

A GENERALIZED LEVERMORE-POMRANING CLOSURE FOR STOCHASTIC MEDIA TRANSPORT PROBLEMS

Shawn D. Pautz* and **Brian C. Franke**
Sandia National Laboratories
Albuquerque, NM 87185-1179
sdpautz@sandia.gov; bcfrank@sandia.gov

ABSTRACT

Stochastic media transport problems have long posed challenges for accurate modeling. Brute force Monte Carlo or deterministic sampling of realizations can be expensive in order to achieve the desired accuracy. The well-known Levermore-Pomraning (LP) closure is very simple and inexpensive, but is inaccurate in many circumstances. We propose a generalization to the LP closure that may help bridge the gap between the two approaches. Our model consists of local calculations to approximately determine the relationship between ensemble-averaged angular fluxes and the corresponding averages at material interfaces. The expense and accuracy of the method are related to how “local” the model is and how much local detail it contains. We show through numerical results that our approach is more accurate than LP for benchmark problems, provided that we capture enough local detail. Thus we identify two approaches to using ensemble calculations for stochastic media calculations: direct averaging of ensemble results for transport quantities of interest, or indirect use via a generalized LP equation to determine those same quantities; in some cases the latter method is more efficient. However, the method is subject to creating ill-posed problems if insufficient local detail is included in the model.

Key Words: radiation transport, stochastic media, Levermore-Pomraning

1 INTRODUCTION

Various transport problems of practical interest involve background media consisting of a mixture of two or more well-characterized materials whose spatial distribution is known only in a statistical sense. Examples of such problems include the transport of solar radiation through cloudy atmosphere and the neutron distribution in pebble bed reactors. Given knowledge of the statistical distribution of the materials in relevant physical realizations, the problem of transport through such stochastic media consists of determining the statistical distribution of angular fluxes and derived quantities such as dose in these realizations. For example, one may wish to determine the mean, variance, and maximum reactivity of a pebble bed reactor in order to ensure criticality safety during operation regardless of the physical arrangement of the pebbles. In the present work we are interested in efficient and accurate calculation of mean quantities.

In principle one could generate physical realizations from a statistical description of the stochastic media, perform calculations on each realization, and then examine the ensemble results. Such an approach suffers from two problems. First, all but the simplest of realizations may be too complex to directly model. Second, the number of realizations required for good statistics may be computationally prohibitive. Therefore it is desirable to construct other methods that can yield the same information at less cost.

* To whom correspondence should be addressed

One such method that has been applied with varying success is the Levermore-Pomraning (LP) approach [1-5]. This approach uses a simple approximation to close the stochastic transport equation, yielding a set of equations that are equivalent to the non-stochastic Boltzmann equation with an extra transition term coupling different materials. In the special case of Markovian media with no scattering the LP model is exact. For other problems the LP treatment may be inaccurate. Unfortunately to date the LP model has not been shown to be part of a richer set of closures that could be employed to increase its accuracy.

In the present work we revisit the assumption made to produce the LP closure, namely that ensemble-averaged fluxes at material interfaces are equal to ensemble averages away from such interfaces. We derive formal expressions to relate the ensemble-averaged angular fluxes to the ones conditioned on the presence of a material interface; this entails the use of an inverse transport problem. These expressions may be discretized to produce a finite ensemble of non-stochastic transport problems whose solution yields an LP-like closure specific to a particular stochastic problem. The accuracy and expense of the method are governed by the fineness of the closure discretization. The LP closure itself is obtained in a certain limit of this approach.

The rest of the paper is organized as follows. In Section 2 we first show how the LP equations are obtained from the exact stochastic equations. We then derive a formal expression for the exact closure. In Section 3 we show one method by which we may discretize that expression. We present results in Section 4 for the more general discretized closure for a set of benchmark problems. In Section 5 we present conclusions and ideas for future work.

