$
  - ODE constraints: $g = \|\boldsymbol{a}(t) - \boldsymbol{a}^*(t)\|$.

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# Appendix: Accounting for modal truncation

**Stabilization algorithm:** returns stabilizing rotation matrix $\boldsymbol{X}$.

**Inputs:** Initial guess $\eta^{(0)} = \text{tr}(L(1:n, 1:n))$ $(\boldsymbol{X} = \boldsymbol{I}_{(n+p)\times n})$, ROM size $n$ and $p \geq 1$, ROM matrices associated with the first $n+p$ most energetic POD modes, convergence tolerance $TOL$, maximum number of iterations $k_{max}$.

for $k = 0, \cdots, k_{max}$

Solve constrained optimization problem on Stiefel manifold:

$$\underset{\boldsymbol{X}^{(k)} \in \mathcal{V}_{(n+p),n}}{\text{minimize}} \quad -\text{tr}\left(\boldsymbol{X}^{(k)\mathrm{T}} \boldsymbol{I}_{(n+p)\times n}\right)$$

$$\text{subject to} \quad \text{tr}(\boldsymbol{X}^{(k)\mathrm{T}} \boldsymbol{L} \boldsymbol{X}^{(k)}) = \eta^{(k)}.$$

Construct new Galerkin matrices using (4).

Integrate numerically new Galerkin system.

Calculate "modal energy" $E(t)^{(k)} = \sum_i^n (a(t)_i^{(k)})^2$.

Perform linear fit of temporal data $E(t)^{(k)} \approx c_1^{(k)} t + c_0^{(k)}$, where $c_1^{(k)}$ =energy growth.

Calculate $\epsilon$ such that $c_1^{(k)}(\epsilon) = 0$ (no energy growth) using root-finding algorithm.

Perform update $\eta^{(k+1)} = \eta^{(k)} + \epsilon$.

if $||c_1^{(k)}|| < TOL$

$\boldsymbol{X} := \boldsymbol{X}^{(k)}$.

terminate the algorithm.

end

end

# Applications

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**CPU times (CPU-hours) for offline and online computations**

| | Procedure | Low Re Cavity | Moderate Re Cavity |
|---|---|---|---|
| | FOM # of DOF | 288,250 | 243,750 |
| offline | Time-integration of FOM | 72 hrs | 179 hrs |
| offline | Basis construction (size $n+p$ ROM) | 0.88 hrs | 3.44 hrs |
| offline | Galerkin projection (size $n+p$ ROM) | 5.44 hrs | 14.8 hrs |
| offline | Stabilization | 14 sec | 170 sec |
| | ROM # of DOF | 4 | 20 |
| online | Time-integration of ROM | 0.16 sec | 0.83 sec |
| | Online computational speed-up | 1.6e6 | 7.8e5 |

- Stabilization is *fast* ($O$(sec) or $O$(min)).
- Significant *online computational speed-up*!