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Bounded extremum seeking for angular velocity actuated control of nonholonomic unicycle

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SUMMARY

We study control of the angular-velocity actuated nonholonomic unicycle, via a simple, bounded extremum seeking controller which is robust to external disturbances and measurement noise. The vehicle performs source seeking despite not having any position information about itself or the source, able only to sense a noise corrupted scalar value whose extremum coincides with the unknown source location. In order to control the angular velocity, rather than the angular heading directly, a controller is developed such that the closed loop system exhibits multiple time scales and requires an analysis approach expanding the previous work of Kurzweil, Jarnik, Sussmann, and Liu, utilizing weak limits. We provide analytic proof of stability and demonstrate how this simple scheme can be extended to include position-independent source seeking, tracking, and collision avoidance of groups on autonomous vehicles in GPS-denied environments, based only on a measure of distance to an obstacle, which is an especially important feature for an autonomous agent.

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KEY WORDS: nonlinear; robust; extremum seeking; unicycle; autonomous systems

1. INTRODUCTION

Motivation Extremum seeking (ES) has had many applications [1] with unknown/uncertain [2, 3] and discrete-time [4] systems, such as enhancing mixing in magnetohydrodynamic channel flows [5], controlling Tokamak plasmas [6], and recently, utilizing a multivariable Newton-based scheme, for the power optimization of photovoltaic micro-converters [7].

The work in [8] developed Extremum Seeking for Control (ESC) to stabilize unknown, open-loop unstable systems by applying the extremum seeker as the controller itself. ESC has been demonstrated in hardware [9] and studied for inverted pendulum stabilization [10], and in a general form on manifolds [11]. In [12] a bounded form of ESC was developed, in which the control efforts and parameter update rates have analytically known bounds, for the simultaneous optimization and stabilization of systems of the form

$$\dot{x} = f(x,t) + g(x,t)u(y), \tag{1}$$

$$y = \psi(x,t) + n(t), \tag{2}$$

where $f(x,t)$ is unknown and possibly destabilizing, $g(x,t)$ is unknown and possible changes sign as a function of state and time, $\psi(x,t)$ is an analytically unknown function to be minimized, $n(t)$ is additive noise, and $y(t)$ is a noise-corrupted measurement of $\psi(x,t)$. The controller

$$u_i = e_i \sqrt{\alpha\omega_i} \cos(\omega_i t + ky), \tag{3}$$

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applied to system (1), (2), where e_i is the ith basis vector in $\mathbb{R}^n$ and $\omega_i \neq \omega_j$, results in average system dynamics

$$\dot{\bar{x}} = f(\bar{x}, t) - g(\bar{x}, t) g^T(\bar{x}, t) \frac{k\alpha}{2} \left(\nabla \psi(\bar{x}, t)\right)^T, \ \bar{x}(0) = x(0), \tag{4}$$

which, due to the fact that $g(\bar{x}, t) g^T(\bar{x}, t) \geqslant 0$, performs a gradient descent of the unknown function $\psi(x, t)$ without requiring a knowledge of the sign of $g(x, t)$ or the gradient of $\psi(x, t)$, in which the influence of random noise $n(t)$ is, on average, removed. The bounded ESC approach is especially useful for digital implementation and for extremely noisy systems, as was recently demonstrated in hardware [13] and studied analytically for a large class of non-differentiable and discontinuous systems [14].

The nonholonomic unicycle is an important nonlinear system, which has been the topic of several studies, including path following in the presence of parametric uncertainties [15] and time-varying controllers for exponential tracking for nonholonomic systems in chained form [16]. In this work, we study an angular velocity actuating controller for a nonholonomic system, providing proof of the convergence of a bounded ESC approach to unicycle control, which is robust to measurement noise and un-modeled environmental disturbances.

Results of the paper Consider the autonomous control of a source-seeking vehicle of the form

$$\dot{x} = \sqrt{\alpha\omega} \cos(\theta), \qquad \dot{y} = \sqrt{\alpha\omega} \sin(\theta), \tag{5}$$

with a controller of the form

$$\dot{\theta} = \omega + k\omega^2 (\hat{J}(x, y, t) - \eta), \ \theta(0) = \theta_0 \tag{6}$$

$$\dot{\eta} = -\omega^2 \left(\eta - \hat{J}(x, y, t)\right), \ \eta(0) = \eta_0 \tag{7}$$

where $\hat{J}(x, y, t) = J(x, y, t) + n(t)$ is a noise-corrupted measurement of an unknown scalar function, $J(x, y, t)$, to be minimized, θ_0 is an unknown initial orientation and η_0 is an arbitrary initial state. The closed loop system dynamics are, on average

$$\dot{\bar{x}} = -\frac{k\alpha}{2} \frac{\partial J(\bar{x}, \bar{y}, t)}{\partial \bar{x}}, \quad \bar{x}(0) = x(0), \tag{8}$$

$$\dot{\bar{y}} = -\frac{k\alpha}{2} \frac{\partial J(\bar{x}, \bar{y}, t)}{\partial \bar{y}}, \quad \bar{y}(0) = y(0), \tag{9}$$

a gradient descent of the unknown, time-varying function. This type of controller was conjectured in [12] without stability analysis.

