

Representing SAR complex image pixels

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ABSTRACT

Synthetic Aperture Radar (SAR) images are often complex-valued to facilitate specific exploitation modes. Furthermore, these pixel values are typically represented with either real/imaginary (also known as I/Q) values, or as Magnitude/Phase values, with constituent components comprised of integers with limited number of bits. For clutter energy well below full-scale, Magnitude/Phase offers lower quantization noise than I/Q representation. Further improvement can be had with companding of the Magnitude value.

Keywords: SAR, complex, image, pixels

1 INTRODUCTION

The digital processing of radar data for a number of airborne Intelligence, Surveillance, and Reconnaissance (ISR) radar modes entails images or maps typically of range versus range-rate, sometimes termed “Doppler.” Individual range-Doppler cells are rendered as elements in a 2-dimensional array, often displayed as pixels in an image format. We will hereafter term these complex elements generically as “pixels” regardless of any display limitations.

Digital processing allows these pixels to manifest phase information as well as magnitude information; corresponding to complex values. That is, each pixel is a complex number. Such pixel values require description with two independent components, such as real and imaginary components, also known as In-phase (I) and Quadrature-phase (Q) components, or perhaps Magnitude and Phase components. This is true of Synthetic Aperture Radar (SAR), Inverse SAR (ISAR), and Moving Target Indicator (MTI) range-Doppler maps. The ability to process phase as well as magnitude information is essential for some image exploitation algorithms, including Coherent Change Detection (CCD), Interferometric SAR (InSAR, or IFSAR), and multi-channel MTI processing.

Furthermore, data size limitations typically cause standard image formats to describe the complex pixel values with pairs of integer values, rather than floating point values. Consequently there are inherent dynamic range issues with the precision limitations that integer descriptions impose upon pixel values. However, for a fixed number of bits per pixel, the precise format of the complex number will impact the dynamic range that is achievable. Quite simply, some formats are ‘better’ than others, especially for the kinds of radar target data the radar is likely to encounter.

In this report, we will confine our analysis to complex pixel encoding for a SAR image of uniform distributed land clutter. In particular, we are interested in the relative performance of integer encoding of I/Q pixel values versus Magnitude/Phase pixel values. In particular, we desire the clutter to be adequately represented to facilitate exploitation as described above. We will further discuss alternate encoding schemes for the pixel magnitude to enhance the dynamic range that is achievable. We will generally ignore any system noise effects, and assume that the only perturbation of pixel values is due to quantization itself. Our measure will be the ratio of mean clutter energy to quantization noise energy in the pixel, which we will term as the Clutter-to-Noise Ratio (CNR).

As background to the following discussion, we offer the following reference material.

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The underlying nature of uniform distributed clutter is described by Oliver and Quegan.¹

Typical clutter reflectivity values are described by Long² and also by Ulaby and Dobson.³

The topic of radar complex image encoding is very related to radar complex image compression, which seeks to minimize, or at least reduce, the number of bits necessary to store or transmit the image data without unacceptable loss in precision and/or accuracy. A paper by Eichel and Ives discusses SAR complex image compression.⁴

The impact on various complex pixel encoding schemes on SAR CCD is discussed in a report by Thompson.⁵

This paper abridges an earlier more comprehensive report.⁶

2 CLUTTER MODEL

We shall model uniform clutter in a pixel as a complex random variable, with In-phase (I) and Quadrature-phase (Q) components modelled as independent Gaussian random variables. That is, the pixel value is given as

$$z(m, n) = X + jY = \text{complex pixel value}, \quad (1)$$

where

$$\begin{aligned} m &= \text{pixel row number,} \\ n &= \text{pixel column number,} \\ X &= \text{pixel In-phase value (random variable), and} \\ Y &= \text{pixel Quadrature-phase value (random variable).} \end{aligned} \quad (2)$$

The Probability Density Distribution (PDF) functions for the individual independent random variables are Gaussian, and described by

$$\begin{aligned} f_X(x) &= \frac{1}{\sqrt{2\pi}\sigma} e^{-\left(\frac{x}{\sigma}\right)^2}, \text{ and} \\ f_Y(y) &= \frac{1}{\sqrt{2\pi}\sigma} e^{-\left(\frac{y}{\sigma}\right)^2}, \end{aligned} \quad (3)$$

where σ = standard deviation of the random variable. Since these components are independent of each other, the joint PDF can be written as

$$f_{X,Y}(x, y) = f_X(x) f_Y(y). \quad (4)$$

We note that the very same pixel can be written in Magnitude-Phase format as

$$z(m, n) = R e^{jP}, \quad (5)$$

where

$$\begin{aligned} R &= \text{pixel magnitude value (random variable), and} \\ P &= \text{pixel phase value (random variable).} \end{aligned} \quad (6)$$