2 THEORY

The statistical transport equation governing transport in binary stochastic media is given by (adapted from [4]):

$$\begin{aligned} & \vec{\Omega} \cdot \nabla \langle \psi_i(\vec{r}, \vec{\Omega}) \rangle + \sigma_{t,i} \langle \psi_i(\vec{r}, \vec{\Omega}) \rangle \\ &= \frac{\sigma_{s,i}}{4\pi} \int d\vec{\Omega}' \langle \psi_i(\vec{r}, \vec{\Omega}') \rangle + \lambda_i^{-1} (\langle \psi_{s,j}(\vec{r}, \vec{\Omega}) \rangle - \langle \psi_{s,i}(\vec{r}, \vec{\Omega}) \rangle) \end{aligned} \quad (1)$$

where $\vec{\Omega}$ is the direction of particle travel, i and j are material indexes, $\langle \psi_i(\vec{r}, \vec{\Omega}) \rangle$ is the ensemble-averaged angular flux conditioned on material i being at location \vec{r} , $\langle \psi_{s,i}(\vec{r}, \vec{\Omega}) \rangle$ is the same average conditioned on an interface (surface) occurring at \vec{r} such that the particles leave material i , $\sigma_{t,i}$ and $\sigma_{s,i}$ are the total and scattering cross sections in material i , and λ_i is the mean chord length of material i . This equation is formally exact for arbitrary mixing statistics. Unfortunately an additional set of equations is needed to relate the two types of conditional averages.

2.1 LP Closure

In the LP approach the following assumption is made:

$$\langle \psi_{s,i}(\vec{r}, \vec{\Omega}) \rangle \approx \langle \psi_i(\vec{r}, \vec{\Omega}) \rangle \quad (2)$$

That is, the LP closure assumes that the ensemble-averaged angular flux for a particular direction and location, conditioned on the presence of a material interface, is equal to the ensemble

average without the condition of a material interface. Substitution of Equation (2) into Equation (1) yields the LP equation:

$$\begin{aligned} & \vec{\Omega} \cdot \nabla \langle \psi_i(\vec{r}, \vec{\Omega}) \rangle + \sigma_{t,i} \langle \psi_i(\vec{r}, \vec{\Omega}) \rangle \\ & = \frac{\sigma_{s,i}}{4\pi} \int d\vec{\Omega}' \langle \psi_i(\vec{r}, \vec{\Omega}') \rangle + \lambda_i^{-1} (\langle \psi_j(\vec{r}, \vec{\Omega}) \rangle - \langle \psi_i(\vec{r}, \vec{\Omega}) \rangle) \end{aligned} \quad (3)$$

This equation is identical in form to the linear Boltzmann equation except that there is one equation for each material with a coupling term akin to a delta-scattering function. Equation (3) often can be implemented in standard transport codes with minor modifications.

In order to examine the assumption made by the LP closure we first rewrite Equation (2) as

$$[\langle \psi_s(\vec{r}, \vec{\Omega}) \rangle] \approx I[\langle \psi(\vec{r}, \vec{\Omega}) \rangle] \quad (4)$$

where I is the identity matrix and we now express the angular fluxes in vector form in both the angular and material spaces. The LP closure states that the relationship between the two vectors of fluxes is the identity matrix.

2.2 Generalized Closure

To obtain a generalized closure we make the ansatz

$$[\langle \psi_s(\vec{r}, \vec{\Omega}) \rangle] = R(\vec{r})[\langle \psi(\vec{r}, \vec{\Omega}) \rangle] \quad (5)$$

where R is a “response” matrix relating the two sets of flux averages, of which the LP closure is a special approximation. Our task is to find some way to determine R . Given an exact R we could recast Equation (1) into a form similar to Equation (3) but without any approximations.

