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LLNL-TR-717001

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January 6, 2017

This work performed under the auspices of the U.S. Department of Energy by Lawrence Livermore National Laboratory under Contract DE-AC52-07NA27344.

The Super Gaussian Laser Intensity Profile in HYDRA's 3D Laser Ray Trace Package (LLNL-TR-XXXXXX)

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January 6, 2017

In this note, the laser focal plane intensity profile for a beam modeled using the 3D ray trace package in HYDRA is determined. First, the analytical model is developed followed by a practical numerical model for evaluating the resulting computationally intensive normalization factor for all possible input parameters.

1 The Analytical Model

The `superg3d` card in HYDRA is used to specify the focal plane laser profile. Formally, the power distribution — or to think of it another way, the probability of launching a ray directed at the point (x, y) in the focal plane — is given in the HYDRA manual as

$$P\left(r\left(x,y\right)\right) = re^{-r^n} \tag{1}$$

where

$$r\left(x,y\right) = \sqrt{\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2}}.$$

The notation has been simplified for the purposes of this document according to σ_x=`spotx`, σ_y=`spoty`, and n=`npower` from the HYDRA manual. This distribution is not normalized. That is, this is only relative.

Given this power (probability) formalism, the intensity distribution may be derived. The functional form is evidently super Gaussian:

$$I = I_0 \exp\left(-\left[\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2}\right]^{n/2}\right).$$

The complexity lies in deriving the required normalization I_0 so that the units are properly W/cm^2 and the values accurately reflect the incident power and beam spot sizes. To do this, note that the area integral of the intensity over the elliptical (`bm_model`=±3) or rectangular (`bm_model`=±4) beam spot must equal the total laser power P_0. In HYDRA, P_0 is the product of the user specified `pmult` from the `superg3d` card and the current interpolated value from the associated `lastimes` card. The normalization I_0 is then determined from the integral relation

$$P_0 = \int dP = \iint I \,\mathrm{d}A = \iint I_0 \exp\left(-\left[\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2}\right]^{n/2}\right) \mathrm{d}x\,\mathrm{d}y,$$

where the integration is carried out over the entire spot: either elliptical or rectangular.

One must be careful in setting the limits of this integration. To avoid launching laser rays with negligible power — or probability — HYDRA truncates the laser spot at three times the spot size. When `bm_model` = ±3, power is sampled within a finite ellipse with semi-major and semi-minor axes of length $3\sigma_x$ and $3\sigma_y$. The total launched power within this ellipse is equal to the total power requested by the user, P_0. When `bm_model`=±4, the entirety of the laser power is instead launched into a rectangle in the focal plane with edges spanning $x \in [-3\sigma_x, 3\sigma_x]$ and $y \in [-3\sigma_y, 3\sigma_y]$ in the lens coordinate system.

To determine the normalization I_0 in the elliptical [`bm_model`=±3] case, the area integral is converted to

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polar coordinates

$$\begin{aligned} P_0 &= \int \mathrm{d}P = \iint I\,\mathrm{d}A = I_0 \int_{-3\sigma_y}^{3\sigma_y} \int_{-3\sigma_x\sqrt{1-y^2/9\sigma_y^2}}^{3\sigma_x\sqrt{1-y^2/9\sigma_y^2}} \exp\left(-\left[\frac{x^2}{\sigma_x^2}+\frac{y^2}{\sigma_y^2}\right]^{n/2}\right)\mathrm{d}x\,\mathrm{d}y \\ &= \sigma_x\sigma_y I_0 \int_{-3}^{3}\int_{-\sqrt{1-y^2/9\sigma_x^2}}^{\sqrt{1-y^2/9\sigma_x^2}} \exp\left(-\left[\frac{x^2}{\sigma_x^2}+\frac{y^2}{\sigma_y^2}\right]\right)\mathrm{d}\left(\frac{x}{\sigma_x}\right)\mathrm{d}\left(\frac{y}{\sigma_y}\right) \\ &= \sigma_x\sigma_y I_0 \int_0^{2\pi}\int_0^3 re^{-r^n}\,\mathrm{d}r\,\mathrm{d}\theta. \end{aligned}$$

Making use of the definition of the incomplete γ function[2], the intensity distribution within the 3σ laser spot is found to be

$$I(x,y) = \left[\frac{P_0}{\sigma_x\sigma_y\left(\frac{2\pi}{n}\right)\gamma\left(\frac{2}{n},3^n\right)}\right]\exp\left(-\left[\frac{x^2}{\sigma_x^2}+\frac{y^2}{\sigma_y^2}\right]^{n/2}\right), \qquad (2)$$

where the normalization factor I_0 is, of course, the factor in square brackets.

