SAND2015-2132C

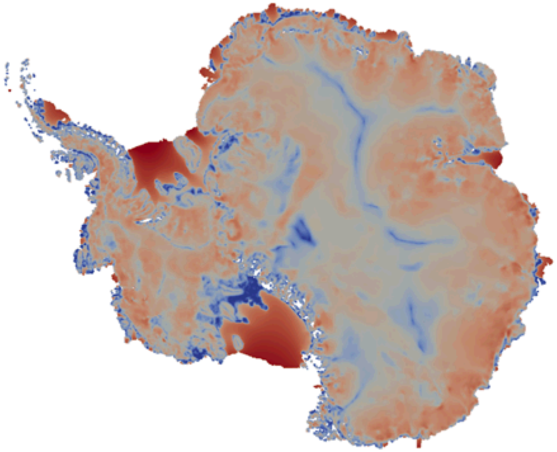
A Hybrid Operator Dependent MG / AMG Approach: Application to Ice Sheet Modeling

Ray Tuminaro, Sandia National Labs

Collaborators

Sandia: Irina Tezaur, Mauro Perego, Andy Salinger, Chris Siefert

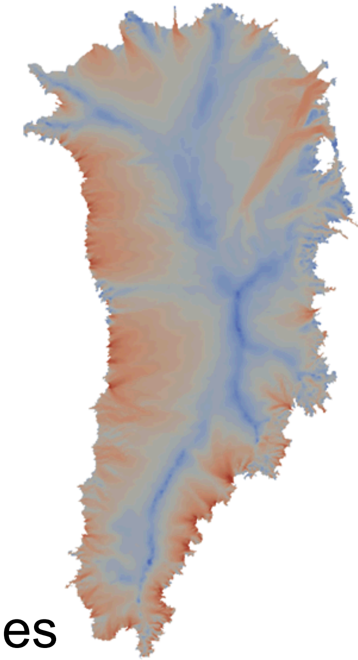
LANL: Steve Price



Sandia National Laboratories is a multi-program laboratory operated by Sandia Corp, a wholly owned subsidiary of Lockheed Martin Corporation, for the U.S. Department of Energy's National Nuclear Security Administration under contract DE-AC04-94AL85000.

Sandia National Laboratories

Outline

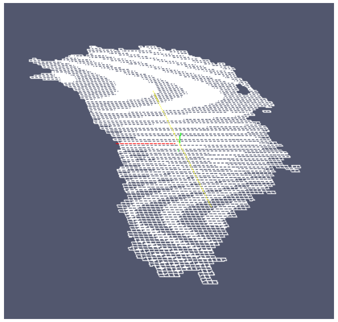


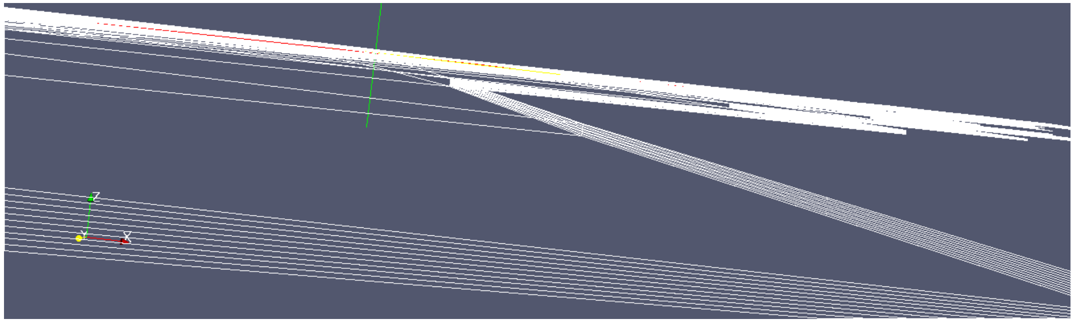
- Ice sheets & motivation
  - equations & meshes
- Linear solver observations
  - ILU solvers & orderings, parallel decompositions, singularities
- Matrix dependent semi-coarsening MG
  - Generalization for extruded meshes and aggressive semi-coarsening
- Hybrid solver
  - AMG on pancake mesh
  - Solver practicalities
- Numerical results

Sandia National Laboratories

New solver motivation

- Ice sheet dynamics are critical in evaluating potential sea level rise
- Lots of uncertain model inputs
  - $\Rightarrow$ many repeated simulations, e.g. UQ, inverse problems
- Linear solver time typically dominates
  - $\Rightarrow$ many linear solvers per simulation, nonlinear with continuation (steady state)
- Equations are simplified Stokes with variable μ
- Main difficulty is anisotropic mesh