In order to develop a model for R we depict an arbitrary realization in Figure 1 for a stochastic transport problem in one-dimensional slab geometry for the “rod” problem [4], which consists of the two directions $\mu = \pm 1$. Here m_l and m_r are the materials at the left and right boundaries, respectively, $m_{c,l}$ and $m_{c,r}$ are the materials to the left and right of an arbitrary interior “central” location, “...” indicates that there are arbitrary material interfaces between the boundary and the central location, Δx_l and Δx_r are the problem thicknesses to the left and right of that location, ψ_{b,l,m_l} and ψ_{b,r,m_r} are the incident boundary fluxes from the left and right, conditioned on the materials at the boundary, and $\psi_{c,l,m_l m_{c,l} m_{c,r} m_r}$ and $\psi_{c,r,m_l m_{c,l} m_{c,r} m_r}$ are fluxes from the left and right at the central location. We assume that the boundary fluxes are specified and that there are no internal sources. Given complete knowledge of the material distribution for any particular realization the interior fluxes are also formally known from the standard Boltzmann transport equation.

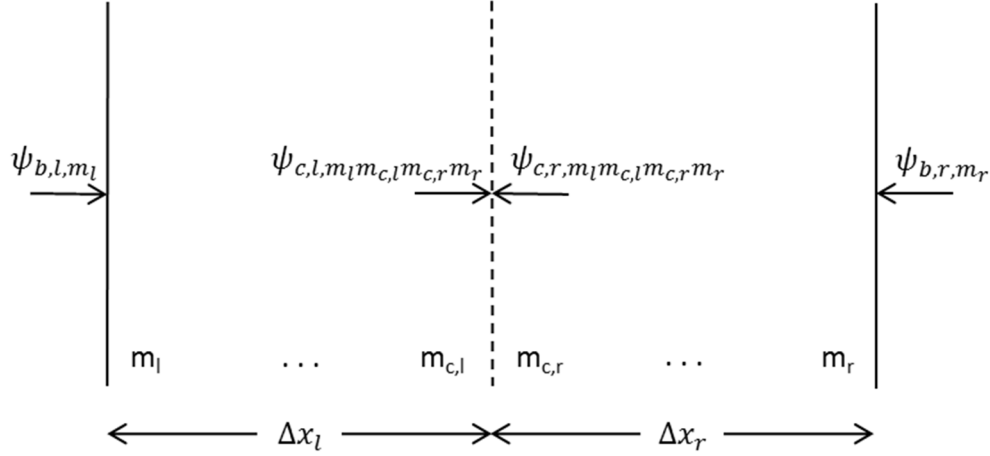


Figure 1. General rod stochastic transport problem.

We define $\langle \psi_{c,l,m_l m_{c,l} m_{c,r} m_r} \rangle$ and $\langle \psi_{c,r,m_l m_{c,l} m_{c,r} m_r} \rangle$ as the ensemble averages of $\psi_{c,l,m_l m_{c,l} m_{c,r} m_r}$ and $\psi_{c,r,m_l m_{c,l} m_{c,r} m_r}$, respectively. We can express these quantities in terms of the boundary fluxes:

$$\begin{bmatrix} \langle \psi_{c,l,m_l m_{c,l} m_{c,r} m_r} \rangle \\ \langle \psi_{c,r,m_l m_{c,l} m_{c,r} m_r} \rangle \end{bmatrix} = R_{m_l m_{c,l} m_{c,r} m_r}(\vec{r}) \begin{bmatrix} \psi_{b,l,m_l} \\ \psi_{b,r,m_r} \end{bmatrix} \quad (6)$$

where $R_{m_l m_{c,l} m_{c,r} m_r}$ is a matrix expressing the transport “response” for the conditional ensemble problem.

