# A Finite Element Method for Nonlocal Mechanics

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## Introduction

- Goal:
  - Develop an accurate, efficient, and robust finite element method for the weak formulation of nonlocal mechanics problems.
- Quadrature challenges:
  - High-dimension (integration over $\mathbb{R}^{2d}$)
  - Discontinuous integrand
  - Complicated shape of support of integrand
  - Singularities in some cases of interest

## Continuous problem

- Strong form: Find $u : \Omega \to \mathbb{R}$ such that

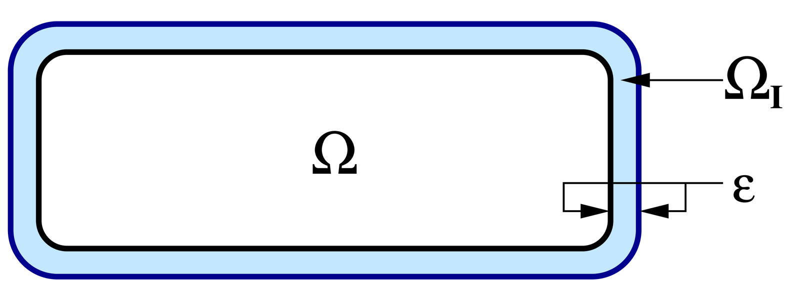
$$2\int_{\Omega\cup\Omega_I}(u(x)-u(x'))\gamma_\varepsilon(x,x')dx' = f(x), \quad \forall x \in \Omega,$$

subject to the Dirichlet volume–constraint

$$u(x) = g(x), \quad \forall x \in \Omega_d \subseteq \Omega_I,$$

and Neumann volume–constraint

$$2\int_{\Omega\cup\Omega_I}(u(x)-u(x'))\gamma_\varepsilon(x,x')dx' = h(x), \quad \text{for } x \in \Omega_n \subseteq \Omega_I.$$




- Weak form: Find $u \in U$ such that

$$\langle u, v\rangle_\gamma = \int_\Omega v(x)f(x)dx + \int_{\Omega_n} v(x)h(x)dx,$$

for all $v \in V$, subject to the Dirichlet volume–constraint.

- Bilinear form:

$$\langle u, v\rangle_\gamma = \int_{\Omega\cup\Omega_I}\int_{\Omega\cup\Omega_I}(u(x)-u(x'))(v(x)-v(x'))\gamma_\varepsilon(x,x')dx'dx$$

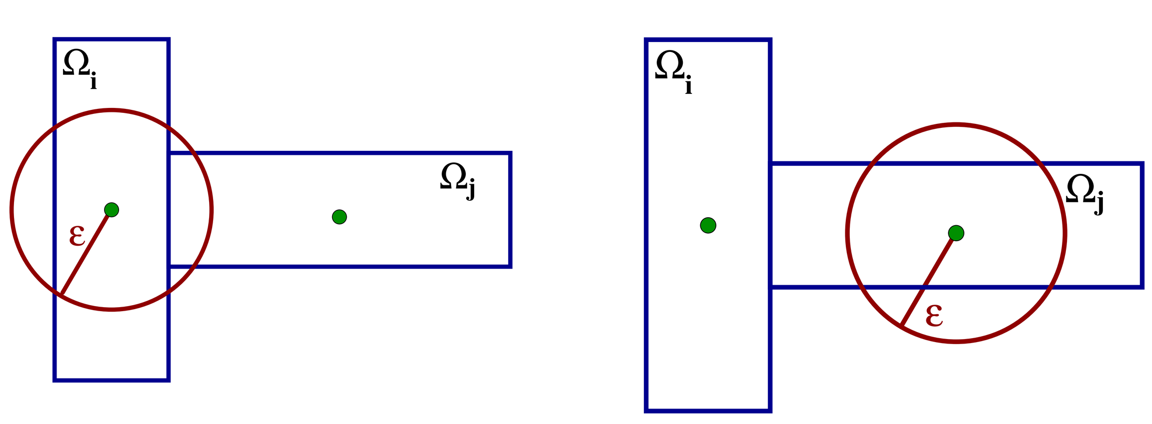
- Energy:

$$E(u) = \frac{1}{2}\langle u, u\rangle_\gamma - \int_\Omega v(x)f(x)dx - \int_{\Omega_n} v(x)h(x)dx$$

- Flux:

$$\text{flux}_{j\to i} = \langle \mathbb{1}_{(\Omega_i)}, u\mathbb{1}_{(\Omega_j)}\rangle_\gamma - \langle \mathbb{1}_{(\Omega_j)}, u\mathbb{1}_{(\Omega_i)}\rangle_\gamma$$

## Strong vs. Weak form




- Does $\Omega_i$ interact with $\Omega_j$?
- Strong Form: Points interact with volumes (asymmetric)
  - "Saved" by low-order quadrature?
- Weak Form: Volumes interact with volumes (symmetric)
  - Requires more advanced quadarture!