- Success with anisotropic-aware geometric MG, but want an algebraic solver

The First-Order Stokes Model for Ice Sheets & Glaciers

- Ice sheet dynamics are given by the ***"First-Order" Stokes PDEs***: approximation* to viscous incompressible ***quasi-static*** Stokes flow with power-law viscosity.

$$\begin{cases} -\nabla \cdot (2\mu \dot{\boldsymbol{\epsilon}}_1) = -\rho g \dfrac{\partial s}{\partial x} \\ -\nabla \cdot (2\mu \dot{\boldsymbol{\epsilon}}_2) = -\rho g \dfrac{\partial s}{\partial y} \end{cases}, \quad \text{in } \Omega$$

$$\dot{\boldsymbol{\epsilon}}_1{}^T = (2\dot{\epsilon}_{11} + \dot{\epsilon}_{22}, \dot{\epsilon}_{12}, \dot{\epsilon}_{13})$$
$$\dot{\boldsymbol{\epsilon}}_2{}^T = (2\dot{\epsilon}_{12}, \dot{\epsilon}_{11} + 2\dot{\epsilon}_{22}, \dot{\epsilon}_{23})$$
$$\dot{\epsilon}_{\mathrm{ij}} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial uj}{\partial xi}\right)$$

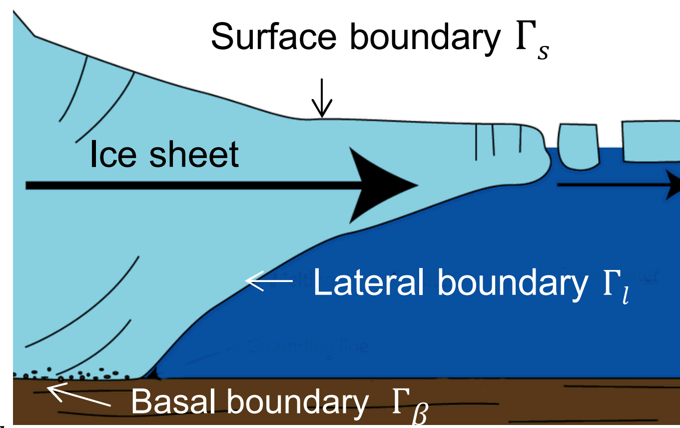
- Viscosity μ is nonlinear function given by "***Glen's law***":

$$\mu = \frac{1}{2} A^{-\frac{1}{n}} \left(\frac{1}{2} \sum_{ij} \dot{\boldsymbol{\epsilon}}_{ij}{}^2\right)^{\left(\frac{1}{2n} - \frac{1}{2}\right)} \qquad (n = 3)$$

- Relevant boundary conditions:
  - ***Stress-free BC:*** $2\mu \dot{\boldsymbol{\epsilon}}_i \cdot \boldsymbol{n} = 0$, on Γ_s
  - ***Floating ice BC:***

$$2\mu \dot{\boldsymbol{\epsilon}}_i \cdot \boldsymbol{n} = \begin{cases} \rho g z \boldsymbol{n}, & \text{if } z > 0 \\ 0, & \text{if } z \leq 0 \end{cases}, \text{ on } \Gamma_l$$

  - ***Basal sliding BC:*** $2\mu \dot{\boldsymbol{\epsilon}}_i \cdot \boldsymbol{n} + \beta ui = 0$, on Γ_β

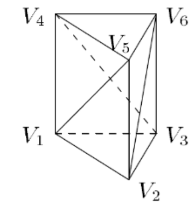


$$\beta = \text{sliding coefficient} \geq 0$$

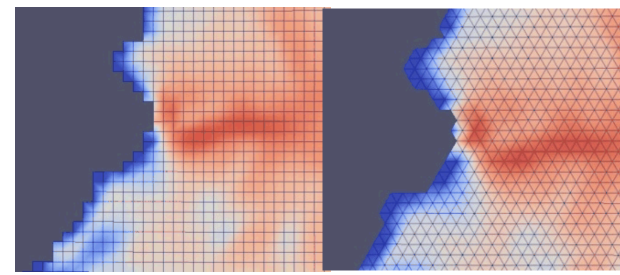
***Assumption:** aspect ratio δ is small and normals to upper/lower surfaces are almost vertical.

