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Notes on the virtual element method for second-order elliptic problems: the low-order conforming formulation

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Abstract

This document contains very simple introductory notes on the Virtual Element Method. Only the low-order conforming formulation is addressed and formulas for the implementation are given.

1. Introduction

1.1. This document contains working annotations on the Virtual Element Method (VEM) for the approximate solution of diffusion problems with variable coefficients. To read this document you are assumed to have familiarity with concepts from the numerical discretization of Partial Differential Equations (PDEs) and, in particular, the Finite Element Method (FEM). This document is *not* an introduction to the FEM, for which many textbooks (also free on the internet) are available. Eventually, this document is intended to evolve into a *tutorial introduction* to the VEM (but this is really a long-term goal).

1.1.1. To ease the exposition, we will refer to the Laplace problem and consider the conforming method for the primal form in two space dimensions.

1.2. Acronyms used in this document:

- **FEM** = Finite Element Method
- **VEM** = Virtual Element Method
- **MFD** = Mimetic Finite Difference method
- **PFEM** = Polygonal Finite Element Method
- **PDE** = Partial Differential Equation

1.3. The Virtual Element Method is a kind of Finite Element Methods where trial and test functions are the solutions of a PDE problem inside each element.

1.3.1. A *naive* approach to implement the VEM would consist in solving each local PDE problem to compute the trial and test functions at least approximately where needed (for example, at the quadrature nodes of the element).

1.4. The key point in the VEM approach is that some elliptic and L^2 polynomial projections of functions and gradients of functions that belong to the finite element space are computable exactly using only their

degrees of freedom. So, the VEM strategy is to substitute test and trial functions in the bilinear forms and linear functionals of the variational formulations with their polynomial projections whenever these latters are computable. In fact, the value of such bilinear forms is exact whenever at least one of the entries is a polynomial function of a given degree. We refer to this exactness property as the *polynomial consistency* or, simply, the *consistency* of the method. This strategy leads to an underdetermined formulation (the global stiffness matrix is rank deficient). In the VEM we fix this issue by adding a stabilization term to the bilinear form. Stabilization terms are required not to upset the exactness property for polynomials and to scale properly with respect to the mesh size and the problem coefficients.

1.4.1. The low-order setting for polygonal cells (in 2D) and polyhedral cells (in 3D) can be obtained as a straightforward generalization of the FEM. The high-order setting deserves special care in the treatment of variable coefficients in order not to loose the optimality of the discretization and in the 3D formulation.

1.5. The shape functions are virtual in the sense that we never compute them explicitly (not even approximately) inside the elements. Note that the polynomial projection of the approximate solution is readily available from the degrees of freedom and we can use it as the numerical solution.

1.6. The major advantage offered by the VEM is in the great flexibility in admitting unstructured polygonal and polyhedral meshes.

1.7. The VEM can be seen as an evolution of the MFD method. Background material (and a few historical notes) will be given in the final section of this document.

2. Meshes

2.1. Meshes with polygonal & polyhedral cells.

The VEM inherits the great flexibility of the MFD method and is suitable to approximate PDE problems on unstructured meshes of cells with very general geometric shapes. The mesh flexibility is a major feature of this kind of discretization, as we often need that computational meshes should be easily adaptable to:

- the geometric characteristics of the domain;
- the solution.

2.1.1. Examples of 2D meshes used in academic problems are given in Figure 1.

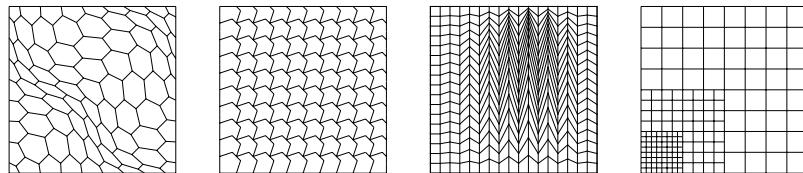


Figure 1: Convex and non-convex cells; highly deformed cells; local adaptive refinements (AMR, hanging nodes).