We now define additional ensemble averages:

$$\begin{aligned} \langle \psi_{l,1} \rangle &= p_{1111} \langle \psi_{c,l,1111} \rangle + p_{1112} \langle \psi_{c,l,1112} \rangle + p_{1121} \langle \psi_{c,l,1121} \rangle + p_{1122} \langle \psi_{c,l,1122} \rangle + \\ &\quad p_{2111} \langle \psi_{c,l,2111} \rangle + p_{2112} \langle \psi_{c,l,2112} \rangle + p_{2121} \langle \psi_{c,l,2121} \rangle + p_{2122} \langle \psi_{c,l,2122} \rangle \\ \langle \psi_{r,1} \rangle &= p_{1111} \langle \psi_{c,l,1111} \rangle + p_{1112} \langle \psi_{c,l,1112} \rangle + p_{1211} \langle \psi_{c,l,1211} \rangle + p_{1212} \langle \psi_{c,l,1212} \rangle + \\ &\quad p_{2111} \langle \psi_{c,l,2111} \rangle + p_{2112} \langle \psi_{c,l,2112} \rangle + p_{2211} \langle \psi_{c,l,2211} \rangle + p_{2212} \langle \psi_{c,l,2212} \rangle \\ \langle \psi_{l,2} \rangle &= p_{1211} \langle \psi_{c,l,1211} \rangle + p_{1212} \langle \psi_{c,l,1212} \rangle + p_{1221} \langle \psi_{c,l,1221} \rangle + p_{1222} \langle \psi_{c,l,1222} \rangle + \\ &\quad p_{2211} \langle \psi_{c,l,2211} \rangle + p_{2212} \langle \psi_{c,l,2212} \rangle + p_{2221} \langle \psi_{c,l,2221} \rangle + p_{2222} \langle \psi_{c,l,2222} \rangle \\ \langle \psi_{r,2} \rangle &= p_{1121} \langle \psi_{c,l,1121} \rangle + p_{1122} \langle \psi_{c,l,1122} \rangle + p_{1221} \langle \psi_{c,l,1221} \rangle + p_{1222} \langle \psi_{c,l,1222} \rangle + \\ &\quad p_{2121} \langle \psi_{c,l,2121} \rangle + p_{2122} \langle \psi_{c,l,2122} \rangle + p_{2221} \langle \psi_{c,l,2221} \rangle + p_{2222} \langle \psi_{c,l,2222} \rangle \end{aligned} \quad (7)$$

where $p_{m_l m_{c,l} m_{c,r} m_r}$ is the conditional probability of realizations of the appropriate material configurations and $\langle \psi_{d,m} \rangle$ is the ensemble-averaged flux conditioned on material m upstream from the central location for particles moving in direction d . Equations (6) and (7) may be combined to produce the following:

$$\begin{bmatrix} \langle \psi_{l,1} \rangle \\ \langle \psi_{r,1} \rangle \\ \langle \psi_{l,2} \rangle \\ \langle \psi_{r,2} \rangle \end{bmatrix} = R_a(\vec{r}) \begin{bmatrix} \psi_{b,l,1} \\ \psi_{b,r,1} \\ \psi_{b,l,2} \\ \psi_{b,r,2} \end{bmatrix} \quad (8)$$

where the subscript ‘‘a’’ indicates the matrix is derived from all material configurations.

A corresponding set of ensemble-averaged fluxes conditioned on the presence of a material interface at the central location may be defined:

$$\begin{aligned} \langle \psi_{s,l,1} \rangle &= p_{1121} \langle \psi_{c,l,1121} \rangle + p_{1122} \langle \psi_{c,l,1122} \rangle + p_{2121} \langle \psi_{c,l,2121} \rangle + p_{2122} \langle \psi_{c,l,2122} \rangle \\ \langle \psi_{s,r,1} \rangle &= p_{1211} \langle \psi_{c,l,1211} \rangle + p_{1212} \langle \psi_{c,l,1212} \rangle + p_{2211} \langle \psi_{c,l,2211} \rangle + p_{2212} \langle \psi_{c,l,2212} \rangle \\ \langle \psi_{s,l,2} \rangle &= p_{1211} \langle \psi_{c,l,1211} \rangle + p_{1212} \langle \psi_{c,l,1212} \rangle + p_{2211} \langle \psi_{c,l,2211} \rangle + p_{2212} \langle \psi_{c,l,2212} \rangle \\ \langle \psi_{s,r,2} \rangle &= p_{1121} \langle \psi_{c,l,1121} \rangle + p_{1122} \langle \psi_{c,l,1122} \rangle + p_{2121} \langle \psi_{c,l,2121} \rangle + p_{2122} \langle \psi_{c,l,2122} \rangle \end{aligned} \quad (9)$$