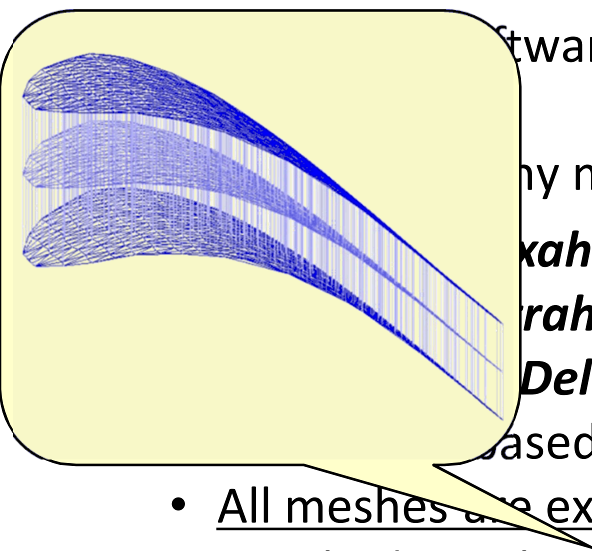
Sandia National Laboratories

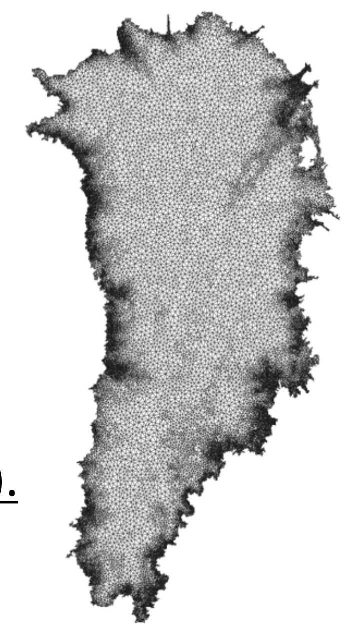
Algorithmic Choices for *Albany/FELIX* Discretization & Meshes



- **<u>Discretization:</u>** unstructured grid finite element method (FEM)
  - Can handle readily complex geometries.
  - Natural treatment of stress boundary conditions.
  - Enables regional refinement/unstructured meshes.
  - ftware and algorithms.







- ny mesh but interested specifically in
  - xahedral meshes (compatible with *CISM*).
  - rahedral meshes (compatible with *MPAS*)
  - ***Delaunay triangle*** meshes with regional
  ased on gradient of surface velocity.
- <u>All meshes a… extruded (structured) in vertical direction as tetrahedra or hexahedra (wedge elements did not work well).</u>

Solver observations

Sometimes ILU works great …

AMG works poorly on these problems …

Sometimes ILU is really bad …

Sometimes $A\ u = b$ is easy … sometimes hard

Floating ice is real trouble for all linear solvers …

Meshes with many vertical points are problematic

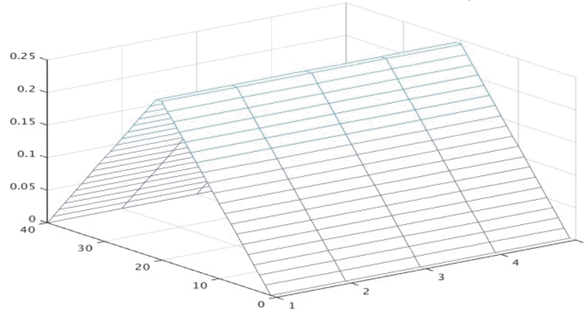
Locality & thin domains

Consider $u_{xx} + u_{yy} = f$ $\quad 0 \le x \le 1 \;,\quad 0 \le y \le \varepsilon$