We note that these are related to the I/Q coordinates as

$$R = \sqrt{X^2 + Y^2}, \text{ and}$$

$$P = \text{atan}\left(\frac{Y}{X}\right). \quad (7)$$

These are also random variables, but with PDF functions given as Rayleigh and Uniform respectively, described by

$$f_R(r) = \frac{r}{\sigma^2} e^{-\frac{1}{2}\left(\frac{r}{\sigma}\right)^2} \text{ for } r \geq 0, \text{ and}$$

$$f_P(p) = \frac{1}{2\pi} \text{rect}\left(\frac{p}{2\pi}\right), \quad (8)$$

where

$$\text{rect}(u) = \begin{cases} 1 & u \leq 1/2 \\ 0 & \text{else} \end{cases}. \quad (9)$$

Similarly to the I/Q coordinates, since these components are independent of each other, their joint PDF can be written as

$$f_{R,P}(r, p) = f_R(r) f_P(p). \quad (10)$$

We note that the clutter model exhibits circular symmetry in the complex plane. Note that the average pixel clutter power is

$$C = 2\sigma^2. \quad (11)$$

3 QUANTIZATION ERROR ANALYSIS

Let us now define a quantized version of the pixel value as having constituent components

$$x_i = \text{quantized pixel In-phase value, and}$$

$$y_i = \text{quantized pixel Quadrature-phase value,} \quad (12)$$

where i = index value into some finite set of allowable quantized values. Corresponding magnitude and phase values are

$$r_i = \sqrt{x_i^2 + y_i^2}, \text{ and}$$

$$p_i = \text{atan}\left(\frac{y_i}{x_i}\right). \quad (13)$$

A corresponding quantized complex pixel value is then

$$z_i(m, n) = x_i + jy_i = r_i e^{jp_i} = \text{quantized complex pixel value.} \quad (14)$$

The quantized values for these components are limited to some set of allowable values. Generally, the allowable values are not independent of each other. We will assume the particular member of the set that is chosen will be that member that is nearest the actual un-quantized value being considered for conversion (quantization). We define the quantization error as the distance between the true value and its quantized value, namely

$$\varepsilon_i = |z(m, n) - z_i(m, n)| = \text{quantization error,} \quad (15)$$

where the quantized value is that which is nearest the true value. The Mean-Squared-Error (MSE) due to quantization error is the statistical measure in which we are interested. We calculate this for an individual member of the set of allowable quantization values as

$$MSE_i = \text{mean}(\varepsilon_i^2) = \text{MSE quantization error for } z_i(m, n), \quad (16)$$

where the mean is calculated over the space of pixel complex values before any quantization. We perform this mean calculation as follows. Consider an area in the complex plane around some allowed quantized pixel value with index i , where all points in this area are assigned to quantized values (x_i, y_i) or equivalent. We define this area as

$$A_i = \text{area in complex plane quantized to value } (x_i, y_i). \quad (17)$$

This is illustrated in Figure 1. We define the overall mean squared quantization error then as

$$MSE = \sum_i \int \int_{A_i} \varepsilon_i^2 f_{X,Y}(x, y) dx dy = \text{mean squared quantization error}. \quad (18)$$

For the purposes of this report, we will assume that $f_{X,Y}(x, y)$ is relatively constant over any particular area A_i . In fact we will assume a conditional Uniform distribution over A_i which allows us to approximate the probability of selecting a coordinate within area A_i as

$$\text{Prob}(A_i) \approx f_{X,Y}(x_i, y_i) A_i. \quad (19)$$

This may also be written as

$$\text{Prob}(A_i) \approx f_{R,P}(r_i, p_i) A_i. \quad (20)$$

Note that the sum of probabilities for all pixels is one, that is

$$\sum_i \text{Prob}(A_i) = 1. \quad (21)$$

The mean squared quantization error is then

$$MSE \approx \sum_i MSE_i \text{Prob}(A_i), \quad (22)$$

where the local MSE is the variance in the quantization error, that is MSE_i = the local MSE within and over the area A_i . In any case, the Clutter-to-Noise Ratio (CNR) for quantization noise is given by

$$CNR = \frac{C}{MSE} = \frac{2\sigma^2}{MSE}. \quad (23)$$

We now consider two special cases.