where $\langle \psi_{s,d,m} \rangle$ is the ensemble-averaged flux conditioned on the presence of a material interface and conditioned on material m upstream from the central location for particles moving in direction d . Equations (6), (8), and (9) may be combined to produce the following:

$$\begin{bmatrix} \langle \psi_{s,l,1} \rangle \\ \langle \psi_{s,r,1} \rangle \\ \langle \psi_{s,l,2} \rangle \\ \langle \psi_{s,r,2} \rangle \end{bmatrix} = R_s(\vec{r}) \begin{bmatrix} \psi_{b,l,1} \\ \psi_{b,r,1} \\ \psi_{b,l,2} \\ \psi_{b,r,2} \end{bmatrix} = R_s(\vec{r}) R_a^{-1}(\vec{r}) \begin{bmatrix} \langle \psi_{l,1} \rangle \\ \langle \psi_{r,1} \rangle \\ \langle \psi_{l,2} \rangle \\ \langle \psi_{r,2} \rangle \end{bmatrix} \equiv R(\vec{r}) \begin{bmatrix} \langle \psi_{l,1} \rangle \\ \langle \psi_{r,1} \rangle \\ \langle \psi_{l,2} \rangle \\ \langle \psi_{r,2} \rangle \end{bmatrix} \quad (10)$$

where the subscript ‘‘s’’ indicates the matrix is derived from only those material configurations in which a material surface/interface occurs at the central location.

Thus we have recast the ansatz of Equation (5) as a formal closure involving a forward stochastic problem (conditioned on the presence of an interface at \vec{r}) and an inverse stochastic problem. Substitution of Equation (10) into Equation (1) yields a generalized form of the LP equation for rod geometry:

$$\begin{aligned} & \vec{\Omega} \cdot \nabla \langle \psi_i(\vec{r}, \vec{\Omega}) \rangle + \sigma_{t,i} \langle \psi_i(\vec{r}, \vec{\Omega}) \rangle \\ &= \frac{\sigma_{s,i}}{4\pi} \int d\vec{\Omega}' \langle \psi_i(\vec{r}, \vec{\Omega}') \rangle + \lambda_i^{-1} \left(\left(R(\vec{r}) \begin{bmatrix} \langle \psi_{l,1} \rangle \\ \langle \psi_{r,1} \rangle \\ \langle \psi_{l,2} \rangle \\ \langle \psi_{r,2} \rangle \end{bmatrix} \right)_{j,\vec{\Omega}} - \left(R(\vec{r}) \begin{bmatrix} \langle \psi_{l,1} \rangle \\ \langle \psi_{r,1} \rangle \\ \langle \psi_{l,2} \rangle \\ \langle \psi_{r,2} \rangle \end{bmatrix} \right)_{i,\vec{\Omega}} \right) \end{aligned} \quad (11)$$

2.3 Discussion of General Closure

In the above analysis we have shown a formal way to express the relationship between ensemble-averaged angular fluxes at material interfaces and the unconditionally-averaged fluxes. This allows us to write a formally exact LP-like equation. We have not necessarily shown that these expressions improve our ability to approximately solve stochastic media problems. One

may solve numerous stochastic realizations and use them to directly construct ensemble averages of the desired transport quantities of interest (e.g. transmitted flux), or one may use those realizations to construct approximations to R for use in Equation (11) to indirectly compute the quantities of interest; in either case one must solve a potentially large number of non-stochastic transport problems.

We observe, however, that the generalized LP approach may be a valuable addition to the set of available stochastic transport methods. We hope that it stimulates additional theoretical insight for stochastic problems. In some cases this approach may require fewer realizations for indirect (rather than direct) evaluation of the transport quantities of interest. Alternatively one could precompute accurate but expensive values for R for repeated application of Equation (11), for example in uncertainty quantification (UQ) studies. Such an application would be analogous to the generation and use of multigroup cross sections.