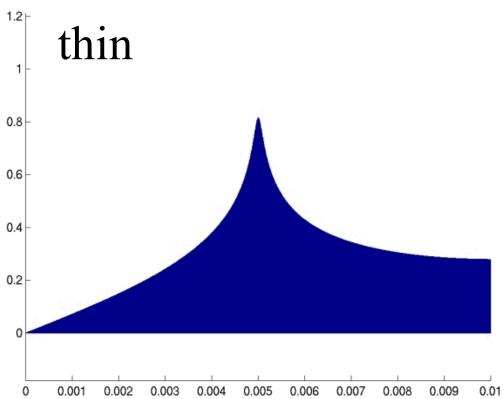
a) $u_y(x,0) = u_y(x, \varepsilon) = 0$

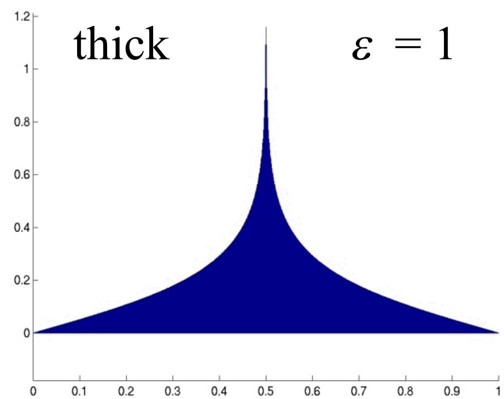
$u(0,y) = u(1,y) = 0$

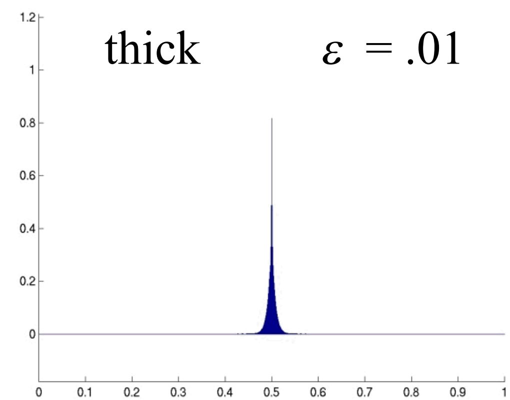
$f(x,y) = \delta(.5,y)$



b) $u(x,0) = 0,\; u_y(x, \varepsilon) = 0$ $\quad u(0,y) = u(1,y) = 0,\; f(x,y) = \delta(.5, \varepsilon/2)$







model ice problem :

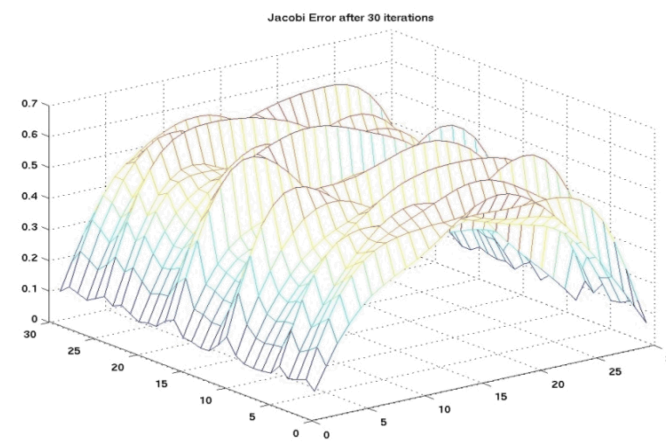
Mesh	small β ILU its.
60 x 60 x 10	7.8
120 x 120 x 10	8.5
240 x 240 x 10	13.9

Mesh	large β ILU its.
60 x 60 x 10	6.6
120 x 120 x 10	6.6
240 x 240 x 10	7.2

Anisotropic Phenomena & Solvers

small eigenvalues associated with weak direction

Model problem: $\varepsilon\, u_{xx} + u_{yy} = f$



MG perspective: hard to smooth errors in weak direction

Remedies: * *fix smoother (e.g., line smoothing)*
* *coarsen only in direction that are easy to smooth*

- *This talks takes the 2nd approach, but does employ line smoothing to allow for aggressive semi-coarsening*
  - *vertical coarsening until just one layer … then horizontal coarsening*

Solver Observations Revisited

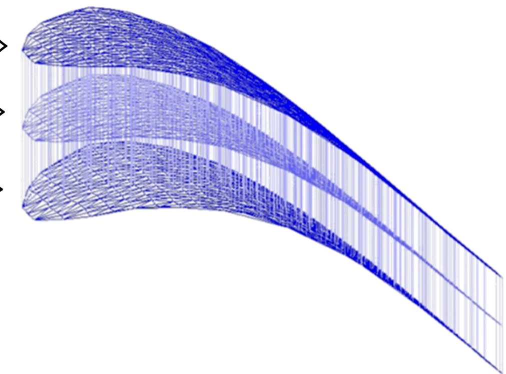
ILU & Anisotropic Problems

- **Vertical line Jacobi & vertical line GS can work well**
- **ILU can work well if vertical coupling accurately captured**
  - **layer-wise ordering best with finite elements**

