HEP Final Report

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2 Miracles in Scattering Amplitudes: from QCD to Gravity, Anastasia Volovich, anastasia_volovich@brown.edu, 805 2597994

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4 Brief description of accomplishments

The goal of my research project "Miracles in Scattering Amplitudes: from QCD to Gravity" involves deepening our understanding of gauge and gravity theories by exploring hidden structures in scattering amplitudes and using these rich structures as much as possible to aid practical calculations.

4.1 Symbology of Scattering Amplitudes

In my earlier work together with Goncharov, Spradlin and Vergu we pioneered the application of a branch of mathematics known as theory of motives (in particular the notion of a symbol) to the computations in quantum gauge theories which by now became the technique widely used by the researchers around the world. Our technology, now known as symbology, has been applied for wide range of examples, ranging from amplitudes in QCD and $\mathcal{N} = 4$ to form-factor computation. Our paper now has over 250 citations.

The symbol for all two-loop MHV amplitudes in planar super-Yang-Mills theory is known, but explicit analytic formulas for the amplitudes are hard to come by except in special limits where things simplify, such as multi-Regge kinematics. By applying symbology in [10] we obtained a formula for the leading behavior of the imaginary part (the Mandelstam cut contribution) of this amplitude in multi-Regge kinematics for any number of gluons. Our result predicted a simple recursive structure which agreed with a direct BFKL computation carried out in a parallel publication.

Infrared divergences in scattering amplitudes arise when a loop momentum ℓ becomes collinear with a massless external momentum p. In gauge theories, it is known that the L-loop logarithm of an amplitude has much softer infrared singularities than the L-loop amplitude itself. In [9] we argued that planar amplitudes in $\mathcal{N} = 4$ super-Yang-Mills theory enjoy softer than expected behavior as $\ell \parallel p$ already at the level of the integrand. Moreover, we conjectured that the four-point integrand can be uniquely determined, to any loop-order, by imposing the correct soft-behavior of the logarithm together with dual conformal invariance and dihedral symmetry. We used these simple criteria to determine explicit formulae for the four-point integrand through seven-loops, finding perfect agreement with previously known results through five-loops. As an input to this calculation we enumerated all four-point dual conformally invariant (DCI) integrands through seven-loops, an analysis which is aided by several graph-theoretic theorems we proved about general DCI integrands at arbitrary loop-order. The six- and seven-loop amplitudes received non-zero contributions from 229 and 1873 individual DCI diagrams respectively.

4.2 Cluster Algebra Structure in Scattering Amplitudes

In a 2013 paper with Goncharov, Golden, Spradlin and Vergu, we established a surprising and deep connection between the cluster algebra and scattering amplitudes in $\mathcal{N} = 4$ Yang-Mills [6]. The paper has received over 50 citations. The abundance of functional identities amongst generalized polylogarithm functions precludes the existence of any particular preferred or canonical functional representation for general multi-loop amplitudes (the only, and important, exception is GSVV formula). While it remains an important outstanding problem in physics to determine explicit formulas for more general amplitudes, from the mathematical point of view doing so has no more value than picking some particular representative of a cohomology class. Instead of studying amplitudes as functions, we therefore advocated to study their canonically defined essential motivic content. First, we introduced motivic amplitudes–objects which contain all of the essential mathematical content of scattering amplitudes in planar SYM theory in a completely canonical way, free from the ambiguities inherent in any attempt to choose particular functional representatives. We found that the cluster structure on the kinematic configuration space $Conf_n(P^3)$ underlies the structure of motivic amplitudes. Specifically, we computed explicitly the coproduct of the two-loop seven-particle MHV motivic amplitude $A_{7,2}$ and found that like the previously known six-particle amplitude, it depends only on certain preferred coordinates known in the mathematics literature as cluster X-coordinates on $Conf_n(P^3)$. We also found intriguing relations between motivic amplitudes and the geometry of generalized associahedrons, to which cluster coordinates have a natural combinatoric connection. For example, the obstruction to $A_{7,2}$ being expressible in terms of classical polylogarithms is most naturally represented by certain quadrilateral faces of the appropriate associahedron. We also found and proved the first known functional equation for the trilogarithm in which all 40 arguments are cluster X-coordinates of a single algebra. In this respect it is similar to Abel's 5-term dilogarithm identity.