2. THEORETICAL BACKGROUND

Recall that a sequence of functions $\{f_k\} \subset L^2[0, 1]$ is said to converge weakly to f in $L^2[0, 1]$, denoted $f_k \to f$, if

$$\lim_{k\to\infty} < f_k, g >= \lim_{k\to\infty} \int_0^1 f_k(\tau) g(\tau) d\tau = \int_0^1 f(\tau) g(\tau) d\tau =< f, g >, \quad \forall g \in L^2[0, 1].$$

Theorem 1 ([14, 17–20]*)*
Consider the vector-valued system

$$\dot{x} = f(x, t) + g(x, t) u(y, t), \qquad y = \psi(x, t) + n(t), \tag{10}$$

where $x \in \mathbb{R}^n$, and the functions

$$f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n, \ g : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^{n\times n}, \ \psi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}, \ n : \mathbb{R} \to \mathbb{R}, \tag{11}$$

are unknown. Assume that f and g are twice continuously differentiable with respect to x and assume that the value y of $\psi(x,t)$ is available for measurement. Consider a controller $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^n$, given by

$$u(y,t) = \sum_{i=1}^{m} k_i(y,t) h_{i,\omega}(t), \quad k_i : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^n, \tag{12}$$

where the functions $k_i(y,t)$ are continuously differentiable and the functions $h_{i,\omega}(t)$ are piece-wise continuous. System (10), (12) has the following equivalent closed-loop form

$$\dot{x}(t) = f(x,t) + \sum_{i=1}^{m} b_i(x,t) h_{i,\omega}(t), \tag{13}$$

$$b_i(x,t) = g(x,t) k_i(\psi(x,t),t). \tag{14}$$

Suppose that the integrals of the functions $h_{i,\omega}(t)$ satisfy the uniform limits

$$\lim_{\omega\to\infty} H_{i,\omega}(t) = \lim_{\omega\to\infty} \int_{t_0}^{t} h_{i,\omega}(\tau) d\tau = 0, \tag{15}$$

and the weak limits

$$h_{i,\omega}(t) H_{j,\omega}(t) \to \lambda_{i,j}(t). \tag{16}$$

Consider also the average system related to (13), (14) as follows:

$$\dot{\bar{x}} = f(\bar{x},t) - \sum_{i,j=1}^{n} \lambda_{i,j}(t) \frac{\partial b_i(\bar{x},t)}{\partial \bar{x}} b_j(\bar{x},t), \quad \bar{x}(0) = x(0). \tag{17}$$

For any compact set $K \subset \mathbb{R}^n$, any $t_0, T \in \mathbb{R}_{\geqslant 0}$, and any $\delta > 0$, there exists $\omega^\star$ such that for each $\omega > \omega^\star$, the trajectories $x(t)$ and $\bar{x}(t)$ of (13) and (17), satisfy

$$\max_{x,\bar{x}\in K \; t\in[t_0,t_0+T]} \|x(t) - \bar{x}(t)\| < \delta. \tag{18}$$

Furthermore, for any $x^\star(t) \in K$, with bounded derivatives,

$$\lim_{t\to\infty} \|\bar{x}(t) - x^\star(t)\| = 0 \Longrightarrow \lim_{t\to\infty} \|x(t) - x^\star(t)\| < \delta.$$

In other words, uniform asymptotic stability of (17) over K implies the $\frac{1}{\omega}$-semiglobal practical uniform asymptotic stability of (13), where the $\frac{1}{\omega}$ term indicates that the region of convergence is made arbitrarily small by choosing arbitrarily large ω, a form of stability studied in [19].

3. 2D ANGULAR VELOCITY ACTUATED VEHICLE

Consider a vehicle in a GPS-denied environment, unaware of its orientation, whose goal it is to reach the location of the minimum of the analytically unknown function $J(x,y,t)$, whose noise-corrupted value, $\hat{J}(x,y,t) = J(x,y,t) + n(t)$ is available for measurement. This formulation is more representative of a real world scenario and more challenging than the simpler direct-angle actuation system as studied in [12], the filtering introduces a third time scale to the problem dynamics. We begin with a detailed analysis for $\hat{J}(x,y)$ and then extend the results to $\hat{J}(x,y,t)$.