ILU(0) U factor ⇒

ILU(0) mixed ⇒

ILU(0) L factor ⇒



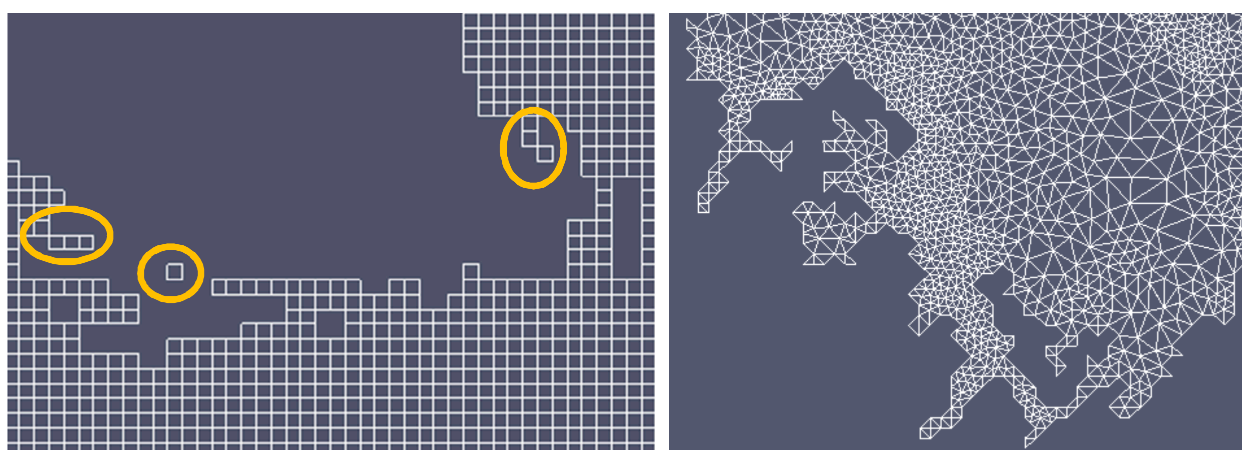
  - **DD+ILU : 2D vertical strip domains**

- **ILU can be very poor if vertical coupling not captured!**

Singularities

floating ice + islands or hinges

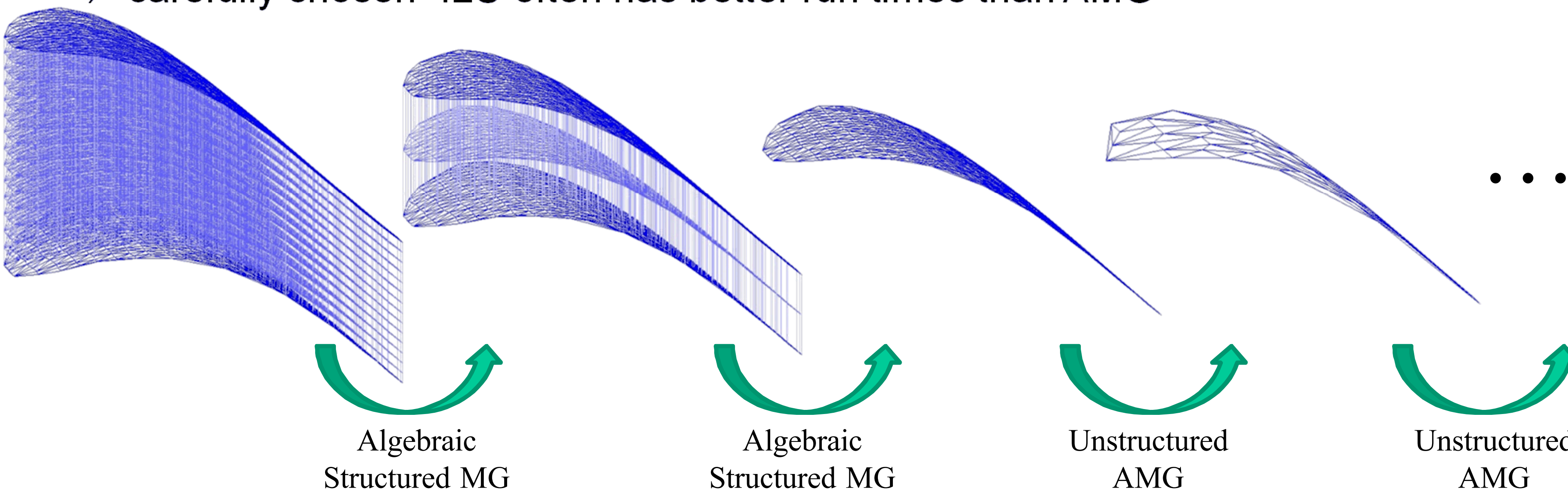
⇒ singular matrix



An Algebraic Multi-Level Approach

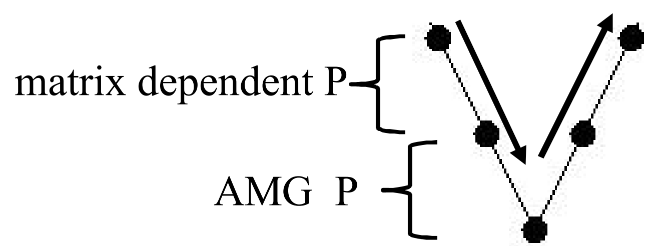
Standard AMG poorly handles bad aspect ratios!