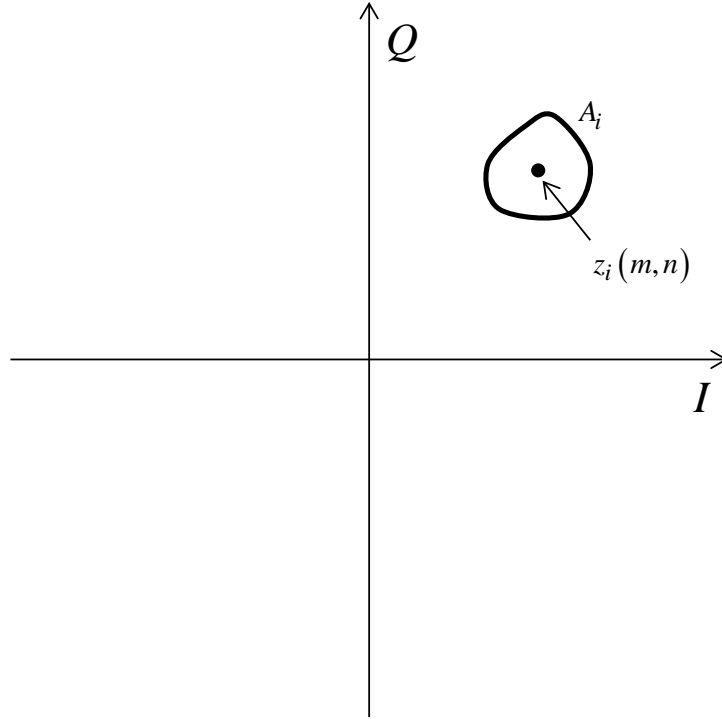


Figure 1. Generalized quantization process.

3.1 Uniform I/Q Quantization

Consider a rectangular area in the complex plane that is centered at (x_i, y_i) , and is defined by

$$\begin{aligned} \Delta x &= \text{the span of the area in the I direction, and} \\ \Delta y &= \text{the span of the area in the Q direction.} \end{aligned} \quad (24)$$

This is illustrated in Figure 2. The area can then be calculated as

$$A_i = \Delta x \Delta y. \quad (25)$$

Since we will presume that $f_{X,Y}(x, y)$ is constant over this area, then we have essentially a uniform distribution of the quantization error over the area A_i . The variance in the quantization error is then identified as

$$MSE_i = \frac{\Delta x^2}{12} + \frac{\Delta y^2}{12} = \text{variance in quantization error over the area } A_i, \quad (26)$$

The overall MSE for uniform I/Q quantization is then calculated as

$$MSE_{IQ} \approx \sum_i \left(\frac{\Delta x^2}{12} + \frac{\Delta y^2}{12} \right) \text{Prob}(A_i) \approx \frac{\Delta x^2}{12} + \frac{\Delta y^2}{12}. \quad (27)$$