We also observe that the current work bears some relation to other recent work. In [6] the authors proposed using auxiliary Monte Carlo calculations to obtain a multiplication factor in Equation (2), or equivalently to replace the identity matrix in Equation (4) with a different diagonal matrix. In our work we do not restrict R to be a diagonal matrix, and in Section 4 our numerical results show that in general R has non-zero off-diagonal entries.

One particular strength of the generalized LP approach is its potential use with subgrid models. The Δx_l and Δx_r in Figure 1 need not necessarily sum to the full width of the problem of interest; one could restrict attention to a smaller region in the vicinity of each point of interest. The computation of individual realizations would be less expensive, and fewer may be needed due to the reduced complexity of the problem. This is analogous to the use of detailed assembly-level calculations for less detailed whole-core analyses.

We do note one important special limit and one special case of the generalized LP approach with subgrid models. In the limit of $\Delta x_l \rightarrow \Delta x_r \rightarrow 0$ we obtain the original LP closure, since then $R_a = R_s = I$. Thus we may view LP not as a *sui generis* approximation to stochastic transport problems, but rather as the simplest member of a class of closures of arbitrary accuracy and expense. Furthermore, in the special case of purely absorbing Markovian materials we also obtain the original LP closure, since then $R_a = R_s$, which yields $R = I$. It is already widely known that the LP closure is exact in this case, which is consistent with our generalized approach.

3 DISCRETIZATION

In the previous section we derived a generalized LP closure which depends on various conditional ensemble averages. We now discuss how we may obtain such averages.

One technique is to use Monte Carlo sampling of realizations and then to apply the needed ensemble averaging. In order to compute R_s one must be careful to generate realizations with material interfaces at the desired locations. We have tried this approach without much success; our solutions of Equations (10) and (11) behaved erratically. We believe this may be due to the uncorrelated nature of the sampling we employed. It might be necessary to somehow correlate the selection of realizations that have an interface at desired locations with those that do not so that the statistical errors in R_a and R_s do not dominate, especially if the two matrices are similar.

A different technique is to use deterministic generation of realizations. In a companion paper [7] we describe how one can generate realizations with interfaces at particular spatial quadrature points. This method also calculates the probabilities of each realization so that ensemble averages may be computed. One advantage is that one can select quadrature points so that interfaces sometimes occur at desired locations, which is necessary for the determination of R_s . The errors in R_a and R_s will also be correlated. It is this method that we will employ to approximately determine the general LP response matrices.

4 NUMERICAL RESULTS

In order to test our modified algorithm we reexamine the benchmark problems first reported in [4]. These problems consist of nine different combinations of binary media and mixing statistics for three different slab widths. The problems are monoenergetic in one-dimensional slab geometry; we restrict our attention to the rod problems. The problems are driven by an isotropic flux on the left boundary. All scattering is isotropic. The reflected and transmitted currents are the transport quantities examined.

Using the methods described in [7], we create ensembles of realizations of varying structures. We generate the transport results for individual realizations with the Sceptre deterministic code [8] using its discretization of the first-order form of the linear monoenergetic Boltzmann equation, controlling the iterative errors to be less than 10^{-7} and spatial errors to be less than 10^{-6} [9]. The spatial quadrature for the interface locations is a simple bisection rule with uniform weights which creates equal-sized regions. These results were used to calculate discrete approximations to R . We have generated results for all cases and slab widths.