- strength-of-connection algorithms help, but convergence rates still disappointing
- FE matrix complicates strength-of-connection
- bad scaling of Robin condition and non-uniform vertical spacing problematic
- $\Rightarrow$ “carefully chosen” ILU often has better run times than AMG



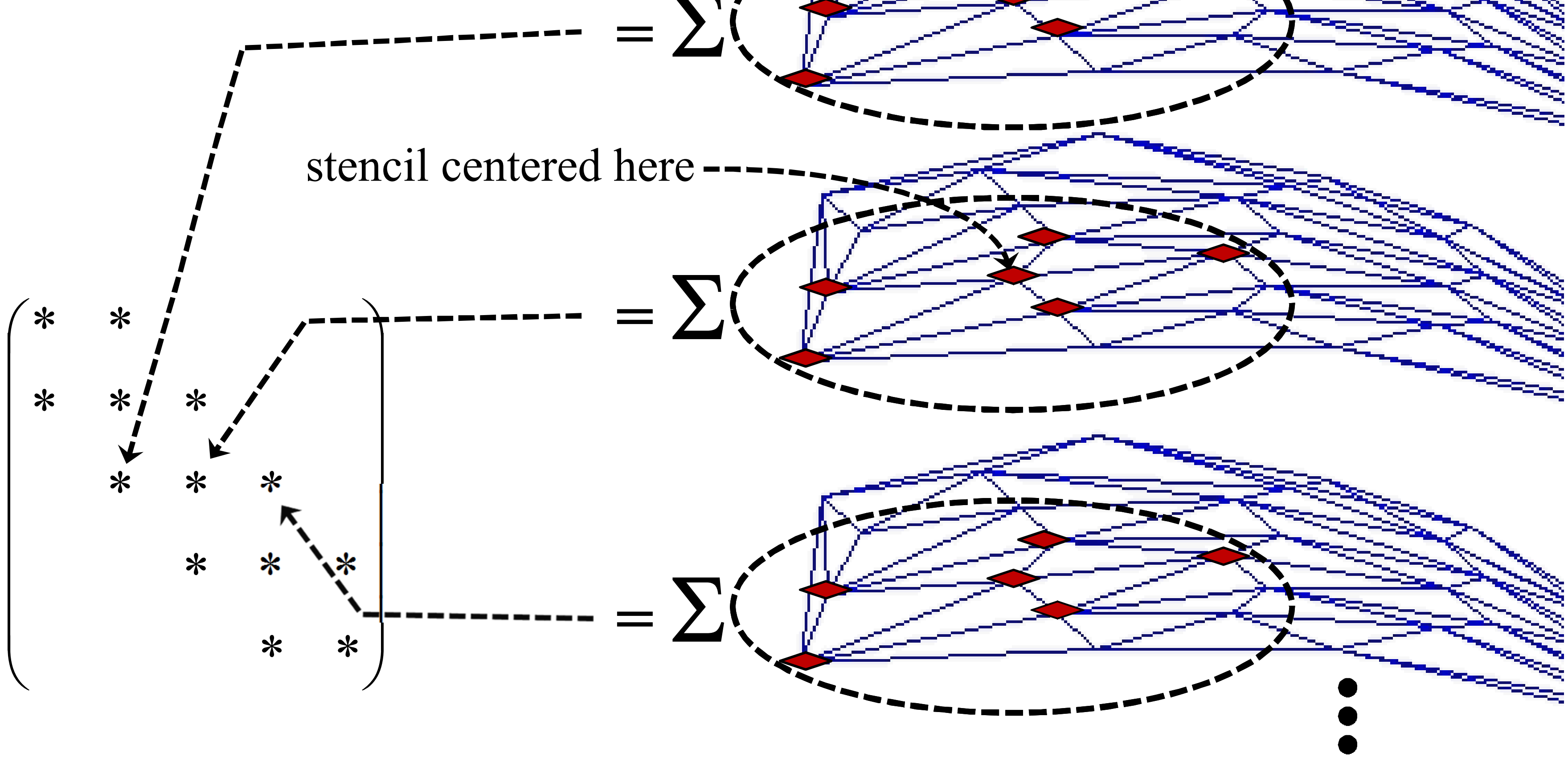
Idea:

- Apply algebraic structured MG (matrix dependent) to 1st coarsen vertically
- Apply traditional SA-AMG on one layer problem to further coarsen