Theorem 2
Consider the system shown in Figure 1, with dynamics:

$$\dot{x} = \sqrt{\alpha\omega}\cos(\theta) = g_{\omega,x}(x,y,t) \tag{19}$$

$$\dot{y} = \sqrt{\alpha\omega}\sin(\theta) = g_{\omega,y}(x,y,t) \tag{20}$$

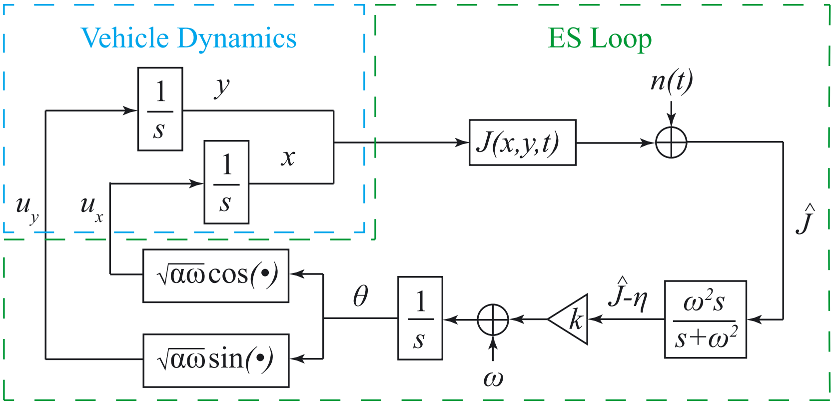

Figure 1. Velocity actuated ES control scheme.

$$\dot{\theta} = \omega + k\omega^2(\hat{J} - \eta), \ \theta(0) = \theta_0 \tag{21}$$

$$\dot{\eta} = -\omega^2 \left(\eta - \hat{J}\right), \ \eta(0) = \eta_0 \tag{22}$$

where θ_0 is an unknown initial orientation, and η_0 is an arbitrary initial state. Consider also the average system

$$\dot{\bar{x}} = -\frac{k\alpha}{2}\frac{\partial J}{\partial \bar{x}}, \quad \bar{x}(0) = x(0), \tag{23}$$

$$\dot{\bar{y}} = -\frac{k\alpha}{2}\frac{\partial J}{\partial \bar{y}}, \quad \bar{y}(0) = y(0). \tag{24}$$

When satisfying all necessary continuity and differentiability requirements of Theorem 1, for any compact set $K \subset \mathbb{R}^n$, any $t_0, T \in \mathbb{R}_{\geqslant 0}$, and any $\delta > 0$, there exists $\omega^\star$ such that for each $\omega > \omega^\star$, the trajectories $(x(t), y(t))$ and $(\bar{x}(t), \bar{y}(t))$ satisfy

$$\max_{(x,y),(\bar{x},\bar{y})\in K \ t\in[t_0,t_0+T]} \|(x(t), y(t)) - (\bar{x}(t), \bar{y}(t))\| < \delta. \tag{25}$$

Remark 1
In the analysis that follows, it becomes apparent that the values of the arbitrary initial conditions θ_0 and η_0 are irrelevant, when we make the simplification:

$$\sin^2\left(k\hat{J} + \theta_0 + k\eta_0\right) + \cos^2\left(k\hat{J} + \theta_0 + k\eta_0\right) = 1,$$

therefore for notational convenience, and without loss of generality, from now on we set both to 0. Furthermore, in the averaged dynamics, all of the surviving terms containing $\hat{J}$ are of the form $\nabla\hat{J}(x, y, t) = \nabla(J(x, y, t) + n(t)) = \nabla J(x, y, t)$; therefore, we work with J instead of $\hat{J}$ throughout the proof.

Proof
Combining (21) and (22) we can expand θ as follows:

$$\theta = \omega t + \underbrace{k\omega^2 e^{-\omega^2 t}\int_0^t e^{\omega^2\tau} J(x(\tau), y(\tau))d\tau}_{I_1} + \theta_0 + k\eta_0. \tag{26}$$

Integrating the term I_1 by parts gives

$$I_1 = kJ(t) - kJ(0)e^{-\omega^2 t} - \underbrace{ke^{-\omega^2 t}\int_0^t e^{\omega^2\tau}\frac{\partial J}{\partial x}\sqrt{\omega}\cos(\theta)d\tau}_{I_{1,1}} - \underbrace{ke^{-\omega^2 t}\int_0^t e^{\omega^2\tau}\frac{\partial J}{\partial y}\sqrt{\omega}\sin(\theta)d\tau}_{I_{1,2}}. \tag{27}$$

Therefore, on the compact set K, the term I_1 has bound

$$|I_1| \leqslant k\left(|J(t)| + |J(0)| + \left\|\frac{\partial J}{\partial x}\right\| + \left\|\frac{\partial J}{\partial y}\right\|\right)\sqrt{\omega}t. \tag{28}$$