## Nonlocal kernel

$$\gamma_\varepsilon(x,x') = \frac{1}{\varepsilon^2 \,\mathrm{Vol}(P_{N,\varepsilon})}\begin{cases}\psi(\|x-x'\|), & \|x-x'\| \le \varepsilon,\\ 0, & \text{otherwise}\end{cases}$$

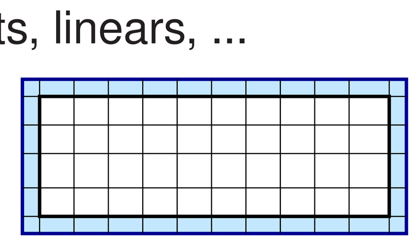
Case 1: $\psi(r) = C$ (no smoothing)

Case 2: $\psi(r) = \frac{C}{r^{d+2s}}$ (smoothing; $0 < s < 1$)

## Discrete problem

- Approximation: Discontinuous constants, linears, ...

$$u_h = \sum_{i=1}^{N}\alpha_i\phi_i(x), \qquad \overline{\Omega\cup\Omega_I} = \bigcup_{i=1}^{N}\overline{\Omega_i}$$



- System of equations

$$\begin{bmatrix} A & B\\ B^T & 0\end{bmatrix}\begin{bmatrix}\vec{\alpha}\\ \vec{\beta}\end{bmatrix} = \begin{bmatrix}\vec{b}\\ \vec{c}\end{bmatrix}, \qquad \beta_k = \text{Lagrange multipliers}$$

$$A_{i,j} = \langle\phi_i,\phi_j\rangle_\gamma, \quad B_{i,k} = \int_{\Omega_d}\phi_i(x)\phi_{j(k)}(x)\,dx,$$

$$b_i = \int_\Omega \phi_i(x)f(x)\,dx + \int_{\Omega_n}\phi_i(x)h(x)\,dx, \quad c_k = \int_{\Omega_d}\phi_{j(k)}(x)g(x)\,dx$$

## Assembly

- Task: Compute

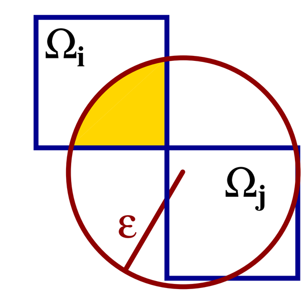
$$A_{i,j} = \int_{\Omega\cup\Omega_I}\int_{\Omega\cup\Omega_I}(\phi_i(x)-\phi_i(x'))(\phi_j(x)-\phi_j(x'))\gamma(x,x')dx'dx$$

- Difficulty: Support of integrand is complicated

$$((\Omega_i\times\Omega_j)\cap B_\varepsilon(\Omega_i,\Omega_j))\cup((\Omega_j\times\Omega_i)\cap B_\varepsilon(\Omega_j,\Omega_i)),$$

where

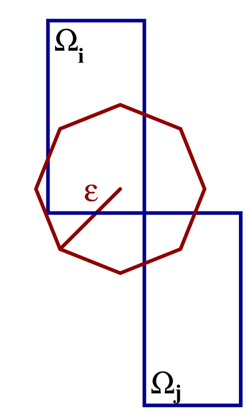
$$B_\varepsilon(\Omega_i,\Omega_j) := \{(x,x')\in\Omega_i\times\Omega_j \mid \|x-x'\|\le\varepsilon\}$$



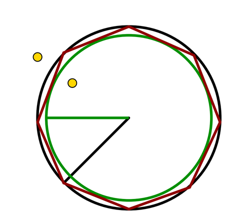

Curved boundaries in $\mathbb{R}^4$ (for 2D) or $\mathbb{R}^6$ (for 3D)

## Quadrature

- Approach: Subdivide (or approx. subdivide) the region of integration into polygonal subregions and apply Gauss quadrature in each subregion.