Table I. Matrix R for case 1

Δx	location							
	center				left boundary			
0.1	$\begin{bmatrix} 1.00313 & -0.00464 & -0.00306 & 0.00457 \\ -0.00464 & 1.00313 & 0.00457 & -0.00306 \\ -0.02570 & 0.15479 & 1.02536 & -0.15444 \\ 0.15479 & -0.02570 & -0.15444 & 1.02536 \end{bmatrix}$				$\begin{bmatrix} 1.00337 & -0.00009 & -0.00337 & 0.00009 \\ -0.01306 & 1.00307 & 0.01298 & -0.00299 \\ -0.03047 & 0.02769 & 1.03047 & -0.02769 \\ 0.21662 & -0.02785 & -0.21635 & 1.02759 \end{bmatrix}$			
1	$\begin{bmatrix} 0.52998 & -0.02478 & 0.66238 & -0.17187 \\ -0.02478 & 0.52998 & -0.17187 & 0.66238 \\ 0.06335 & 0.19044 & 0.75991 & -0.01250 \\ 0.19044 & 0.06335 & -0.01250 & 0.75990 \end{bmatrix}$				$\begin{bmatrix} 1.03240 & -0.00271 & -0.03156 & 0.00190 \\ -0.14695 & 0.46473 & -0.03174 & 0.70810 \\ -0.29650 & 0.10827 & 1.26310 & -0.07598 \\ 0.42104 & 0.07548 & -0.31849 & 0.82529 \end{bmatrix}$			
10	$\begin{bmatrix} 0.32325 & -0.03510 & 2.29299 & -1.63250 \\ -0.03510 & 0.32325 & -1.63250 & 2.29299 \\ -0.02831 & 0.25740 & -0.54531 & 1.28868 \\ 0.25740 & -0.02831 & 1.28867 & -0.54530 \end{bmatrix}$				$\begin{bmatrix} 1.04513 & -0.00767 & -0.01727 & -0.01880 \\ -0.23824 & 0.30503 & -1.01312 & 1.88455 \\ -0.35468 & 0.23039 & 0.51859 & 0.56439 \\ 0.82264 & 0.00544 & -0.48904 & 0.67734 \end{bmatrix}$			

We show one set of results for R for case 1 in Table I, which corresponds to Table 1 in [4]. These results are for full-grid models; the boundaries of Figure 1 are equal to that of the full stochastic problem. Results are shown in two locations: the center of the problem and also the

spatial quadrature location (potential interface location) closest to the left boundary. These results are from the most accurate calculations we have performed, namely a spatial quadrature with 15 points and a maximum number of “pseudointerfaces” [7] of 15.

We note some trends in the above and similar results:

- The matrices are not the identity matrix. For thin problems they are closest to being the identity matrix, but as problem thicknesses increase they can diverge greatly from the LP closure.
- The response matrices at the center have block symmetry (within roundoff error) which we expect, but they lose symmetry as one moves towards the boundaries. Matrices towards the boundary can be significantly different than those at the center.
- Individual entries can be significantly greater than unity, particularly for thicker problems.

Regarding the final observation above, we must note that for many thicker problems the results are not at all resolved [7]. The mean distances between interfaces in the continuous problem can be much smaller than those in our discrete model. We believe that this will produce larger computed matrix entries than the correct ones, since variations in coarse realizations (due to conditional averaging) will have a greater effect on transport results than those from realizations with much finer interfacial structures. That is, discrete versions of R_a and R_s will have greater differences from each other than the correct ones.

In order to test the use of response matrices in the generalized LP equation we modified the Sceptre code (which has an LP treatment) to make use of general response matrices. We also provided for the use of spatially-dependent matrices to test boundary effects. We obtained generalized LP results for all of the test problems and for varying accuracies of R . We do not show results here, but we make some observations of the results:

- For relatively simple problems (in terms of average number of interfaces) the accuracy of the general LP method for transport quantities of interest is often comparable to or better than direct calculation of those quantities from the ensemble of realizations. The accuracy is measured with respect to benchmark results [4,9].
- In some cases the results from both very coarse and very fine calculations of R were much better than those from ensembles of intermediate expense; we do not have an explanation for the accuracy of the coarse results.
- In many cases when R was underresolved (as measured by the effective average distance between interfaces) the general LP calculation was unstable when solved with source iteration, presumably because the problem was ill-posed. As we noted earlier, individual matrix entries can be quite large in these situations. Furthermore when we apply Equation (11) we use the original values of λ_i , which will be smaller than the discrete ones for underresolved calculations of R . We have not attempted to use the discrete values in our general LP calculations.