By considering the time-scale $\sigma = \sqrt{\omega}t$, we can rewrite θ as

$$\theta = \sqrt{\omega}\sigma + I_1, \tag{29}$$

$$|I_1| \leqslant k\left(|J(t)| + |J(0)| + \left\|\frac{\partial J}{\partial x}\right\| + \left\|\frac{\partial J}{\partial y}\right\|\right)\sigma, \tag{30}$$

with the bound on $|I_1|$ independent of ω. Therefore, by the Riemann-Lebesgue lemma, the term

$$g_\omega(x,y,t) = \begin{bmatrix} g_{\omega,x}(x,y,t) \\ g_{\omega,y}(x,y,t) \end{bmatrix}, \tag{31}$$

whose components, on the σ time scale, can be expanded in the form

$$\cos\left(\sqrt{\omega}\sigma\right)\cos\left(I_1\right) - \sin\left(\sqrt{\omega}\sigma\right)\sin\left(I_1\right), \tag{32}$$

weakly, uniformly converges to zero, which implies that

$$\lim_{\omega\to\infty} G_\omega(x,y,t) = \lim_{\omega\to\infty}\int_0^t g_\omega(x,y,\tau)d\tau = 0 \tag{33}$$

uniformly, which satisfies requirement (15) of Theorem 1. By the same argument, the terms $I_{1,1}$ and $I_{1,2}$ uniformly converge to zero as ω approaches infinity, clearly so does the term $kJ(0)e^{-\omega^2 t}$. Therefore, in order to consider the limit of the type (16) in Theorem 1, we consider the equivalent limiting representation of G_ω as follows:

$$\lim_{\omega\to\infty} G_\omega(x,y,t) = \lim_{\omega\to\infty}\int_0^t \begin{bmatrix}\sqrt{\alpha\omega}\cos(\theta) \\ \sqrt{\alpha\omega}\sin(\theta)\end{bmatrix}d\tau = \lim_{\omega\to\infty}\int_0^t \underbrace{\begin{bmatrix}\sqrt{\alpha\omega}\cos(\omega\tau + kJ(\tau)) \\ \sqrt{\alpha\omega}\sin(\omega\tau + kJ(\tau))\end{bmatrix}}_{A_\omega}d\tau. \tag{34}$$

Applying trigonometric identities we expand A_ω as

$$A_\omega = \begin{bmatrix}\sqrt{\alpha\omega}\cos(\omega\tau)\cos(kJ) - \sqrt{\alpha\omega}\sin(\omega\tau)\sin(kJ) \\ \sqrt{\alpha\omega}\cos(\omega\tau)\sin(kJ) + \sqrt{\alpha\omega}\sin(\omega\tau)\cos(kJ)\end{bmatrix}. \tag{35}$$

Therefore, to calculate the average system, as in (17), we need to find a function $h(x,y,t)$ such that

$$\lim_{\omega\to\infty}\int_0^t DG_\omega(x,y,\tau)g_\omega(x,y,\tau)d\tau = -\int_0^t h(x,y,\tau)d\tau, \tag{36}$$

uniformly with ω, where $D = \frac{\partial}{\partial x}$. By the previous arguments, the limit we are taking is equal to the limit

$$\lim_{\omega\to\infty}\int_0^t\left(\int_0^\tau DA_\omega(x,y,s)ds\right)A_\omega(x,y,\tau)d\tau. \tag{37}$$

Note that we were free to pass the derivative through the integral because all of the functions considered are continuous. Direct calculation shows that

$$DA_\omega(x,y,s) = \begin{bmatrix} \frac{\partial A_x}{\partial x} & \frac{\partial A_y}{\partial x} \\ \frac{\partial A_y}{\partial x} & \frac{\partial A_y}{\partial y} \end{bmatrix}, \tag{38}$$

where

$$\frac{\partial A_x}{\partial x} = -k\frac{\partial J}{\partial x}\sqrt{\alpha\omega}\cos(\omega s)\sin(kJ) - k\frac{\partial J}{\partial x}\sqrt{\alpha\omega}\sin(\omega s)\cos(kJ), \tag{39}$$

$$\frac{\partial A_x}{\partial y} = -k\frac{\partial J}{\partial y}\sqrt{\alpha\omega}\cos(\omega s)\sin(kJ) - k\frac{\partial J}{\partial y}\sqrt{\alpha\omega}\sin(\omega s)\cos(kJ), \tag{40}$$

$$\frac{\partial A_y}{\partial x} = k\frac{\partial J}{\partial x}\sqrt{\alpha\omega}\cos(\omega s)\cos(kJ) - k\frac{\partial J}{\partial x}\sqrt{\alpha\omega}\sin(\omega s)\sin(kJ), \tag{41}$$

$$\frac{\partial A_y}{\partial y} = k\frac{\partial J}{\partial y}\sqrt{\alpha\omega}\cos(\omega s)\cos(kJ) - k\frac{\partial J}{\partial y}\sqrt{\alpha\omega}\sin(\omega s)\sin(kJ). \tag{42}$$

