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FINITE ELEMENT APPROXIMATION OF THE EXTENDED FLUID-STRUCTURE INTERACTION (EXFSI) PROBLEM

Bhuiyan Shameem Mahmood Ebna Hai †
Research Fellow
Dept. of Mechanical Engineering
Helmut Schmidt University
Hamburg, Germany.

Markus Bause
Professor
Dept. of Mechanical Engineering
Helmut Schmidt University
Hamburg, Germany.

Paul Kuberry
Research Fellow
Center for Computing Research
Sandia National Laboratories ‡
Albuquerque, NM, USA.

ABSTRACT

This contribution is the second part of three papers on Adaptive Multigrid Methods for the eXtended Fluid-Structure Interaction (eXFSI) Problem, where we introduce a monolithic variational formulation and solution techniques. To the best of our knowledge, such a model is new in the literature. This model is used to design an on-line structural health monitoring (SHM) system in order to determine the coupled acoustic and elastic wave propagation in moving domains and optimum locations for SHM sensors. In a monolithic nonlinear fluid-structure interaction (FSI), the fluid and structure models are formulated in different coordinate systems. This makes the FSI setup of a common variational description difficult and challenging. This article presents the state-of-the-art in the finite element approximation of FSI problem based on monolithic variational formulation in the well-established arbitrary Lagrangian Eulerian (ALE) framework. This research focuses on the newly developed mathematical model of a new FSI problem, which is referred to as extended Fluid-Structure Interaction (eXFSI) problem in the ALE framework. The eXFSI is a strongly coupled problem of typical FSI with a coupled wave propagation problem on the fluid-solid interface (WpFSI). The WpFSI is a strongly coupled problem of acoustic and elastic wave equations, where wave propagation problems automatically adopts the boundary conditions from the FSI problem at each time step. The ALE approach provides a simple but powerful procedure to couple solid deformations with fluid flows by a monolithic solution algorithm. In such a setting, the fluid problems are transformed to a fixed reference configuration by the ALE mapping. The goal of this work is the development of concepts for the efficient numerical solution of eXFSI problem, the analysis of various fluid-solid mesh motion techniques and comparison of different second-order time-stepping schemes. This work consists of the investigation of different time stepping scheme formulations for a nonlinear FSI problem coupling the acoustic/elastic wave propagation on the fluid-structure interface. Temporal discretization is based on finite differences and is formulated as a one step-θ scheme, from which we can consider the following particular cases: the implicit Euler, Crank-Nicolson, shifted Crank-Nicolson and the Fractional-Step-θ schemes. The nonlinear problem is solved with a Newton-like method where the discretization is done with a Galerkin finite element scheme. The implementation is accomplished via the software library package DOPELIB based on the `deal.II` finite element library for the computation of different eXFSI configurations.

Keywords: *Fluid-structure interaction (FSI), wave propagation in fluid-structure interface (WpFSI), Galerkin finite element method, arbitrary Lagrangian Eulerian framework*

†Corresponding author; shameem.ebna.hai@hsu-hh.de

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INTRODUCTION

Fluid-structure interactions (FSI) problems have a historical and practical importance and have to be taken into account in the design of many engineering systems such as aircraft wings, wind turbine, internal combustion engine, tower, bridge, etc. Subsequently, for the last several decades they have been the subject of intensive research, yet the numerical approximation and simulation of fluid-structure interactions remains an indisputably challenging topic with an immense number of unresolved problems and issues. While the numerical analysis of the coupled system in terms of well-posedness and convergence is typically restricted to simple model problems, a plethora of insight has been gained over the years from numerical simulations. Established methods like the Immersed Boundary method or the Arbitrary Lagrangian-Eulerian (ALE) method [1–6] have been successfully applied to a wide range of applications, including for example: biomechanics, mechanical engineering, aero-elasticity, and aero-acoustics. Nevertheless, there are yet a good number of engineering problems, where most of the established methods fail or come to a constraint. Problems are caused, for example, by structural deformations or contact problems, stiff couplings, extreme parameters or extreme computational complexity. In the last few years, a number of novel methods and approaches have been developed to tackle such problems, many of which are still the subject of ongoing research.

An area of research on its own is the development of efficient solvers for the underlying linear systems of equations. The high intricacy of authentic real world applications calls for algorithms that include adaptivity in time and space, model reduction, as well as parallelization. In the case of strong couplings, the coupled system of equations can be badly conditioned, such that the design of efficient solvers is a challenge.

In recent times, solving FSI problems with the monolithic approach by FEM has become an attractive numerical methodology to perform complex fluid flow simulations with elastic structural deformation. The FSI simulation allows one to study the convective flow, pressure fields, and deformation by solving the nonlinear multi-physics problem. A number of previous studies can be found in the literature, in which the reliability and accuracy of the FSI models for simulating different benchmark problems has been extensively studied. R. Rannacher et al. [1–3] performed FSI problems in ALE, fully Eulerian and fully Lagrangian coordinates for the FSI-1, 2 and 3 benchmark test cases. T. Richter et al. [1–4] developing the numerical technique and goal oriented error estimation for FSI simulation in fully Eulerian coordinates. T. Wick et al. [3, 6] performed studies on FSI where they introduce goal-oriented mesh adaptivity and mesh movement techniques. T. Wick has considered the FSI problems with the monolithic approach to do

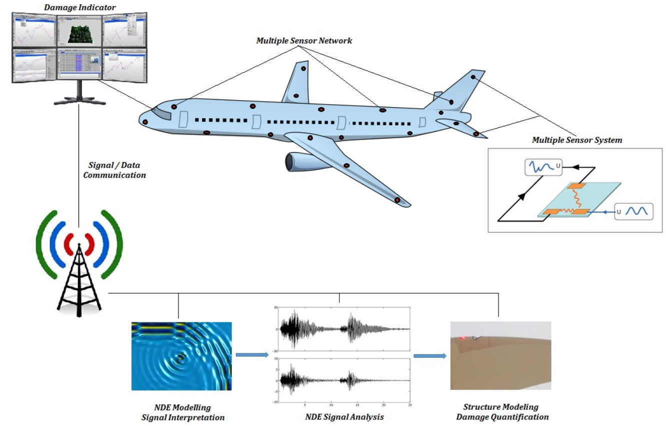


FIGURE 1: A typical on-line structural health monitoring (SHM) system.

a heart valve simulation. B.S.M. Ebna Hai et al. [8–10, 17] has developed a new FSI model, which is referred to as eXtended Fluid-Structure Interaction (eXFSI). The eXFSI is used to design on-line structural health monitoring system (SHM) (see Figure-1). The eXFSI is a strongly coupled version of a typical FSI problem with a wave propagation problem coupled on the fluid-structure interface, in which the wave propagation problem is posed on the moving mesh which is automatically adopted from the FSI problem at each time step.

This research focuses on the recently developed mathematical model of a new FSI problem, which is referred to as eXtended Fluid-Structure Interaction (eXFSI) problem in the ALE framework. This model is utilized to design an on-line structural health monitoring (SHM) system in order to determine the coupled acoustic and elastic wave propagation in moving domains and optimum locations for SHM sensors. The eXFSI is a one-direction coupling of typical FSI with a wave propagation problem on the fluid-structure interface, where wave propagation problems are solved on the moving mesh which is automatically adopted from the typical FSI problem at each time step. The ALE approach provides a simple procedure to couple solid deformations with fluid flows by a monolithic solution algorithm. In such a setting, the fluid problems are transformed to a reference configuration by the ALE mapping. The main goal of this research is to apply high order finite element methods (FEM) for the elastic wave propagation in a composite material under FSI effect. We present the state-of-the-art in computational methods and techniques for the elastic wave propagation with FSI. The traditional FSI physics are solved as a monolithic system which is then used by the elastic wave propagation. By solving the FSI in a monolithic way, we avoid the difficulties with the added-mass effect [18] that can arise in iterative partitioned solution approaches such as Dirichlet-Neumann coupling [19].

While we choose to solve the monolithic FSI problem and then pass the solution to the WpFSI system, the eXFSI approach does not prohibit us from using partitioned methods to solve the FSI system.

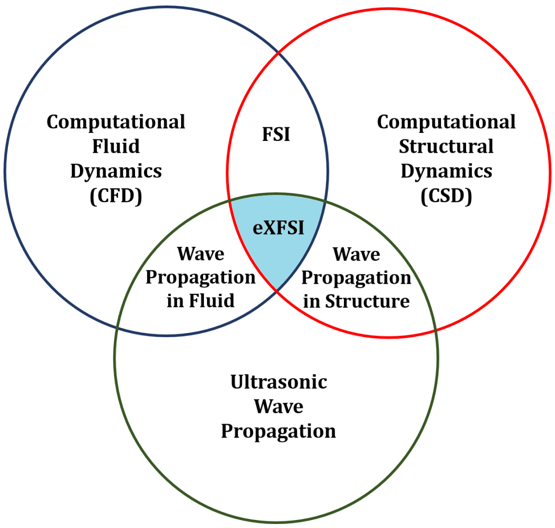


FIGURE 2: The eXtended Fluid-Structure Interaction (eXFSI) problem.