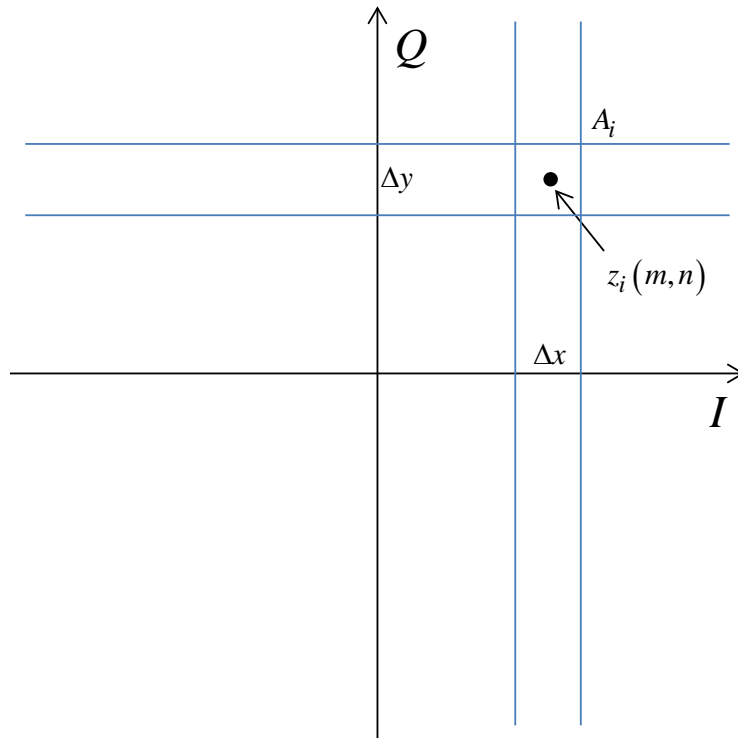


Figure 2. Uniform I/Q Quantization model.

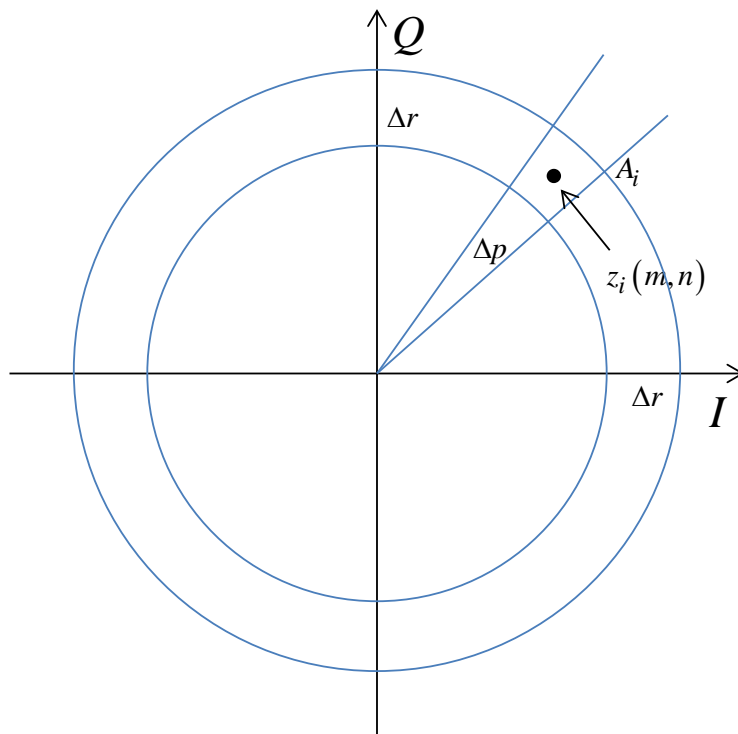


Figure 3. Uniform Magnitude/Phase Quantization model.

Now consider that the spans of the quantization area can be written in terms of full-scale values and number of bits as

$$\Delta x = \frac{2\sigma_{fs}}{2^{b_I}}, \text{ and}$$

$$\Delta y = \frac{2\sigma_{fs}}{2^{b_Q}}, \quad (28)$$

where σ_{fs} = full-scale signal value, b_I = number of bits to represent I value, and b_Q = number of bits to represent Q value. The factor of “2” in the numerator of the equations accounts for allowing the signal to span the range $(-\sigma_{fs}, \sigma_{fs})$ in each of the I and Q dimensions. We are also assuming that full-scale signals are substantially larger than the clutter level, that is $\sigma_{fs} \gg \sigma$. This is quite typical for SAR images. Combining these allows us to write

$$MSE_{IQ} \approx \frac{\sigma_{fs}^2}{3} \left(\frac{1}{2^{2b_I}} + \frac{1}{2^{2b_Q}} \right). \quad (29)$$

The CNR is then calculated as

$$CNR = \frac{C}{MSE_{IQ}} = 6 \left(\frac{\sigma^2}{\sigma_{fs}^2} \right) \left(\frac{1}{2^{2b_I}} + \frac{1}{2^{2b_Q}} \right)^{-1}. \quad (30)$$

We observe that bits in I versus bits in Q have equivalent impact on MSE. Consequently, there is no advantage in trading one for the other.