Integrating, by parts, the $\cos(\omega s)$ term of (39), gives

$$\begin{aligned}&\sqrt{\frac{\alpha}{\omega}}\sin(\omega\tau)\left(-k\frac{\partial J}{\partial x}\sin(kJ)\right) + \int_0^\tau \sqrt{\alpha}k\left(\frac{\partial^2 J}{\partial^2 x} + \frac{\partial^2 J}{\partial y\partial x}\right)\cos(\theta)\sin(kJ)ds \\ &+ \int_0^\tau \sqrt{\alpha}k^2\frac{\partial J}{\partial x}\left(\frac{\partial J}{\partial x}\sin(\theta) + \frac{\partial J}{\partial y}\cos(\theta)\right)\sin(kJ)ds.\end{aligned} \tag{43}$$

By the previous arguments, all of the integral terms in (43) uniformly, weakly converge to 0. Furthermore, integrating, by parts, the $\sin(\omega s)$ part of (39), gives (neglecting the integral terms):

$$\left(\sqrt{\frac{\alpha}{\omega}}\cos(\omega\tau) - \underbrace{\sqrt{\frac{\alpha}{\omega}}}_{\xi}\right)\left(k\frac{\partial J}{\partial x}\cos(kJ)\right). \tag{44}$$

In what follows, we will see that only the combination of terms of the form $\cos^2(\omega t)$ and $\sin^2(\omega t)$, will survive and all those with mixed or only single $\cos(\omega\tau)$ or $\sin(\omega\tau)$ dependence, will, by the Riemann Lebesgue lemma, converge to zero, so from now on, we neglect terms such as ξ as well. Performing integration by parts on all of the components of $DA_\omega(x,y,s)$ and keeping only those as described previously, we end up with the matrix $B_\omega(x,y,\tau)$, where

$$B_{1,1} = -\sqrt{\frac{\alpha}{\omega}}\sin(\omega\tau)\left(k\frac{\partial J}{\partial x}\sin(kJ)\right) + \sqrt{\frac{\alpha}{\omega}}\cos(\omega\tau)\left(k\frac{\partial J}{\partial x}\cos(kJ)\right), \tag{45}$$

$$B_{1,2} = -\sqrt{\frac{\alpha}{\omega}}\sin(\omega\tau)\left(k\frac{\partial J}{\partial y}\sin(kJ)\right) + \sqrt{\frac{\alpha}{\omega}}\cos(\omega\tau)\left(k\frac{\partial J}{\partial y}\cos(kJ)\right), \tag{46}$$

$$B_{2,1} = \sqrt{\frac{\alpha}{\omega}}\sin(\omega\tau)\left(k\frac{\partial J}{\partial x}\cos(kJ)\right) + \sqrt{\frac{\alpha}{\omega}}\cos(\omega\tau)\left(k\frac{\partial J}{\partial x}\sin(kJ)\right), \tag{47}$$

$$B_{2,2} = \sqrt{\frac{\alpha}{\omega}}\sin(\omega\tau)\left(k\frac{\partial J}{\partial y}\cos(kJ)\right) + \sqrt{\frac{\alpha}{\omega}}\cos(\omega\tau)\left(k\frac{\partial J}{\partial y}\sin(kJ)\right). \tag{48}$$

Before we perform the matrix multiplication $B_\omega(x, y, \tau)A_\omega(x, y, \tau)$, we consider the fact that mixed terms of the form $\cos(\omega t)\sin(\omega t)$ weakly converge to zero; therefore, we omit all but the $\cos^2(\omega t)$ and $\sin^2(\omega t)$ terms, which weakly converge to $\frac{1}{2}$. Thus, we may break up the vector A and the matrix B as:

$$A = \sqrt{\alpha\omega}\sin(\omega t)A_s + \sqrt{\alpha\omega}\sin(\omega t)A_c, \quad B = \sqrt{\frac{\alpha}{\omega}}\sin(\omega t)B_s + \sqrt{\frac{\alpha}{\omega}}\cos(\omega t)B_c \tag{49}$$

and consider only the products

$$\alpha\cos^2(\omega t)B_cA_c + \alpha\sin^2(\omega t)B_sA_s, \tag{50}$$

which weakly converge to

$$\frac{\alpha}{2}\left(B_cA_c + B_sA_s\right). \tag{51}$$

The matrix multiplication gives

$$B_cA_c = \begin{bmatrix} k\frac{\partial J}{\partial x}\cos^2(kJ) + k\frac{\partial J}{\partial y}\cos(kJ)\sin(kJ) \\ k\frac{\partial J}{\partial x}\cos(kJ)\sin(kJ) + k\frac{\partial J}{\partial y}\sin^2(kJ) \end{bmatrix}, \tag{52}$$

$$B_sA_s = \begin{bmatrix} k\frac{\partial J}{\partial x}\sin^2(kJ) - k\frac{\partial J}{\partial y}\cos(kJ)\sin(kJ) \\ -k\frac{\partial J}{\partial x}\cos(kJ)\sin(kJ) + k\frac{\partial J}{\partial y}\cos^2(kJ) \end{bmatrix}. \tag{53}$$

