EFFICIENT ALGORITHM FOR A NONLINEAR TRANSIENT VIBRATIO... SAND2016-0576C

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Summary: The Galerkin method is a powerful computational tool that is successfully applied to many engineering problems. In the recently published monograph [1], V. M. Fridman provides a consistent treatment of the nonlinear vibration problems by using the Newton-Kantorovich approach to linearize nonlinear operators and the Galerkin method to solve the resulting linear problem. In the same book, an original approach to the solution of the linear transient vibration problem also is proposed.

In this paper, we seek to combine both techniques to develop a method for the solution of a nonlinear transient vibration problem. First, we derive equations of the linearized problem using the Newton-Kantorovich method. To facilitate application of the Galerkin method, a new unknown function is introduced. A system of linear equations resulting from the projection conditions of the Galerkin method is derived and solved numerically. The solution is used to perform the next step of the Newton-Kantorovich method. Certain properties of the equations allow for the efficient numerical implementation of the procedure.

NEWTON-KANTOROVICH METHOD FOR A NONLINEAR VIBRATION PROBLEM

For simplicity, the method will be presented as it applies to the damped vibration of a single degree of freedom system with nonlinear stiffness subjected to an external force. An Extension to more complex systems is straightforward. We seek to find a solution to the initial value problem:

$$\ddot{u} + g\dot{u} + c(u) - p = 0, \tag{1}$$

where $u = u(t)$ – mass displacement, t – time, $p = p(t) -$ applied force , $g\dot{u},\ c(u), p\ -$ are ratios of the damping force, nonlinear force in the spring, and external force, respectively, to the mass. For convenience, Eq.(1) is replaced by a system of the two equations with appropriate initial conditions:

$$\begin{matrix} \dot{u} - v = 0, \\ \dot{v} + gv + c(u) - p = 0 \end{matrix} \quad \begin{pmatrix} u(0) \\ \mathrm{v}(0) \end{pmatrix} = \begin{pmatrix} U \\ V \end{pmatrix} \tag{2}$$

which can be written in compact form as,

$$\dot{w} = L(w), \quad L(w) = A(w) + f \tag{3}$$

$$w = \begin{pmatrix} u \\ v \end{pmatrix},\ f = \begin{pmatrix} 0 \\ p \end{pmatrix},\ A(w) = \begin{pmatrix} v \\ -c(u) - gv \end{pmatrix} \tag{4}$$

We also introduce $F(w) = \dot{w} - L(w)$ so that the Eq. (3) takes the form:

$$F(w) = 0 \tag{5}$$

The Newton-Kantorovich method [2] is employed to reduce the solution of the nonlinear Eq. (5) to a sequence of linear problems. Ensuing the Newton-Kantorovich method, the following iterative process is constructed,

$$F(w_{\mathrm{n}}) + F'(w_{\mathrm{n}})(w_{n+1} - w_{\mathrm{n}}) = 0, \tag{6}$$

where $F'(w_{\mathrm{n}})$ denotes Frechet derivative of the operator $F(z)$ and

$$F'(w_{\mathrm{n}})w_{n+1} = \dot{w}_{n+1} - L'(w_n)w_{n+1} = \begin{pmatrix} \dot{u}_{n+1} \\ \dot{v}_{n+1} \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -c'(u_n) & -g \end{pmatrix} \begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix},\quad c' = \frac{dc(u)}{du} \tag{7}$$

The iterative process defined by the equations will, under certain conditions [2], converge to the solution of Eq. (5). After substituting (3), (4), and (7), Eq. (6) takes form:

$$\begin{pmatrix} \dot{u}_{n+1} - v_{n+1} \\ \dot{v}_{n+1} + c(u_n) + c'(u_n)(u_{n+1} - u_n) + gv_{n+1} \end{pmatrix} = \begin{pmatrix} 0 \\ p \end{pmatrix} \tag{8}$$

The linearized equations in (8) can be solved numerically using ordinary differential equation solution methods. We use the Galerkin method to solve the system in Eq. (8). A driving force is the ability to obtain a functional form of the solution.

GALERKIN METHOD FOR THE LINEARIZED EQUATION

The solution of the linearized equation (8) needs to be determined at each step of the iterative procedure. Application of the Galerkin method for that purpose was extensively studied by Krasnoselskii et al [3]. In the case of periodic oscillations, it is convenient to construct the solution as a Fourier series and use Galerkin projection conditions to find unknown coefficients in the expansion. Unfortunately, that idea can't be applied directly to the case of transient vibration.