Motivated by the cluster structure of two-loop scattering amplitudes in $\mathcal{N} = 4$ Yang-Mills theory in [5] we defined "cluster polylogarithm functions". We found that all such functions of weight 4 are made up of a single simple building block associated to the A_2 cluster algebra. Adding the requirement of locality on generalized Stasheff polytopes, these A_2 building blocks arrange themselves to form a unique function associated to the A_3 cluster algebra. This A_3 function manifests all of the cluster algebraic structure of the two-loop n-particle MHV amplitudes for all n, and we used it to provide an explicit representation for the most complicated part of the $n = 7$ amplitude as an example.

In order to develop a computational framework which exploits the connection between scattering amplitudes and cluster structure, in [2] we showed how to construct bases of Goncharov polylogarithm functions, at any weight, whose symbol alphabet consists of cluster coordinates on the A_n cluster algebra. Using such a basis we presented a new expression for the 2-loop 6-particle NMHV amplitude which makes some of its cluster structure manifest.

4.3 BMS Symmetry and Scattering Amplitudes

Strominger proposed that a certain infinite-dimensional subgroup of the Bondi, van der Burg, Metzner, Sachs (BMS) supertranslation group is an exact symmetry of the quantum gravity S-matrix. Weinberg's soft theorem is a Ward identity for this subgroup. It was further conjectured by Cachazo and Strominger that there is a new soft graviton theorem which states that the sub-leading term in the soft graviton expansion of the graviton scattering amplitude is also universal. They proved it using BCFW recursion relations at tree level. This has resurrected a lot of interest in the study of subleading soft behavior of gravitons and gluons. Schwab and I investigated soft and subleading theorems in both Yang-Mills and gauge theories, providing a universal formula in arbitrary dimension [4]. We used the compact CHY integral formula for tree-level scattering amplitudes in arbitrary dimension. This was a very surprising result since the original conjecture of the universal subleading soft factors was based on the BMS symmetry principle which was only available in four dimensions. Our work with Schwab sparked follow-up work specifically treating subleading soft theorems and symmetries in higher dimensions as well as the connection to ambitwistor string theory. Our paper has already received over 50 citations.

In [3] we investigated the tree-level S-matrix in gauge theories and open superstring theory with several soft particles. We showed that scattering amplitudes with two or three soft gluons of non-identical helicities behave universally in the limit, with multi-soft factors which are not the product of individual soft gluon factors. The results were obtained from the BCFW recursion relations in four dimensions, and further extended to arbitrary dimensions using the CHY formula. We also found new soft theorems for double soft limits of scalars and fermions in $\mathcal{N} = 4$ and pure $\mathcal{N} = 2$ SYM. Finally, we showed that the double-soft-scalar theorems can be extended to open superstring theory without receiving any α' corrections.

4.4 Landau Singularities of Scattering Amplitudes

A long-standing goal of the S-matrix program is to be able to construct formulas for the scattering amplitudes of a quantum field theory based on general principles and physical constraints. As amplitudes are expected to be holomorphic functions of the kinematic data, with isolated poles and branch cuts, a thorough understanding of their analytic structure is clearly of great importance towards this goal. Landau formulated a simple set of equations whose solutions parameterize the locus, in the space of kinematic data, where a given Feynman integral can develop branch points.

Maldacena, Simmons-Duffin and Zhiboedov pointed that for amplitudes of generalized poly-logarithm type, there should evidently be a close connection between symbol entries and solutions of the Landau equations. In [1] we applied the Landau equations, whose solutions parameterize the locus of possible branch points, to the one- and two-loop integrals relevant to MHV amplitudes in planar $\mathcal{N} = 4$ super-Yang-Mills theory. We identified which of the Landau singularities appear in the symbols of the amplitudes, and which do not. We observed that all of the symbol entries in the two-loop MHV amplitudes are already present as Landau singularities of one-loop pentagon integrals. These were the first steps in exploring a potentially fruitful connection between symbology and the Landau equations, with many open questions left for future research.