In the rectangular beam case — that is, when `bm_model`=±4 —

$$\begin{aligned} P_0 &= I_0\int_{-3\sigma_y}^{3\sigma_y}\int_{-3\sigma_x}^{3\sigma_x}\exp\left(-\left[\frac{x^2}{\sigma_x^2}+\frac{y^2}{\sigma_y^2}\right]^{n/2}\right)\mathrm{d}x\,\mathrm{d}y \\ &= 4\sigma_x\sigma_y I_0\left\{\int_0^{\pi/4}\int_0^{3/\cos\theta} re^{-r^n}\,\mathrm{d}r\,\mathrm{d}\theta + \int_{\pi/4}^{\pi/2}\int_0^{3/\sin\theta} re^{-r^n}\,\mathrm{d}r\,\mathrm{d}\theta\right\} \\ &= 8\sigma_x\sigma_y I_0\int_0^{\pi/4}\int_0^{3/\cos\theta} re^{-r^n}\,\mathrm{d}r\,\mathrm{d}\theta. \end{aligned}$$

Again making use of the incomplete γ function and rearranging, the intensity profile of the rectangular super Gaussian beam mode is given by

$$I(x,y) = \left[\frac{P_0}{\sigma_x\sigma_y\left(\frac{8}{n}\right)\int_0^{\pi/4}\gamma\left(\frac{2}{n},\left[\frac{3}{\cos(\theta)}\right]^n\right)\mathrm{d}\theta}\right]\exp\left(-\left[\frac{x^2}{\sigma_x^2}+\frac{y^2}{\sigma_y^2}\right]^{n/2}\right) \qquad (3)$$

where the symmetry of the underlying super Gaussian beam has been invoked to break the polar angle integration into eight equal pieces.

2 The Numerical Model

The total power P_0, spot sizes σ_x (`spotx`) and σ_y (`spoty`), and super Gaussian exponent n (`npower`) are specified by the user, and with the coordinate of interest (x, y) known, the intensity $I(x, y)$ is easily computed, in principle, from Eq. 2 or 3. The prefactors I_0 are, however, computationally prohibitive to evaluate directly as numerical integrals. To ameliorate this for `bm_model`=±3, HYDRA makes use of the logarithmic polynomial and Padé approximant fits

$$\left(\frac{2\pi}{n}\right)\gamma\left(\frac{2}{n},3^n\right) = \begin{cases} 2\pi\exp\left(c_0+c_1n+c_2n^2+c_3n^3+c_4n^4+c_5n^5\right) & n<1 \\ \dfrac{a_0+a_1n+a_2n^2+a_3n^3+a_4n^4+a_5n^5}{1+b_1n+b_2n^2+b_3n^3+b_4n^4+b_5n^5} & n\geq 1 \end{cases} \qquad (4)$$

where the coefficients are defined in Tables 1 and 2. The Padé approximant is accurate to 0.003% for $1 \leq n \leq 40$ and is good to better than 0.0115% for $1 \leq n \leq 25000$, and the exponential expression is good to better than 0.00153% for $0.02 \leq n \leq 1$. When `bm_model`=±4, the Padé approximants

$$\left(\frac{8}{n}\right)\int_0^{\pi/4}\gamma\left(\frac{2}{n},\left[\frac{3}{\cos\theta}\right]^n\right)\mathrm{d}\theta = \begin{cases} \dfrac{\alpha_0+\alpha_1n+\alpha_2n^2+\alpha_3n^3+\alpha_4n^4+\alpha_5n^5}{1+\beta_1n+\beta_2n^2+\beta_3n^3+\beta_4n^4+\beta_5n^5} & n<2 \\ \dfrac{a_0+a_1n+a_2n^2+a_3n^3+a_4n^4+a_5n^5}{1+b_1n+b_2n^2+b_3n^3+b_4n^4+b_5n^5} & n\geq 2 \end{cases}. \qquad (5)$$

[2] $\gamma(a,x) \equiv \int_0^x t^{a-1}e^{-t}\,\mathrm{d}t$

are used. The coefficients are defined in Tables 2 and 3. The $n \geq 2$ Padé approximant is accurate to 0.006% for $2 \leq n \leq 40$ and is good to better than 0.0115% for $2 \leq n \leq 25000$, and the $n < 2$ Padé approximant is good to better than 0.0059% for $0.02 \leq n \leq 2$. The Padé approximants have been derived using the eigenvalue method of Curtis and Osborne [1], and the exponential fit by polynomial least squares fitting to the logarithm of the incomplete γ function integral.