2.1.2. Examples of 3D meshes used in academic problems are given in Figure 2.

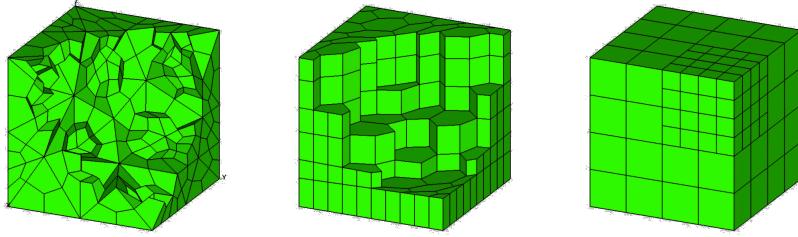


Figure 2: Random hexahedra; prismatic meshes; local adaptive refinements (AMR, hanging nodes).

3. Formulations

To introduce the basic ideas of the virtual element method, we first consider the case of the Laplace equation. Let $\Omega \subset \mathbb{R}^d$ ($d = 2$) be a polygonal domain with boundary Γ . We use standard notation on Sobolev (and Hilbert) spaces, norms, and semi-norms. Hereafter, $H_g^1(\Omega)$ denotes the affine Sobolev space of functions in $L^2(\Omega)$ with first derivatives in $L^2(\Omega)$, and boundary trace equal to g ; $H_0^1(\Omega)$ is the subspace of $H^1(\Omega)$ corresponding to $g = 0$. Also, we assume that $f \in L^2(\Omega)$ and $g \in H^{\frac{1}{2}}(\Gamma)$.

3.1. Differential formulation. *Find $u \in H^1(\Omega)$ such that:*

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= g \quad \text{on } \Gamma. \end{aligned}$$

3.2. Variational formulation. *Find $u \in H_g^1(\Omega)$ such that:*

$$\mathcal{A}(u, v) := \int_{\Omega} \nabla u \cdot \nabla v \, dV = \int_{\Omega} f v \, dV =: (f, v) \quad \forall v \in H_0^1(\Omega).$$

3.3. Finite element formulation. *Find $u_h \in V_{h,g} \subset H_g^1(\Omega)$ such that:*

$$\mathcal{A}(u_h, v_h) := \int_{\Omega} \nabla u_h \cdot \nabla v_h \, dV = \int_{\Omega} f v_h \, dV =: (f, v_h) \quad \forall v_h \in V_{h,0} \subset H_0^1(\Omega),$$

where $V_{h,g}$ and $V_{h,0}$ are finite dimensional affine and linear functional spaces, dubbed *the finite element spaces* (see Section 4 for the formal definition).

3.4. A short guide to a happy life (with VEM)

Let P be a cell of a mesh partitioning Ω_h of domain Ω . The construction of the VEM follows these four logical steps.

- (i) Define the functions of the virtual element space $V_h(P)$ as the solution of a PDE problem;
- (ii) Split the virtual element space as the direct sum of a the polynomial subspace $\mathbb{P}_m(P)$ of the polynomials of degree up to m and its complement:

$$V_h(\mathcal{P}) = [\text{polynomials}] \oplus [\text{non-polynomials}].$$

At this point, it is natural to introduce the polynomial projection operator $\Pi_m^\nabla : V_h(\mathcal{P}) \rightarrow \mathbb{P}_m(\mathcal{P})$ and reformulate the splitting above as:

$$V_h(\mathcal{P}) = \Pi_m^*(V_h(\mathcal{P})) \oplus (1 - \Pi_m^*)(V_h(\mathcal{P})).$$

(iii) Define the virtual element approximation of the local bilinear form as the sum of two terms, respectively related to the consistency and the stability of the method:

$$\int_{\mathcal{P}} \nabla u_h \cdot \nabla v_h \, dV \approx [\text{CONSISTENCY}] + [\text{STABILITY}] =: \mathcal{A}_{h,\mathcal{P}}(u_h, v_h).$$

(iv) Prove the convergence theorem:

$$[\text{CONSISTENCY}] + [\text{STABILITY}] \Rightarrow [\text{CONVERGENCE}]$$

4. Extending the linear conforming FEM from triangles to polygonal cells

4.1. The linear conforming Galerkin FEM on triangles. For every cell \mathcal{P} , we define the **local finite element space** as:

$$V_{h,\mathcal{P}} := \left\{ v_h \in H^1(\mathcal{P}) \mid v_h \in \mathbb{P}_1(\mathcal{P}) \right\}.$$

Local spaces glue gracefully and yield the **global** finite element space:

$$V_h := \left\{ v_h \in H^1(\Omega) \mid v_h|_{\mathcal{P}} \in V_{h,\mathcal{P}} \ \forall \mathcal{P} \in \Omega_h \right\} \subset H^1(\Omega),$$

which is a conforming approximation of $H^1(\Omega)$.

