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# geometric optics and WKB method for electromagnetic wave propagation in an inhomogeneous plasma near cutoff

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## Introduction

This report outlines the theory underlying electromagnetic (EM) wave propagation in an unmagnetized, inhomogeneous plasma. The inhomogeneity is given by a spatially non-uniform plasma electron density $n_e(\mathbf{r})$, which will modify the wave propagation in the direction of the gradient $\nabla n_e(\mathbf{r})$.

This density gradient will cause the wave group velocity direction to change, and, depending on the wave frequency, be responsible for a reversal in the propagation direction (reflection).

We are particularly interested in the reflection characteristics of an EM wave as it propagates toward a cutoff point in the plasma. That is, a point where the wavenumber vanishes, $k \to 0$, beyond which, the E-field is evanescent. Calculations are presented in one dimension, but should apply to the component of the wave vector in a three dimensional geometry.

It is assumed that the EM propagation can be solved using ray theory, under the WKB [1, 2, 3, 4] approximation.

## wave propagation in homogeneous plasmas

To start, we will review EM wave propagation in an unmagnetized isotropic, homogeneous plasma [5, 6]. Homogeneous in this sense means that the plasma properties (say electron density $n_e$) are uniform; while isotropic implies that the plasma properties do not depend on a given direction (hence the lack of an externally applied magnetic field). Consider an EM wave propagating through this plasma. Spatial uniformity implies

$$\nabla n_e(\mathbf{r}) = 0 \tag{1}$$

that is, no spatial variation in electron density. The wave frequency $\omega$ will be assumed high enough that the massive ions motions can be neglected. Assume a time harmonic dependence

$$\mathbf{\Gamma}(\mathbf{r}, t) \sim \mathbf{\Gamma}(\mathbf{r}) e^{-i\omega t} \tag{2}$$

where boldface type denotes a vector.

The electron motion is given by the Lorentz equation ($\mathbf{F} = m\mathbf{a}$) with no magnetic field

$$\mathbf{v}_1(\mathbf{r}) = -\frac{q}{i\omega m}\mathbf{E}_1(\mathbf{r}) \tag{3}$$

where the subscript 1 denotes an oscillating wave quantity (that is, we have no DC electric or magnetic fields), and $q$ and $m$ are the electron charge and mass. The current density due to these electrons is

$$\mathbf{j}_1(\mathbf{r}) = -q n_e \mathbf{v}_1(\mathbf{r}) = -\frac{n_e q^2}{i\omega m} = -\epsilon_0 \frac{\omega_p^2}{i\omega} \tag{4}$$

Here, $\epsilon_0$ is the permittivity of free space and $\omega_p^2$ is the electron plasma frequency, which is directly proportional to the electron density. This current density is used in Ampere's law

$$\nabla \times \mathbf{B}_1(\mathbf{r}) = -\frac{\omega_p^2}{i\omega c^2}\mathbf{E}_1(\mathbf{r}) - \frac{i\omega}{c^2}\mathbf{E}_1(\mathbf{r}) = -\frac{i\omega}{c^2}\left(1 - \frac{\omega_p^2}{\omega^2}\right)\mathbf{E_1}(\mathbf{r}) \tag{5}$$

which is then used in the curl of Faraday's law to get the wave equation

$$\nabla \times \nabla \times \mathbf{E_1}(\mathbf{r}) = \frac{\omega^2}{c^2}\left(1 - \frac{\omega_p^2}{\omega^2}\right)\mathbf{E_1}(\mathbf{r}) \tag{6}$$

Let

$$\mathsf{n}^2 = 1 - \frac{\omega_p^2}{\omega^2} \tag{7}$$

such that

$$\nabla \times \nabla \times \mathbf{E_1}(\mathbf{r}) = \frac{\omega^2}{c^2}\mathsf{n}^2\mathbf{E_1}(\mathbf{r}) \tag{8}$$

The left hand side of this equation is simplified using a vector identity to get

$$\nabla(\nabla \cdot \mathbf{E_1}(\mathbf{r})) - \nabla^2\mathbf{E_1}(\mathbf{r}) - \frac{\omega^2}{c^2}\mathsf{n}^2\mathbf{E_1}(\mathbf{r}) = 0 \tag{9}$$