The first part of this project consists of the mathematical modelling of ultrasonic wave propagation in the deformable elastic structure (see Figure-2) at each time step with fluid flow, as well as error estimation and adaptive meshing. Here, the choice of appropriate fluid mesh movement is very important and challenging in FSI problem in the "arbitrary Lagrangian-Eulerian" (ALE) framework. We emphasize that the ALE framework is a standard framework for solving fluid-structure interaction. The crucial issue in this framework is the construction of the fluid mesh motion. In this study, we use the biharmonic operator (in a mixed formulation) for the mesh motion. It has the advantage of enabling large deformations of the structure, but has increased computational cost. These ingredients lead to a solvable semi-linear form of the coupled setting on the continuous level. We will also consider the investigation of different time stepping scheme formulations for an elastic wave equation with a nonlinear fluid-structure interaction problem coupling the incompressible Navier-Stokes equations with a hyperelastic solid based on the Arbitrary Lagrangian-Eulerian (ALE) framework.

In this part of this project paper, we restrict ourselves to a numerical study of the mathematical modelling for different test cases. The main target of this research is to study numerical analysis of finite element approximation (FEM) to the monolithic wave propagation problems coupled on the fluid-structure interface (WpFSI) and eXtended Fluid-Structure Interaction (eXFSI) problems. Numerical simulation of the solution to eXFSI problems, where the dynamics of fluid flows dominates, poses a formidable challenge to even the most advanced numerical techniques. Currently, eXFSI simulations are at the forefront of ongoing work in computational fluid dynamics (CFD). Typically, the fluid and the structure equations are modelled in different coordinate systems, making a common solution approach challenging. Fluid flows are modelled in an Eulerian framework, whereas the structure is treated in Lagrangian coordinates. This work focuses on a monolithic approach in which all equations are solved simultaneously. Employing the strongly coupled approach, the interface conditions, the continuity of velocity, the normal stresses and the boundary conditions are automatically satisfied at each time step. Here, variational formulation of a coupled monolithic problem is an inevitable prerequisite for gradient based optimization methods with rigorous goal oriented error estimation and mesh adaptation. However, this coupling leads to the additional nonlinear behavior of the overall system.

Afterwards, for the temporal discretization of the mathematical model we use finite difference method and apply the one step-θ scheme, the particular cases of which are Crank-Nicolson, shifted Crank-Nicolson, implicit Euler, and the fractional-step-θ schemes [2–7]. Spatial discretization is realized by a standard Galerkin finite element approach by using the Q_2^c element for structural discretization, Q_2^c/P_1^{dc} for fluid flows, elastic and acoustic wave propagation on the fluid-structure interface [1–6, 11–13]. The solution of the nonlinear discretized system can be achieved with a Newton-like method, which provides robust and rapid convergence. We will analyze variational space-time methods for the wave equation [8–10]. The ALE approach provides a simple, but powerful, procedure to couple fluid flows with solid deformations by a monolithic solution algorithm. In such a setting, the fluid equations are transformed to a fixed reference configuration via the ALE mapping. We use a direct solver to solve the linear systems, although a preconditioner could be used to make Krylov subspace solvers practical. Nevertheless, we extract the block-structure to expose the inner sub-structure of the linear system matrix.

For numerical simulation, we consider three examples that are taken from [1–6]. Our main target is to study the impact of FSI on modelling and simulation of wave propagation in composite materials in the ALE framework by using a monolithic approach. The FSI and the wave propagation problems are solved by using the in-house `C++` code based on the differential equations and optimization libraries DOPELIB [14], which is a flexible modularized high-level algorithms toolbox based on the finite element library `deal.II` [15]. The state of the art software can be used to solve stationary and non-stationary

PDE problems, optimal control problems constrained by PDEs, and also can do the Dual-Weighted-Residual approach for goal-oriented error estimation. This method is used for the mesh adaption during the computation.

MATHEMATICAL MODELS

Notations and Functional spaces

In this project, we aim to solve second order hyperbolic equations coupled with a nonstationary fluid-structure interaction problem in arbitrary Lagrangian-Eulerian coordinates, where the mesh motion model is based on the solution of a biharmonic equation. To formulate the system of elasticity, let us assume that $\Omega \subset \mathbb{R}^d, d = 2,3$, is a bounded and convex domain at time $t = 0$ with Lipschitz boundary $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$, where $\partial\Omega_D$, $\partial\Omega_N$ denote Dirichlet and Neumann boundaries, respectively. The computational domain $\Omega := \Omega(t)$ is split into two time-dependent subdomains $\Omega_f(t)$ (for a homogeneous, Newtonian and incompressible fluid) and $\Omega_s(t)$ (for a compressible hyperelastic structure), and the sensors of SHM systems would be optimally located in $\Omega_s(t)$. Both domains depend on time and their common boundary $\partial\Omega_i(t) = \partial\Omega_f(t) \cap \partial\Omega_s(t)$, where

$$\partial\Omega_f(t) = \partial\Omega_{fD}(t) \cup \partial\Omega_{fN}(t) \cup \partial\Omega_i(t)$$
$$\partial\Omega_s(t) = \partial\Omega_{sD}(t) \cup \partial\Omega_{sN}(t) \cup \partial\Omega_i(t).$$

We begin this paper with some basic notation. For d a positive integer representing dimension, let $X \subset R^d$ denote an arbitrary bounded Lipschitz domain with boundary ∂X. As usual, let $L^2(X)$ denote the space of square integrable functions on X and define $\mathbf{L}^2(X) = (L^2(X))^d$. We will also utilize the standard Lebesgue space $L^p(X)$ where $1 \leqslant p \leqslant \infty$ that consists of measurable functions u, which are Lebesgue-integrable to the p-th power and their vectorial counterpart $\mathbf{L}^p(X)$. The set $L^p(X)$ forms a Banach space with the norm $|u|_{L^p(X)}$. The Sobolev space $W^{m,p}(X)$, $m \in N$, $1 \leqslant p \leqslant \infty$ is the space of functions in $L^p(X)$ that have distributional derivatives of order up to m, which belong to $L^p(X)$. For $p = 2, H^m(X) := W^{m,2}(X)$ is a Hilbert space equipped with the norm $\|.\|_{H^m(X)}$ [1–3]. Finally, the subspace $W^{m,p}(X)$ of functions indicate with zero trace on ∂X are denoted by $W_0^{m,p}(X)$. Specifically, Hilbert space with zero trace on ∂X is defined as $H_0^1(X) = \{u \in H^1(X) : u|_{\Gamma_D} = 0,\ where\ \Gamma_D = \partial X_D\}$, where ∂X_D is that part of the boundary ∂X at which Dirichlet boundary conditions are imposed. So for given set X, let us consider the Lebesque space $L_X := L^2(X)$, and $L_X^0 := L^2(X)/\mathbb{R}$. The functions in L_X with first-order distributional derivatives in L_X make up the Sobolev space $H^1(X)$. Furthermore, we can use the function spaces $V_X := H^1(X)^d$, $V_X^0 := H_0^1(X)^d$, and for time-dependent functions

$$\mathscr{L}_X := L^2[0,T;L_X], \qquad \mathscr{V}_X := L^2[0,T;V_X] \cap H^1[0,T;V_X^*],$$
$$\mathscr{L}_X^0 := L^2[0,T;L_X^0], \qquad \mathscr{V}_X^0 := L^2[0,T;V_X^0] \cap H^1[0,T;V_X^*],$$

where, V_X^* is the dual of V_X.

Governing Equations

eXtended Fluid-structure interactions (eXFSI) are defined as the ultrasonic wave propagation on the interaction of a deformable or movable structure with surrounding or an internal fluid flow, which describe the coupled dynamics of fluid mechanics and structural mechanics.

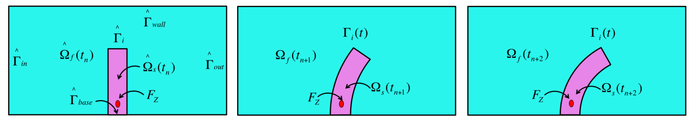


FIGURE 3: Typical eXFSI computational domain.