4.1.1. We take into account the essential boundary conditions through the affine space

$$V_{h,g} := \left\{ v_h \in V_{h,\mathcal{P}} \mid v_h|_{\Gamma} = g \right\}$$

for the given boundary function $g \in H^{\frac{1}{2}}(\Gamma)$. We take $g = 0$ in the definition above and consider the linear subspace $V_{h,0} \subset V_h$. The affine space $V_{h,g}$ and the linear space $V_{h,0}$ are respectively used for the trial and the test functions in the variational formulation.

4.1.2. The **degrees of freedom** of a virtual function $v_h \in V_h$ are the values at mesh vertices. Unisolvence can be proved and conformity of the global space is obvious.

4.2. An extension of the FEM to polygonal cells. We extend the FEM to polygonal meshes by redefining the *local finite element space* $V_{h,\mathcal{P}}$ on the polygonal cell \mathcal{P} . To this end, we require that

- the degrees of freedom of each function are its vertex values; therefore, the dimension of this functional space on cell \mathcal{P} is equal to the number of vertices of the cell: $\dim V_{h,\mathcal{P}} = N_{\mathcal{P}}^V$;

- on triangles $V_{h,P}$ must coincide with the linear Galerkin finite element space. This condition implies that $V_{h,P}$ contains the linear polynomials $1, x, y$ and all their linear combinations;
- the local spaces $V_{h,P}$ *glue gracefully* to give a **conformal** finite element space V_h of functions globally defined on Ω .

4.3. Shape functions on polygons. The construction of a set of shape functions defined on P and satisfying the requirements listed above is hard if we assume that P may have a general polygonal geometric shape. The challenge here is to define $V_{h,P}$ by specifying the minimal information about its functions so that a *computable variational formulation* is possible. To achieve this task, we specify the behavior of the functions of $V_{h,P}$ on ∂P , for example by assuming that their trace on each polygonal boundary is a piecewise polynomial of an assigned degree. Then, we characterize them inside P in a very indirect way by assuming that they solve a partial differential equation.

4.4. The local finite element space on a generic polygonal cell. We define the local finite element space $V_{h,P}$ as the span of a set of shape functions $\varphi_i \in H^1(P)$, which are associated with the cell vertices v_i . A possible construction of these shape functions is given in the following three steps.

1. For each vertex v_i consider the boundary function δ_i such that:
 - $\delta_i(v_j) = 1$ if $i = j$, and 0 otherwise;
 - δ_i is continuous on ∂P ;
 - δ_i is linear on each edge.
2. Then, set $\varphi_i|_{\partial P} = \delta_i$;
3. Finally, consider the function φ_i that is the **harmonic lifting** of the function δ_i inside P .

Eventually, we set:

$$V_{h,P} := \text{span}\{\varphi_1, \dots, \varphi_{N^P}\}$$

4.5. The harmonic lifting.

We define the shape function φ_i associated with vertex i as the harmonic function on P having δ_i as trace on boundary ∂P . Formally, $\varphi_i \in H^1(P)$ is the solution of the following elliptic problem:

$$\begin{aligned} -\Delta \varphi_i &= 0 && \text{in } P, \\ \varphi_i &= \delta_i && \text{on } \partial P. \end{aligned}$$

4.5.1. Properties of φ_i .

Note that:

- the functions $\{\varphi_i\}$ are linearly independent (in the sense of the Gramian determinant);
- if $w_h \in V_{h,P}$, then $w_h = \sum_{i=1}^{N^P} w_h(v_i) \varphi_i$;

- linear polynomials belong to $V_{h,\mathbb{P}}$, i.e., $1, x, y \in V_{h,\mathbb{P}}$;
- the trace of each φ_i on each edge of $\partial\mathbb{P}$ only depends on the vertex values of that edge. Consequently, the local spaces $V_{h,\mathbb{P}}$ glue together giving a conformal finite element space $V_h \subset H_0^1(\Omega)$.

4.5.2. Moreover, we can easily prove that:

- if \mathbb{P} is a triangle, we recover the \mathbb{P}_1 Galerkin elements;
- if \mathbb{P} is a square, we recover the \mathbb{Q}_1 bilinear elements.