4.5 Mellin Space Techniques

I also investigated Mellin space techniques for scattering amplitudes and correlation functions computations. The computation of CFT correlation functions via Witten diagrams in AdS space can be simplified via the Mellin transform. Recently a set of Feynman rules for tree-level Mellin space amplitudes has been proposed for scalar theories. In [11] we derived these rules by explicitly evaluating all of the relevant Witten diagram integrals for the scalar ϕ^n theory. We also checked that the rules reduce to the usual Feynman rules in the flat space limit.

Motivated by the utility of Mellin space for representing conformal correlators in AdS/CFT, in [8] we studied its suitability for representing dual conformal integrals of the type which appear in perturbative scattering amplitudes in super-Yang-Mills theory. We discussed Feynman-like rules for writing Mellin amplitudes for a large class of integrals in any dimension, and found explicit representations for several familiar toy integrals. However we showed that the power of Mellin space is that it provides simple representations even for fully massive integrals, which except for the single case of the 4-mass box have not yet been computed by any available technology. Mellin space is also useful for exhibiting differential relations between various multi-loop integrals, and we showed that certain higher-loop integrals may be written as integral operators acting on the fully massive scalar n-gon in n dimensions, whose Mellin amplitude is exactly 1. Our chief example was a very simple formula expressing the 6-mass double box as a single integral of the 6-mass scalar hexagon in 6 dimensions.

In [7] we studied single and multi-loop conformal integrals, such as the ones appearing in dual conformal theories in flat space. Using Mellin amplitudes, a large class of higher loop integrals can be written as simple integro-differential operators on star integrals: one-loop n-gon integrals in n dimensions. These are known to be given by volumes of hyperbolic simplices. We explicitly computed the five-dimensional pentagon integral in full generality using Schläfli's formula. Then, as a first step to understanding higher loops, we used spline technology to construct explicitly the $6d$ hexagon and $8d$ octagon integrals in two-dimensional kinematics. The fully massive hexagon and octagon integrals are then related to the double box and triple box integrals respectively. We commented on the classes of functions needed to express these integrals in general kinematics, involving elliptic functions and beyond.

4.6 Invited Talks

I was invited to give a number of talks at various conferences and universities. I also spent my sabbatical year and several summers as a visiting associate scientist in the CERN theory group.

1. Workshop on Current Themes in High Energy Physics and Cosmology, Niels Bohr Institute, Copenhagen, Aug. 16

2. Nordita program on Aspects of Scattering Amplitudes, Stockholm, Jun. 16
3. Isaac Newton Institute for Mathematical Sciences program on Gravity, Twistors and Amplitudes, Cambridge, UK, Jun. 16
4. Quarks-2016, May 16 (declined)
5. Flat Holography, Simons Center Workshop, Apr. 16
6. MHV @ 30 Workshop, Fermilab, Mar. 16
7. IAS Seminar, Feb. 16 (declined)
8. Miami 2015, Dec. 15 (declined)
9. Hidden symmetries and integrability methods in Super-Yang-Mills theories and their string duals workshop, Montreal, Aug. 15
10. Lectures at the International Summer School on Theoretical Problems of Physics of Fundamental Interactions, Zelenogorsk, Russia, Jul. 15
11. Amplitudes, Motives and Beyond workshop, Mainz, Jun. 15
12. AndyFest, Harvard, Jul. 15 (declined)
13. Perimeter Institute summer school, Summer 15 (declined)
14. URI Colloquium, Spring 15 (postponed)
15. Math department, Brown, Spring 15 (postponed)
16. The interrelation between mathematical physics, number theory and non-commutative geometry workshop, Vienna, Mar. 15 (declined)
17. Miami 2014, Miami, Dec. 14
18. IAS Program on Scattering Amplitudes, Hong Kong, Nov. 14 (declined)
19. Grassmannian Geometry of Scattering Amplitudes workshop, Caltech, Nov. 14
20. New geometric structures in scattering amplitudes workshop, Oxford, Sep. 14
21. CERN, Aug. 2014
22. Amplitudes 2014, Jun. 14 (declined)
23. Integrable Structures in Scattering Amplitudes workshop, Durham, Apr. 14
24. Yale University, Apr. 14
25. IAS seminar, Jan. 14 (declined)
26. Max Planck Institute, Munich, Dec. 13 (cancelled)