We are interested only in EM wave propagation, that is a wave whose electric field is perpendicular to the direction of propagation and is not due to charge bunching, thus

$$\nabla \cdot \mathbf{E_1}(\mathbf{r}) = 0 \tag{10}$$

From this point onward, we will drop the 1 subscript since we are only dealing with oscillating wave fields. Eqn. 9 is

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\mathbf{E}(\mathbf{r}) + \frac{\omega^2}{c^2}\mathsf{n}^2\mathbf{E}(\mathbf{r}) = 0 \tag{11}$$

Consider the scalar wave equation for $E_x(\mathbf{r})$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)E_x(\mathbf{r}) + \frac{\omega^2}{c^2}\mathsf{n}^2E_x(\mathbf{r}) = 0 \tag{12}$$

Our restriction of a homogeneous plasma, along with a time harmonic form for wave quantities leads to the well known plane wave solution to this equation

$$E_x(\mathbf{r}) = A_0e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} + A_1e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \tag{13}$$

$$= A_0e^{i\theta} + A_1e^{-i\theta} \tag{14}$$

where any other solution can be constructed as a superposition of these plane waves. Furthermore, the coefficients in Eqn. 12 are constant, that is

$$\nabla n_e \sim \nabla\omega_p^2 = 0 \tag{15}$$

so we can Fourier analyze in all spatial dimensions in Eqn. 12 to get

$$\left(-k_x^2 - k_y^2 - k_z^2 + \frac{\omega^2}{c^2}\mathsf{n}^2\right)\tilde{E}_x(k) = 0 \tag{16}$$

where $\tilde{E}$ is the Fourier amplitude, and the solution for the wavevector $\mathbf{k}$ is the dispersion relation

$$|\mathbf{k}| = k = \sqrt{k_x^2 + k_y^2 + k_z^2} = \frac{\omega}{c}\mathsf{n} = \frac{\omega}{c}\sqrt{1 - \frac{\omega_p^2}{\omega^2}} \tag{17}$$

Moving in the direction of $\pm k$, along a surface of constant phase $\theta$, we find the phase velocity from

$$\frac{\partial\theta}{\partial t} = \pm\frac{\partial}{\partial t}\left(\mathbf{k}\cdot\mathbf{r} - \omega t\right) = 0 \tag{18}$$

which results in

$$v_p = \pm\frac{\partial \mathbf{r}}{\partial t} = \pm\frac{\omega}{\mathbf{k}} \tag{19}$$

The refractive index $\mathsf{n}$, which can be a vector, seen by an EM wave propagating in a medium is defined as the ratio of the speed of light to the phase velocity of the wave in that medium

$$\mathsf{n} = \frac{c\mathbf{k}}{\omega} \tag{20}$$

So we see immediately from Eqn. 7 that the index of refraction seen by the EM wave in the plasma for this case is

$$\mathtt{n} = |\mathsf{n}| = \sqrt{1 - \frac{\omega_p^2}{\omega^2}} \tag{21}$$

which is a scalar quantity given that the plasma is homogeneous.

EM wave propagation in this case is basically identical to EM wave propagation in uniform media. Waves travel in straight trajectories. If the wave frequency is less than the plasma frequency, $\mathsf{n}$ and thus $\mathbf{k}$ become pure imaginary constants, and we will have non-propagating, evanescent wave fields. The EM wave is then considered to be cutoff.

## wave propagation in non-homogeneous plasmas: WKB method

Now consider an unmagnetized isotropic, *non-homogeneous* plasma where plasma properties change spatially. We will focus on a plasma that has a nonuniform electron density, that is $n_e = n_e(\mathbf{r})$. This spatial nonuniformity restricts our ability to use a spatial Fourier analysis to obtain the solution for the EM wave propagation and fields. We assume time dependence and drop the $e^{-i\omega t}$ term from this point onward. For simplicity, we will restrict the variation in the electron density to be in the $x$-direction only

$$n_e(\mathbf{r}) = n_e(x) \tag{22}$$

The analysis [7, 5, 6] parallels Eqns. 2 - 12 except that now $n_e$ has a nonzero gradient in the $x$-direction, so that Eqn. 12 becomes

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) E_x(\mathbf{r}) + \frac{\omega^2}{c^2}\mathsf{n}(x)^2 E_x(\mathbf{r}) = 0 \tag{23}$$

where

$$\mathsf{n}(x) = \sqrt{1 - \frac{\omega_p^2(x)}{\omega^2}} \tag{24}$$

Recall that Fourier analysis is restricted to constant coefficient equations, thus we can only apply Fourier techniques to the $y$- and $z$- directions in Eqn. 23.