To formulate the typical eXFSI problem (see Figure-3) we consider a linear elastic rheology for the heterogeneous solid, while the fluid is assumed to be of homogeneous density. We restrict ourselves to isotropic materials, although the method can also handle anisotropic materials accurately. Let us consider the following definitions: the reference domains are denoted by $\hat{\Omega}_f$ and $\hat{\Omega}_s$, respectively, with their common interface $\partial\hat{\Omega}_i$ at time, $t = 0$. The fluid external boundaries $\partial\Omega_{f_{in}}$ and $\partial\Omega_{f_{out}}$ are supposed to be fixed. The corresponding outward normal vectors to the fluid and solid boundaries are denoted by n_f and n_s, respectively. The variational ALE formulation of the fluid part is transformed from its Eulerian description into an arbitrary Lagrangian-Eulerian framework and stated on the (arbitrary) reference domain $\hat{\Omega}_f$, while the structure part is formulated in Lagrangian coordinates on the domain $\hat{\Omega}_s$, where $\hat{\Omega} = \hat{\Omega}_f \cup \partial\hat{\Omega}_i \cup \hat{\Omega}_s$. Moreover, here we solve the Laplace equation for the definition of the ALE mapping. Here, the continuity of velocity requires $\hat{v}_f = \hat{v}_s$ and $\hat{u}_f = \hat{u}_s$ across the common fluid-structure interface on $\partial\hat{\Omega}_i(t) = \partial\hat{\Omega}_f(t) \cap \partial\hat{\Omega}_s(t)$. The ALE mapping is denoted by $\hat{T}$ and transforms the reference configuration $\hat{\Omega}_f$ of the fluid to the physical domain $\Omega_f(t)$. Furthermore, any function $\hat{q} \in \hat{\Omega}$ can be defined by:

$$q(x) = \hat{q}(\hat{x}) \qquad \text{with} \quad x \quad = \hat{T}(\hat{x},t). \tag{1}$$

In this FSI problem, the principal unknowns are the fluid domain displacement (mesh motion) $\hat{u}_f : \hat{\Omega}_f \times \mathbb{R}^+ \mapsto \mathbb{R}^3$, the fluid velocity $\hat{v}_f : \hat{\Omega}_f \times \mathbb{R}^+ \mapsto \mathbb{R}^3$, the fluid pressure $\hat{p} : \hat{\Omega}_f \times \mathbb{R}^+ \mapsto \mathbb{R}$ and the structure displacement $\hat{u}_s : \hat{\Omega}_s \times \mathbb{R}^+ \mapsto \mathbb{R}^3$. Thus $\hat{u}_f = Ext(\hat{u}_s|_{\partial\Omega_i(t)})$, the ALE mapping is defined through the extension of the structural displacement (for large mesh deformations without remeshing) into the fluid domain and are determined by solving the following biharmonic equation

$$\begin{aligned} \Delta^2 \hat{u}_f &= 0 && \text{in } \hat{\Omega}_f \\ \hat{u}_f = \partial_n \hat{u}_f &= 0 && \text{on } (\partial\hat{\Omega}_{f_{in}} \cup \partial\hat{\Omega}_{f_{out}}) \\ \hat{u}_f &= \hat{u}_s && \text{on } \partial\hat{\Omega}_i \\ \partial_n \hat{u}_f &= \partial_n \hat{u}_s && \text{on } \partial\hat{\Omega}_i. \end{aligned} \tag{2}$$

The ALE map is constructed by solving a mixed formulation of the biharmonic equation, where we introduce an auxiliary variable $\hat{w} = -\hat{\Delta}\hat{u}$ and obtain

$$\begin{aligned} \hat{w} &= -\hat{\Delta}\hat{u} && \text{in } \hat{\Omega} \\ -\hat{\Delta}\hat{w} &= 0 && \text{in } \hat{\Omega}_f \\ \hat{u}_f = \partial_n \hat{u}_f &= 0 && \text{on } (\partial\hat{\Omega}_{f_{in}} \cup \partial\hat{\Omega}_{f_{out}}) \\ \hat{u}_f &= \hat{u}_s && \text{on } \partial\hat{\Omega}_i \\ \partial_n \hat{u}_f &= \partial_n \hat{u}_s && \text{on } \partial\hat{\Omega}_i. \end{aligned} \tag{3}$$

The ALE approach belongs to interface-tracking methods in which the mesh is moved such that it fits in all time steps with the FSI-interface. However, this leads to a degeneration of the ALE map. Methods to circumvent such as degeneration as long as possible are re-meshing techniques or to use (as suggested here) a biharmonic mesh motion technique.

$$\hat{F} := I + \hat{\nabla}\hat{u}, \quad \hat{J} = \det(\hat{F}).$$

In the structure domain, $\hat{T}$ takes the place of the Lagrangian-Eulerian coordinate transformation, while in the fluid domain, $\hat{T}$ has no physical meaning but serves as ALE mapping. Let us define the density $\hat{\rho}$ and the Cauchy stress tensor $\hat{\sigma}$ for the whole domain by

$$\hat{\rho}(\hat{x}) = \begin{cases} \hat{\rho}_f(\hat{x}),\ \hat{x} \in \hat{\Omega}_f \\ \hat{\rho}_s(\hat{x}),\ \hat{x} \in \hat{\Omega}_s \cup \partial\hat{\Omega}_i \end{cases} \quad \hat{\sigma}(\hat{x}) = \begin{cases} \hat{\sigma}_f(\hat{x}),\ \hat{x} \in \hat{\Omega}_f \\ \hat{\sigma}_s(\hat{x}),\ \hat{x} \in \hat{\Omega}_s \cup \partial\hat{\Omega}_i \end{cases}$$

Problem 1 - The FSI problem in ALE framework:

This model concentrates on the coupling between the Navier-Stokes and elastodynamics equation for incompressible fluid and compressible elastic materials.

Find $\{\hat{v}, \hat{u}, \hat{w}, \hat{p}\} \in \{\hat{v}^D + \hat{\mathscr{V}}^0_{\hat{\Omega}}\} \times \{\hat{u}^D + \hat{\mathscr{V}}^0_{\hat{\Omega}}\} \times \hat{\mathscr{V}} \times \hat{\mathscr{L}}_{\hat{\Omega}}$, *such that* $\hat{v}(0) = \hat{v}^0$ *and* $\hat{u}(0) = \hat{u}^0$, *for almost all time steps* t:

$$\begin{aligned} &(\hat{\rho}\hat{J}(\hat{F}^{-1}(\hat{v} - \partial_t\hat{u}) \cdot \hat{\nabla})\hat{v}), \hat{\phi}^v)_{\hat{\Omega}_f} \\ &+(\hat{J}\hat{\rho}\partial_t\hat{v}, \hat{\phi}^v)_{\hat{\Omega}_f} - \langle \hat{g}, \hat{\phi}^v \rangle_{\partial\hat{\Omega}_N} + (\hat{\rho}\partial_t\hat{v}, \hat{\phi}^v)_{\hat{\Omega}_s} \\ &+(\hat{J}\hat{\sigma}\hat{F}^{-T}, \hat{\nabla}\hat{\phi}^v)_{\hat{\Omega}_f} + (\hat{J}\hat{\sigma}\hat{F}^{-T}, \hat{\nabla}\hat{\phi}^v)_{\hat{\Omega}_s} = 0 \quad \forall \hat{\phi}^v \in \hat{V}^0_{\hat{\Omega}}, \\ &(\hat{\alpha}_u\hat{w}, \hat{\phi}^w)_{\hat{\Omega}_f} + (\hat{\alpha}_u\hat{\nabla}\hat{u}, \hat{\nabla}\hat{\phi}^w)_{\hat{\Omega}_f} + (\hat{\alpha}_u\hat{\nabla}\hat{w}, \hat{\nabla}\hat{\phi}^w)_{\hat{\Omega}_s} = 0 \quad \forall \hat{\phi}^w \in \hat{V}_{\hat{\Omega}}, \\ &(\hat{\rho}(\partial_t\hat{u} - \hat{v}), \hat{\phi}^u)_{\hat{\Omega}_s} + (\hat{\alpha}_u\hat{\nabla}\hat{w}, \hat{\nabla}\hat{\phi}^u)_{\hat{\Omega}_f} = 0 \quad \forall \hat{\phi}^u \in \hat{V}^0_{\hat{\Omega}}, \\ &(\hat{div}\,(\hat{J}\hat{F}^{-1}\hat{v}), \hat{\phi}^p)_{\hat{\Omega}_f} + (\hat{p}, \hat{\phi}^p)_{\hat{\Omega}_s} = 0 \quad \forall \hat{\phi}^p \in \hat{L}_{\hat{\Omega}}, \end{aligned} \tag{4}$$

with the densities $\hat{\rho}_f$ *and* $\hat{\rho}_s$, *the viscosity* ν_f, *the Lamé parameters* μ_s, λ_s *and the deformation gradient* $\hat{F}$, *and its determinant* $\hat{J}$. *The stress tensors for the fluid and structure are implemented by*

$$\begin{aligned} \hat{\sigma}_f &= -\hat{p}I + \hat{\rho}_f\nu_f(\hat{\nabla}\hat{v}\hat{F}^{-1} + \hat{F}^{-T}\hat{\nabla}\hat{v}^T), \\ \hat{\sigma}_s &= \hat{F}(2\mu_s\hat{E} + \lambda_s tr\hat{E}I) \end{aligned} \tag{5}$$

with $\hat{E} = \frac{1}{2}(\hat{F}^T\hat{F} - I)$,

where we notice that this problem is driven by a Dirichlet inflow condition. In this formulation, for momentum equations, integration by parts in both subdomains yields the boundary term on $\hat{\Gamma}_i$ as:

$$(\hat{n}_f.(\hat{J}\hat{\sigma}_s\hat{F}^{-T}), \hat{\phi}^v)_{\hat{\Gamma}_i} + (\hat{n}_s.(\hat{J}\hat{\sigma}_f\hat{F}^{-T}), \hat{\phi}^v)_{\hat{\Gamma}_i} = 0. \tag{6}$$

By omitting this boundary integral jump over $\partial\hat{\Omega}_i$ the weak continuity of the normal stresses becomes an implicit condition of the fluid-structure interaction problem.