After cancelation and simplification based on the trigonometric identity $\sin^2(\cdot) + \cos^2(\cdot) = 1$, we are left with

$$\frac{\alpha}{2}\left(B_cA_c + B_sA_s\right) = \frac{k\alpha}{2}\left(\nabla J\right)^T. \tag{54}$$

Summing up, we have

$$\lim_{\omega\to\infty}\int_0^t \left(DG_\omega(x, y, \tau)\right)g_\omega(x, y, \tau)d\tau = \lim_{\omega\to\infty}\int_0^t \left(\int_0^\tau DA_\omega(x, y, s)ds\right)A_\omega(x, y, \tau)d\tau$$
$$= \int_0^t \frac{\alpha}{2}\left(B_cA_c + B_sA_s\right)d\tau = \int_0^t \frac{k\alpha}{2}\left(\nabla J\right)^T d\tau. \tag{55}$$

Therefore, by Theorem 1 the trajectory $(x(t), y(t))$ of system (19)–(22) uniformly converges to the trajectory $(\bar{x}(t), \bar{y}(t))$, of the system

$$\dot{\bar{x}} = -\frac{k\alpha}{2}\frac{\partial J}{\partial \bar{x}}, \quad \bar{x}(0) = x(0), \tag{56}$$

$$\dot{\bar{y}} = -\frac{k\alpha}{2}\frac{\partial J}{\partial \bar{y}}, \quad \bar{y}(0) = y(0). \tag{57}$$

We now return to the dynamics of $\eta(t)$ as in (22). We define the error $\eta_e = \eta - J$ and consider

$$\dot{\eta}_e = -\omega^2\eta_e + \dot{J}, \tag{58}$$

which we expand as

$$\dot{\eta}_e = -\omega^2\eta_e + \frac{\partial J}{\partial x}\sqrt{\alpha\omega}\cos(\theta) + \frac{\partial J}{\partial y}\sqrt{\alpha\omega}\sin(\theta). \tag{59}$$

We consider the time scale $\tau = \omega t$, and rewrite the η_e dynamics as

$$\frac{\partial \eta_e}{\partial \tau} = -\omega \eta_e + \sqrt{\frac{\alpha}{\omega}} \left(\frac{\partial J}{\partial x} \cos(\theta) + \frac{\partial J}{\partial y} \sin(\theta) \right), \tag{60}$$

which uniformly converge to the dynamics of $\frac{\partial \bar{\eta}_e}{\partial \tau} = -\omega \bar{\eta}_e$; therefore, as $(x(t), y(t))$ converges to $(x^\star, y^\star)$, $\eta(t)$ converges to $J(x^\star, y^\star)$. □

The previous result can be easily extended to the same system tracking a time-varying source based on a scalar measurement of the form $J(x, y, t)$ and also experiencing external disturbances. We consider a function $f(x, y, t) = \left(f_x(x, y, t), f_y(x, y, t)\right)^T$, over a compact set $(x, y) \in K \subset \mathbb{R}^2$, which is continuous with respect to t and Lipschitz continuous with respect to (x, y). If the function $J(x, y, t)$ has a global minimum at $(x^\star(t), y^\star(t)) \in K\ \forall t$, such that the location of the minimum point has bounded velocity $|\dot{x}^\star(t)|, |\dot{y}^\star(t)| < M$, and

$$\nabla J|_{(x^\star(t), y^\star(t))} = 0, \tag{61}$$

$$\nabla J \neq 0,\ \forall (x(t), y(t)) \neq (x^\star(t), y^\star(t)). \tag{62}$$

We consider a system like the one previously, with external disturbances, of the form

$$\dot{x} = f_x(x, y, t) + \sqrt{\alpha\omega} \cos(\theta), \tag{63}$$

$$\dot{y} = f_y(x, y, t) + \sqrt{\alpha\omega} \sin(\theta). \tag{64}$$

We define the error variables $e_x(t) = x(t) - x^\star(t)$ and $e_y(t) = y(t) - y^\star(t)$ and show, by the same proof as the previous example, that the the trajectory of the error system of (63)–(64) uniformly converges to the trajectory of

$$\dot{\bar{e}}_x = f_x(\bar{e}_x + x^\star, \bar{e}_y + y^\star, t) - \frac{k\alpha}{2} \frac{\partial J}{\partial \bar{e}_x} + \dot{x}^\star(t), \tag{65}$$

$$\dot{\bar{e}}_y = f_y(\bar{e}_x + x^\star, \bar{e}_y + y^\star, t) - \frac{k\alpha}{2} \frac{\partial J}{\partial \bar{e}_y} + \dot{y}^\star(t). \tag{66}$$