$$\left(\frac{\partial^2}{\partial x^2} - k_y^2 - k_z^2\right) \tilde{E}_x(x, k_y, k_z) + \frac{\omega^2}{c^2}\mathsf{n}(x)^2 \tilde{E}_x(x, k_y, k_z) = 0 \tag{25}$$

where, now, $\tilde{E}_x(x, k_y, k_z)$ is the Fourier transform in the $y$- and $z$- directions. We can write this equation as

$$\left(\frac{\partial^2}{\partial x} + k_x^2(x)\right) \tilde{E}_x(x, k_y, k_z) = 0 \tag{26}$$

where

$$k_x^2(x) = \frac{\omega^2}{c^2}\left(1 - \frac{\omega_p^2(x)}{\omega^2}\right) - k_y^2 - k_z^2 \tag{27}$$

We can write Eq. 26 as

$$\tilde{E}_x''(x) + k_x^2(x)\tilde{E}_x(x) = 0 \tag{28}$$

where a prime ($'$) denotes $\partial/\partial x$ and the functional dependence of the Fourier amplitudes on $k_y, k_z$ is assumed.

Equation 26 cannot be solved exactly for an arbitrary functional form of $k_x(x)$. However, we were able to find the exact solutions to the *homogeneous* form of this equation (Eqn. 16), when $k_x(x) \to k_x =$ *constant*, using the superposition of plane wave functions (Eqn. 13). Knowing this, we can say that if we restrict the medium to vary slowly, plane wave-like solutions could still apply. Let the variation in the plasma change over a scale length $L$ such that $\lambda << L$. We thus require that $k_x(x)$ varies little over a wavelength and assume that the solution to $\tilde{E}_x(x)$ in Eqn. 26 is almost plane-wave

$$\tilde{E}_x(x) = A(x)e^{i\phi(x)x} \tag{29}$$

where $A(x)$ is an amplitude, $\phi(x)x$ is known as the eikonal, and the amplitude varies slowly compared to the eikonal over a wavelength. We then approximate a plane-wave solution

$$\frac{\partial \phi(x)}{\partial x} \simeq \pm k_x \tag{30}$$

in equation 26 and assume a solution of the form given in Eqn. 29

$$\tilde{E}_x(x) = A(x)e^{i\phi(x)x} \tag{31}$$

The first and second derivatives of $\tilde{E}_x(x)$ are

$$\tilde{E}' = e^{i\phi x}\left[iA(\phi x)' + A'\right] \tag{32}$$

$$\tilde{E}'' = e^{i\phi x}\left[A'' + 2iA'(\phi x)' + iA(\phi x)'' - A(\phi x)'^2\right] \tag{33}$$

So that Eqn. 28 is

$$\overbrace{A''}^{1} + \overbrace{2iA'(\phi x)'}^{2} + \overbrace{iA(\phi x)''}^{3} - \overbrace{A(\phi x)'^2}^{4} + \overbrace{k_x^2(x)A}^{5} = 0 \tag{34}$$

where variations in the plasma and wave fields (derivatives) are on the spatial scale length $L$. We want to order the terms in the equation by their spatial scale relative to $L$. First, note that

$$\phi' = \mathcal{O}\left(\frac{\phi}{L}\right), \qquad \frac{x}{L} = \mathcal{O}(1)$$

and, by our earlier assumptions (Eqn. 30), the ordering for $k_x$ is

$$k_x \to \mathcal{O}\left(\frac{\phi}{L}\right) \to \mathcal{O}(\phi)$$

note also that the ordering for $(\phi x)'$ goes as

$$(\phi x)' = \phi + \phi' x \simeq \phi(1 + x/L) = \mathcal{O}(\phi) \tag{35}$$

Then the ordering for the terms in Eqn. 34 is

$$\begin{array}{cccc}
1 \to & A'' & \simeq & \frac{A}{L^2} & \sim L^{-2} \\
2 \to & 2iA'(\phi x)' & \simeq & \frac{A\phi}{L} & \sim L^{-1} \\
3 \to & iA(\phi x)'' & \simeq & \frac{A\phi}{L} & \sim L^{-1} \\
4 \to & -A(\phi x)'^2 & \simeq & -A\phi^2 & \sim L^0 \\
5 \to & Ak_x^2(x) & \simeq & A\phi^2 & \sim L^0
\end{array}$$

The WKB approximation consists of neglecting the $L^{-2}$ term and equating terms proportional to $L^{-1}$ and $L^0$ to zero separately.