As a final demonstration of our generalized LP method we examine its use in subgrid methods. In Tables II-V we report the errors of our method for all test cases using response matrices computed from realizations whose thicknesses ($\Delta x_l + \Delta x_r$) are equal to or less than the true stochastic problem. As noted earlier a thickness of zero is in fact the original LP method.

We used the most accurate calculation of R that we had generated in each case. We also used only the matrix at the center of the problem and ignored potential boundary effects. In almost every case the use of a thin subgrid model ($\Delta x_l + \Delta x_r = 0.1$) has less error than the original LP method. As the subgrid model thickness increases the results can improve more, but eventually the error increases and/or the calculation becomes unstable.

Table II. Errors of subgrid models for reflection for slab thickness of 1.

case	$\Delta x_l + \Delta x_r$		
	0 (LP)	0.1	1
1	-0.084	-0.038	-0.169
2	-0.103	-0.098	0.229
3	-0.091	-0.046	-0.185
4	-0.10	-0.06	< 0.01
5	-0.028	-0.025	< 0.001
6	-0.11	-0.07	< 0.01
7	< 0.01	< 0.01	< 0.01
8	-0.01	-0.01	< 0.01
9	-0.01	< 0.01	< 0.01

Table III. Errors of subgrid models for transmission for slab thickness of 1.

case	$\Delta x_l + \Delta x_r$		
	0 (LP)	0.1	1
1	0.025	0.012	0.051
2	< 0.001	-0.008	0.365
3	0.024	0.011	0.064
4	0.014	0.008	0.001
5	0.002	0.002	< 0.001
6	0.010	0.006	< 0.001
7	0.003	0.002	< 0.001
8	< 0.01	< 0.01	< 0.01
9	0.004	0.003	< 0.001

Table IV. Errors of subgrid models for reflection for slab thickness of 10.

case	$\Delta x_l + \Delta x_r$			
	0 (LP)	0.1	1	10
1	-0.101	-0.076	0.041	---
2	-0.28	-0.28	-0.26	---
3	-0.157	-0.132	-0.033	---
4	-0.24	-0.18	-0.06	-0.17
5	-0.245	-0.244	-0.226	-0.223
6	-0.330	-0.272	-0.141	-0.156
7	-0.113	-0.106	-0.069	-0.017
8	-0.32	-0.31	-0.28	0.04
9	-0.212	-0.200	-0.135	-0.016

Table V. Errors of subgrid models for transmission for slab thickness of 10.

case	$\Delta x_l + \Delta x_r$			
	0 (LP)	0.1	1	10
1	0.47	0.35	-0.14	---
2	-0.04	-0.11	-0.43	---
3	0.36	0.27	-0.23	---
4	0.29	0.22	0.07	0.19
5	-0.03	-0.03	-0.05	-1.03
6	0.11	0.10	0.08	0.73
7	0.33	0.30	0.20	0.05
8	< 0.01	< 0.01	-0.02	0.15
9	0.19	0.17	0.11	0.11

5 CONCLUSIONS

In the present work we have reexamined the theory behind the LP closure and created a generalized framework for closures to the stochastic transport equations involving response matrices. We have shown that the LP closure is a special case of this framework in the limit of a subgrid model of zero thickness. We have shown one method of computing the general response matrices of varying accuracy and expense. Numerical results show that the general LP approach with subgrid models for the response matrices can be more accurate than the original LP method provided that the models are sufficiently resolved.

In future work we want to examine the use of subgrid models more fully. We would like to know how large and detailed such models should be for the desired accuracy. We would like to understand how problem boundaries affect the response matrices and whether this could lead to more efficient generation of position-dependent matrices and/or corrections. We also want to extend this method to higher order angular quadratures, multigroup transport, and multidimensional geometries, which would increase the complexity of the model.

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