Problem 2 - The wave propagation problem (WpFSI) in ALE framework:

In this model we concentrate on the coupled problems of the acoustic wave equation and elastic wave equations for incompressible fluid and compressible elastic materials, respectively. The fluid is governed by the conservation of mass and momentum. We consider a linear elastic rheology for the heterogeneous solid, while the fluid is assumed to be of homogeneous density. We restrict ourselves to isotropic materials, although the method can also handle anisotropic materials accurately.

Find $\{\hat{v}_w, \hat{u}_w\} \in \{\hat{v}^D_w + \hat{\mathscr{V}}^0_{\hat{\Omega}}\} \times \{\hat{u}^D_w + \hat{\mathscr{V}}^0_{\hat{\Omega}}\}$, *such that* $\hat{v}_w(0) = \hat{v}^0_w$ *and* $\hat{u}_w(0) = \hat{u}^0_w$, *for almost all* $t \in I$ *it holds that:*

$$\begin{aligned} &(\hat{J}\hat{\rho}\partial_t\hat{v}_w, \hat{\phi}^{v_w})_{\hat{\Omega}_f} - (\hat{J}\hat{\rho}(\hat{F}^{-1}\partial_t\hat{u} \cdot \hat{\nabla})\hat{v}_w, \hat{\phi}^{v_w})_{\hat{\Omega}_f} \\ &+(c^2\hat{J}\hat{\rho}(\hat{F}^{-1}\hat{\nabla} \cdot \hat{u}_w)\hat{F}^{-T}, \hat{\nabla}\hat{\phi}^{v_w})_{\hat{\Omega}_f} - \langle \hat{g}_w, \hat{\phi}^{v_w} \rangle_{\hat{\Gamma}_N} \\ &+(\hat{J}\hat{\rho}\partial_t\hat{v}_w, \hat{\phi}^{v_w})_{\hat{\Omega}_s} - (\hat{J}\hat{\rho}(\hat{F}^{-1}\partial_t\hat{u} \cdot \hat{\nabla})\hat{v}_w, \hat{\phi}^{v_w})_{\hat{\Omega}_s} \\ &+(\hat{J}\hat{\rho}\sigma\hat{F}^{-T}, \hat{\nabla}\hat{\phi}^{v_w})_{\hat{\Omega}_s} - (\hat{J}\hat{f}_s, \hat{\phi}^{v_w})_{\hat{\Omega}_s} = 0 \quad \forall \hat{\phi}^{v_w} \in \hat{V}^0_{\hat{\Omega}}, \\ &(\hat{J}\hat{\rho}(\partial_t\hat{u}_w - (\hat{F}^{-1}\partial_t\hat{u} \cdot \hat{\nabla})\hat{u}_w - \hat{v}_w), \hat{\phi}^{u_w})_{\hat{\Omega}_s} \\ &+(\hat{J}\hat{\rho}(\partial_t\hat{u}_w - (\hat{F}^{-1}\partial_t\hat{u} \cdot \hat{\nabla})\hat{u}_w - \hat{v}_w), \hat{\phi}^{u_w})_{\hat{\Omega}_f} = 0 \quad \forall \hat{\phi}^{u_w} \in \hat{V}^0_{\hat{\Omega}}, \end{aligned} \tag{7}$$

with a disc-shaped force, $\hat{f}_s(x,t)$ *with the radius* r_{f_s} *=7.5 mm*

$$\hat{f}_s(x,t) = \begin{bmatrix} r(t)cos(\varphi(x)) \\ r(t)sin(\varphi(x)) \end{bmatrix}, \quad x \in \hat{\Omega}_{f_s},$$

with

$$\hat{\Omega}_{f_s} = \{\mathbf{x} \in \Omega \mid x_1^2 + x_2^2 \leq r_{f_s}^2\}$$

and the time-depended radius,

$$r(t) := (\sin(t) - \sin(t - \frac{n_s}{f_c})) \cdot (1 - \cos(2\pi\frac{f_c}{n_s}t)) \cdot \frac{1}{4}\cos(2\pi f_c t)$$

where $f_c = 100\ kHz$, $n_s = 5$ *and*

$$\varphi(x) = \begin{cases} \arctan\left(\frac{x_2}{x_1}\right), & x_1 > 0 \text{ and } x_2 \geq 0, \\ \frac{\pi}{2}, & x_1 = 0 \text{ and } x_2 > 0, \\ \pi + \arctan\left(\frac{x_2}{x_1}\right), & x_1 < 0, \\ \frac{3\pi}{2}, & x_1 = 0 \text{ and } x_2 < 0, \\ 2\pi + \arctan\left(\frac{x_2}{x_1}\right), & x_1 > 0 \text{ and } x_2 < 0. \end{cases}$$

However, we want to impose Dirichlet-boundary values to guarantee continuity $\hat{u}_{w_f} = \hat{u}_{w_s}$ on $\partial\hat{\Omega}_i$. This way, the equations satisfy the condition of continuity of displacements, and are coupled over the two domains. In the discretization and solution schemes we have to carefully treat the additional boundary terms which are needed to satisfy the continuity of normal stresses. In the coupled formulation, for momentum equations, integration by parts in both subdomains yields the boundary term on $\partial\hat{\Omega}_i$ as:

$$\begin{aligned} &(\hat{n}_s.(\hat{J}\hat{\sigma}_s\hat{F}^{-T}), \hat{\phi}^{v_w})_{\partial\hat{\Omega}_i} \\ &+(\hat{n}_f.(c^2\hat{J}\hat{\rho}_f(\hat{\nabla}\hat{v}_{w_f}\hat{F}^{-1})\hat{F}^{-T}), \hat{\phi}^{v_w})_{\partial\hat{\Omega}_i} = 0 \end{aligned} \tag{8}$$

By omitting this boundary integral jump over $\partial\hat{\Omega}_i$ the weak continuity of the normal stresses becomes an implicit condition of the eXtended fluid-structure interaction (eXFSI) problem.

Problem 3 - eXtended Fluid-Structure Interaction (eXFSI):

Find $\{\hat{v},\hat{u},\hat{w},\hat{p},\hat{v}_w,\hat{u}_w\} \in \{\hat{v}^D + \hat{\mathcal{V}}^0_{\hat{\Omega}}\} \times \{\hat{u}^D + \hat{\mathcal{V}}^0_{\hat{\Omega}}\} \times \hat{\mathcal{V}} \times \hat{\mathcal{L}}_{\hat{\Omega}} \times \{\hat{v}^D_w + \hat{\mathcal{V}}^0_{\hat{\Omega}}\} \times \{\hat{u}^D_{w_f} + \hat{\mathcal{V}}^0_{\hat{\Omega}}\}$, *such that* $\hat{v}(0) = \hat{v}^0$, $\hat{u}(0) = \hat{u}^0$, $\hat{v}_w(0) = \hat{v}^0_w$ *and* $\hat{u}_w(0) = \hat{u}^0_w$, *for almost all* $t \in I$ *it holds that:*