4.6. The low-order local virtual element space. The **low-order** local virtual element space ($m = 1$) is given by:

$$V_h^1(\mathbb{P}) = \left\{ v_h \in H^1(\mathbb{P}) : \Delta v_h = 0, \begin{array}{l} v_{h|e} \in \mathbb{P}_1(e) \quad \forall e \in \partial\mathbb{P}, \\ v_{h|\partial\mathbb{P}} \in C^0(\partial\mathbb{P}) \end{array} \right\}$$

4.6.1. Extension of the conforming virtual space to the high-order case. The **high-order** local VE space ($m \geq 2$) reads as:

$$V_h^m(\mathbb{P}) = \left\{ v_h \in H^1(\mathbb{P}) : \Delta v_h \in \mathbb{P}_{m-2}(\mathbb{P}), \begin{array}{l} v_{h|e} \in \mathbb{P}_m(e) \quad \forall e \in \partial\mathbb{P}, \\ v_{h|\partial\mathbb{P}} \in C^0(\partial\mathbb{P}) \end{array} \right\}$$

4.6.2. The local spaces glue gracefully to provide a *conforming* approximation of $H^1(\Omega)$. A natural way to guarantee the last property in the previous definitions, i.e., $v_{h|\partial\mathbb{P}} \in C^0(\partial\mathbb{P})$, is to consider the *vertex values* in the set of the degrees of freedom.

4.7. The Harmonic Finite Element Method. Using the shape function of the previous construction, we can define a Polygonal Finite Element Method (PFEM), which we may call the *Harmonic FEM*. This method formally reads as: *Find $u_h \in V_h$ such that*

$$\mathcal{A}(u_h, v_h) = F_h(v_h) \quad \text{for all } v_h \in V_h,$$

where (as usual)

$$\mathcal{A}(u_h, v_h) = \int_{\Omega} \nabla u_h \cdot \nabla v_h \, dV,$$

and $F_h(v_h)$ is some given approximation of $\int_{\Omega} f v \, dV$, which we assume computable from the given function f by using only the vertex values of v_h . A possible approximation for the right-hand side is given in the following step.

4.7.1. Low-order approximation of the right-hand side. The low-order approximation of the forcing term is given by:

$$(f, v_h) = \sum_P \int_P f v_h \, dV \approx \sum_P \bar{v}_{h,P} \int_P f \, dV =: F_h(v_h) \quad \bar{v}_{h,P} = \frac{1}{N_P} \sum_{i=1}^{N_P} v_h(\mathbf{x}_i)$$

4.7.2. In the literature, harmonic shape functions are used, for example, in computer graphics (see, for example, [116]) and elasticity (see [47]).

4.8. Mesh assumptions. Under *reasonable* assumptions on the mesh and the sequence of meshes for $h \rightarrow 0$, the harmonic finite element discretization of an elliptic problem enjoys the usual convergence properties. We do not enter into the details here; among these properties we just mention that

- all geometric objects must scale properly with the characteristic length of the mesh; for example, $|P| \simeq h^2$, $|\mathbf{e}| \simeq h$, etc;
- each polygon must be *star-shaped* with respect to an internal ball of points \Rightarrow *interpolation estimates hold* (see [50, 79]); more generally, each polygon can be the union of a *uniformly bounded* number of star-shaped subcells.

4.9. Computability issues. The Harmonic FEM is a very nice method that works on polygonal meshes and has a solid mathematical foundation. This method is expected to be second order accurate when it approximates sufficiently smooth solutions. In such a case, the approximation error is expected to scale down proportionally to h in the energy norm and proportionally to h^2 in the L^2 norm. However, we need to compute the shape functions to compute the integrals forming the stiffness matrix

$$\mathcal{A}(\varphi_i, \varphi_j) = \int_{\Omega} \nabla \varphi_i \cdot \nabla \varphi_j \, dV$$

and the term on the right-hand side

$$F_h(\mathbf{v}_h) = \int_{\Omega} f \varphi_i \, dV.$$

A numerical approximation of the shape functions inside P is possible for example by partitioning the cell in simplexes and then applying a standard FEM. However, this approach is rather expensive since it requires the solution of a Laplacian problem on element P for each one of its vertices.