27. Physics and Mathematics of Scattering Amplitudes, Stony Brook, Nov. 13 (declined)
28. Brandeis University, Oct. 2013
29. Aegean Summer School, Greece, Sep. 13 (declined)
30. QCD workshop, Paris, Jun 13 (declined)
31. Queen Mary College, London, Jun 13
32. Humboldt University, Berlin, May 13
33. Niels Bohr Institute, Copenhagen, May 13
34. Zurich Theoretical Physics Colloquium, Apr. 13
35. Amplitudes 2013, Tegernsee, Germany, Apr. 13
36. CERN Theory Colloquium, Nov. 12
37. The multi-Regge limit of Scattering Amplitudes workshop, University of Madrid, Oct. 12 (declined)
38. Indian Strings Meeting 2012, Puri, Dec. 12 (declined)
39. Scattering without Spacetime, CTS-TIFR, Bangalore, Sep. 12 (declined)
40. The Geometry of Scattering Amplitudes, Banff, Canada, Aug 12 (declined)
41. Workshop Scattering Amplitudes: from QCD to maximally supersymmetric Yang- Mills theory and back, Trento, Italy, Jul. 12 (declined)
42. String-Math 2012, Bonn, Germany, Jul. 12
43. Madrid ICMAT workshop Periods and Motives, Spain , Jul. 12 (declined)
44. Quarks 2012, Russia, Jun. 12 (declined)
45. Ginzburg Conference on Physics, Moscow, May 12 (declined)
46. Conference "$\mathcal{N} = 4$ Super Yang-Mills Theory, 35 Years After", Caltech, Apr. 12
47. Isaac Newton Institute Workshop on Scattering Amplitudes, Cambridge, UK, Apr. 12
48. Perimeter Institute, Canada, 2012 (postponed)
49. Simons Workshop Mathematical Approaches to Quantum Field Theory, Stony Brook, Jan. 12 (declined)
50. Northeast Conference for Undergraduate Women in Physics, Yale, Jan.12
51. Miami 2011, Florida, Dec. 11
52. Amplitudes 2011, Michigan, Nov. 11

53. INT workshop "Frontiers of QCD," Seattle, Sep. 11

54. Harvard University, Sep. 11

55. Strings 2011, Uppsala, Sweden, Jun. 11

4.7 Conferences Organized

1. Scientific Advisory Committee, Amplitudes 2018, SLAC, 2018

2. Co-organizer MHV @ 30: Amplitudes and Modern Applications, Fermilab, Mar. 2016

3. Co-organizer of AndyFest: A Celebration of the Science of Andrew Strominger, Harvard, July 2015

4. Co-organizer of the *New England String Meetings*, Nov. 2011, Nov. 2014, Nov. 2015

5. Co-organizer of CERN Theory Institute on Scattering Amplitudes, with J. Drummond, H. Johansson, N. Lambert, M. Spradlin, Jul. 2013

6. Session co-organizer at the *Twelfth Workshop on Non-Perturbative QCD*, Paris, Jun. 2013

7. Co-organizer of the IHES workshop "Amplitudes and Periods", IHES, Paris, with A. Goncharov, G. Korchemsky, M. Spradlin and P. Vanhove, Dec. 2012

8. Co-Organizer of the three month scientific program in Kavli Institute for Theoretical Physics, Santa Barbara, *The Harmony of Scattering Amplitudes: from QCD to Gravity,* together with N. Arkani-Hamed, Z. Bern, T. Gehrmann and F. Petriello, 2011

5 Publications

1. T. Dennen, M. Spradlin and A. Volovich, "Landau Singularities and Symbology: One- and Two-loop MHV Amplitudes in SYM Theory," JHEP **1603**, 069 (2016) [arXiv:1512.07909 [hep-th]].