3.2 Uniform Magnitude/Phase Quantization

Now consider a segment of an annular ring in the complex plane that is centered at (r_i, p_i) , and is defined by

$$\Delta r = \text{the span of the area in the magnitude direction, and}$$

$$\Delta p = \text{the span of the area in the phase direction.} \quad (31)$$

This is illustrated in Figure 3. For small spans in both magnitude and phase, the area can then be calculated as

$$A_i \approx \Delta r r_i \Delta p. \quad (32)$$

Since we will presume that $f_{R,P}(r, p)$ is constant over this area, then we have essentially a uniform distribution of the quantization error over the area A_i . The variance in the quantization error is then identified as

$$MSE_i \approx \frac{\Delta r^2}{12} + \frac{r_i^2 \Delta p^2}{12} = \text{variance in quantization error,} \quad (33)$$

The overall MSE for uniform Magnitude/Phase quantization is then calculated as

$$MSE_{RP} \approx \sum_i \left(\frac{\Delta r^2}{12} + r_i^2 \frac{\Delta p^2}{12} \right) \text{Prob}(A_i). \quad (34)$$

This can be expanded and then simplified to

$$MSE_{RP} \approx \frac{\Delta r^2}{12} \sum_i \text{Prob}(A_i) + \frac{\Delta p^2}{12} \sum_i r_i^2 \text{Prob}(A_i) \approx \frac{\Delta r^2}{12} + \frac{\Delta p^2}{12} (2\sigma^2). \quad (35)$$

Now consider that the spans of the quantization area can be written as

$$\Delta r = \frac{\sigma_{fs}}{2^{b_R}}, \text{ and}$$

$$\Delta p = \frac{2\pi}{2^{b_P}}, \quad (36)$$

where σ_{fs} = full-scale signal value, b_R = number of bits to represent Magnitude value, and b_P = number of bits to represent Phase value. We are also still assuming that full-scale signals are substantially larger than the clutter level, that is $\sigma_{fs} \gg \sigma$. Recall that this is quite typical for SAR images. Combining these allows us to write

$$MSE_{RP} \approx \frac{\sigma_{fs}^2}{12} \left(\left(\frac{1}{2^{2b_R}} \right) + 4\pi^2 \left(\frac{2\sigma^2}{\sigma_{fs}^2} \right) \left(\frac{1}{2^{2b_P}} \right) \right). \quad (37)$$

The CNR is then calculated as

$$CNR = \frac{C}{MSE_{RP}} = 12 \left(\frac{2\sigma^2}{\sigma_{fs}^2} \right) \left(\left(\frac{1}{2^{2b_R}} \right) + 4\pi^2 \left(\frac{2\sigma^2}{\sigma_{fs}^2} \right) \left(\frac{1}{2^{2b_P}} \right) \right)^{-1}. \quad (38)$$

We observe that bits in Magnitude have a different impact than bits in Phase. For low clutter levels, the phase bits have severely diminished influence, and this can be approximated as

$$CNR \approx 12 \left(\frac{2\sigma^2}{\sigma_{fs}^2} \right) 2^{2b_R}. \quad (39)$$

This also suggests that phase bits can be traded for magnitude bits to some degree. For example, for a mean clutter level 50 dB below full-scale, that is $2\sigma^2/\sigma_{fs}^2 = 10^{-5}$, equivalent impact allows about 6 fewer phase bits than magnitude bits.⁶

3.3 Comments on Comparing the Two

We can examine the relative MSE for the two complex pixel representations by looking at their ratio, namely

$$\frac{MSE_{RP}}{MSE_{IQ}} \approx \frac{\left(\frac{1}{2^{2b_R}} + 4\pi^2 \left(\frac{2\sigma^2}{\sigma_{fs}^2} \right) \frac{1}{2^{2b_P}} \right)}{4 \left(\frac{1}{2^{2b_I}} + \frac{1}{2^{2b_Q}} \right)}. \quad (40)$$

For the same number of bits in all components (i.e. I, Q, Magnitude, and Phase), this simplifies to

$$\frac{MSE_{RP}}{MSE_{IQ}} \approx \frac{1 + 4\pi^2 \left(\frac{2\sigma^2}{\sigma_{fs}^2} \right)}{8} \approx \frac{1 + 40 \left(\frac{2\sigma^2}{\sigma_{fs}^2} \right)}{8}. \quad (41)$$

