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Episode IV:
Calibration-free characterization

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Sandia National Laboratories

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Introduction

What do we want? **Fault Tolerance!**

- (worst-case) error metrics
- more than process tomography…
- more than benchmarking…

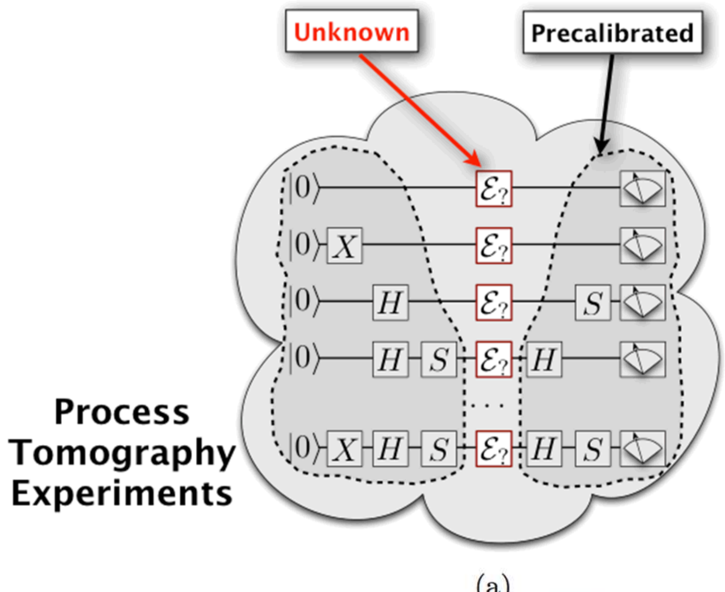
Then what do we need?

And how will we get it? **Calibration-free characterization!**

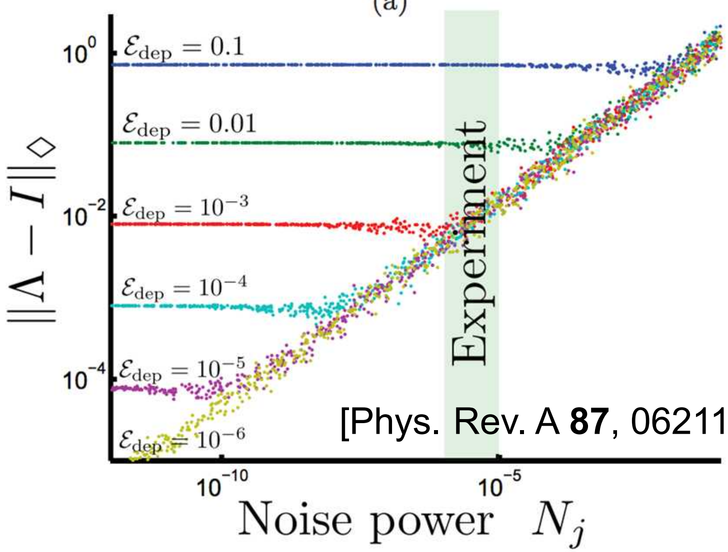
- Overview
  - Process tomography is insufficient (bad assumption that states are perfect, i.e. "calibrated")
  - We need "calibration-free" ways of extracting tomographic information.
  - In this episode, we'll review & contrast several modern methods:
    - Randomized Benchmarking Tomography
    - Robust Phase Estimation
    - **Gate Set Tomography**

Inadequacy of Process Tomography

- When the states and measurements used to interrogate the system are generated by gates that have systematic error (practically unavoidable), **process tomography is very inaccurate**.



- **(Right)** Error in process tomography estimate of identity gate when:
  - Depolarized SPAM (E_{dep})
  - Gaussian noise on measurement outcomes (N_j)



[Phys. Rev. A **87**, 062119]

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2009: Initial recognition of gap btwn tomography theory & exp.; importance of error bars

Relative tomography of an unknown quantum state, Mogilevtsev, Řeháček, and Hradil, Phys. Rev. A **79**, 020101(R) (Feb 2009)

Quantum tomographic reconstruction with error bars: a Kalman filter approach, Audenaert and Scheel, New J. Phys. 11 023028 (Feb 2009)

2012-2013: Initial "self-calibration" ideas, initial gate-set-tomography (GST) work

Self-calibration for self-consistent tomography, Mogilevtsev, Řeháček, and Hradil, New J. Phys. 14 095001 (Sept 2012)

Self-calibrating tomography for angular Schmidt modes in spontaneous parametric down-conversion, Straupe et al., Phys. Rev. A **87**, 042109 (April 2013)

Self-consistent quantum process tomography, Merkel et al., Phys. Rev. A **87**, 062119 (June 2013)

Self-calibrating tomography for multidimensional systems, Quesada, Brańczyk, and James, Phys. Rev. A **87**, 062118 (June 2013)

Tomography of a spin qubit in a double quantum dot, Takahashi, Bartlett, and Doherty, Phys. Rev. A **88**, 022120 (August 2013)

Robust, self-consistent, closed-form tomography of quantum logic gates on a trapped ion qubit, Blume-Kohout et al., arXiv:1310.4492 (2013)

2014: Several more new types of tomography; initial randomized-benchmarking-tomography (RBT) work

Robust Extraction of Tomographic Information via Randomized Benchmarking, Kimmel et al., Phys. Rev. X **4**, 011050 (March 2014)

Self-consistent tomography of the state-measurement Gram matrix, Stark, Phys. Rev. A **89**, 052109 (May 2014)

Quantum model averaging, Ferrie, New J. Phys. 16, 093035 (Sept 2014)

Randomized benchmarking with confidence, Wallman and Flammia, New J. Phys. 16 103032 (Oct 2014)

2015: More process tomography scrutiny; RBT in action, initial robust-phase-estimation (RPE) work

Systematic Errors in Current Quantum State Tomography Tools, Schwemmer et al., Phys. Rev. Lett. **114**, 080403 (Feb 2015)

Demonstration of robust quantum gate tomography via randomized benchmarking, Johnson et al., New J. Phys. 17 113019 (Nov 2015)

Robust calibration of a universal single-qubit gate set via robust phase estimation, Kimmel, Low, and Yoder, Phys. Rev. A **92**, 062315 (Dec 2015)

Desirable qualities

of a characterization method

- **Tomographic completeness**: can a method characterize the entire quantum process?
- **Efficiency**: how fast does the error decrease with each additional measurement (or other relevant unit, e.g. "hour in the lab")
  - Stochastic scaling- $1/\sqrt{N_{\text{resources}}}$ should be minimally expected.
  - Heisenberg scaling- $1/(\text{max. sequence length})$ is the best one can expect.
- **Error bars**: Does a method put error bars on its estimate?
- **Simplicity**: Is a method straightforward to implement?

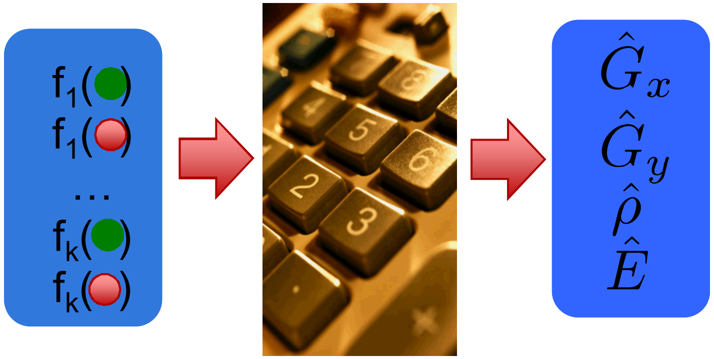
Comparison of mainstream methods

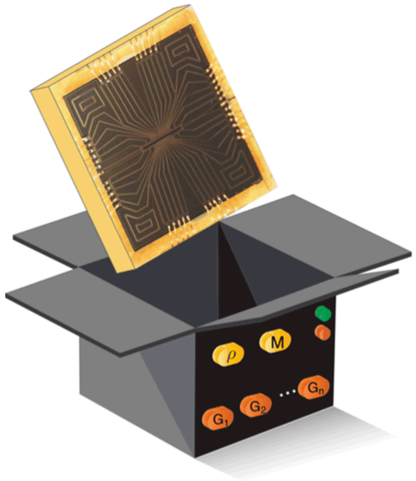
Method	Completeness	Efficiency	Error bars	Simplicity
Randomized Benchmarking Tomography (RBT)	**Only unital part of processes.**	**Non-Heisenberg**	**yes, via bootstrapping**	**Simple analysis** **(some work to construct sequences)**
Robust Phase Estimation (RPE)	**Only several specific parameters (e.g. the “phase”).**	**Heisenberg**	**yes, via bootstrapping**	**Simple analysis (trig.$^{-1}$ funcs)**
Gate Set Tomography (GST)	**Complete.**	**Heisenberg**	**computable**	**Fairly complex analysis.**

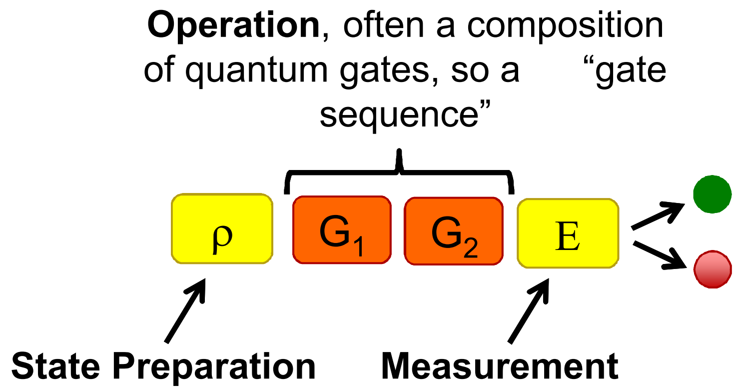
We focus on this method

Gate sequences, the common language

- Each characterization method utilizes experimental data.
- Each experiment (in this talk) consists of:
  - State preparation
  - (possibly) some operation on the state
  - Measurement
- Represented pictorially by:







Randomized Benchmarking Tomography

- **Main idea**: Use randomized benchmarking (RB) to estimate the *unital* part of **any** process matrix.
- **Reference**: Kimmel et al. Phys. Rev. X 4, 011050 (2014)
- Pros:
  - Estimates almost the entire process matrix (the unital part)
  - Error bars can be obtained by repeating by bootstrapping.
  - Analysis procedure is simple; very similar to doing RB.
- Cons:
  - Accuracy does not scale well with the number & length of gate sequences, and cannot even make use of long sequences b/c of fast decay.
  - Error bars are somewhat ad-hoc and have subtleties.