$$\begin{aligned}
&(\hat{J}\hat{\rho}\partial_t\hat{v}, \hat{\phi}^v)_{\hat{\Omega}_f} + (\hat{\rho}_f\hat{J}(\hat{F}^{-1}(\hat{v} - \partial_t\hat{u})\cdot\hat{\nabla})\hat{v}, \hat{\phi}^v)_{\hat{\Omega}_f} \\
&\quad + (\hat{J}\hat{\sigma}\hat{F}^{-T}, \hat{\nabla}\hat{\phi}^v)_{\hat{\Omega}_f} + (\hat{\rho}\partial_t\hat{v}, \hat{\phi}^v)_{\hat{\Omega}_s} \\
&\quad + (\hat{J}\hat{\sigma}\hat{F}^{-T}, \hat{\nabla}\hat{\phi}^v)_{\hat{\Omega}_s} - \langle\hat{g}, \hat{\phi}^v\rangle_{\hat{\Gamma}_N} = 0 \quad \forall\hat{\phi}^v \in \hat{V}^0_{\hat{\Omega}}, \\
&(\hat{J}\hat{\rho}\partial_t\hat{v}_w, \hat{\phi}^{v_w})_{\hat{\Omega}_f} - (\hat{J}\hat{\rho}(\hat{F}^{-1}\partial_t\hat{u}\cdot\hat{\nabla})v_w, \hat{\phi}^{v_w})_{\hat{\Omega}_f} \\
&\quad + (c^2\hat{J}\hat{\rho}(\hat{F}^{-1}\hat{\nabla}\cdot\hat{v}_w)\hat{F}^{-T}, \hat{\nabla}\hat{\phi}^{v_w})_{\hat{\Omega}_f} \\
&\quad + (\hat{J}\hat{\rho}_s\partial_t\hat{v}_w, \hat{\phi}^{v_w})_{\hat{\Omega}_s} - \langle\hat{g}_w, \hat{\phi}^{v_w}\rangle_{\hat{\Gamma}_N} \\
&\quad + (\hat{J}\hat{\rho}\sigma_s\hat{F}^{-T}, \hat{\nabla}\hat{\phi}^{v_w})_{\hat{\Omega}_s} - (\hat{J}\hat{f}_s, \hat{\phi}^{v_w})_{\hat{\Omega}_s} \\
&\quad - (\hat{J}\hat{\rho}(\hat{F}^{-1}\partial_t\hat{u}\cdot\hat{\nabla})v_w, \hat{\phi}^{v_w})_{\hat{\Omega}_s} = 0 \quad \forall\hat{\phi}^{v_w} \in \hat{V}^0_{\hat{\Omega}}, \\
&(\hat{\alpha}_u\hat{w}, \hat{\phi}^w)_{\hat{\Omega}_f} + (\hat{\alpha}_u\hat{\nabla}\hat{u}, \hat{\nabla}\hat{\phi}^w)_{\hat{\Omega}_f} + (\hat{\alpha}_u\hat{\nabla}\hat{w}, \hat{\nabla}\hat{\phi}^w)_{\hat{\Omega}_s} = 0 \quad \forall\hat{\phi}^w \in \hat{V}_{\hat{\Omega}}, \\
&\hat{\rho}(\partial_t\hat{u} - \hat{v}, \hat{\phi}^u)_{\hat{\Omega}_s} + (\hat{\alpha}_u\hat{\nabla}\hat{w}, \hat{\nabla}\hat{\phi}^u)_{\hat{\Omega}_f} = 0 \quad \forall\hat{\phi}^u \in \hat{V}^0_{\hat{\Omega}}, \\
&(\hat{div}(\hat{J}\hat{F}^{-1}\hat{v}), \hat{\phi}^p)_{\hat{\Omega}_f} + (\hat{p}, \hat{\phi}^p)_{\hat{\Omega}_s} = 0 \quad \forall\hat{\phi}^p \in \hat{L}_{\hat{\Omega}}, \\
&(\hat{J}\hat{\rho}(\partial_t\hat{u}_w - (\hat{F}^{-1}\partial_t\hat{u}\cdot\hat{\nabla})\hat{u}_w - \hat{v}_w), \hat{\phi}^{u_w})_{\hat{\Omega}_s} \\
&\quad + (\hat{J}\hat{\rho}(\partial_t\hat{u}_w - (\hat{F}^{-1}\partial_t\hat{u}\cdot\hat{\nabla})\hat{u}_w - \hat{v}_w), \hat{\phi}^{u_w})_{\hat{\Omega}_f} = 0 \quad \forall\hat{\phi}^{u_w} \in \hat{V}^0_{\hat{\Omega}},
\end{aligned} \tag{9}$$

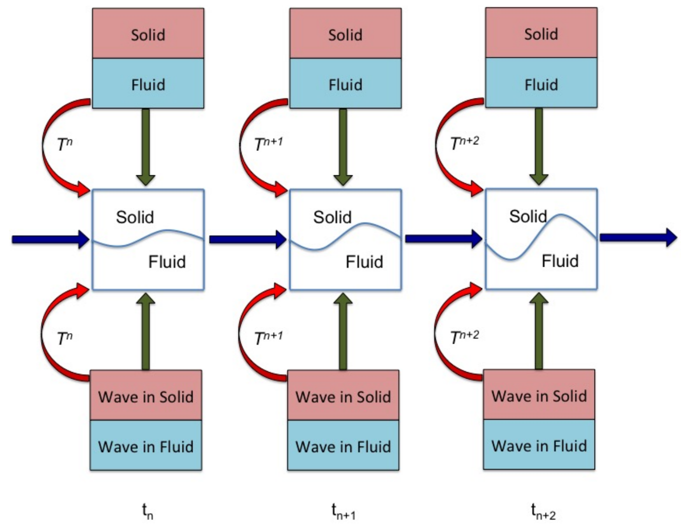


FIGURE 4: Arbitrary transformation approach: Typical eXFSI problem in the ALE framework. [9]

In this case, we will use a monolithic approach in which all equations are solved simultaneously (see Figure-4). Employing a strongly coupled approach, the interface conditions, the continuity of velocity and the normal stresses, are automatically achieved at each time step. Here, a coupled monolithic variational formulation is an inevitable prerequisite for gradient based optimization methods, for rigorous goal oriented error estimation and mesh adaptation. However, this coupling leads to additional nonlinear behaviour of the overall system.

GALERKIN FORMULATION

In this section, we briefly comment on temporal and spatial discretization and explain the solution process of the nonlinear problem. Finally, we give a short account on the form of the linear equation system, which must be solved in each Newton step. In fact, because that the fluid equations have been transformed on a fixed reference configuration, the whole problem is solved therein (instead of moving the fluid mesh explicitly).

Temporal and spatial discretization

For arguments $\hat{U} = \{\hat{v},\hat{u},\hat{w},\hat{p},\hat{v}_w,\hat{u}_w\}$ and $\hat{\phi} = \{\hat{\phi}^v,\hat{\phi}^u,\hat{\phi}^w,\hat{\phi}^p,\hat{\phi}^{v_w},\hat{\phi}^{u_w}\} \in \hat{\mathcal{X}}^0$, where

$$\begin{aligned} \hat{\mathcal{X}}^0 := \{\hat{\phi} \in \{\hat{v}^D + \hat{\mathcal{V}}^0_{\hat{\Omega}}\} \times \{\hat{u}^D + \hat{\mathcal{V}}^0_{\hat{\Omega}}\} \times \hat{\mathcal{V}} \times \hat{\mathcal{L}}_{\hat{\Omega}} \\ \times \{\hat{v}^D_w + \hat{\mathcal{V}}^0_{\hat{\Omega}}\} \times \{\hat{u}^D_w + \hat{\mathcal{V}}^0_{\hat{\Omega}}\}, \\ \hat{\phi}^v(0) = \hat{\phi}^u(0) = \hat{\phi}^{v_w}(0) = \hat{\phi}^{u_w}(0) = 0\} \end{aligned}$$

we introduce the space-time semilinear form as:

Find $\hat{U} \in \hat{U}^D + \hat{\mathscr{X}}^0$, such that

$$\int_0^T \hat{A}(\hat{U})(\hat{\phi})dt = 0 \quad \forall \hat{\phi} \in \hat{\mathscr{X}}^0, \tag{10}$$

where $\hat{U}^D$ is an appropriate extension of the Dirichlet boundary and initial data.

The time integral is defined in an abstract sense such that the equation holds for almost all time steps. The semi-linear form $\hat{A}(\hat{U})(\hat{\phi})$ is defined by

$$\begin{aligned}
\hat{A}(\hat{U})(\hat{\phi}) &= (\hat{J}\hat{\rho}_f \partial_t \hat{v}, \hat{\phi}^v)_{\hat{\Omega}_f} + (\hat{\rho}_f \hat{J}(\hat{F}^{-1}(\hat{v} - \partial_t \hat{u}) \cdot \hat{\nabla})\hat{v}, \hat{\phi}^v)_{\hat{\Omega}_f} \\
&+ (\hat{J}\hat{\sigma}_f \hat{F}^{-T}, \hat{\nabla}\hat{\phi}^v)_{\hat{\Omega}_f} + (\hat{\rho}_s \partial_t \hat{v}, \hat{\phi}^v)_{\hat{\Omega}_s} \\
&+ (\hat{J}\hat{\sigma}_s \hat{F}^{-T}, \hat{\nabla}\hat{\phi}^v)_{\hat{\Omega}_s} - \langle \hat{g}, \hat{\phi}^v \rangle_{\hat{\Gamma}_N} \\
&+ (\hat{J}\hat{\rho}_f \partial_t \hat{v}_w, \hat{\phi}^{v_w})_{\hat{\Omega}_f} - (\hat{J}\hat{\rho}_f (\hat{F}^{-1}\partial_t \hat{u} \cdot \hat{\nabla}) v_w, \hat{\phi}^{v_w})_{\hat{\Omega}_f} \\
&+ (c^2 \hat{J}\hat{\rho}_f (\hat{F}^{-1}\hat{\nabla} \cdot \hat{v}_w)\hat{F}^{-T}, \hat{\nabla}\hat{\phi}^{v_w})_{\hat{\Omega}_f} \\
&+ (\hat{J}\hat{\rho}_s \partial_t \hat{v}_w, \hat{\phi}^{v_w})_{\hat{\Omega}_s} - \langle \hat{g}_w, \hat{\phi}^{v_w} \rangle_{\hat{\Gamma}_N} \\
&+ (\hat{J}\hat{\rho}_s \sigma_s \hat{F}^{-T}, \hat{\nabla}\hat{\phi}^{v_w})_{\hat{\Omega}_s} - (\hat{J}\hat{f}_s, \hat{\phi}^{v_w})_{\hat{\Omega}_s} \\
&- (\hat{J}\hat{\rho}_s (\hat{F}^{-1}\partial_t \hat{u} \cdot \hat{\nabla}) v_w, \hat{\phi}^{v_w})_{\hat{\Omega}_s} \\
&+ (\hat{\alpha}_u \hat{w}, \hat{\phi}^w)_{\hat{\Omega}_f} + (\hat{\alpha}_u \hat{\nabla}\hat{u}, \hat{\nabla}\hat{\phi}^w)_{\hat{\Omega}_f} + (\hat{\alpha}_u \hat{\nabla}\hat{w}, \hat{\nabla}\hat{\phi}^w)_{\hat{\Omega}_s} \\
&+ \hat{\rho}_s (\partial_t \hat{u} - \hat{v}, \hat{\phi}^u)_{\hat{\Omega}_s} + (\hat{\alpha}_u \hat{\nabla}\hat{w}, \hat{\nabla}\hat{\phi}^u)_{\hat{\Omega}_f} \\
&+ (\hat{J}\hat{\rho}_s (\partial_t \hat{u}_w - (\hat{F}^{-1}\partial_t \hat{u} \cdot \hat{\nabla})\hat{u}_w - \hat{v}_w), \hat{\phi}^{u_w})_{\hat{\Omega}_s} \\
&+ (\widehat{\mathrm{div}}\,(\hat{J}\hat{F}^{-1}\hat{v}), \hat{\phi}^p)_{\hat{\Omega}_f} + (\hat{p}, \hat{\phi}^p)_{\hat{\Omega}_s} \\
&+ (\hat{J}\hat{\rho}_f (\partial_t \hat{u}_w - (\hat{F}^{-1}\partial_t \hat{u} \cdot \hat{\nabla})\hat{u}_w - \hat{v}_w), \hat{\phi}^{p_w})_{\hat{\Omega}_f}
\end{aligned} \tag{11}$$