2. D. Parker, A. Scherlis, M. Spradlin and A. Volovich, "Hedgehog bases for A_n cluster polylogarithms and an application to six-point amplitudes," JHEP **1511**, 136 (2015) [arXiv:1507.01950 [hep-th]].

3. A. Volovich, C. Wen and M. Zlotnikov, "Double Soft Theorems in Gauge and String Theories," JHEP **1507**, 095 (2015) [arXiv:1504.05559 [hep-th]].

4. B. U. W. Schwab and A. Volovich, "Subleading Soft Theorem in Arbitrary Dimensions from Scattering Equations," Phys. Rev. Lett. **113**, no. 10, 101601 (2014) [arXiv:1404.7749 [hep-th]].

5. J. Golden, M. F. Paulos, M. Spradlin and A. Volovich, "Cluster Polylogarithms for Scattering Amplitudes," J. Phys. A **47**, no. 47, 474005 (2014) [arXiv:1401.6446 [hep-th]].

6. J. Golden, A. B. Goncharov, M. Spradlin, C. Vergu and A. Volovich, “Motivic Amplitudes and Cluster Coordinates,” JHEP **1401**, 091 (2014) [arXiv:1305.1617 [hep-th]].

7. D. Nandan, M. F. Paulos, M. Spradlin and A. Volovich, “Star Integrals, Convolutions and Simplices,” JHEP **1305**, 105 (2013) [arXiv:1301.2500 [hep-th]].

8. M. F. Paulos, M. Spradlin and A. Volovich, “Mellin Amplitudes for Dual Conformal Integrals,” JHEP **1208**, 072 (2012) [arXiv:1203.6362 [hep-th]].

9. J. L. Bourjaily, A. DiRe, A. Shaikh, M. Spradlin and A. Volovich, “The Soft-Collinear Bootstrap: N=4 Yang-Mills Amplitudes at Six and Seven Loops,” JHEP **1203**, 032 (2012) [arXiv:1112.6432 [hep-th]].

10. A. Prygarin, M. Spradlin, C. Vergu and A. Volovich, “All Two-Loop MHV Amplitudes in Multi-Regge Kinematics From Applied Symbology,” Phys. Rev. D **85**, 085019 (2012) [arXiv:1112.6365 [hep-th]].

11. D. Nandan, A. Volovich and C. Wen, “On Feynman Rules for Mellin Amplitudes in AdS/CFT,” JHEP **1205**, 129 (2012) [arXiv:1112.0305 [hep-th]].

12. M. Spradlin and A. Volovich, “Symbols of One-Loop Integrals From Mixed Tate Motives,” JHEP **1111**, 084 (2011) [arXiv:1105.2024 [hep-th]].

13. L. Lipatov, A. Prygarin and H. J. Schnitzer, “The Multi-Regge limit of NMHV Amplitudes in N=4 SYM Theory,” JHEP **1301** (2013) 068 [arXiv:1205.0186 [hep-th]].

14. J. Bartels, A. Kormilitzin, L. N. Lipatov and A. Prygarin, Phys. Rev. D **86** (2012) 065026 [arXiv:1112.6366 [hep-th]].

15. M. F. Paulos, “JuliBootS: a hands-on guide to the conformal bootstrap,” arXiv:1412.4127 [hep-th].

16. J. Golden and M. F. Paulos, “No unitary bootstrap for the fractal Ising model,” JHEP **1503**, 167 (2015) [arXiv:1411.7932 [hep-th]].

17. S. G. Avery and M. F. Paulos, “Universal Bounds on the Time Evolution of Entanglement Entropy,” Phys. Rev. Lett. **113**, no. 23, 231604 (2014) [arXiv:1407.0705 [hep-th]].

18. M. F. Paulos and B. U. W. Schwab, “Cluster Algebras and the Positive Grassmannian,” JHEP **1410**, 31 (2014) [arXiv:1406.7273 [hep-th]].