Notice that for $n \gtrsim 2$ one can make use of the limits

$$\left(\frac{2\pi}{n}\right)\gamma\left(\frac{2}{n},3^n\right) \xrightarrow{n\gtrsim 2} \left(\frac{2\pi}{n}\right)\Gamma\left(\frac{2}{n}\right) \xrightarrow{n\to\infty} \pi$$

$$\left(\frac{8}{n}\right)\int_0^{\pi/4}\gamma\left(\frac{2}{n},\left[\frac{3}{\cos\theta}\right]^n\right)\mathrm{d}\theta \xrightarrow{n\gtrsim 2} \left(\frac{2\pi}{n}\right)\Gamma\left(\frac{2}{n}\right) \xrightarrow{n\to\infty} \pi,$$

which is why the bm_model $= \pm 3$ and ± 4 normalization factors use the same approximant for large n. Physically, this results because the beam becomes sufficiently centrally peaked that there is effectively no intensity at or beyond the 3σ boundary. Incidentally, the second ($n \to \infty$) limit is also the reason that the polynomial orders for the numerator and denominator are the same.

The small n limits may also be analytically determined. The limit as n tends zero of the `bm_model`$=\pm 3$ normalization factor is first computed by making use of the Maclaurin series of the exponential function in computing the γ function integral:

$$\begin{aligned}
\lim_{n\to 0}\frac{2\pi}{n}\int_0^{3^n} t^{2/n-1}e^{-t}\,\mathrm{d}t &= \lim_{n\to 0}\left(\frac{2\pi}{n}\right)\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\int_0^{3^n} t^{2/n-1+m}\,\mathrm{d}t \\
&= \lim_{n\to 0}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\frac{18\pi\cdot 3^{mn}}{2-n+mn} \\
&= 9\pi\sum_{m=0}^{\infty}\frac{(-1)^m}{m!} \\
&= 9\pi e^{-1}.
\end{aligned}$$

Numerically, this is a value approximately equal to 10.40154614811 83. Similarly, the `bm_model`$=\pm 4$ normalization factor limit may be computed using the same technique:

$$\begin{aligned}
\lim_{n\to 0}\frac{8}{n}\int_0^{\pi/4}\int_0^{(3/\cos\theta)^n} t^{2/n-1}e^{-t}\,\mathrm{d}t\,\mathrm{d}\theta &= \lim_{n\to 0}\left(\frac{8}{n}\right)\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\int_0^{\pi/4}\left[\int_0^{(3/\cos\theta)^n} t^{2/n-1+m}\,\mathrm{d}t\right]\mathrm{d}\theta \\
&= \lim_{n\to 0}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\frac{72\cdot 3^{mn}}{2-n+mn}\int_0^{\pi/4}(\sec\theta)^{2+mn}\,\mathrm{d}\theta \\
&= 36\left\{\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\right\}\left\{\int_0^{\pi/4}\sec^2\theta\,\mathrm{d}\theta\right\} \\
&= 36\left\{e^{-1}\right\}\{1\} \\
&= 36e^{-1}
\end{aligned}$$

for a numerical value of 13.2436598821719. The numerical errors of the approximations in Eqs. 4 and 5 in the limits as $n \to 0$ are 0.003% and 0.012%, respectively. In addition to this relatively small error even to $n = 0$, these functional forms do not require the evaluation of factorials of large numbers or infinity as most built-in γ function formulations do.

With the intensity normalization factors I_0 calculated, the laser beam intensity from HYDRA's 3D laser ray trace package in the focal plane for both super Gaussian beam models is now fully specified by Eqs. 2–5 for all values of the super Gaussian order n (`npower`), spot sizes σ_x and σ_y (`spotx` and `spoty`), and laser power P_0 (`pmult`$\times$`lastimes`).

References

[1] A. Curtis and M. R. Osborne. The construction of minimax rational approximations to functions. *The Computer Journal*, 9:286, 1966.

Table 1: Fitting coefficients for an elliptical beam [bm_model=±3] with npower < 1.

c_0	0.50410935866033502
c_1	-0.59960370647330696
c_2	-0.17079985465238293
c_3	-0.03114525147846556
c_4	0.07913248428147572
c_5	-0.00375285384930669

Table 2: Fitting coefficients for an elliptical beam [bm_model=±3] with npower ≥ 1 and for a rectangular beam [bm_model=±4] with npower ≥ 2.

a_0	6.001911243595717		
a_1	-5.394377353542378	b_1	-1.764463010691547
a_2	3.286001028025678	b_2	2.400240048677038
a_3	0.179793458490833	b_3	-1.050131004267564
a_4	-1.204662992085034	b_4	-0.105429575820128
a_5	0.751902653309727	b_5	0.239310679629733

Table 3: Fitting coefficients for a rectangular beam [bm_model=±4] with npower < 2.

α_0	13.2452217950014166		
α_1	-20.3075205938825718	β_1	-0.80259438691477270
α_2	15.3824084867438717	β_2	0.57488756423204912
α_3	-5.9000243145711257	β_3	-0.14368538649811363
α_4	1.2533918458702606	β_4	0.04139209383610994
α_5	-0.0753919083195495	β_5	0.00804441340039306