Temporal discretization is based on finite differences and the one step-θ schemes. The derivation for our set of equations was made in [14]. Spatial discretization in the reference configuration $\hat{\Omega}$ is treated by a conforming Galerkin finite element scheme, leading to a finite dimensional subspace $\hat{\mathscr{X}}_h \subset \hat{\mathscr{X}}$. The discrete spaces are based on the Q_2^c/P_1^{dc} element for the fluid problem and wave propagation problem. The structure problem is discretized by the Q_2^c element.

Time and spatial discretization end at each single time step in a nonlinear quasi-stationary problem:

$$\hat{A}(\hat{U}_h^n)(\hat{\phi}) = \hat{F}(\hat{\phi}) \quad \forall \hat{\phi} \in \hat{\mathscr{X}}_h, \tag{12}$$

which is solved with a Newton-like method. Given an initial Newton guess $\hat{U}_h^{n,0}$, find for $j = 0, 1, 2, \ldots$ the update $\delta\hat{U}_h^n$ of the linear defect-correction problem

$$\begin{aligned}
\hat{A}'(\hat{U}_h^{n,j})(\delta\hat{U}_h^{n,j}, \hat{\phi}) &= -\hat{A}(\hat{U}_h^n)(\hat{\phi}) + \hat{F}(\hat{\phi}), \\
\hat{U}_h^{n,j+1} &= \hat{U}_h^{n,j} + \lambda\delta\hat{U}_h^n.
\end{aligned} \tag{13}$$

In this algorithm, $\lambda \in (0,1]$ is used as damping parameter for line search iterations. A crucial role for (highly) nonlinear problems includes the appropriate determination of λ. A simple strategy is to modify the update step in (9) as follows: For given $\lambda \in (0,1)$ determine the minimal $l^* \in \mathbb{N}$ via $l = 0, 1, \ldots, N_l$, such that

$$\begin{aligned}
R(\hat{U}_{h,l}^{n,j+1}) &< R(\hat{U}_{h,l}^n), \\
\hat{U}_{h,l}^{n,j+1} &= \hat{U}_h^{n,j} + \lambda^l \delta\hat{U}_h^n.
\end{aligned} \tag{14}$$

For the minimal l, we set

$$\hat{U}_h^{n,j+1} := \hat{U}_{h,l^*}^{n,j+1} \tag{15}$$

In this context, the nonlinear residual $R(\cdot)$ is defined as

$$R(\hat{U}_h^n) := \max_i \{\hat{A}(\hat{U}_h^n)(\hat{\phi}_i) - \hat{F}(\hat{\phi}_i)\} \quad \forall \hat{U}_h^n \in \hat{\mathscr{X}}_h, \tag{16}$$

where $\hat{\phi}_i$ denotes the nodal basis of $\hat{\mathscr{X}}_h$. The directional derivative $\hat{A}'(\hat{U})(\delta\hat{U}, \hat{\phi})$ that is utilized previously, is defined as Gateaux derivative. The application to a semi-linear form reads:

$$\begin{aligned}
\hat{A}'(\hat{U})(\delta\hat{U}, \hat{\phi}) &:= \lim_{\varepsilon \mapsto 0} \frac{1}{\varepsilon} \{\hat{A}(\hat{U} + \varepsilon\delta\hat{U})(\hat{\phi}) - \hat{A}(\hat{U})(\hat{\phi})\} \\
&= \frac{d}{d\varepsilon} \hat{A}_h(\hat{U} + \varepsilon\delta\hat{U})(\hat{\phi})\,|_{\varepsilon=0}
\end{aligned} \tag{17}$$

Now we discuss an example of one specific directional derivative that includes all of the necessary steps. We also refer the reader to other discussions on the exact derivation of the Jacobian. Let us consider a part of the fluid convection term in ALE coordinates. As part of a semi-linear form, it holds

$$\hat{A}_{conv}(\hat{U})(\hat{\phi}) = (\hat{\rho}_f \hat{J}(\hat{F}^{-1}\hat{v} \cdot \hat{\nabla})\hat{v}, \hat{\phi}^v)_{\hat{\Omega}_f} = (\hat{\rho}_f \hat{\nabla}\hat{v}\hat{J}\hat{F}^{-1}\hat{v}, \hat{\phi}^v)_{\hat{\Omega}_f} \tag{18}$$

In this case only for FSI problem, the directional derivative $\hat{A}_{conv}^{'}(\hat{U})(\hat{\phi})$ in the direction $\hat{\delta}\hat{U} = \{\delta\hat{v}, \delta\hat{u}, \delta\hat{p}\}$ is given by

$$\begin{aligned}
\hat{A}_{conv}(\hat{U})(\hat{\phi}) &= (\hat{\nabla}\delta\hat{v}\hat{J}\hat{F}^{-1}\hat{v}, \hat{\phi}^v) + (\hat{\nabla}\hat{v}(\hat{J}\hat{F}^{-1})^{'}(\delta\hat{u})\hat{v}, \hat{\phi}^v) \\
&+ (\hat{\nabla}\hat{v}\hat{J}\hat{F}^{-1}\delta\hat{v}, \hat{\phi}^v)
\end{aligned} \tag{19}$$

and for WpFSI problem, the directional derivative $\hat{A}_{conv}^{'}(\hat{U}_w)(\hat{\phi})$ in the direction $\hat{\delta}\hat{U}_w = \{\delta\hat{v}_w, \delta\hat{u}_w\}$ is given by

$$\begin{aligned}
\hat{A}_{conv}(\hat{U}_w)(\hat{\phi}) &= (\hat{\nabla}\delta\hat{v}_w\hat{J}\hat{F}^{-1}\hat{v}_w, \hat{\phi}^{v_w}) \\
&+ (\hat{\nabla}\hat{v}_w(\hat{J}\hat{F}^{-1})^{'}(\delta\hat{u}_w)\hat{v}_w, \hat{\phi}^{v_w})
\end{aligned} \tag{20}$$

In the paper, we restrict our considerations to a two-dimensional example because the inverse of the deformation matrix can easily be stated in explicit form. Explicitly, the deformation matrix reads:

$$\hat{F} = I + \hat{\nabla}\hat{u} = \begin{pmatrix} I + \hat{\partial}_1\hat{u}_1 & \partial_2\hat{u}_1 \\ \partial_1\hat{u}_2 & I + \hat{\partial}_2\hat{u}_2 \end{pmatrix}, \tag{21}$$

which brings us to

$$\hat{J}\hat{F}^{-1} = \begin{pmatrix} I + \hat{\partial}_2\hat{u}_2 & -\partial_2\hat{u}_1 \\ -\partial_1\hat{u}_2 & I + \hat{\partial}_2\hat{u}_2 \end{pmatrix}, \tag{22}$$

and its directional derivative in direction $\delta\hat{u} = (\delta\hat{u}_1, \delta\hat{u}_2)$:

$$(\hat{J}\hat{F}^{-1})'(\delta\hat{u}) = \begin{pmatrix} \hat{\partial}_2\delta\hat{u}_2 & -\partial_2\delta\hat{u}_1 \\ -\partial_1\delta\hat{u}_2 & \hat{\partial}_2\delta\hat{u}_2 \end{pmatrix}, \quad (23)$$

This expression is part of the second term shown in Equation-19 and 20. The remaining expressions for directional derivatives can be derived in an analogous way. With these ingredients the Jacobian is built explicitly to identity optimal Newton convergence. For more details on computation of the directional derivatives on the interface, we refer to [7].