5. The virtual approximation of the bilinear form $\mathcal{A}(\varphi_i, \varphi_j)$

The virtual element approach follows a different strategy by resorting to the so-called *virtual construction* of the bilinear form. Roughly speaking, we assume that the bilinear form \mathcal{A} of the Harmonic FEM is approximated by a virtual bilinear form \mathcal{A}_h . Since the bilinear form \mathcal{A} can be split into elemental contributions \mathcal{A}_P

$$\mathcal{A}(v_h, w_h) = \sum_P \mathcal{A}_P(v_{|P}, w_{|P}) = \sum_P \int_P \nabla v \cdot \nabla w \, dV,$$

we assume that \mathcal{A}_h is split into elemental contributions $\mathcal{A}_{h,P}$. Each $\mathcal{A}_{h,P}$ is a local approximation of the corresponding \mathcal{A}_P . Therefore, we have that:

$$\mathcal{A}_h(v_h, w_h) = \sum_P \mathcal{A}_{h,P}(v_h|_P, w_h|_P), \quad \text{with } \mathcal{A}_{h,P} \approx \mathcal{A}_P.$$

5.1. Now, we give *two* conditions that must be satisfied by each local bilinear form: **consistency** and **stability**. These two conditions are inherited from the MFD formulation and as for this latter they guarantee the convergence of the method.

5.2. Consistency. Consistency is an exactness property for linear polynomials. Formally, for all $q \in \mathbb{P}_1(P)$ and for all $v_h \in V_{h,P}$:

$$\mathcal{A}_{h,P}(v_h, q) = \mathcal{A}(v_h, q).$$

5.3. Stability. There exist two positive constants α^* and α_* independent of P , such that

$$\alpha_* \mathcal{A}_P(v_h, v_h) \leq \mathcal{A}_{h,P}(v_h, v_h) \leq \alpha^* \mathcal{A}_P(v_h, v_h).$$

5.4. Convergence Theorem.

Theorem 1.1. Assume that for each polygonal cell P the bilinear form $\mathcal{A}_{h,P}$ satisfies the properties of consistency and stability introduced above. Let $u_h \in V_h$ be such that

$$\mathcal{A}_h(u_h, v_h) = F_h(v_h) \quad \text{for all } v_h \in V_h.$$

Then,

$$\boxed{\|u - u_h\|_{H^1(\Omega)} \leq Ch\|u\|_{H^2(\Omega)}}$$

Proof. See [18]. □

5.5. A good starting point to build $\mathcal{A}_{h,P}$... leading to a rank-deficient stiffness matrix.

5.5.1. We know the functions v_h of $V_{h,P}$ only on the boundary of P and we can compute the *exact value* of the quantity

$$\overline{\nabla v_h} := \frac{1}{|P|} \int_P \nabla v_h \, dV$$

using only the vertex values. In fact,

$$\int_P \nabla v_h \, dV = \int_{\partial P} v_h \mathbf{n}_P \, dS = \sum_{i=1}^{N^P} \left(\int_{\mathbf{e}_i} v_h \, dS \right) \mathbf{n}_{P,i} = \sum_{i=1}^{N^P} \frac{v_h(\mathbf{v}_i) + v_h(\mathbf{v}_{i+1})}{2} |\mathbf{e}_i| \mathbf{n}_{P,i}$$

∇v_h is a constant vector in \mathbb{R}^2 .

5.5.2. Now, we are really tempted to say that

$$\mathcal{A}_P(\varphi_i, \varphi_j) := \int_P \nabla \varphi_i \cdot \nabla \varphi_j \, dV \approx \int_P \overline{\nabla \varphi_i} \cdot \overline{\nabla \varphi_j} \, dV =: \mathcal{A}_{h,P}(\varphi_i, \varphi_j)$$

Note that if P is a triangle we obtain the stiffness matrix of the linear Galerkin FEM. However, $\mathcal{A}_{h,P}(\varphi_i, \varphi_j)$ would have *rank 2* for any kind of polygons, thus leading to a *rank deficient* approximation for \mathcal{A}_h on any mesh that is not (strictly) triangular!