- Method:
  1. Use polyhedral approx. of spherical horizon
  2. Describe convex region, $\tilde{B}_\varepsilon(\Omega_i,\Omega_j)$, with linear inequality constraints
  3. Delaunay "triangulate" region with `Qhull`
  4. Apply standard/Gauss quadrature rules on each high-dimensional simplex



- Cost reduction:
  1. Use symmetric polyhedral approx. of spherical horizon
  2. Combine simplicial quadrature rules to form a simple low-complexity rule for each pair of elements
  3. Bound polyhedral regions with spheres for identification of completely interacting/noninteracting elements

## Quadrature accuracy

$$\left|\int_{\Omega_i}\int_{\Omega_j}F(x,x')(S_\varepsilon(x-x')-P_{N,\varepsilon}(x-x'))dx'dx\right| \le \|F\|_\infty C(N,\varepsilon)$$

where

$$C(N,\varepsilon) \propto \varepsilon^d\frac{1}{N^k} \quad \text{with} \quad k = 2,1 \quad \text{for} \quad d = 2,3$$

## Quadrature derivation complexity

- Number of linear inequality constraints
  1. Element distance bounded bounding spheres

$$|\text{Facets }\Omega_i| + |\text{Facets }\Omega_j|$$

  2. Bounding sphere test fails

$$|\text{Facets }\Omega_i| + |\text{Facets }\Omega_j| + |\text{Facets }P_{N,\varepsilon}|$$

- Overall complexity

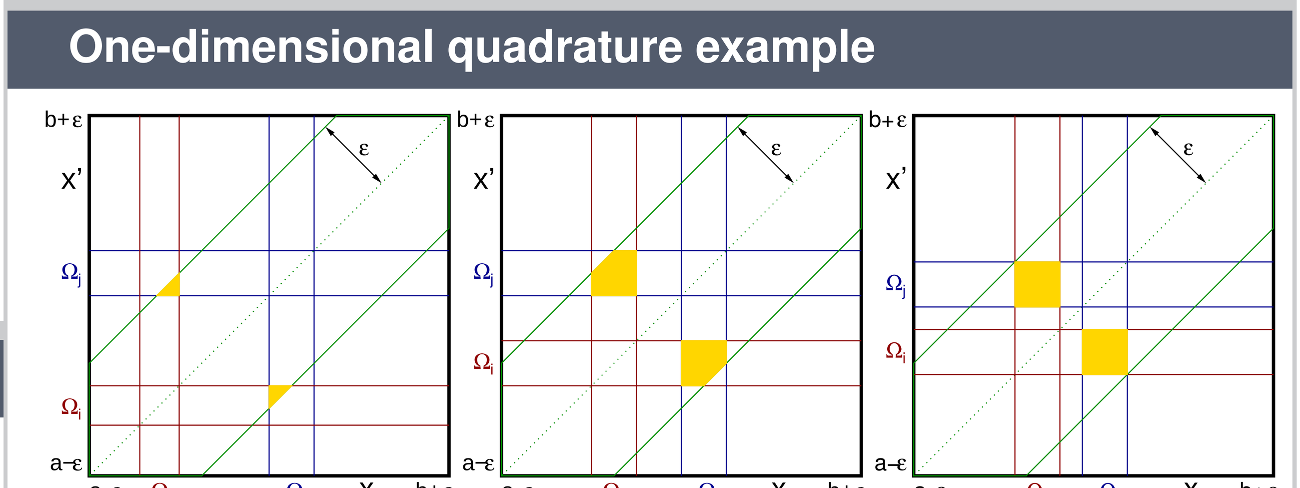
$$O\left(\text{vert}_{\text{in}}\,\text{facet}_{\text{out}}/\text{vert}_{\text{out}}\right)$$

## Simplicial quadrature in $\mathbb{R}^n$

| Degree | Quadrature Pts. | $\lvert\mathbb{P}_k(\mathbb{R}^n)\rvert$ |
|---|---|---|
| $k=1$ | 1 | $n+1$ |
| $k=2$ | $n+1$ | $(n+2)(n+1)/2$ |
| $k=3$ | $n+2$ | $(n+3)(n+2)(n+1)/6$ |
| $k$ | ? | $(n+k)!/(n!\,k!)$ |

See: N. J. Walkington, 2000

## One-dimensional quadrature example




- Support is not a simple Cartesian product
- Can precompute weight, $w_c$, and centroid, $(x_c, x_c')$.

$$w_c = \int_{\Omega_i}\int_{\Omega_j}\gamma(x,x'),$$

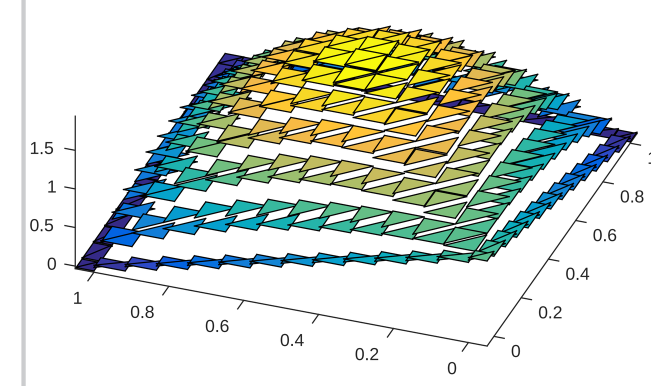
$$w_c x_c = \int_{\Omega_i}\int_{\Omega_j}x\gamma(x,x')dxdx', \quad w_c x_c' = \int_{\Omega_i}\int_{\Omega_j}x'\gamma(x,x')dxdx',$$

for each pair $\Omega_i$ and $\Omega_j$.