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \tau_x & \lambda_{xx} & \lambda_{xy} & \lambda_{xz} \\ \tau_y & \lambda_{yx} & \lambda_{yy} & \lambda_{yz} \\ \tau_z & \lambda_{zx} & \lambda_{zy} & \lambda_{zz} \end{pmatrix}$$

unital part: G'

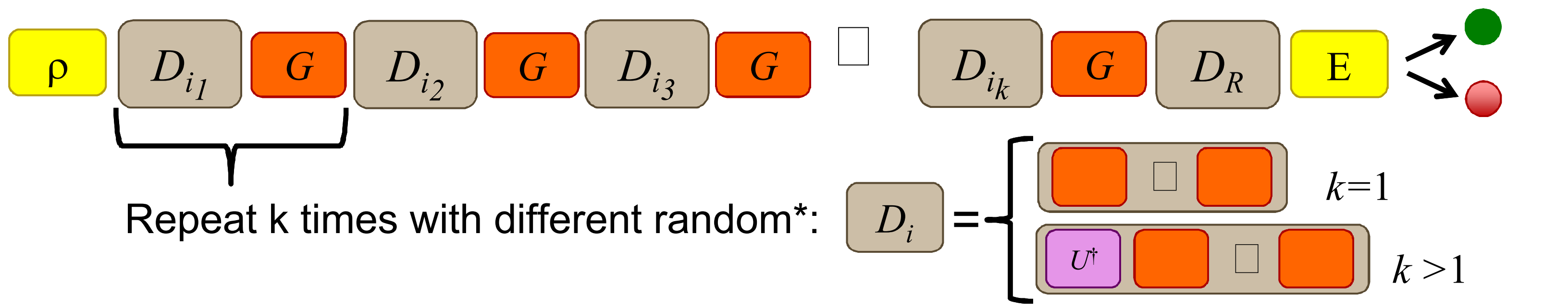
- Prep for what's next:
  - recall RB yields <F(U,G)> for unknown G and unitary U:

$$\overline{F}(G,U) = \int d\mu(\psi) \langle \psi | (U^+ \circ G(|\psi\rangle\langle\psi|)) | \psi \rangle$$

Randomized Benchmarking Tomography

How to do RBT:

- Find $G \circ N$ by fitting data from the gate sequences (for different k):

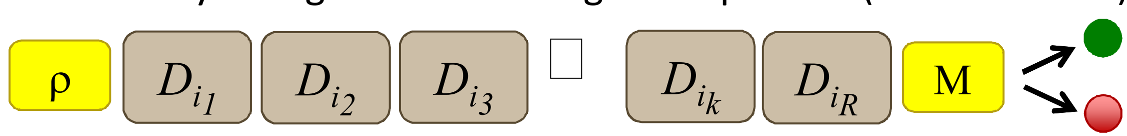


*assume:
1) $D_i = N \circ D_i^{perfect}$
2) D_R s.t. $\rho_i = \rho_f$ when $G=U$

Fit data(k) to $A_0 p^k + B_0$ → $p = \dfrac{d\,\bar{F}(G \circ N, U) - 1}{d-1}$ → $\boldsymbol{G \circ N}$

{U} spans cliffords

- Find N by fitting data from the gate sequences (for different k):



Fit data(k) to $A_0 p^k + B_0$ → $p = \dfrac{d\,\bar{F}(N, U) - 1}{d-1}$ → $\boldsymbol{N}$ → $\boldsymbol{G'}$

Use the result:

$G' = (G \circ N)' \circ (N')^{-1}$

(X' =unital part of X)

Randomized Benchmarking Tomography

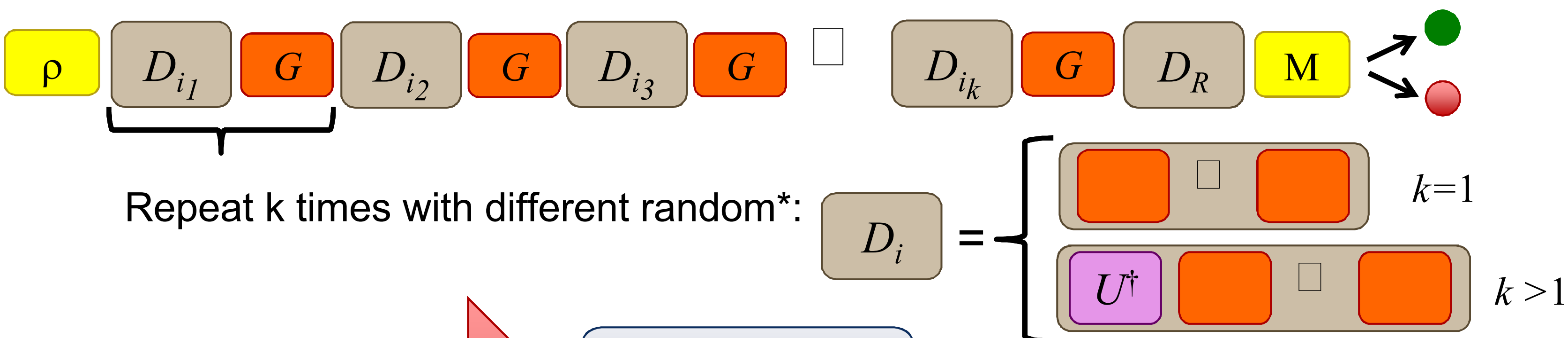
- **Main Idea:** use RB to estimate the unital part of **any** process matrix.
- RB gives access to average Fidelity between unknown process G and unitary U:

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \tau_x & \lambda_{xx} & \lambda_{xy} & \lambda_{xz} \\ \tau_y & \lambda_{yx} & \lambda_{yy} & \lambda_{yz} \\ \tau_z & \lambda_{zx} & \lambda_{zy} & \lambda_{zz} \end{pmatrix}$$

unital part

$$\bar{F}(G,U) = \int d\mu(\psi) \langle \psi | (U^{+} \circ G(|\psi\rangle\langle\psi|)) | \psi \rangle$$

  - Find $F\,(G,U)$ by fitting data from the gate sequences (for different k):



Fit data(k) to $A_0 p^k + B_0$

$$p = \frac{d\,\bar{F}(G \circ N, U) - 1}{d - 1}$$

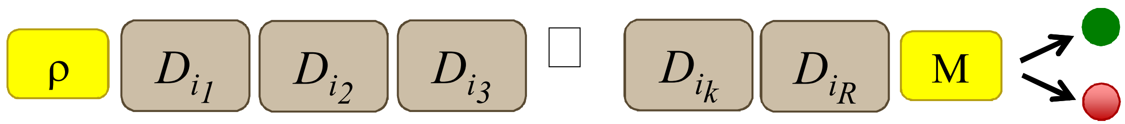
*assume:

- $D_i = N \circ D_i^{perfect}$
- D_R s.t. $\rho_{i=}\rho_f$ when $G=U$

- Knowing $F(G \circ N, U)$ between G and a spanning set of Clifford gates lets **you determine $G \circ N$**

Randomized Benchmarking Tomography

- **(continued…)** Performing the same sequences without G:



gives an estimate for just the noise process $\boldsymbol{N}$, and using the result:

$$G' = (G \circ N)' \circ (N')^{-1}$$

(where X' means the unital part of X)

One obtains an **estimate for G', the *unital* part of G.**

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \tau_x & \lambda_{xx} & \lambda_{xy} & \lambda_{xz} \\ \tau_y & \lambda_{yx} & \lambda_{yy} & \lambda_{yz} \\ \tau_z & \lambda_{zx} & \lambda_{zy} & \lambda_{zz} \end{pmatrix}$$

G’

- **Error bars** can be obtained by repeating by bootstrapping.