5.6. The local projection operator Π_m^∇ . We define the *local projection operator* for each polygonal cell P

$$\Pi_m^\nabla : V_{h,P} \longrightarrow \mathbb{P}_1(P)$$

that has the two following properties:

(i) it approximates the gradients using only the vertex values:

$$\nabla(\Pi_m^\nabla v_h) = \overline{\nabla v_h};$$

(ii) it preserves the linear polynomials:

$$\Pi_m^\nabla q = q \quad \text{for all } q \in \mathbb{P}_1(P).$$

5.7. The bilinear form $\mathcal{A}_{h,P}$. We start writing that

$$\mathcal{A}_{h,P}(u_h, v_h) = \mathcal{A}_{h,P}(\Pi_m^\nabla u_h, v_h) + \mathcal{A}_{h,P}(u_h - \Pi_m^\nabla u_h, v_h).$$

With an easy computation it can be shown that

$$\mathcal{A}_{h,P}(\Pi_m^\nabla u_h, v_h) = \mathcal{A}_P(\Pi_m^\nabla u_h, \Pi_m^\nabla v_h) := \mathcal{A}_{h,P}^0(u_h, v_h)$$

and

$$\mathcal{A}_{h,P}((I - \Pi_m^\nabla)u_h, v_h) = \mathcal{A}_P((I - \Pi_m^\nabla)u_h, (I - \Pi_m^\nabla)v_h) \longrightarrow \mathcal{A}_{h,P}^1(u_h, v_h).$$

We will set:

$$\mathcal{A}_{h,P} = \mathcal{A}_{h,P}^0 + \mathcal{A}_{h,P}^1 = \text{CONSISTENCY} + \text{STABILITY}$$

5.8. The consistency term $\mathcal{A}_{h,P}^0$. The bilinear form $\mathcal{A}_{h,P}^0$ provides a sort of “*constant gradient approximation*” of the stiffness matrix. In fact, $\mathcal{A}_{h,P}^0$ ensures the *consistency condition*: $\mathcal{A}_{h,P}(v_h, q) = \mathcal{A}_P(v_h, q)$ for all $q \in \mathbb{P}_1(P)$; in fact,

$$\begin{aligned} \mathcal{A}_{h,P}^0(v_h, q) &= \int_P \overline{\nabla v_h} \cdot \overline{\nabla q} \, dV = |P| \overline{\nabla v_h} \cdot \overline{\nabla q} = \left(\int_P \nabla v_h \right) \cdot \overline{\nabla q} \, dV \\ &= \int_P \nabla v_h \cdot \overline{\nabla q} \, dV = \int_P \nabla v_h \cdot \nabla q \, dV = \mathcal{A}_P(v_h, q). \end{aligned}$$

5.8.1. The remaining term is zero because $(I - \Pi_m^\nabla)q = 0$ if $q \in \mathbb{P}_1(\mathcal{P})$.

5.9. The stability term $\mathcal{A}_{h,\mathcal{P}}^1$.

We need to correct $\mathcal{A}_{h,\mathcal{P}}^0$ in such a way that:

- consistency is not upset;
- the resulting bilinear form is stable;
- the correction is computable using only the degrees of freedom!

In the founding paper it is shown that we can substitute the (non computable) term $\mathcal{A}_{\mathcal{P}}((I - \Pi_m^\nabla)u_h, (I - \Pi_m^\nabla)v_h)$ with

$$\mathcal{A}_{h,\mathcal{P}}^1(u_h, v_h) := \mathcal{S}_{h,\mathcal{P}}((I - \Pi_m^\nabla)u_h, (I - \Pi_m^\nabla)v_h)$$

where $\mathcal{S}_{h,\mathcal{P}}$ can be **any symmetric and positive definite bilinear form** that behaves (asymptotically) like $\mathcal{A}_{\mathcal{P}}$ on the kernel of Π_m^∇ .

Hence:

$$\mathcal{A}_{h,\mathcal{P}}(u_h, v_h) := \underbrace{\mathcal{A}_{\mathcal{P}}(\Pi_m^\nabla u_h, \Pi_m^\nabla v_h)}_{\text{CONSISTENCY}} + \underbrace{\mathcal{S}_{h,\mathcal{P}}((I - \Pi_m^\nabla)u_h, (I - \Pi_m^\nabla)v_h)}_{\text{STABILITY}}$$

6. Implementation of the local virtual bilinear form in four steps

6.1. The central idea of the VEM is that we can compute the polynomial projections of the virtual functions and their gradients exactly using only their degrees of freedom. So, to implement the VEM we introduce the matrix representation of such projection, i.e., the projection matrix. The projection matrix represents the projection of the shape functions with respect to a given monomial basis (or with respect to the same basis of shape functions).