From the $L^0$ terms, we have

$$k_x^2(x) = (\phi x)'^2 \tag{36}$$

$$(\phi x)' = \pm k_x(x) \tag{37}$$

$$\phi x = \pm \int k_x(x) dx \tag{38}$$

and from the $L^{-1}$ terms

$$2A'(\phi x)' = -A(\phi x)'' \tag{39}$$

$$\frac{A'}{A} = -\frac{1}{2}\frac{(\phi x)''}{(\phi x)'} = -\frac{1}{2}\frac{k_x'}{k_x} \tag{40}$$

$$(\ln A)' = -\frac{1}{2}(\ln k)' \tag{41}$$

$$A = \frac{C}{\sqrt{k_x(x)}} \tag{42}$$

so that the solution for $\tilde{E}_x(x)$ is

$$\tilde{E}_x(x) = \frac{1}{\sqrt{k_x(x)}} \left( C_1 e^{+i\int k_x(x)dx} + C_2 e^{-i\int k_x(x)dx} \right) \tag{43}$$

which represents incoming and outgoing wave contributions to the E-field.

Recall that we neglected terms of order $L^{-2}$ in Eq. 34. A more formal statement of this is

$$\left| \frac{1}{k_x(x)} \frac{dk_x(x)}{dx} \right| \ll k_x(x) \tag{44}$$

which is basically stating that the change of wavelength over a wavelength must be small. Thus, this method fails when $k_x(x) \to 0$ or when $k'(x)_x \to \infty$; that is, when the EM wave approaches a cutoff or resonance.

# wave propagation in a non-homogeneous plasma near a cutoff

We now consider EM wave propagation in an unmagnetized, nonhomogeneous plasma near a cutoff, $k_x(x = x_0) \to 0$, where the WKB form of the wave fields is valid on one side of the cutoff location. This is shown in Figure 0.1 where a wave is propagating from the right

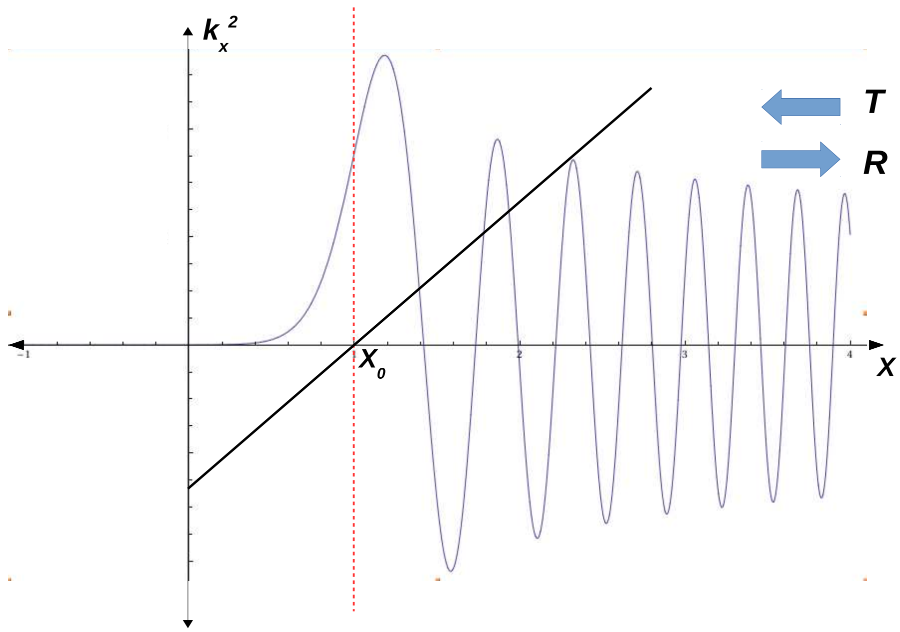


Figure 0.1.: EM wave incident from the right along a linear gradient in $k_x^2(x) \sim n_e(x)$ (black line). The resulting wave E-field is shown in light blue.