SOLUTION TECHNIQUES OF THE LINEAR SYSTEM

After discretization and linearization, we solve in each Newton step a linearized problem, to achieve the solution of the (originally) nonlinear problem. In this work, we upgrade these ideas to the unsteady case and the biharmonic fluid mesh-motion model. Specifically, we are interested in the block-structure of the semi-linear form. To simplify the notation in this section, we omit the "hats" because it is clear that we are still working in the reference configuration $\hat{\Omega}$. The global linear equation system (eXFSI), which has the following form in each Newton step:

$$A\delta U = B \quad (24)$$

where A denotes a block matrix and for the coupled system can be described as-

$$A = \begin{bmatrix} A_{FSI} & 0 \\ 0 & A_{WpFSI} \end{bmatrix}. \quad (25)$$

Here,

$$A_{FSI} = \begin{bmatrix} \frac{M_{vv}}{k} + N_{vv} + L_{vv} + \frac{M_{vv}}{k} & E_{vu} + S_{vu} & 0 & B_{vp} \\ M_{uv} & \frac{M_{uu}}{k} & \sigma_w L_{uw} & 0 \\ 0 & \sigma_u L_{wu} & \sigma_u M_{ww} & 0 \\ B_{vp}^T & S_{pu} & 0 & M_{pp} \end{bmatrix}, \quad (26)$$

$$A_{WpFSI} = \begin{bmatrix} \frac{M_{vv}}{k} + L_{vv} + \frac{M_{vv}}{k} & E_{v_w u_w} + S_{v_w u_w} \\ M_{v_w u_w} & \frac{M_{u_w u_w}}{k} \end{bmatrix},$$

and

$$\delta U = \begin{bmatrix} \delta v \\ \delta u \\ \delta w \\ \delta p \\ \delta v_w \\ \delta u_w \end{bmatrix}, \quad B = \begin{bmatrix} b_v(t_{n+1}, t_n) \\ b_u(t_{n+1}, t_n) \\ b_w(t_{n+1}, t_n) \\ b_p(t_{n+1}, t_n) \\ b_{v_w}(t_{n+1}, t_n) \\ b_{u_w}(t_{n+1}, t_n) \end{bmatrix}. \quad (27)$$

Here A is a block-diagonal matrix, where the second part (A_{WpFSI}) of this block matrix depends on the time-dependent behavior of the first part (A_{FSI}) of the block matrix. In the first block, we have in the fluid domain the mass term M_{vv}, the convection term N_{vv}, the Laplacian L_{vv} and a mass matrix M_{vv} in the structural part. In the next block in the upper row, we find the elasticity of the structure and coupling terms in the fluid domain. In the last block, we have the gradient matrix B_{vp}. In the second row, in the first two blocks, we find again two mass terms M_{uv} and M_{uu} in the structural domain. Next, we detect a Laplacian L_{uw} due to the biharmonic fluid mesh motion. In the third row, we start in the second block with a Laplacian L_{wu} because of the mesh motion model. The same reason holds for the appearance of the next mass term M_{ww}. In the last row, we find the (negative) transposed divergence matrix B_{vp}^T. Then, we have again a coupling term S_{pu} and in the last entry of the system matrix, we have the pressure mass matrix in the structure domain. Except for the diffusion parameters of the mesh motion model and the time step k, we have omitted all other parameters. Specifically, the θ parameter of the time-stepping scheme is not shown. We recall, that all terms of the former time step are hidden in the right hand side vector B. However, we observe immediately that two terms are zero on the diagonal in the system matrix. This lack must be resolved if one were to solve this problem using an iterative solver in conjunction with a Block-Schur preconditioner.

The philosophy of the linear solver is to treat the coupled system in a monolithic manner as long as possible. Usually two different time discretization scheme and time step size are required to solve eXFSI problem as far as wave propagation is faster than material deformation. It's possible to solve eXFSI by using one type time discretization scheme and time step size, but the solution cost will be extremely high. The representation (26) shadows one important fact. Namely, that all terms are defined on two different domains Ω_f and Ω_s, which has consequences for an appropriate construction of a preconditioner. Consequently, we split the system into fluid variables and structure variables, which is not shown here to the convencience for the reader. To solve system 24, one could try to find a preconditioner such that

$$P^{-1}A\delta U = P^{-1}B \quad (28)$$

If we find appropriate entries for P^{-1} such that the condition number of $P^{-1}A$ is moderate, then the whole systems would converge in a few iterations. Specifically, one could consider using geometric multigrid method to solve the Laplace-dominated blocks in the preconditioner. In this linear system (eXFSI) we have two types of FSI problems (FSI and WpFSI), where FSI gives us material deformation (boundary condition for each time step) and based on the new boundary condition we solve wave propagation problem (WpFSI) in the system (see Figure-4). In this case, for each time step, first we solve the time dependent FSI problem and move forward the new mesh line that moves with the solid-fluid interface as the new boundary conditions for WpFSI problem and solve it in the reference domain. The resulting linear subproblems could then be solved by the "Generalized Minimal Residual (GMRES)" method with preconditioning by a geometric multigrid method with block-ILU smoothing.

As we have two types of FSI problems in this system, we are suggesting one apply the partitioning in the multigrid smoother. Brummelen et al. [16] showed that a partitioned smoother with exact solution of the two subproblems is a perfect smoother for a certain class of fluid-structure interactions. Perfect here implies, that the convergence rate will go to zero for increasing number of mesh-levels. Also from observation its easy to understand that the role of the multigrid smoother is not that of finding a global solution, but it's only intent is to locally smooth high frequency error contributions. Here, global coupling conditions must not be resolved. The application of various preconditioners to an iterative solver such as GMRES will be the subject of future research, and we used a direct solver with UMFPACK for our numerical studies in this work.

NUMERICAL EXAMPLE

The aim of this research is to explore and understand the behavior of engineering artifacts in extreme environments. To achieve the main ambition of this work, we split this research into two parts. The first part we consider is to determine the effect of fluid flow around a elastic beam and study the displacement of a control point $A(t)$ under incompressible fluid flow to understand the structural deformation with time. The second part of this research focuses on the FSI effect on an engineering model (aircraft wing, pipeline, or wind turbine) to identify the list of critical design points to implementing a Damage Identification Strategy (DIS). The main target is to study the elastic structural deformation under fluid flow in order to understand the variational boundary conditions which may avail to solve eXFSI problems for designing an on-line structural health monitoring system (SHM).

Configuration test model

The computational domain is designed based on the $2D$ WpFSI and eXFSI problem and it is determined by the following characteristics:

Test model for WpFSI problem

- The computational domain has the length $L = 1.0$ and height $H = 0.71$.
- The 'L' shape liquid domain is located in the central area of the solid domain.
- Three disc-shaped forces $f_{s_1}(x,t)$ at (0.5,0.6), $f_{s_2}(x,t)$ at (0.2,0.2) and $\hat{f}_{s_3}(x,t)$ at (0.8,0.2) with the radius r_{f_s} =0.0075 when t=0.

Test model for eXFSI problem

- The computational domain has the length $L = 2.5$ and height $H = 0.41$.
- The elastic beam has width $w = 0.2$ and height $h = 0.2$ and the bottom end is attached to the wall.

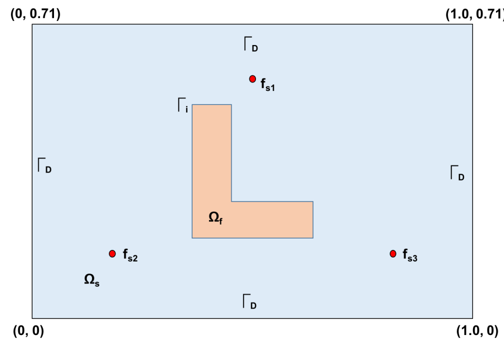


FIGURE 5: Configuration for $\dot{2}$D WpFSI test model with three disc-shaped forces (f_{s_1}, f_{s_2} and f_{s_3})

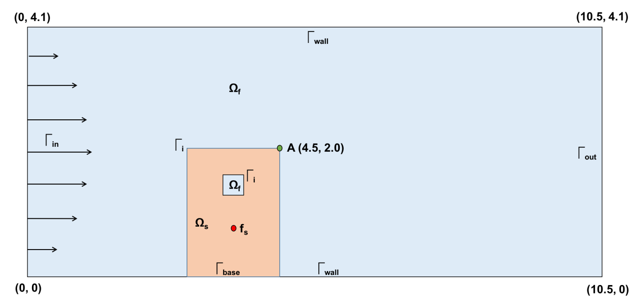


FIGURE 6: Configuration for 2D eXFSI test model with a disc-shaped force f_s

- The disc-shaped force ($f_s(x,t)$) at (0.35,0.05) with the radius r_{f_s} =0.0075 when t=0.
- The control points are fixed at the trailing edge of the structure with $A(t)|_{t=0} = (0.45, 0.2)$, measuring deflections due to FSI effect.

Material properties

This work is concerned with numerical approximation of the FSI effect on a St. Venant-Kirchhoff (STVK) compressible elastic material model. This model is suitable for large displacements with moderate strains.

The elasticity of material structures is characterized by the Poisson ratio ν_s and the Young modulus E_{Y_s}. The relationship of two material parameters μ_s and λ_s is given by:

$$\begin{aligned} \nu_s &= \frac{\lambda_s}{2(\lambda_s+\mu_s)}, \quad & E_{Y_s} &= \mu_s\frac{3\lambda_s+2\mu_s}{\lambda_s+\mu_s}, \\ \mu_s &= \frac{E_{Y_s}}{2(1+\nu_s)}, \quad & \lambda_s &= \frac{\nu_s E_{Y_s}}{(1+\nu_s)(1-2\nu_s)}, \end{aligned} \tag{29}$$

where for compressible material $\nu_s < \frac{1}{2}$ and incompressible material $\nu_s = \frac{1}{2}$. And the fluid is assumed to be incompressible and Newtonian.