6.2. Step 1. To compute the projection matrix we need two special matrices called B and D. Matrices B and D are constructed from the basis of the polynomial subspace (the scaled monomials). All other matrices are derived from straightforward calculations involving B and D. The procedure is exactly the same for the conforming and nonconforming method, except for the definition of matrix B.

6.2.1. Matrix D. To start the implementation of the VEM, we consider the matrices B and D. The columns of matrix D are the degrees of freedom of the scaled monomials. Since the scaled monomials are the same for both the conforming and nonconforming case, the definition of matrix D is also the same. Matrix D is given by:

$$D = \begin{bmatrix} 1 & \frac{x_1 - x_{\mathcal{P}}}{h_{\mathcal{P}}} & \frac{y_1 - y_{\mathcal{P}}}{h_{\mathcal{P}}} \\ 1 & \frac{x_2 - x_{\mathcal{P}}}{h_{\mathcal{P}}} & \frac{y_2 - y_{\mathcal{P}}}{h_{\mathcal{P}}} \\ \dots \\ 1 & \frac{x_n - x_{\mathcal{P}}}{h_{\mathcal{P}}} & \frac{y_n - y_{\mathcal{P}}}{h_{\mathcal{P}}} \end{bmatrix}.$$

6.2.2. Matrix B. The column of matrix B are the right-hand sides of the projection problem. Matrix B is given by:

$$B = \frac{1}{2h_P} \begin{bmatrix} \frac{1}{n} & \frac{1}{n} & \dots & \frac{1}{n} \\ |e_1|n_{x,1} + |e_n|n_{x,n} & |e_2|n_{x,2} + |e_1|n_{x,1} & \dots & |e_n|n_{x,n} + |e_{n-1}|n_{x,n-1} \\ |e_1|n_{y,1} + |e_n|n_{y,n} & |e_2|n_{y,2} + |e_2|n_{y,1} & \dots & |e_n|n_{y,n} + |e_{n-1}|n_{y,n-1} \end{bmatrix}.$$

6.3. Step 2. Using these matrices, we compute matrix $G = BD$.

6.4. Step 3. Then, we solve the **projection problem** in algebraic form: $G\Pi_m^\nabla = B$ for matrix Π_m^∇ , which is the representation of the projection operator Π_m^∇ .

6.5. Step 4. Using matrices Π_m^∇ and $\Pi_m^{\nabla,\phi} = D\Pi_m^\nabla$ we build the **stiffness matrix**:

$$M = (\Pi_m^\nabla)^T \tilde{G} \Pi_m^\nabla + \nu_P (I - \Pi_m^{\nabla,\phi})^T (I - \Pi_m^{\nabla,\phi})$$

where \tilde{G} is matrix G with the first row set to zero, ν_P is a scaling coefficient.

7. Background material on numerical methods for PDEs using polygonal and polyhedral meshes

7.1. The **conforming VEM** for the Poisson equation in primal form as presented in the *founding paper* [18] is a *variational reformulation* of the nodal Mimetic Finite Difference method of References [51] (low-order case) and [29] (arbitrary order case).

7.1.1. The **nonconforming VEM** for the Poisson equation in primal form as presented in the *founding paper* [13] is the variational reformulation of the arbitrary-order accurate nodal MFD method of Reference [104].

7.1.2. Incremental extensions are for different type of applications (including parabolic problems) and formulations (mixed form of the Poisson problem), hp refinement, and also different variants (arbitrary continuity, discontinuous variants). A complete and detailed review of the literature on MFD and VEM is out of the scope of this collection of notes. However, we list below a few references that can be of interest to a reader willing to know more.

7.1.3. Compatible discretization methods have a long story. A review can be found in the conference paper [48]. A recent overview is also found in the first chapter of the Springer book [31] on the MFD method for elliptic problem and the recent paper [122].

7.1.4. The state of the art on these topics is well represented in two recent **special issues** on numerical methods for PDEs on unstructured meshes [42, 41].

7.1.5. Comparisons of different methods are found in the **conference benchmarks** on 2D and 3D diffusion problems with anisotropic coefficients, see [86].