into a region where the plasma density changes enough to cutoff the wave. Using the same assumptions as before, we start with Eqn. 28

$$\tilde{E}_x''(x) + k_x^2(x)\tilde{E}_x(x) = 0 \tag{45}$$

In the immediate neighborhood of the cutoff, we can Taylor expand $k_x^2$

$$k_x^2(x) \approx k_x^2(x_0) + \frac{d}{dx}k_x^2(x)\bigg|_{x_0}(x - x_0) + \cdots \tag{46}$$

but $k_x^2(x_0) \to 0$, so we have

$$k_x^2(x) \simeq \frac{d}{dx}k_x^2(x)\bigg|_{x_0}(x - x_0) \tag{47}$$

Also in this neighborhood of the cutoff point, we can approximate a linear plasma density profile, that is since

$$k_x^2(x) = \frac{\omega^2}{c^2}\left(1 - \frac{\omega_p^2}{\omega^2}\right) = f[n_e(x)] \tag{48}$$

we can say

$$\frac{d}{dx}k_x^2(x)\bigg|_{x_0} = \text{constant} = \beta^2 \tag{49}$$

Then from Eqn. 47 we have

$$k_x^2(x) \simeq \beta^2(x - x_0) \tag{50}$$

and the wave equation (Eqn. 28) becomes

$$\frac{d^2\tilde{E}_x}{dx^2} + \beta^2(x - x_0)\tilde{E}_x = 0 \tag{51}$$

If we make the substitution

$$z = -\beta^{2/3}(x - x_0) \tag{52}$$

the wave equation becomes Airy's equation [8]

$$\frac{d^2\tilde{E}_x}{dz^2} - z\tilde{E}_x = 0 \tag{53}$$

with solutions given by linear combinations of the Airy functions Ai(z) and Bi(z)

$$\tilde{E}_x(z) = D_1\text{Ai}(z) + D_2\text{Bi}(z) \tag{54}$$

In order to couple these solutions to the region where the WKB solutions are valid, we will need the asymptotic forms of these functions ($z \to \pm\infty$)

$$\begin{aligned}
\mathrm{Ai}(z) &= \frac{1}{2\sqrt{\pi} z^{1/4}}\; e^{-\zeta} && (55)\\
\mathrm{Ai}(-z) &= \frac{1}{\sqrt{\pi} z^{1/4}} \left[\sin\left(\zeta + \frac{\pi}{4}\right) - \cos\left(\zeta + \frac{\pi}{4}\right)\right] && (56)\\
\mathrm{Bi}(z) &= \frac{1}{\sqrt{\pi} z^{1/4}}\; e^{+\zeta} && (57)\\
\mathrm{Bi}(-z) &= \frac{1}{\sqrt{\pi} z^{1/4}} \left[\sin\left(\zeta + \frac{\pi}{4}\right) + \cos\left(\zeta + \frac{\pi}{4}\right)\right] && (58)
\end{aligned}$$

with

$$\zeta = \frac{2}{3} z^{3/2}$$

As $x \to \infty$ ($z \to -\infty$), the wave field is given by the WKB form in Eq. 43

$$\tilde{E}_x(x) = \frac{C_1}{\sqrt{k_x(x)}} e^{+i\int k_x(x)dx} \;+\; \frac{C_2}{\sqrt{k_x(x)}} e^{-i\int k_x(x)dx} \tag{59}$$

or, assuming a reflection coefficient $R$ (note sign convention in exponential)

$$\tilde{E}_x(x) = \frac{C_0}{\sqrt{k_x(x)}} e^{-i\int k_x(x)dx} \;+\; \frac{C_0 R}{\sqrt{k_x(x)}} e^{+i\int k_x(x)dx} \tag{60}$$

and this field must be matched to the Airy form of the field near the cutoff. Note that at as $x \to -\infty$ ($z \to \infty$), Bi($z$)$\to \infty$. We must therefore set $D_2 \to 0$ in Eq. 54 so that the Airy field solution at a distance far to the right of $x_0$ is (using a trigonometric sum identity for $\sin\theta + \cos\theta$ in Eqn. 56) is

$$\tilde{E}_x(z \to -\infty) \simeq \frac{C}{\sqrt{\pi} z^{1/4}} \sin\left[\frac{2}{3}(-z)^{3/2} + \frac{\pi}{4}\right] \tag{61}$$