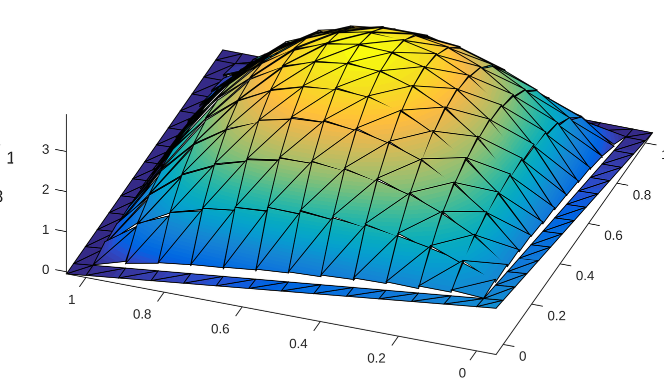
- One-point is sufficient for discontinuous linears ($i \ne j$)
- Higher-order and $i = j$ requires more points

## Numerical results

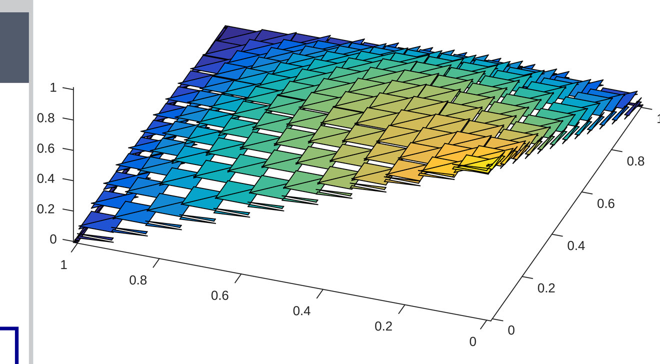
- The nonlocal kernel, $\gamma_\varepsilon$, is piecewise constant
- $h = \frac{1}{12}$, $f = 75$, $C = \frac{9}{4}$



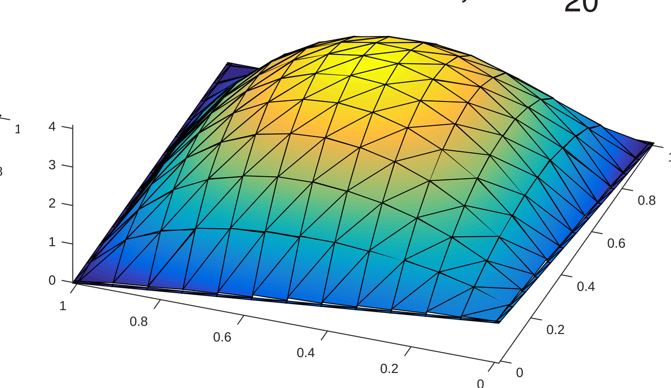
Piecewise Constant, $\varepsilon = \frac{1}{20}$



Piecewise Linear, $\varepsilon = \frac{1}{20}$



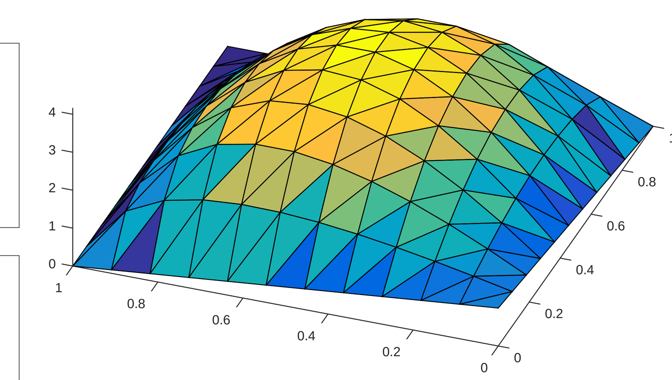
Piecewise Constant, $\varepsilon = \frac{1}{100}$



Piecewise Linear, $\varepsilon = \frac{1}{100}$

Convergence $\varepsilon \to 0$
- Piecewise Constant = No
- Piecewise Linear = Yes