Matrix Dependent Semi-Coarsening P's

- **Collapse 3D stencils to 1D vertical stencils**

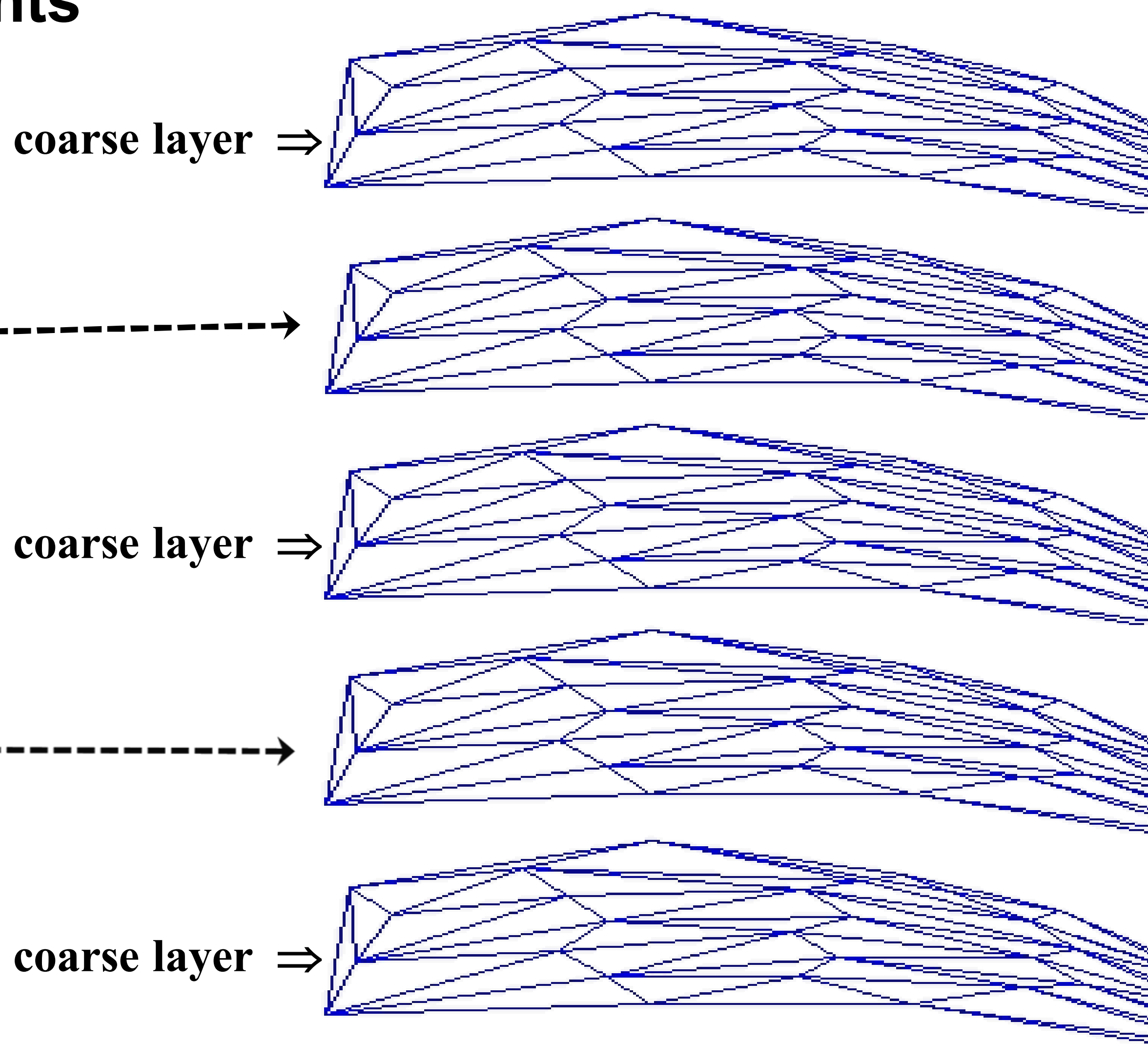


- **Collect into tri-diagonal (or band) systems**
  - **one for each vertical line**

Matrix Dependent Semi-Coarsening

- **Solve tri-diagonal systems for prolongator coefficients**

$$
\begin{pmatrix}
1 & 0 & & & \\
* & * & * & & \\
 & 0 & 1 & 0 & \\
 & & * & * & * \\
 & & & 0 & 1
\end{pmatrix}
\begin{pmatrix} 0 \\ \\ 1 \\ \\ 0 \end{pmatrix}
=
\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}
$$



Practicalities

- Line Jacobi & line GS relaxation to allow for aggressive semi-coarsening

- Implemented in Trilinos/ML
- Almost in Trilinos/MueLu

- Only non-algebraic information needed
  - must provide line orientation information of extruded mesh
  - Can specify layer-wise or vertical column ordering