Because the velocities $|\dot{x}^\star|$ and $|\dot{y}^\star|$ are bounded, and the function $f(x, y, t)$ is bounded on the compact set K, for any $\delta > 0$, by choosing arbitrarily large values of $k\alpha$ and ω, we may ultimately bound $(\bar{x}, \bar{y})$ within a δ neighborhood of $(x^\star, y^\star)$.

3.1. Source tracking and obstacle avoidance

In practice, especially in electronics, there are many circumstances in which various parameters must maintain distinct values, such as, for example, switching time settings for the transistors of an H-bridge, whose coincidence would result in a short circuit. Another, more intuitive example is in automatic collision avoidance between autonomous vehicles, or between vehicles and obstacles [21, 22]. Here, we briefly consider the application of Theorem 1 for collision avoidance between multiple parameters. We simulate the simple case of two vehicles, each seeking a separate source, while avoiding collisions with one another. Towards this goal, we utilize the ability of the ES approach to minimize measurable, but analytically unknown navigation functions, such as $V = \|x(t) - r(t)\|$, where $r(t)$ is an a priori unknown trajectory we wish to track, to whom we can detect the distance V. We consider two vehicles, with trajectories $v_1(t) = (x_1(t), y_1(t))$ and $v_2(t) = (x_2(t), y_2(t))$, whose dynamics are as follows:

$$\dot{x}_i = \sqrt{\alpha_i \omega_i} \cos(\theta_i), x_1(0) = -2, x_2(0) = 2, \tag{67}$$

$$\dot{y}_i = \sqrt{\alpha_i \omega_i} \sin(\theta_i),\, y_1(0) = y_2(0) = 0, \tag{68}$$

$$\dot{\theta}_i = \omega_i + k_i \omega_i^2 (\hat{J}_i - \eta_i), \tag{69}$$

$$\dot{\eta}_i = -\omega_i^2 \eta_i + \omega_i^2 \hat{J}_i, \tag{70}$$

where

$$\hat{J}_1 = \underbrace{(x_1 - 1)^2 + y_1^2}_{\text{stabilization}} + \underbrace{k_p e^{-k_a\left((x_1-x_2)^2+(y_1-y_2)^2\right)}}_{\text{collision avoidance}} + \underbrace{n(t)}_{\text{noise}},$$

$$\hat{J}_2 = \underbrace{(x_2 + 1)^2 + y_2^2}_{\text{stabilization}} + \underbrace{k_p e^{-k_a\left((x_1-x_2)^2+(y_1-y_2)^2\right)}}_{\text{collision avoidance}} + \underbrace{n(t)}_{\text{noise}},$$

and $\omega_1 = 50, \omega_2 = 77, \alpha_1 = \alpha_2 = 1, k_1 = k_2 = \frac{1}{2}, k_p = 2, k_a = 2, n(t) = \sin(20t) + \sin(130t)$.

Note that in the absence of the exponential term, and the noise, the two functions $J_1(x, y)$ and $J_2(x, y)$ have global minimums at $(1, 0)$ and $(-1, 0)$, respectively. The exponential term, which quickly decays and has no influence on system dynamics for large values of $(x_1 - x_2)$ and $(y_1 - y_2)$ comes into effect only when the systems are close together. This exponential term creates a local maximum as the vehicles come together. The result is that the two vehicles avoid collision as they each approach the global minimums of their functions J_1 and J_2. The simulation results are shown in Figure 2, first with the collision avoidance part of the function off ($k_p = 0$) and then on ($k_p = 2$), with and without the noise term $n(t)$.

We also consider a system, whose trajectory, $x(t) = (x_1(t), x_2(t))$, is to follow one trajectory, $r_1(t) = (\cos(t), \sin(t))$ while avoiding another, $r_2(t) = r_1(t) + f(t)(\sin(50t), \cos(50t))$, where $f(t)$ is a function whose value starts around 1, quickly decreases to 0.1, and then returns near 1, resulting in a trajectory which starts far from $r_1(t)$ and then quickly approaches, pushing $x(t)$ away to avoid collision. The system dynamics are the following:

$$\dot{x}_1(t) = \frac{x_1^2(t)}{4} + \sqrt{\alpha\omega} \cos(\theta),$$

$$\dot{x}_2(t) = \frac{x_2^2(t)}{4} + \sqrt{\alpha\omega} \sin(\theta),$$

where the θ dynamics are as the previous example, the $x_i^2/4$ terms are external disturbances, and the function being minimized contains information about a moving source to be tracked as well as local maxima for collision avoidance