- Reference: [PHYS. REV. X 4, 011050 (2014)]

Robust Phase Estimation

- **Main idea**: a characterization protocol for estimating a few (specific) parameters of a set of quantum gates.
- **Reference:** Kimmel, Low, Yoder Phys. Rev. A 92, 062315 (2015);
- Aim is **to estimate a few parameters as efficiently as possible** (contrast with "tomography", which seeks to estimate the *entire* process matrix)
- Pros:
  - Parameters are learned with **optimal efficiency (Heisenberg scaling)**
  - **Non-adaptive** (simple!)
  - Accommodates a additional errors (depol. Noise)
- Cons:
  - Need **bounds** on state preparation
  - Need **bounds** on gate control

Robust Phase Estimation

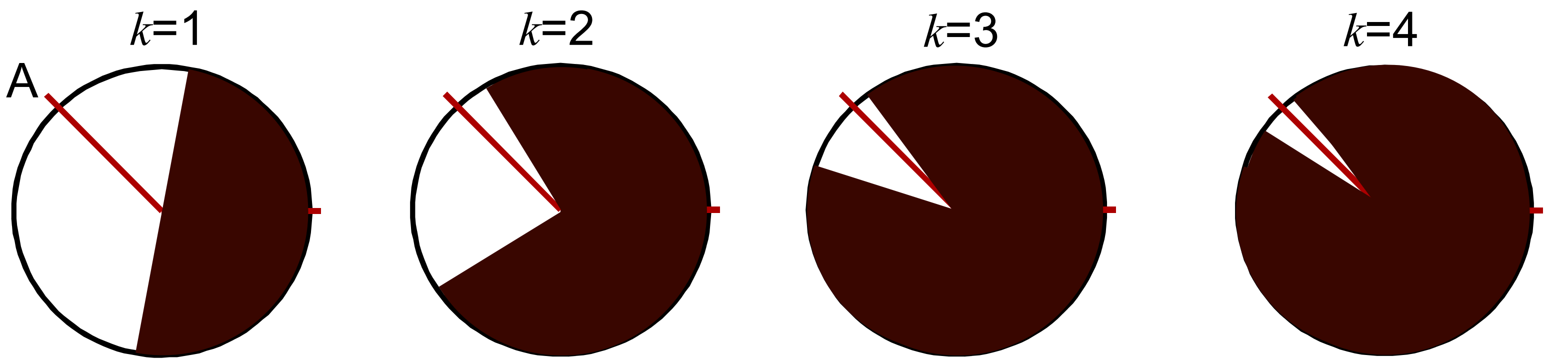
- Main Idea: one can estimate rotation of angle A in [-pi,pi] efficiently when you have access to “coins” with heads-probabilities

$$p_0(A,k) = \frac{1+\cos(kA)}{2} + \delta_0(k),$$
$$p_+(A,k) = \frac{1+\sin(kA)}{2} + \delta_+(k).$$

(k an integer)

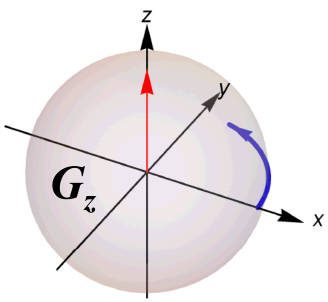
as long as the d errors aren’t too large ($\sup_k\{|\delta_0(k)|,|\delta_+(k)|\} < 1/\sqrt{8}.$)

- Start with k=1 and increment. At each k, can rule our half of remaining angular space.



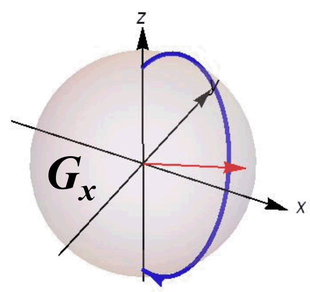
Robust phase estimation

- **Example**; 2 gates (but works for almost any Z,X-like):
  - G_z: rotation about *z-axis by* $\pi/2(1+\alpha)$
  - G_x: rotation about $(cos(\phi)x+sin(\phi)z)$*-axis by* $\pi/2(1+\varepsilon)$



Amplitude of Rotation:

Axis of Rotation:



Amplitude of Rotation:

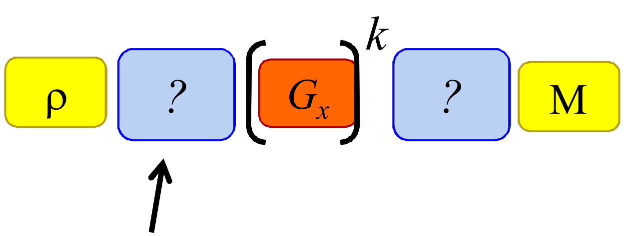
Axis of Rotation:

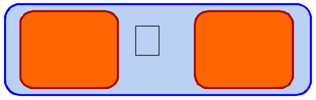
  - G_z case:

$$|\langle +|Z_{\pi/2}(\alpha)^k|+\rangle|^2 = \frac{1+\cos[-k(\pi/2)(1+\alpha)]}{2},$$

$$|\langle +|Z_{\pi/2}(\alpha)^k|\rightarrow\rangle|^2 = \frac{1+\sin[-k(\pi/2)(1+\alpha)]}{2}.$$

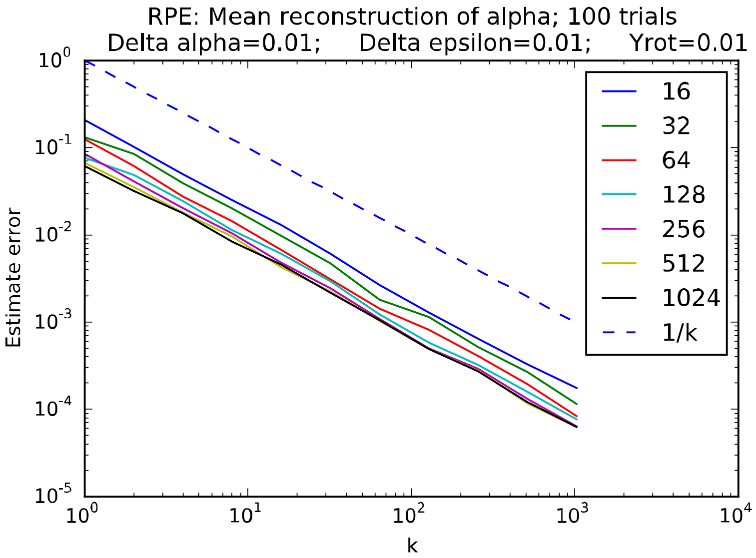
  - Other gates similar…

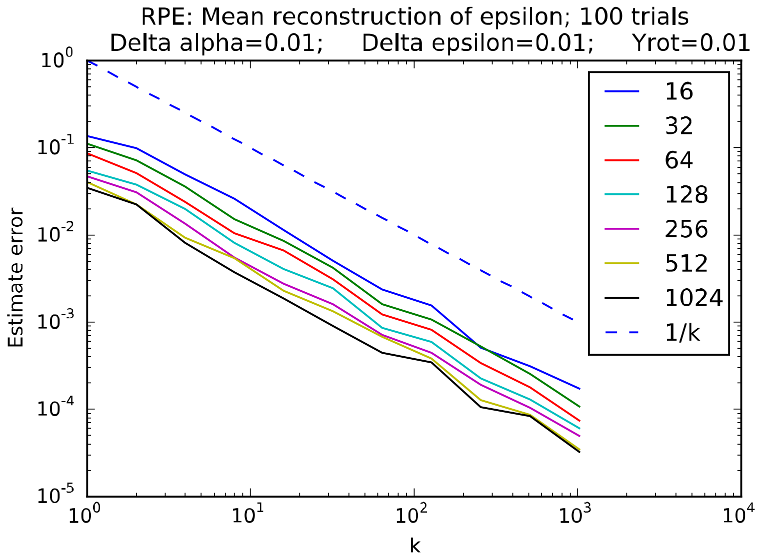


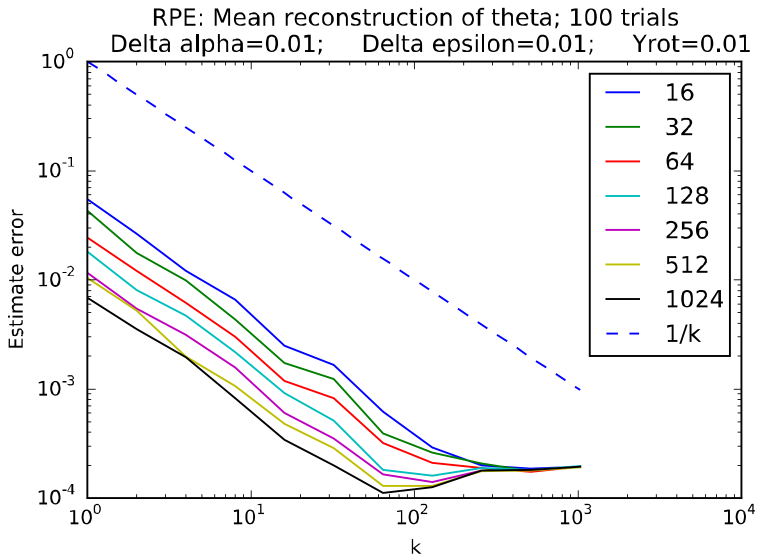


Extra gates may be required in practice for prep/measure.

Robust phase estimation







Numerical simulations demonstrate Heisenberg scaling in accuracy/

Accuracy of $2*10^{-4}$ with only 1056 experiments!

For more details:
E44.06, E44.07
(Tuesday 9:24 AM, 9:26 AM)

Gate Set Tomography

- **Main idea**: Estimates an entire set of gates, along with state preparation and measurement, *very very* accurately – but it takes some work to do it.

- **Originated from ideas in**: Phys. Rev. A **87**, 062119
- **“Modern” GST Reference**: Blume-Kohout et al., arXiv:1310.4492

- Pros:
  - Provides full tomographic estimates
  - Uses long-sequence data efficiently, allowing very high accuracy in estimates.
  - Provides rigorous confidence-interval error bars on “raw” gate estimates as well as derived quantities (e.g. fidelity, diamond-norm,…)
  - Clearly detects model violation (when data is “bad”, i.e., it doesn’t fit *any* qubit model)

- Cons:
  - Analysis of data is complicated; results in lots of estimate data to interpret.