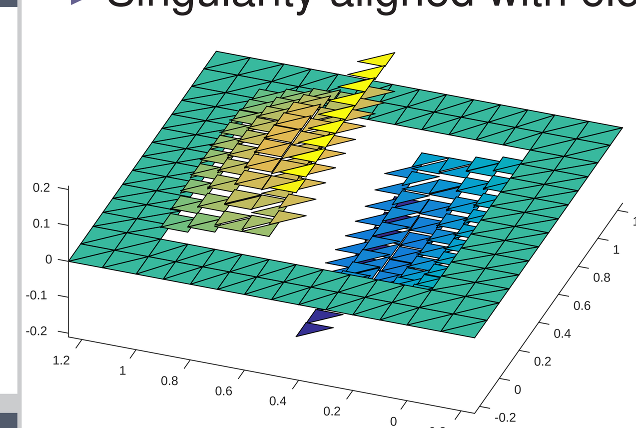
$L_2$ Convergence $h \to 0$
- Piecewise Constant = $O(h)$
- Piecewise Linear = $O(h^2)$



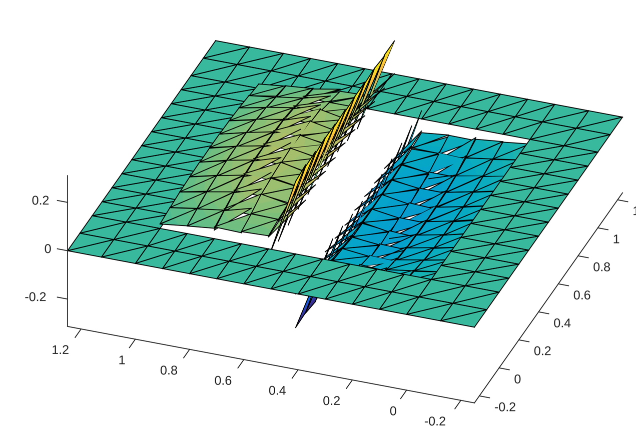
Finite Element Solution

## Numerical results

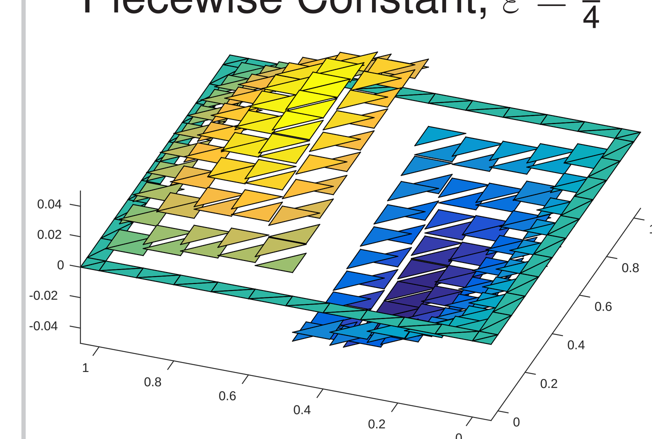
- The nonlocal kernel, $\gamma_\varepsilon$, is piecewise constant
- $h = \frac{1}{10}$, $f = 1/(x_2 - 0.5)$, $C = \frac{9}{4}$
- Singularity aligned with element boundaries



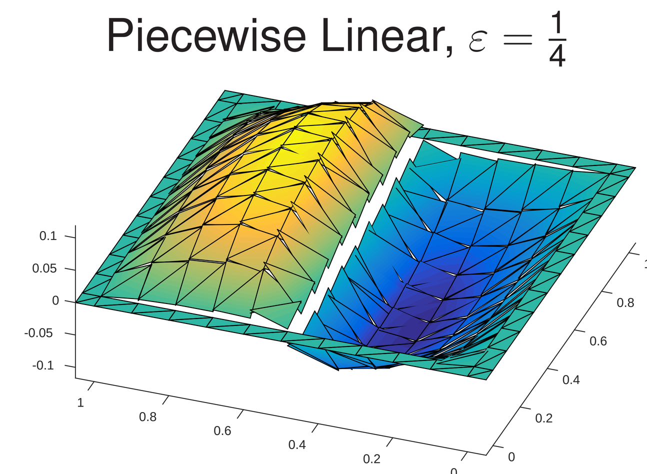
Piecewise Constant, $\varepsilon = \frac{1}{4}$



Piecewise Linear, $\varepsilon = \frac{1}{4}$



Piecewise Constant, $\varepsilon = \frac{1}{20}$



Piecewise Linear, $\varepsilon = \frac{1}{20}$

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