Boundary Conditions

The boundary conditions are as follows (for $2D$ eXFSI):

- A constant parabolic inflow profile is prescribed at the left inlet as
$$v_f(0,y) = 1.5U_m \frac{4y(H-y)}{H^2}, \tag{30}$$
where U_m is the mean inflow velocity and the maximum inflow velocity in $1.5U_m$
- At outlet, zero-stress $\sigma.n = 0$ is realized by using the 'do-nothing' approach in the variational formulation.
- Along the upper and lower boundary, the usual 'no-slip' condition is used for the velocity.

Initial Conditions

The initial conditions are as follows (for $2D$ eXFSI):

$$v_f(t;0,y) = \begin{cases} v_f(0,y)\frac{1-cos(\frac{\pi}{2}t)}{2}, & t < 2.0, \\ v_f(0,y), & t \geq 2.0, \end{cases} \tag{31}$$

and for wave propagation part:

$$v_{w_f}(x,y,t)|_{t=0} = v_{w_s}(x,y,t)|_{t=0} = 0, \tag{32}$$
$$u_{w_f}(x,y,t)|_{t=0} = u_{w_s}(x,y,t)|_{t=0} = 0 \tag{33}$$

NUMERICAL SIMULATION

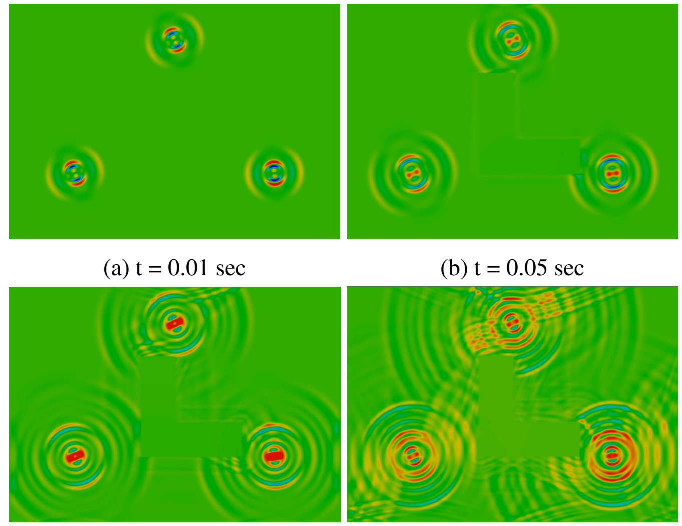
Global refinement is done for every cell and the time discretization in time is done by the well-known fractional-step-θ scheme. The fractional-step-θ is a second order scheme and has a similar work complexity to the Crank-Nicholson scheme.

The WpFSI problem

We present an illustration of the global solution for the displacement field u_x on the top surface of a solid plate with 'L' shaped liquid domain for the simulation times 0.01 s, 0.05 s, 0.1 s and 0.2 s in the Figure-7. Here, for the simulation time t > 0.04 s, the 'L' shaped liquid domain starts to become visible due to different magnitudes of wave propagation. Moreover, when the simulation time t > 0.09 s, we can additionally trace the symmetric wave, which is the reflection of the symmetric wave package from the boundaries.

The eXFSI problem

We introduce three eXFSI test cases that are treated with different inflow velocities (see Table-1). The parameters are chosen such that a visible transient behaviour of the double wedge airfoil can be seen. For the numerical implementation of the monolithic eXFSI problem, we have verified our solution obtained via optimization toolbox DOPELIB with the $2D$ FSI



(a) t = 0.01 sec (b) t = 0.05 sec (c) t = 0.1 sec (d) t = 0.2 sec

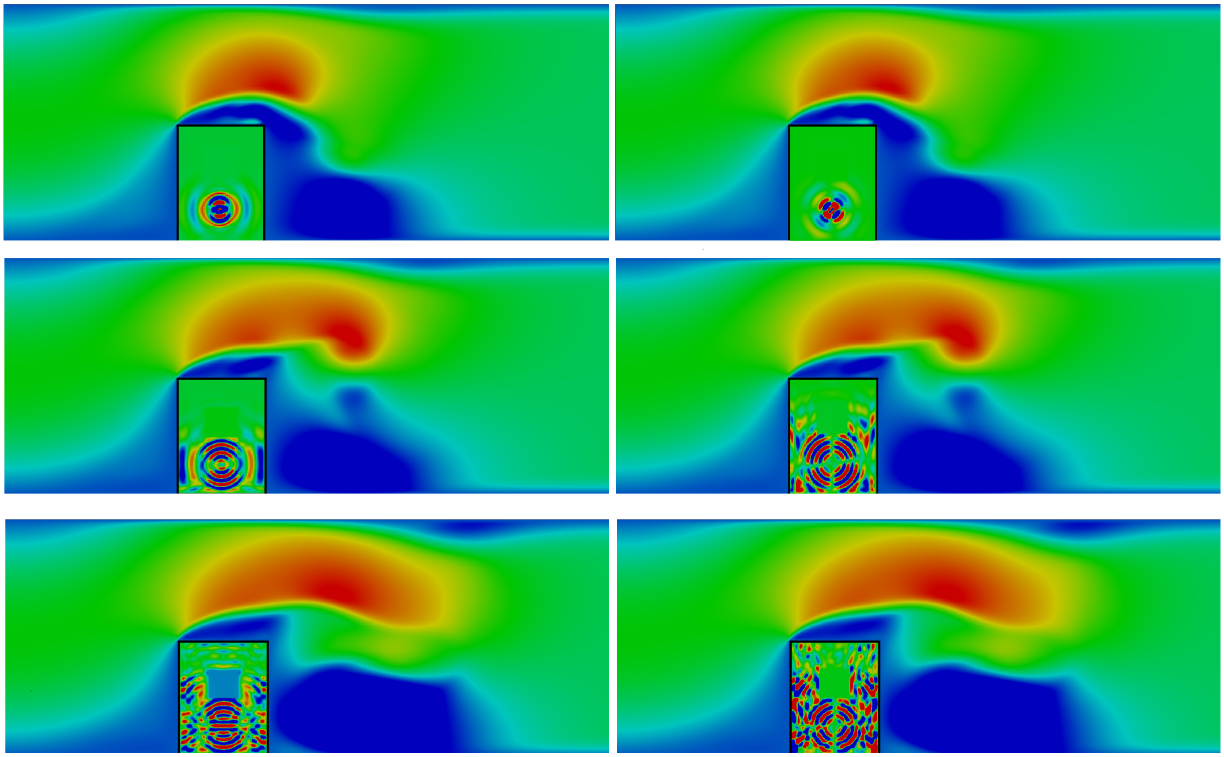
FIGURE 7: An ultrasonic wave propagation in a solid plate with 'L' shape liquid domain with a disc-shaped force ($f_s(x,t)$).

TABLE 1: Parameter setting for the eXFSI test cases

Parameter	eXFSI-1	eXFSI-2	eXFSI-3
Structure model	STVK	STVK	STVK
$\rho_f[10^3\ kgm^{-3}]$	1	1	1
$\rho_s[10^3\ kgm^{-3}]$	1	10	1
$\nu_f[10^{-3}\ m^{-2}s{-}1]$	1	1	1
ν_s	0.4	0.4	0.4
$\mu_s[10^6\ kgm^{-1}s^{-2}]$	0.50	0.50	2.0
$U_m[ms^{-1}]$	0.2	1.0	2.0

benchmark test [1–7] and have found the same results. To ensure a 'fair' comparison of results, we calculate the comparison values using the ALE method. In all cases, a uniform time-step k size is used. Moreover, to ensure the convergence of numerical simulations, different time-step schemes and step sizes are used. The time discretization is done by the "fractional-step-θ scheme" as regards the eXFSI simulation convergence with different time-step sizes. In order to avoid the additional remeshing around the fluid-solid interface, we use the same mesh in all time steps, where the interface motion is tracked accurately by a moving mesh line that moves with the solid-fluid interface.

In Figure-8 we present the global solution for the displacement field u_x and u_y for different time steps, where for the



(a) The displacement field component u_x (b) The displacement field component u_y

FIGURE 8: Global solution, illustration of the displacement field components u_x and u_y in a solid plate with rectangular liquid domains with the fluid velocity field at different times for the eXFSI-1 test case.

simulation time t > 0.025 s, the rectangular liquid domains inside the solid plate start to become visible due to different magnitudes of wave propagation.

CONCLUSION AND REMARKS

The coupled dynamics of an incompressible fluid with nonlinear hyperelastic solids for FSI effect and wave propagation gives very large algebraic equations, where serial computations usually are very costly due to memory limitations and computation time. Our proposed solver is based on a Newton linearization of the fully monolithic system of equations, discretized by a Galerkin finite element method. The numerical simulation result of WpFSI and eXFSI on a 2D model is under review, where these models are utilized to design an off-line and on-line SHM system. However we present WpFSI and eXFSI-1 test case results to give an overview of numerical simulation off-line and on-line SHM system.

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