For average clutter levels well below the full-scale limit, this ratio approaches

$$\frac{MSE_{RP}}{MSE_{IQ}} \rightarrow \frac{1}{8}. \quad (42)$$

This suggests that Magnitude/Phase format approaches a 9 dB advantage over I/Q format with respect to quantization noise for low clutter levels. We observe that 6 dB of this is due to the finer spacing of magnitude increments being limited to the range $[0, \sigma_{fs}]$, and 3 dB of this is due to the phase contribution to MSE disappearing at low clutter levels. This advantage manifests as an improved CNR, which in turn manifests as an improved coherence measure with truth, and ultimately as improved performance of those modes that depend on coherence, such as Interferometric SAR (InSAR or IFSAR), and Coherent Change Detection (CCD).

4 MAGNITUDE COMPANDING

We observe from the previous sections that CNR is improved with finer quantization in the region of the complex plane where signal levels are more likely, which for clutter means at smaller magnitudes. This begs the question “Can we just use finer quantization near the origin of the complex plane where the clutter signal is more likely, and leave the coarser quantization away from the origin where the clutter signal is less likely?” The answer is quite obviously “Yes.” In telephony this is often called “companding,” as a blending of “compressing and expanding” the signal for transmission through a channel with limited dynamic range. The essence of companding is to increase sensitivity to small signals by sacrificing sensitivity to large signals; reducing the ratio of peak to average power. Schemes to accomplish this are discussed in any number of communications texts.^{7,8,9}

The companding can be effected by putting the linear signal through a nonlinear gain stage prior to the channel (compression), and then at the other end applying the inverse of the nonlinear characteristic to the signal (expansion) to recover the original linear signal. If the channel is an Analog-to-Digital Converter (ADC) then the process is described as nonlinear encoding of the signal. If applied to digital data, it might be called Digital Companding. While we could apply companding to I/Q values, the circular symmetry of our clutter suggests that we are best served applying companding to only the magnitude of the clutter signal, and not the phase. We will henceforth only consider companding applied to clutter signal magnitude. Any of the following functions would serve as a compression function prior to quantization

- N^{th} root of magnitude,
- logarithm of magnitude,
- floating-point representation of magnitude (i.e. magnitude with exponent), and
- any monotonic increasing function with diminishing slope.

A number of other specific companding functions also exist, which we will not specifically address here.

Recovering the linear magnitude involves applying the inverse function of these compression functions. This expands the signal to its original magnitude. When expanded, we witness that any additive noise now has power that is signal-magnitude dependent, or more so than without companding. This is true for both quantization noise as well as additive system noise. Consequently, our metric for companding performance will remain the CNR. Table 1 details some examples of various companding functions as well as bit-resolutions for magnitude and phase data. We make some observations as follows.

- Companding increases CNR, more so for lower clutter levels.
- For equal number of magnitude and phase bits and –50 dB clutter with respect to full-scale, companding with a cubic magnitude scaling (companding function) is worth about 4 bits each (8 bits total) with respect to a linear magnitude scaling.
- Even with linear magnitude scaling and –50 dB clutter with respect to full-scale, trading 3 phase bits for 3 magnitude bits yields about a 13 dB increase in CNR.

Table 1. CNR estimates for various bit-resolutions and sample companding functions.

Magnitude Bits	Phase Bits	calculation vs. numerical	Magnitude scaling	Clutter relative to Full-scale (dB)	CNR (dB)
16	16	calculation	linear	-70	37.1214
16	16	numerical	linear	-70	37.1227
16	16	numerical	square root	-70	66.6096
16	16	numerical	cube root	-70	74.5925
16	16	calculation	linear	-50	57.1214
16	16	numerical	linear	-50	57.1216
16	16	numerical	square root	-50	76.4769
16	16	numerical	cube root	-50	80.9259
12	12	calculation	linear	-50	33.0390
12	12	numerical	linear	-50	33.0346
12	12	numerical	square root	-50	52.3910
12	12	numerical	cube root	-50	56.8407
8	8	calculation	linear	-50	8.9566
8	8	numerical	linear	-50	8.4298
8	8	numerical	square root	-50	28.3069
8	8	numerical	cube root	-50	32.7573
19	13	numerical	linear	-50	71.0051
15	9	numerical	linear	-50	46.9219
11	5	numerical	linear	-50	22.8403

How much companding is desired?