Review of Lioville representation

- We can vectorize density matrices, and can do so in the basis of Pauli matrices:

$$\rho = \begin{pmatrix} a & b \\ \bar{b} & a \end{pmatrix} = \rho_I \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \rho_x \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \rho_y \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \rho_z \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Rightarrow \begin{pmatrix} \rho_I \\ \rho_x \\ \rho_y \\ \rho_z \end{pmatrix}$$

- Operators M on density matrices ("super-operators"; "maps") are matrices, which act on density matrix vectors by matrix multiplication:

$$M : \rho \mapsto \rho' \qquad M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \tau_x & \lambda_{xx} & \lambda_{xy} & \lambda_{xz} \\ \tau_y & \lambda_{yx} & \lambda_{yy} & \lambda_{yz} \\ \tau_z & \lambda_{zx} & \lambda_{zy} & \lambda_{zz} \end{pmatrix}$$

- Composition of maps is just matrix multiplication:

Gate Set Tomography

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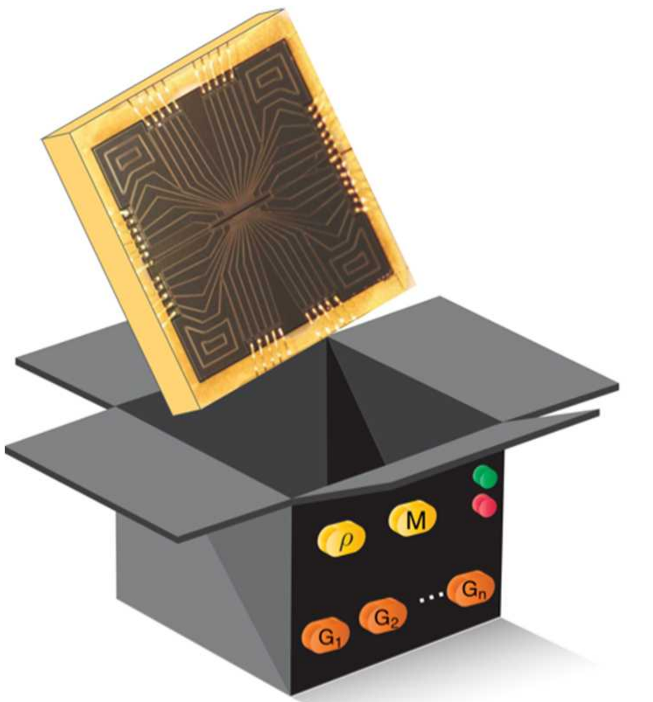
Experiment	Model ("gate set")



"gate sequence"

Press: (x 100) ρ $\underbrace{G_1 \; G_2}$ M → ●, ●

$= n$ ●, $N\text{-}n$ ●

$\boxed{p = n/N \text{ estimates } p_0}$

ρ (prep) $= \begin{pmatrix} \rho_I \\ \rho_x \\ \rho_y \\ \rho_z \end{pmatrix}$

E● (outcome) $= \begin{pmatrix} E_I & E_x & E_y & E_z \end{pmatrix}$

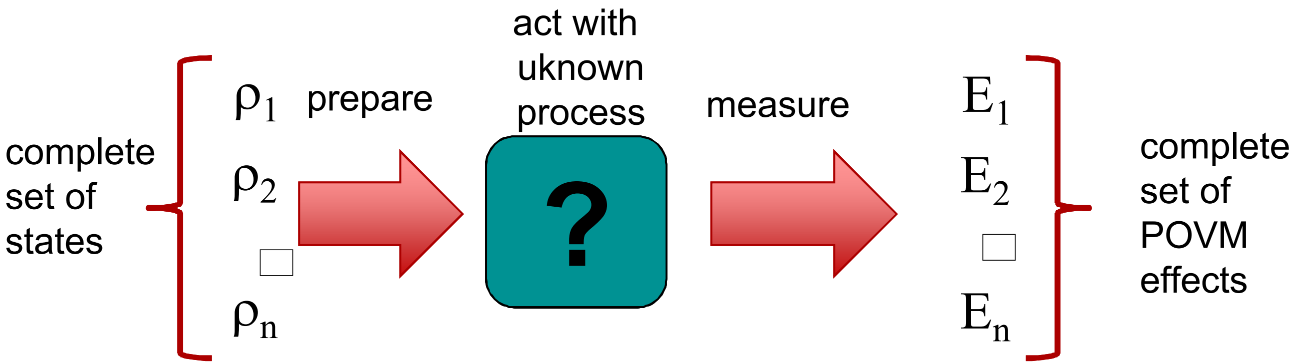
G_i (gate) $= \begin{pmatrix} 1 & 0 & 0 & 0 \\ \tau_x & \lambda_{xx} & \lambda_{xy} & \lambda_{xz} \\ \tau_y & \lambda_{yx} & \lambda_{yy} & \lambda_{yz} \\ \tau_z & \lambda_{zx} & \lambda_{zy} & \lambda_{zz} \end{pmatrix}$

Matrix mult. (note order reversal):

$\langle\langle E_{\bullet} | G_2 G_1 | \rho \rangle\rangle = p_0$

Linear GST: robust process tomography

Process Tomography:



Linear Gate Set Tomography:

Choose "fiducial" gate sequences $\boldsymbol{F_i}$:

$$|\rho_i\rangle\rangle = F_i|\rho\rangle\rangle \qquad \rho_i = \rho\,\overbrace{G_{i_1} \cdots G_{i_n}}^{F_i}$$

$$\langle\langle E_i| = \langle\langle E|F_i \qquad E_i = \underbrace{G_{i_1} \cdots G_{i_n}}_{F_i}\,E$$

And perform experiments:

$$\rho \quad F_i \quad G_k \quad F_j \quad E$$

Estimates: $\langle\langle E|F_j G_k F_i|\rho\rangle\rangle$

$$\rho \quad F_i \quad F_j \quad E \qquad \& \qquad \rho \quad F_i \quad E$$

$$\langle\langle E|F_j F_i|\rho\rangle\rangle \qquad \& \qquad \langle\langle E|F_j F_i|\rho\rangle\rangle$$

Linear GST: (cont.)

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Define "theory" matrices:

$$A = \begin{bmatrix} \langle\langle E_1 | \\ \langle\langle E_2 | \\ \vdots \\ \langle\langle E_n | \end{bmatrix}$$

$$B = \begin{bmatrix} |\rho_1\rangle\rangle & |\rho_2\rangle\rangle & \cdots & |\rho_n\rangle\rangle \end{bmatrix}$$

Define "data" matrices (w/tilde)

$$\tilde{G}_k = \left[f_{ji}^{k} \right]_{ji}$$

$$\tilde{I} = \left[I_{ji} \right]_{ji}$$

$$\tilde{S} = \left[S_i \right]_i$$

Experiments to estimate G_k :

ρ | F_i | G_k | F_j | E

Estimate of $\langle\langle E | F_j G_k F_i | \rho \rangle\rangle = f_{ji}^{k}$

ρ | F_i | F_j | E

Estimate of $\langle\langle E | F_j F_i | \rho \rangle\rangle = I_{ji}$

ρ | F_i | E

Estimate of $\langle\langle E | F_i | \rho \rangle\rangle = S_i$

Then:

is estimated by…

$AB \longrightarrow \tilde{I}$

$B^{-1}A^{-1} \longrightarrow \tilde{I}^{-1}$

$AG_kB \longrightarrow \tilde{G}_k$

$A|\rho\rangle\rangle \longrightarrow \tilde{S}$

$B|\rho\rangle\rangle \longrightarrow \tilde{I}^{-1}\,\tilde{S}$

$\langle\langle E|B \longrightarrow \tilde{S}^{T}$

$B^{-1}G_kB \longrightarrow \tilde{I}^{-1}\tilde{G}_k$

Estimates for $\langle\langle E|B$, $B^{-1}G_kB$, and $B|\rho\rangle\rangle$ constitute an **estimate of the entire gate set!**

Gauge Freedom

Estimates for $\langle\langle E|B$, $B^{-1}G_kB$, and $B|\rho\rangle\rangle$ constitute an **estimate of the entire gate set!**

What??

(we don't know B !)

- Altering a gate set by mapping

$$\langle\langle E| \longrightarrow \langle\langle E|B$$

$$|\rho\rangle\rangle \longrightarrow B|\rho\rangle\rangle$$

$$G_k \longrightarrow B^{-1}G_kB$$

"Gauge Transformation"

Does not, ***for any invertible matrix B***, affect any of the physical probabilities predicted by the gate set. (Because all probabilities are computed by:

$$\langle\langle E|G_s \cdots G_k G_j G_i|\rho\rangle\rangle = p_0$$

- We call this degree of freedom a "**gauge**" freedom. It defines equivalence classes of gate sets (where all the gate set representations in a class correspond to the same physical gate set) .
- Gates are **relational**. You tomograph a gate set, not just a gate…

GST Fiducial Selection

How do we select the F_i fiducial sequences (for use with LGST)?

- Require that $\{\, |\rho_i\rangle\rangle = F_i|\rho\rangle\rangle \}$ and $\{ \langle\langle E_i| = \langle\langle E|F_i \}$ each span the Hilbert-Schmidt space of d x d density matrices.