7.1.6. For the **Mimetic Finite Difference method**, we refer the interested reader to the following works:

- extension to polyhedral cells with curved faces [55];
- general presentation of the MFD method: book [31]; review papers [97, 108]; connection of MFD with other methods [105]; benchmarks [103, 111]; conference paper [112],
- advection-diffusion problems: connection with finite volume schemes [25]; convergence analysis [65];
- diffusion equation in primal form: the low-order formulation [51], post-processing of solution and flux [39]; arbitrary-order of accuracy on polygonal [29, 30] and polyhedral meshes [104]; arbitrary regularity [36, 37]; a posteriori estimators [9];
- diffusion equation in mixed form: founding paper on the low-order formulation: convergence analysis [54] and implementation [56]; with staggered coefficients [114, 107]; arbitrary order of accuracy [96]; monotonicity conditions and discrete maximum/minimum principles [109, 110]; eigenvalue calculation [61]; second-order flux discretization [35]; convergence analysis [27]; post-processing of flux [64]; a posteriori analysis and grid adaptivity [34]; flows in fractured porous media [8];
- steady Stokes equations: the MFD method on polygonal meshes [26]; error analysis [28];
- differential forms and Maxwell eigenvalues [52]; magnetostatics [106];
- quasilinear elliptic problems [11], nonlinear and control problems [6, 7], obstacle elliptic problem [10],
- topology optimization [91].

7.1.7. For the Virtual Element Method, we refer the interested reader to the following works:

- first (founding) paper on the conforming VEM [18];
- diffusion problems in primal form: “hp” refinements [24]
- diffusion problems in mixed form [53, 22]
- geomechanics [2]
- L^2 -projections [1]
- steady Stokes equations: stream function formulation [4]; divergence free methods [33]; mixed VEM for the pseudostressvelocity formulation [58]
- fracture network simulations [90, 46, 45, 43]
- Cahn-Hilliard equation on polygonal meshes [5];
- linear [19] and nonlinear elasticity [32]; plate bending problem [57]; finite deformations [68]; three-dimensional elasticity [93];
- non-conforming formulation: first (founding) paper [13]; Stokes [62]; advection-reaction-equation [67]; bi-harmonic problems [139, 3];
- eigenvalue calculation [94]; Steklov eigenvalue problem [118];

- connection with other methods: PFEM [115, 121]; MFD and BEM-based FEM [63];
- Helmholtz equation [123];
- implementation [20, 129];
- further developments of the conforming virtual formulation [21, 23]; stabilization of the hourglass phenomenon [66]; VEM with arbitrary regularity [40, 36];
- advection-diffusion problems in the advection dominated regime [44];
- a posteriori analysis [38];
- topology optimization [92];
- contact problems [138].

7.2. Polygonal/Polyhedral FEM. The development of numerical methods with such kind of flexibility or independence of the mesh has been one of the major topics of research in the field of numerical PDEs in the last two decades and a number of schemes are currently available from the literature. These schemes are often based on approaches that are substantially different from MFDs or VEMs.

Recently developed discretization frameworks related to general meshes include

- the finite element method using rational basis functions [133, 134, 87] and using generalized polynomial interpolants on polygons [128, 126, 130];
- the finite volume methods (see the review paper [81]) and the connection with the MFD method (see [83]);
- hybrid high-order (HHO) method [76, 78, 77];
- the discontinuous Galerkin (DG) method [79];
- hybridized discontinuous Galerkin (HDG) method [71];
- the weak Galerkin (wG) method [136].

7.2.1. Other examples (extended FEM, partition of unity, meshless, non-local decomposition, etc) can be found in: [12, 14, 15, 16, 17, 47, 48, 75, 85, 88, 89, 95, 101, 117, 124, 125, 127, 131, 137, 74, 113, 60, 132, 59]

7.2.2. Most of these methods use trial and test functions of a rather complicate nature, that often could be computed (and integrated) only in some approximate way. In more recent times several other methods have been introduced in which the trial and test functions are pairs of polynomial (instead of a single non-polynomial function) or the degrees are defined on multiple overlapping meshes: Examples are found in: [49, 69, 71, 72, 84, 119, 120, 135, 135, 73, 80, 82, 98, 99, 100, 102, 70]

8. Next versions. The next incremental developments of this document will cover:

- the high-order VEM formulation;

- the nonconforming formulation;
- connection with the MFD method;
- 3D extensions (enhancement of the virtual element space, etc).

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