This must be matched to Eqn. 60. Substituting for $k_x^2(x)$ from Eqn. 50, this gives, for the WKB solution

$$\begin{aligned}
\tilde{E}_x(x) &= \frac{C_0}{\sqrt{\beta(x-x_0)^{1/2}}} e^{-i\int \beta(x-x_0)^{1/2}dx} \;+\; \frac{C_0 R}{\sqrt{\beta(x-x_0)^{1/2}}} e^{+i\int \beta(x-x_0)^{1/2}dx} && (62)\\
&= \frac{C_0}{\beta^{1/2}(x-x_0)^{1/4}} \left[e^{-i\frac{2}{3}\beta(x-x_0)^{3/2}} \;+\; R\; e^{+i\frac{2}{3}\beta(x-x_0)^{3/2}}\right] && (63)
\end{aligned}$$

Equation 61 can be re-written as

$$\begin{aligned}
\tilde{E}_x(z \to -\infty) &\simeq \frac{\sqrt{2}C_1}{\sqrt{\pi} z^{1/4}} \frac{e^{-i\frac{\pi}{4}}}{2i} \left[e^{i(\frac{2}{3}z^{3/2} + \frac{\pi}{4})} - e^{-i(\frac{2}{3}z^{3/2} + \frac{\pi}{4})}\right] && (64)\\
&= \frac{-iC_1}{\sqrt{2\pi}} \frac{1}{z^{1/4}} \left[e^{-i\frac{2}{3}z^{3/2}} - e^{+i\frac{2}{3}z^{3/2}} e^{-i\frac{\pi}{2}}\right] && (65)\\
&= \frac{C_1}{\sqrt{2\pi}} \frac{1}{\beta^{1/6}(x-x_0)^{1/4}} \left[e^{-i\frac{2}{3}\beta(x-x_0)^{3/2}} - i\; e^{+i\frac{2}{3}\beta(x-x_0)^{3/2}}\right] && (66)
\end{aligned}$$

Now, comparing Eqns. 63 and 66, we see that to within a constant

$$R = -i \tag{67}$$

Indicating total reflection with a $-\pi/2$ phase shift at the cutoff point.

These results can be extended to the case of absorption [7, 9]. For this case, we assume that electrons suffer collisions at a constant rate $\nu_e$ that doesn't vary with spatial coordinate. The plasma index of refraction is then

$$\mathtt{n}^2(x) = \frac{c^2 k_x^2(x)}{\omega^2} \; = \; 1 - \frac{\omega_p^2(x)}{\omega^2}\left(1 + \frac{i\nu_e}{\omega}\right)^{-1} \tag{68}$$

The analysis parallels Eqns. 45 - 66, resulting in a reflection coefficient [9]

$$|R| = \exp\left(-\frac{4}{3}\frac{\nu_e}{c}\Delta x\right) \tag{69}$$

again with a phase shift of $-\pi/2$ radians at the cutoff point. $\Delta x$ is the distance from where the WKB and Airy function solutions are matched to the cutoff point. This reflection results in a standing wave pattern for the E-field on the reflected side of the cutoff (see Fig. 0.1).

# Discussion

We arrived at our results for the reflection characteristics of EM waves propagating in a nonhomogeneous plasma by requiring that there exist a finite region where the Airy solutions and WKB solutions are simultaneously valid. This was accomplished by imposing linearity in $k_x^2(x)$ near the cutoff point. This is adequate in many situations. Furthermore, this problem has been solved for parabolic and $1/x^2$ variations in $k_x^2(x)$ near the cutoff point [7]. If, however, the spatial variation in $k_x^2(x)$ is very different from these before the WKB forms of the E-field solutions are valid, there will not be a region where both WKB *and* Airy forms of the E-field solutions are valid, and accurate matching can not be achieved using this theory.

# Conclusions

We have shown that the reflection of EM waves in a nonhomogeneous plasma near cutoff can be described using reflection coefficients. These coefficients have been solved for collisionless and collisional plasmas.

| | Magnitude | Phase Shift (Radians) |
|---|---|---|
| collisionless | 1 | $-\pi/2$ |
| collsional | $\exp\left(-\frac{4}{3}\frac{\nu_e}{c}\Delta x\right)$ | $-\pi/2$ |

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