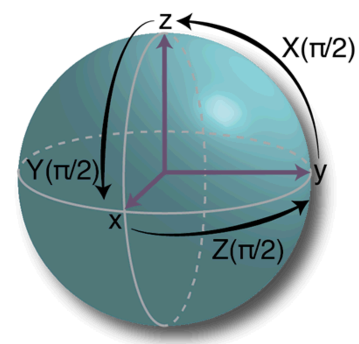
- In general, almost any set of d^2 matrices will be linearly independent (and so span the space) – but we want them to be *as linearly dependent as possible*. This is quantified by the Gram matrix, defined by:

$$\mathrm{Gram}_{ij} = \langle\langle E_i | \rho_j \rangle\rangle$$

(If either set contains linear dependences, the Gram matrix will be rank deficient.)

- For uniform informational completeness, and smallest singular value of the Gram matrix should be as large as possible.
    - Holds when sets for 2-designs
    - One 2-design of 4-elements for single-qubits (SIC POVM)
    - More convenient **1-qubit 2-design with 6-elements**:

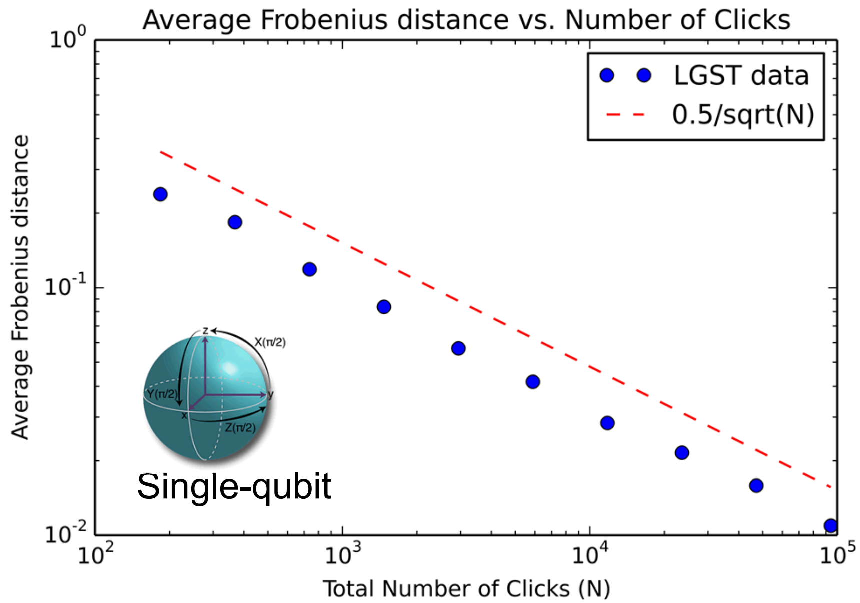
$$\emptyset,\ G_x,\ G_y,\ G_xG_x,\ G_xG_xG_x,\ G_yG_yG_y$$

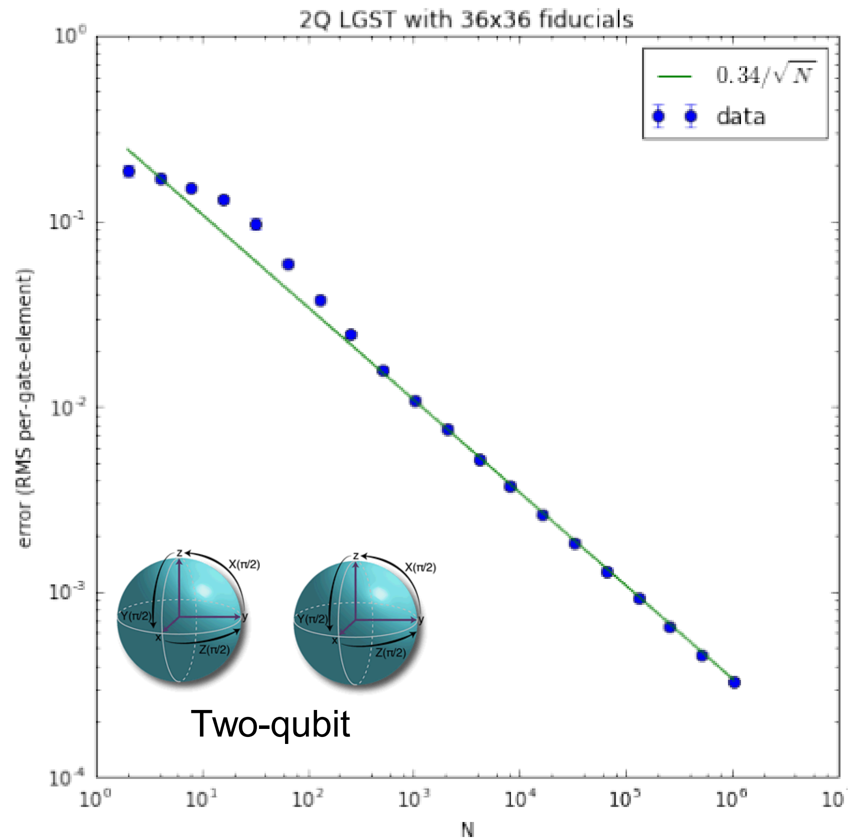


LGST Simulations: $1/\sqrt{N}$ Error scaling

- **Plots** show error between data-generating gate set and estimated gate set, as a function of N (the number of samples).
- Expect $1/\sqrt{N}$ behavior, as uncertainty in the mean of a distribution decreases as $1/\sqrt{N}$.

GST performance on **simulated data** (data generated from a *hidden* gateset)



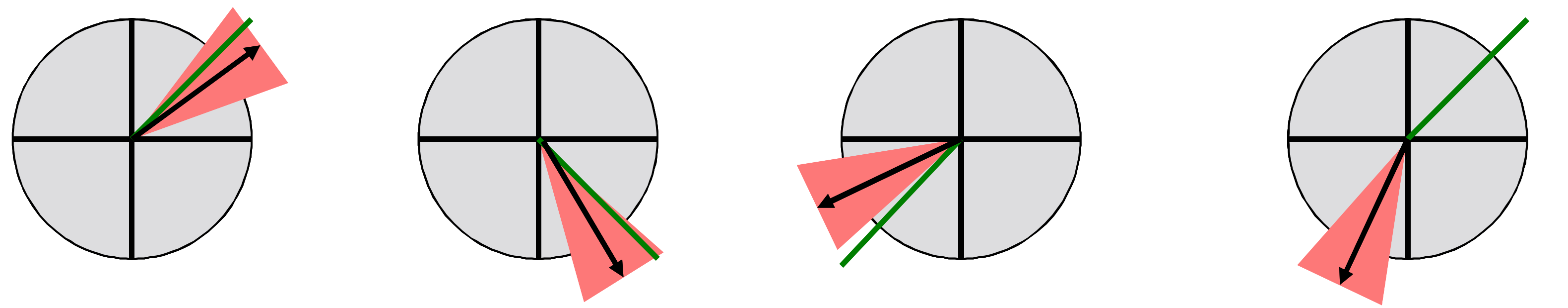


Coherent Error Amplification

- Error amplification – amplify **coherent** errors in gates by repeating them.
- Example: A $\pi/8$-rotation gate which over-rotates by θ:

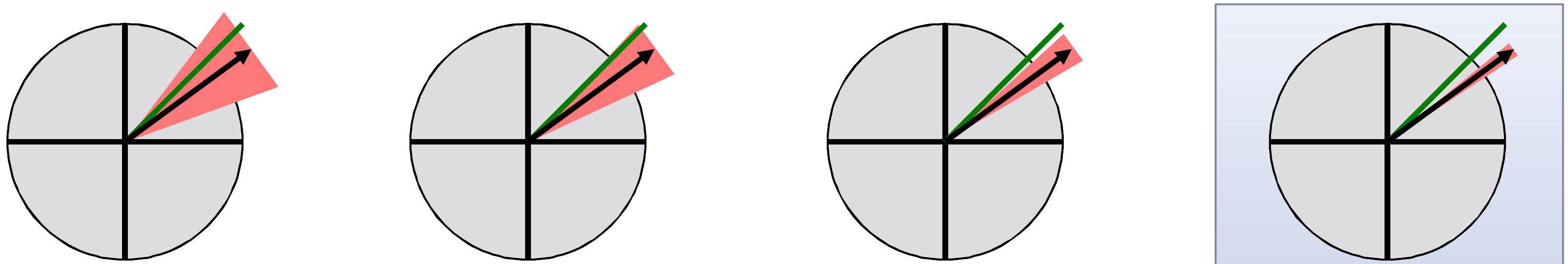
Repeats:	1	2	3	R

Repeated gate:



Estimated Angle:	$(\pi/8+\theta)+2\pi l$ $\pm\alpha/\sqrt{N}$	$(\pi/4+2\theta)+2\pi l$ $\pm\alpha/\sqrt{N}$	$(3\pi/8+3\theta)+2\pi l$ $\pm\alpha/\sqrt{N}$	$(R\pi/8+\mathrm{R}\theta)+2\pi l$ $\pm\alpha/\sqrt{N}$

Single gate:

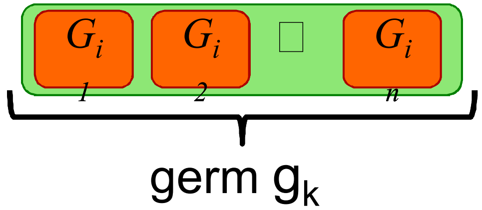


Estimated Angle:	$(\pi/8+\theta)+2\pi l$ $\pm\alpha/\sqrt{N}$	$(\pi/8+\theta)+\pi l$ $\pm\alpha/\left(2\sqrt{N}\right)$	$(\pi/8+\theta)+2\pi l/3$ $\pm\alpha/\left(3\sqrt{N}\right)$	$(\pi/8+\theta)+2\pi l/R$ $\pm\alpha/\left(\mathrm{R}\sqrt{N}\right)$

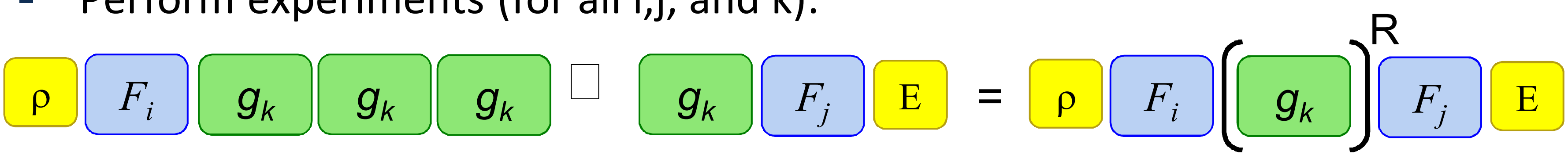
Uncertainty ~ 1/R !