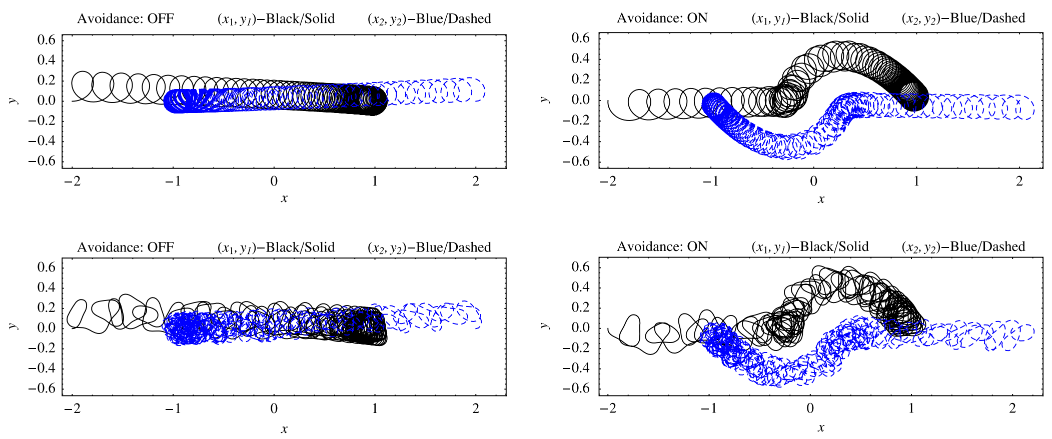

Figure 2. When $k_p = 0$, the trajectories of the vehicles pass through each other on the way towards their respective sources, as shown on top. With $k_p = 2$, the vehicles reach the sources that they are seeking while avoiding a collision on the way. Top two results without noise, bottom with noise.

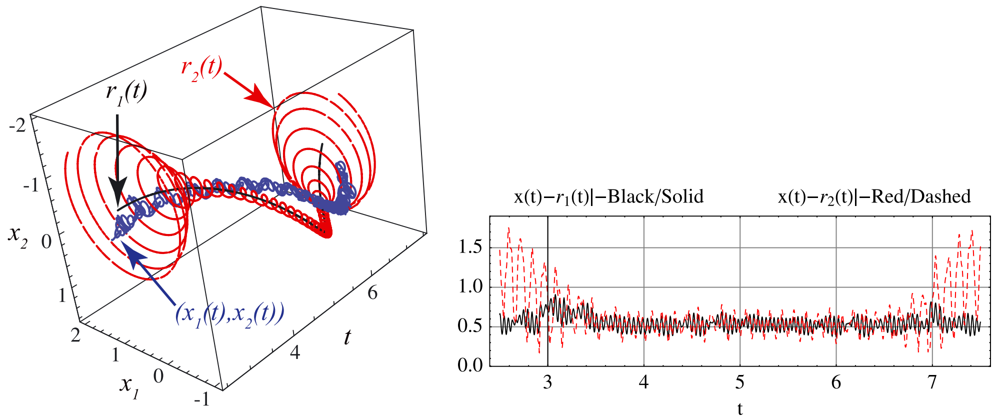

Figure 3. The trajectory $x(t)$ (blue), is chasing the trajectory $r_1(t)$ (black), while avoiding collision with both $r_1(t)$ and $r_2(t)$ (red). We see that a non-zero distance between trajectories, $\|x(t)-r_1(t)\|$ and $\|x(t)-r_2(t)\|$, is always maintained.

$$J(x,t) = \underbrace{k_1\|x-r_1\|^2}_{\text{tracking}} + \underbrace{k_2 e^{-\sigma\|x-r_1\|^2} + k_3 e^{-\sigma\|x-r_2\|^2}}_{\text{collision avoidance}}.$$

The simulation is shown with system parameters as follows: $\omega = 151$, $\alpha = 1$, $k_1 = 2$, $k_2 = 20$, $k_3 = 40$, and $\sigma = 30$. Simulation results are shown in Figure 3.

4. CONCLUSIONS

We studied the dynamics of a system which by design has multiple time scales because of filters which are required in order to enable actuation of the rate of change of the vehicle's heading direction rather than its instantaneously value. Such an approach is more realistic for actual in-hardware implementation. The result is a simple ES control scheme for autonomous vehicles performing source seeking while avoiding collisions with other vehicles or obstacles, in noisy, GPS denied environments. Control is performed despite a lack of location information, all goals are accomplished by sensing noisy scalar functions

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