A modification of the method for the case of linear transient vibration was proposed in [1], where new unknown functions were introduced in a way that allowed recasting the problem in a form suitable for the application of the Galerkin method. We use that approach to solve the initial value problem for the linearized system in (6). First, functions $\varphi(t),\ \psi(t)$ are introduced:

$$\varphi(t) = u(t)e^{-qt}; \quad u(t) = \varphi(t)e^{qt} \ \ and \ \ \psi(t) = v(t)e^{-qt}; \ v(t) = \psi(t)e^{qt}, \ q > 0 \tag{9}$$

Initial conditions for the newly introduced functions are still the same:

$$\varphi(0) = U; \quad \psi(0) = V \tag{10}$$

The transient problem can be solved for a finite but sufficiently large interval $[0;T]$. On that finite, interval functions $\varphi(t)$, $\psi(t)$ and external force can be represented by a Fourier series

$$\varphi(t) = \sum_{m=-M}^{M} \varphi^m e^{im\omega t}, \varphi^m = \frac{1}{T}\int_0^T e^{-im\omega t}\varphi(t)\,dt \tag{11}$$

$$\psi(t) = \sum_{m=-M}^{M} \psi^m e^{im\omega t}, \psi^m = \frac{1}{T}\int_0^T e^{-im\omega t}\psi(t)dt\,, \quad \omega = 2\pi/T \tag{12}$$

$$p(t) = \sum_{m=-M}^{M} p^m e^{im\omega t}, p^m = \frac{1}{T}\int_0^T e^{-im\omega t} p(t)dt \tag{13}$$

The Galerkin method is used to determine coefficients in the expansions (11) and (12). A special choice of the test function proposed in [1] for the linear problem is employed here as well and projection conditions for Eq.(8) are written as:

$$\frac{1}{T}\int_0^T e^{-(q+im\omega)t}\left[\frac{d(\varphi_{n+1}(t)e^{qt})}{dt} - (\psi_{n+1}(t)e^{qt})\right]dt = 0 \tag{14}$$

$$\frac{1}{T}\int_0^T e^{-(q+im\omega)t}\left[\frac{d(\psi_{n+1}(t)e^{qt})}{dt} + c(\varphi_n(t)e^{qt}) + c'(\varphi_n(t)e^{qt})((\varphi_{n+1}(t)-\varphi_n(t))e^{qt}) + g\psi_{n+1}(t)e^{qt}\right]dt \tag{15}$$

The integrals in Eq. (14) and (15) that contain derivatives with respect to time are evaluated by integration by parts, for example:

$$\frac{1}{T}\int_0^T e^{-(q+im\omega)t}\left[\frac{d(\varphi_{n+1}(t)e^{qt})}{dt}\right]dt = \frac{1}{T}[\varphi_{n+1}(T) - \varphi_{n+1}(0)] + \frac{(q+im\omega)}{T}\int_0^T e^{-im\omega t}\varphi_{n+1}(t)dt = 0 \tag{16}$$

As stated in [1], because of (9), $\varphi_{n+1}(T)$ can be made as small as desired in almost any problem by proper choice of q and T. Using the expansion (11) and initial condition (10), the integral in (16) can be reduced to

$$\frac{1}{T}\int_0^T e^{-(q+im\omega)t}\left[\frac{d(\varphi_{n+1}(t)e^{qt})}{dt}\right]dt = -\frac{U}{T} + (q+im\omega)\varphi_{n+1}^m \tag{17}$$

The second term in Eq.(14) is computed as

$$\frac{1}{T}\int_0^T e^{-(q+im\omega)t}[-(\psi_{n+1}(t)e^{qt})]dt = -\frac{1}{T}\int_0^T e^{-im\omega t}\psi_{n+1}(t))dt = -\psi_{n+1}^m \tag{18}$$

After evaluating the integrals in (16) in a similar manner and using c^m, d^{k-m} and r^{k-m} to denote their values, we arrive at the following system of the $2M$ equations for the unknown coefficients ψ_{n+1}^m, φ_{n+1}^m

$$(q+im\omega)\varphi_{n+1}^m - \psi_{n+1}^m - \frac{U}{T} = 0, \quad m = 1, \dots, M \tag{19}$$

$$-\frac{V}{T} + \left(q + im\omega + \frac{g}{T}\right)\psi_{n+1}^m + c^m + \sum_{k=-M}^{M}\varphi_{n+1}^k d^{k-m} - \sum_{k=-M}^{M}\varphi_n^k r^{k-m} = p^m, \quad m = 1, \dots, M \tag{20}$$

This system has some special properties that allow for efficient implementation and simplified numerical calculations. After the solution of the system is found, a new step in the iterative process (6) is performed. The procedure is continued until desired covergence is achieved.

CONCLUSIONS

We developed a method for a solution of the nonlinear transient vibration based on the combination of the Newton-Kantorovich iterations and the Galerkin projection procedure. This method is more efficient than conventional methods based on the numerical integration of the equations of motion. Unlike averaging or harmonic balance methods, a desired accuracy can be achieved simply by increasing the number of the retained harmonics in the projection step and increasing the number of the Newton-Kantorovich iterations.

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