Long-sequence GST

- Use error amplification to amplify **all** possible gate errors by repeating not just the gates themselves but a set of short gate sequences $\{g_k\}$ called “germs”.



- Perform experiments (for all i,j, and k):



- Results in estimates (“frequencies”) for $\langle\langle E|\mathrm{F}_j(g_k)^R\,\mathrm{F}_i|\rho\rangle\rangle$ which we compare with the probabilities predicted by the model using the log-likelihood or χ^2 statistic:

$$\log L = \sum_i N f_i \log(p_i) \qquad \chi^2 = \sum_i N \frac{(p_i - f_i)^2}{p_i}$$

  - N = #samples, f = frequency, p = probability, and i ranges over gate sequences *and* outcomes.
  - Maximizing the likelihood or minimizing χ^2 gives an estimate for the gate set.

Interlude: Germ Selection

Sandia National Laboratories

How does one select a “complete” set of germs?

- **Jacobian** of the germs to power L w.r.t. gate set parameters, should collectively have n singular values which grow linearly with L (indicating they amplify the error given by the corresponding right singular vector), where n is the number of gauge-invariant gateset parameters.
- Define:

$$\nabla^{(L)}_{g_k} \equiv \frac{1}{L}\frac{\partial\left[g_k^L\right]}{\partial \vec{G}}\bigg|_{\vec{G}=\vec{G}_{\text{target}}}$$

we care about infinite-L limit of:
(does it have right singular rank = n?)

$$J = \begin{pmatrix} \nabla^{(L)}_{g_1} \\ \vdots \\ \nabla^{(L)}_{g_n} \end{pmatrix}$$

Gateset parameters (process mx elements)

Elements of g_k product

$$\nabla^{(L)}_{g_1}$$

“**herald vector**”: Did any germ amplify the trial error?

$$= \begin{bmatrix} \nabla^{(L)}_{g_1} \\ \nabla^{(L)}_{g_2} \\ \nabla^{(L)}_{g_3} \end{bmatrix}$$

Trial gateset error

- Insight: infinite-L limit = twirl = projection onto commutant :

$$\nabla^{(L)}_g = \frac{1}{L}\sum_{n=0}^{L-1}\sigma(g)^n\frac{\partial\sigma(g)}{\partial\vec{G}}\sigma(g)^{L-1-n}$$
$$= \left[\frac{1}{L}\sum_{n=0}^{L-1}\sigma(g)^n\nabla^{(1)}_g(\sigma(g)^\dagger)^n\right]\sigma(g)^{-(L-1)}$$

$$\lim_{L\to\infty}\nabla^{(L)}_g = \underbrace{\Pi_{\sigma(g)}\left[\nabla^{(1)}_g\right]}_{\text{projection onto commutant}}$$

Interlude: Likelihood and χ^2 statistics

- The likelihood function: $L = \Pr(\text{data}|\text{model}) = \Pr(\text{data}|\text{gateset})$
  - Often sharply peaked so use logarithm, the "log-likelihood"
  - The log-likelihood approximated to 2nd order using χ^2

- **Example**: N coin flips, f percent come up heads. If our model is a coin with probability p of heads, then

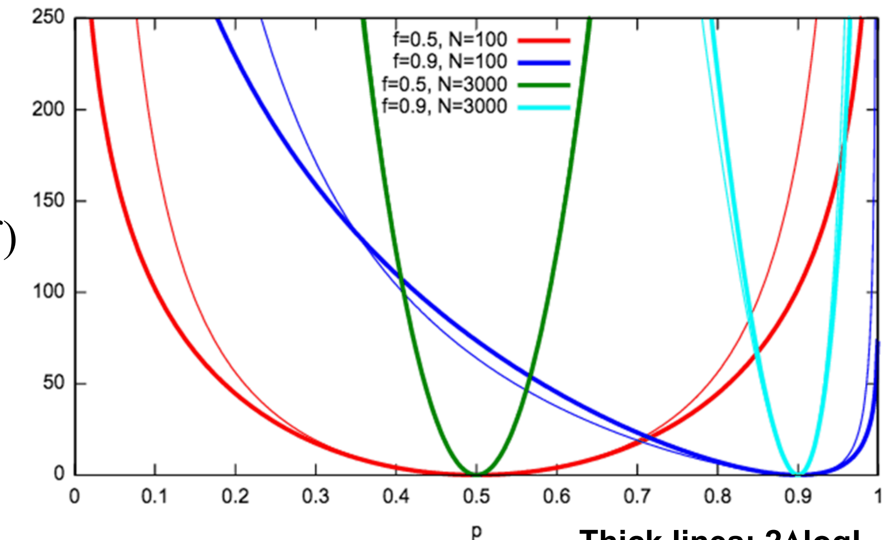
$$L = p^{Nf}(1-p)^{N(1-f)}$$

$$\log L = Nf\log(p) + N(1-f)\log(1-p)$$

$$\max_p(\log L) = Nf\log(f) + N(1-f)\log(1-f)$$

$$\chi^2 = N\frac{(p-f)^2}{p(1-p)}$$

Can check: $2(\max_p(\log L) - \log L) \cong \chi^2$



Thick lines: 2ΔlogL
Thin lines: χ^2

- In GST, we just have lots of coin flips (or dice rolls):

$$\log L = \sum_i Nf_i\log(p_i) \qquad \chi^2 = \sum_i N\frac{(p_i - f_i)^2}{p_i}$$

Long-sequence GST: iterations

- To perform “long sequence GST” we do the following:
  - **Run LGST** to get an initial gate set estimate, $\mathcal{G}_0$
  - **Iteratively** maximize the log-likelihood to obtain sequentially better estimates:

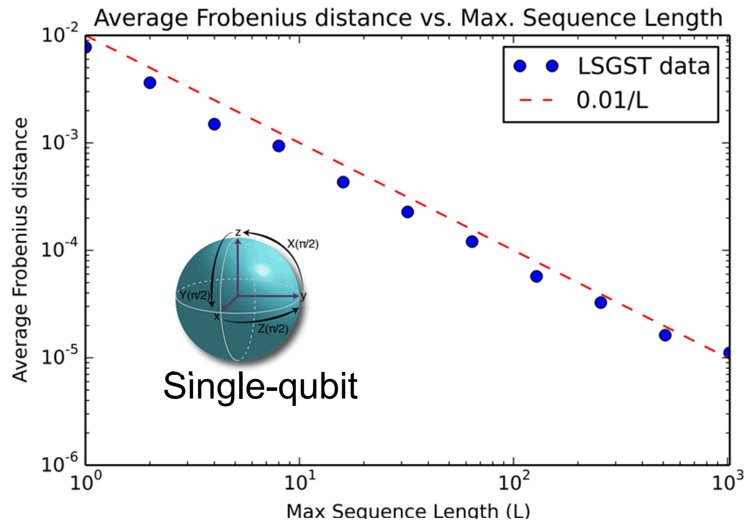
Iteration	Gate Sequences used in log L	Starting gate set	Max. Likelihood Estimate
0	$\rho \; F_i \; g_k \; F_j \; E$	$\mathcal{G}_0$	$\mathcal{G}_1$
1	$\rho \; F_i \; g_k \; g_k \; F_j \; E$	$\mathcal{G}_1$	$\mathcal{G}_2$
2	$\rho \; F_i \; g_k \; g_k \; g_k \; g_k \; F_j \; E$	$\mathcal{G}_2$	$\mathcal{G}_3$
r	$\rho \; F_i \; \left(g_k \right)^{2^r} F_j \; E$	$\mathcal{G}_{(r-1)}$	$\mathcal{G}_r$