The somewhat broader question is “When and how much companding is desired or required?” The answer to this requires as prerequisite two items of knowledge, namely “How much dynamic range do we have?” and “How much dynamic range do we need?” Dynamic range is measured here in dB. We define it here as the difference between full scale and the noise level, presumed to be the quantization noise for this discussion.

Consider magnitude only for now. We recall that with a linear scaling, each magnitude bit yields 6.02 bits of dynamic range. Quantization noise manifests at approximately 11 dB below the Least Significant Bit (LSB), adding 11 dB to the dynamic range *as limited by quantization noise*. Recall that we are ignoring other noise sources in this discussion. Consequently, for linear scaling we identify the dynamic range available as

$$DR_{linear} \approx 6.02 b_R + 11. \quad (43)$$

For example, with 16 bits of magnitude, and neglecting phase, we identify a dynamic range of about 107 dB. If we require more than this, then companding should be considered.

But with companding, quantization noise is clutter signal-level dependent. Therefore dynamic range as defined above is signal dependent when companding is used. However, since our reason to improve dynamic range is generally to allow observation of weaker signals, we will assume for analysis that the signal level of interest is when the RMS clutter level is at the LSB. For 16 bits of magnitude, this is at a clutter level about 96 dB below full-scale. At this clutter signal-level,

we retain a CNR of about 11 dB, due to quantization alone and of course absent any other noise sources. With clutter at this signal-level, based on numerical simulations, a square-root companding function will boost dynamic range to about 53 dB, and a cube-root companding function will yield about 66 dB of dynamic range. This illustrates the improvement that companding offers. We do note that while companding magnitude gives us smaller magnitude steps for smaller signals, we do not get smaller phase steps. However, the quantization error due to phase also depends on magnitude as Eq. (33) suggests. Consequently, for small signals, magnitude still dominates quantization, even with companding.

In addition, with other noise sources perturbing the pixel value, such as system noise, there is essentially no advantage in reducing MSE due to quantization very far below the system noise. As a practical matter, real CNR will be dominated by the larger of quantization noise or system noise.

5 CONCLUSIONS

We summarize the preceding analysis by stating that for complex radar image pixels, a specific number of bits with Magnitude/Phase representation generally offers superior precision and accuracy over an I/Q representation. For low-clutter signals, the advantage is about 9 dB as measured with respect to quantization noise. Furthermore, an equal number of Magnitude and Phase bits will have unequal impact on quantization noise. Consequently, several Phase bits can generally be traded for Magnitude bits to achieved improved (lower) quantization noise. Finally, companding the magnitude of a pixel can significantly reduce the statistical quantization noise for clutter.

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REFERENCES

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- ¹ Christopher Oliver, Shaun Quegan, *Understanding Synthetic Aperture Radar Images*, ISBN089006850X, Artech House, Inc., 1998.
 - ² Maurice W. Long, *Radar Reflectivity of Land and Sea*, second edition, ISBN-13: 978-0890061305 , Artech House, Inc., 1983.
 - ³ Fawwaz T. Ulaby, M. Craig Dobson, *Handbook of Radar Scattering Statistics for Terrain (Artech House Remote Sensing Library)*, ISBN-13: 978-0890063361, Artech House, Inc., 1989.
 - ⁴ Paul Eichel, Robert W. Ives, "Compression of Complex-Valued SAR Images," IEEE Transactions on Image Processing, Vol. 8, No. 10, pp. 1483-1487, October 1999.
 - ⁵ Douglas G. Thompson, *Correlation and Image Compression for Limited-Bandwidth CCD*, Sandia Report SAND2005-3402, Unlimited Release, July 2005.
 - ⁶ Armin W. Doerry, "SAR Image Complex Pixel Representations," Sandia National Laboratories Report SAND2015-2309, Unlimited Release, March 2015.
 - ⁷ K. Sam Shanmugam, *Digital and Analog Communication Systems*, ISBN 0-471-03090-2, John Wiley & Sons, Inc., 1979.
 - ⁸ Wayne Tomasi, *Advanced Electronic Communications Systems*, fifth edition, ISBN 0-13-022126-0, Prentice Hall, Inc., 2001.
 - ⁹ John G. Proakis, Masoud Salehi, *Communications Systems Engineering*, second edition, ISBN 0-13-061793-8, Prentice Hall, Inc., 2002.