    …. or can just provide coordinates and we figure it out
  - Ideally, rigid body mode information would be needed for SA …. but some further code modifications needed to apply xy rotational mode

Sandia National Laboratories

Numerical Results

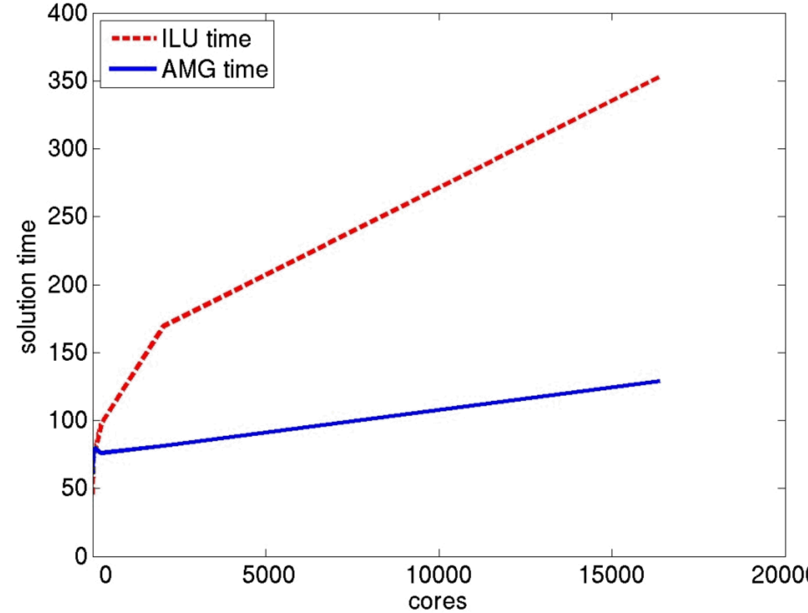
Greenland

slight rise in solution time

- parallel efficiency drops
- its/solve increase slightly

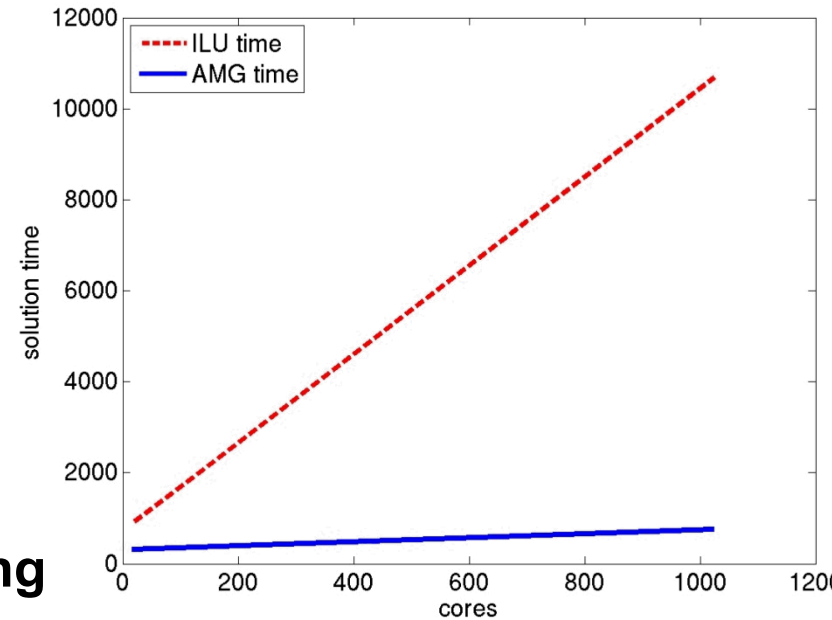
(no mode for SA)

- 8km/5layers → 500m/80layers



Antarctica (regularized β)

- larger benefit over ILU
- 8km/5layers → 2km/80layers???



- nonlinear with homotopy
- # nonlinear steps increases with size
- 14x coarsening rate for semi-coarsening
- V(1,1) line Jacobi with ω = .55 for finer grids, Chebyshev on AMG levels

Conclusions

- thin domains & anisotropic meshes in ice sheet modeling pose problems for standard AMG schemes

- grounded ice is characterized by relative local behavior
  - ILU with proper ordering/partitioning can sometimes work well
  - # of MG levels can be somewhat limited

- matrix dependent extension to extruded meshes proposed

- AMG on 1 layer mesh is not problematic

$\Rightarrow$ almost fully algebraic hybrid multilevel solver

Sandia National Laboratories