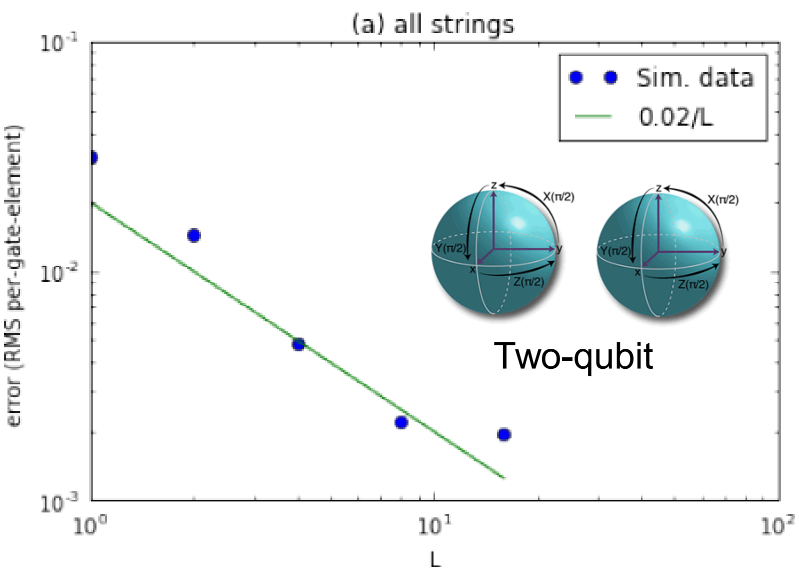
Final GST Estimate

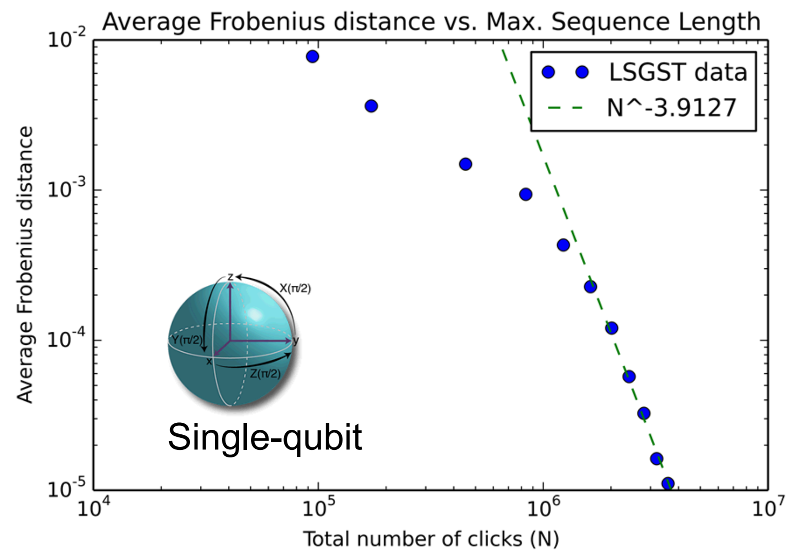
- Why iteratively? To avoid wrong “branch” of solution
- *Technical Point: actually use the χ^2 instead of log L for all but the final iter.

LS-GST Simulations: $1/L$ Error Scaling

- Long-sequence GST on simulated data shows desired $1/L$ scaling, where L=length (or exponent) of germ power.
- Even better scaling ($N^{-3.9}$?) with the total number of experiments.

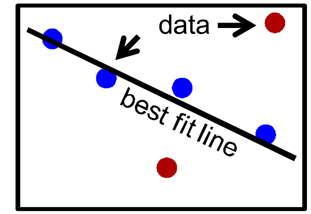




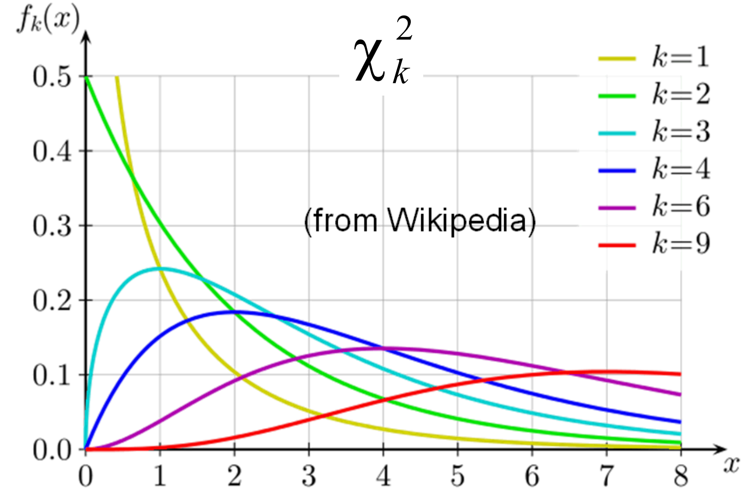


Likelihood ratio test and Wilks Theorem

- When we maximize the log-likelihood, it would be nice to know what values indicate the model is fitting "well".



- **Wilk's theorem** states: $2(\log L_1 - \log L_2)$ will be asymptotically (as $N \to \infty$) χ^2_k - distributed where k equals the difference in dimensionality of the models used to compute L_1 and L_2 when the models are valid.



- In our case we use a "**maximal model**" in which each data point is fit exactly to compute L_1. This model has the ma nd has $n_{\max} \equiv n_{\text{gatestrings}}(n_{\text{outcomes}} - 1)$ parameters

  - Thus, we expect the quantity (readily computable by GST):

$$2\left(\max_p(\log L) - \log L\right)$$

will be χ^2 – distributed with degrees of freedom equal to the difference between $n_{\max}$ and the number of (gauge invariant) gate set parameters.

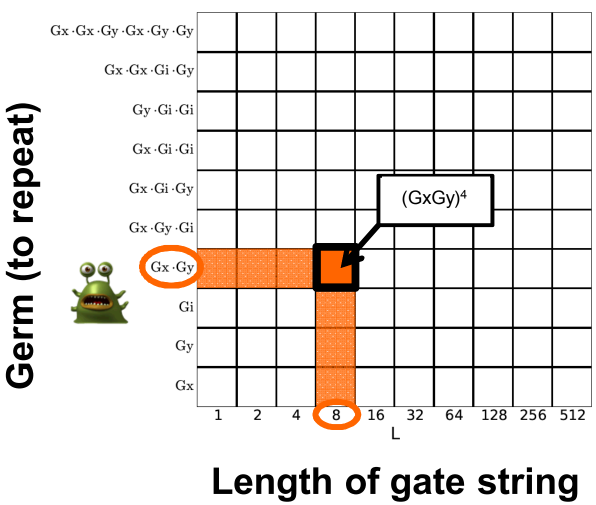
  - We expect the log L contribution of **single sequences** to be χ^2_1-distributed.
  - The extent to which this is not true indicates **model violation**, which, in this way, is easily detected.

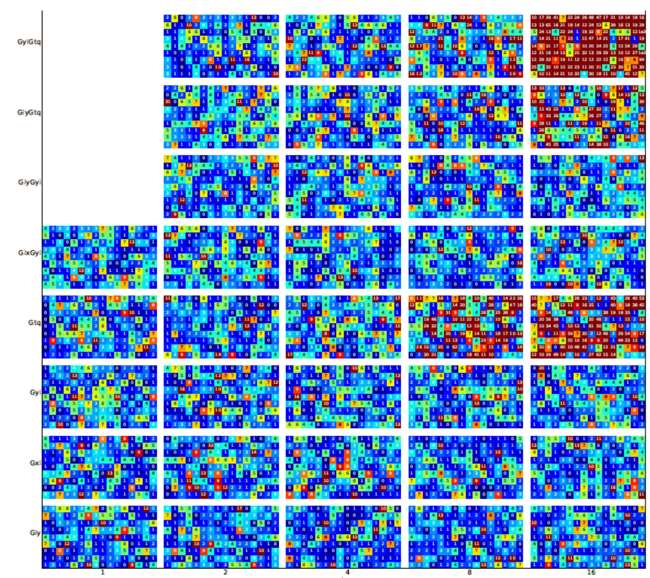
Detecting model violation in GST

- Example summary goodness-of-fit table from a GST report:

L	$2\Delta\log(\mathcal{L})$	k	$2\Delta\log(\mathcal{L})-k$	$\sqrt{2k}$	P	N_s	N_p	Rating
1	1523.267	1485	38.2672	54.49771	0.24	2508	1023	★★★★★
2	4791.623	4752	39.623	97.48846	0.34	5775	1023	★★★★★
4	19047.72	18684	363.7215	193.308	0.03	19707	1023	★★★★
8	52886.24	49356	3530.241	314.1847	0	50379	1023	★★★★
16	97747.73	80028	17719.73	400.07	0	81051	1023	★★★★

- Example "color box plots", showing contributions to $2\left(\max_p(\log L) - \log L\right)$ from **individual** gate sequences:

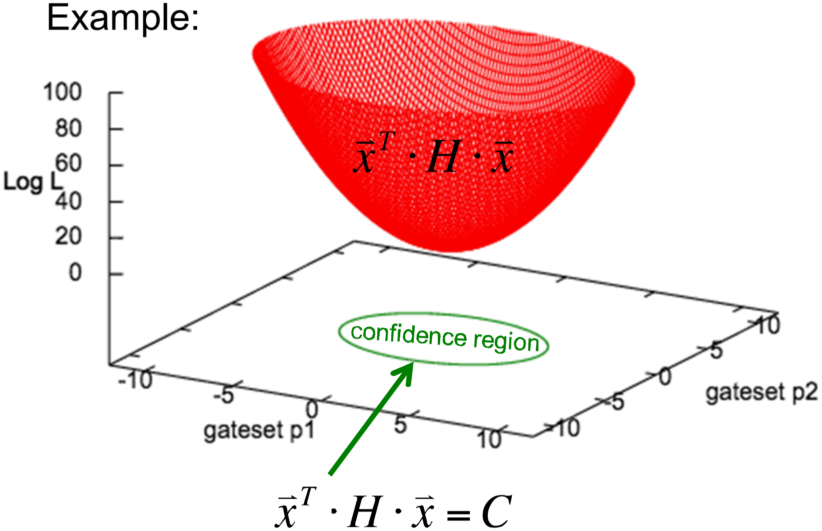




Likelihood-ratio error bars

- Everybody wants error bars…
- GST computes error bars by:
  - approximating the log-likelihood function as being quadratic (ok since we don't constrain it to hard boundaries).
  - Evaluating the Hessian at the maximum-log L point = covariance tensor.
  - Scale the Hessian H appropriately to find a valid 1-α confidence region.

Example:



gate set parameters *relative* to MLE point

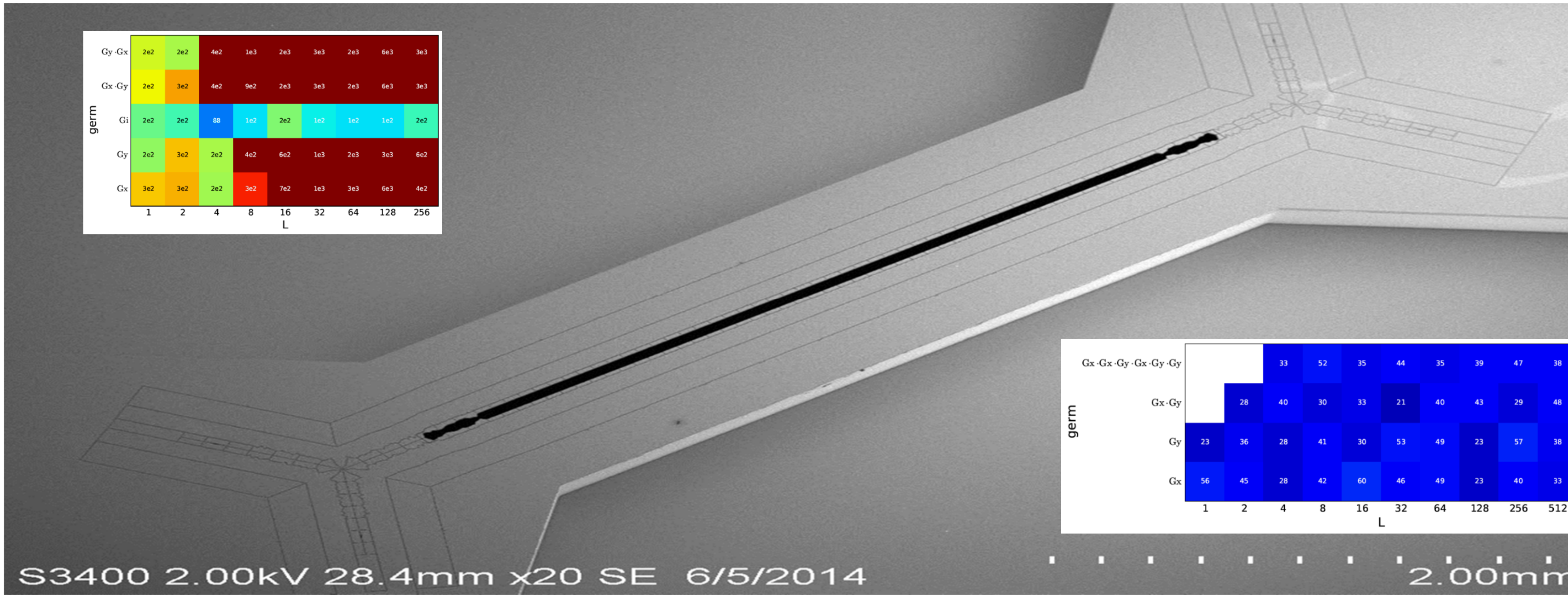
$$\vec{x}^T \cdot H \cdot \vec{x} = C \quad \text{(where } \vec{x} = d\vec{G} = \vec{G} - \vec{G}_0 \text{)}$$

Describes the boundary of a 1-α confidence region when $C = \mathrm{CDF}^{-1}[\chi_k^2](\alpha)$, where k is the dimensionality of the region being constructed.

arxiv:1202.5270

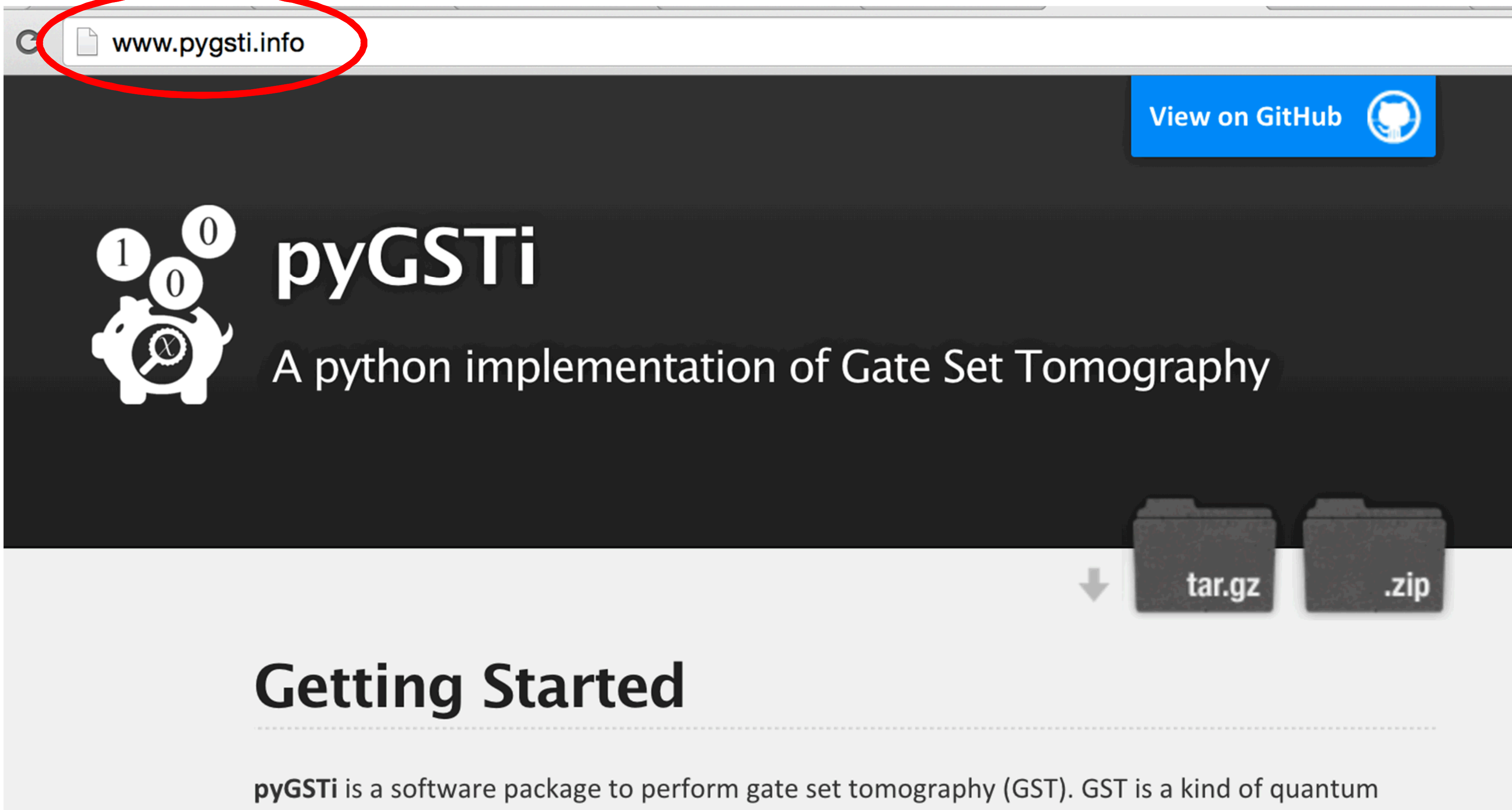
Experimental Demonstrations

- SNL Ion Trap (1Q and 2Q)



- SNL donor qubit
- UNSW Silicon donor single qubit
- Raytheon BBN transmon qubits (1Q and 2Q)
- Wisconsin-Madison double quantum dot
- More…

Open source GST software:



`import pygsti` (today!)

Recap: comparison of mainstream methods

Method	Completeness	Efficiency	Error bars	Simplicity	Sequences
Randomized Benchmarking Tomography (RBT)	Only unital part of processes.	Non-Heisenberg	yes, via bootstrapping	Simple analysis	$\rho \; \left[D_{i_1} \; G \right]^{k} \; D_R \; E$
Robust Phase Estimation (RPE)	Only several specific parameters (e.g. the "phase").	Heisenberg	yes, via bootstrapping	Simple analysis	$\rho \; ? \; \left[G_x \right]^{k} \; ? \; M$
Gate Set Tomography (GST)	Complete.	Heisenberg	computable	Fairly complex analysis.	$\rho \; F_i \; \left(g_k \right)^{2^r} \; F_j \; E$

Summary

- Calibration-free characterization methods are essential to creating fault-tolerant qubits.
- Common currency: gate sequences
- Key idea: exploit **long**-sequences for accuracy

- Many methods exist for calibration-free characterization; we covered a few of the more mainstream ones:
  - **Randomized Benchmarking Tomography** (RBT): Simple to implement & data-drive, but lacks some important advantages of GST.
  - **Robust Phase Estimation** (RPE): lightweight and efficient; intended for rapid characterization of a few gate-set parameters.
  - **Gate Set Tomography** (GST): efficient and tomographically complete; currently the best method for full-gate-set characterization. Complex implementation, but downloading free software is easy.

GST **Myths – a leftover from another of Erik's talks – probably don't need this….**

- **Myth**: GST is only for *good* qubits
  - GST is intended to operate in "harsh environments", and extract as much information as possible from noisy data.
- **Myth**: GST is slow
  - GST takes minutes to run on typical datasets
- **Myth**: GST results are hard to interpret
  - GST reports include detailed explanation of what results mean.