19. S. El-Showk, M. F. Paulos, D. Poland, S. Rychkov, D. Simmons-Duffin and A. Vichi, “Solving the 3d Ising Model with the Conformal Bootstrap II. c-Minimization and Precise Critical Exponents,” J. Stat. Phys. **157**, 869 (2014) [arXiv:1403.4545 [hep-th]].

20. D. Gaiotto, D. Mazac and M. F. Paulos, “Bootstrapping the 3d Ising twist defect,” JHEP **1403**, 100 (2014) [arXiv:1310.5078 [hep-th]].

21. S. El-Showk, M. Paulos, D. Poland, S. Rychkov, D. Simmons-Duffin and A. Vichi, "Conformal Field Theories in Fractional Dimensions," Phys. Rev. Lett. **112**, 141601 (2014) [arXiv:1309.5089 [hep-th]].

22. S. El-Showk and M. F. Paulos, "Bootstrapping Conformal Field Theories with the Extremal Functional Method," Phys. Rev. Lett. **111**, no. 24, 241601 (2013) [arXiv:1211.2810 [hep-th]].

23. M. F. Paulos, "Loops, Polytopes and Splines," JHEP **1306**, 007 (2013) [arXiv:1210.0578 [hep-th]].

24. S. G. Avery and B. U. W. Schwab, "Burg-Metzner-Sachs symmetry, string theory, and soft theorems," Phys. Rev. D **93**, 026003 (2016) [arXiv:1506.05789 [hep-th]].

25. B. U. W. Schwab, "A Note on Soft Factors for Closed String Scattering," JHEP **1503**, 140 (2015) [arXiv:1411.6661 [hep-th]].

26. B. U. W. Schwab, "Subleading Soft Factor for String Disk Amplitudes," JHEP **1408**, 062 (2014) [arXiv:1406.4172 [hep-th]].

27. D. Nandan and C. Wen, "Generating All Tree Amplitudes in N=4 SYM by Inverse Soft Limit," JHEP **1208**, 040 (2012) [arXiv:1204.4841 [hep-th]].

28. J. Golden and M. Spradlin, "A Cluster Bootstrap for Two-Loop MHV Amplitudes," JHEP **1502**, 002 (2015) [arXiv:1411.3289 [hep-th]].

29. J. Golden and M. Spradlin, "An analytic result for the two-loop seven-point MHV amplitude in $\mathcal{N} = 4$ SYM," JHEP **1408**, 154 (2014) [arXiv:1406.2055 [hep-th]].

30. J. Golden and M. Spradlin, "The differential of all two-loop MHV amplitudes in $\mathcal{N} = 4$ Yang-Mills theory," JHEP **1309**, 111 (2013) [arXiv:1306.1833 [hep-th]].

31. J. Golden and M. Spradlin, "Collinear and Soft Limits of Multi-Loop Integrands in N=4 Yang-Mills," JHEP **1205**, 027 (2012) [arXiv:1203.1915 [hep-th]].

32. M. Zlotnikov, "Polynomial reduction and evaluation of tree- and loop-level CHY amplitudes," JHEP **1608**, 143 (2016) [arXiv:1605.08758 [hep-th]].

33. M. Zlotnikov, "Sub-sub-leading soft-graviton theorem in arbitrary dimension," JHEP **1410**, 148 (2014) [arXiv:1407.5936 [hep-th]].

6 Graduate students and postdocs

6.1 Postdoc: Alex Prygarin

Prygarin was a postdoc in Brown in 2011-2012. After Brown, he left to a postdoc position in Weizmann institute, and then accepted a faculty position in Ariel University in Israel. While in Brown he published papers [13],[14],[10].

Prygarin has been investigating analytic properties of the planar scattering amplitudes in the supersymmetric theories and their consistency with reggeon field theory. His research interests were focused on MHV amplitudes in particular kinematic regions, where Mandelstam cuts give a non-vanishing contribution and can be predicted using BFKL approach based on QCD. He was analyzing the connection between two different kinematic regions: collinear and Regge kinematics. These two were shown to overlap for six particle amplitude opening a window for fixing undetermined parameters in existing ansatz as well as gaining a deep physical insight relating the MHV amplitudes to the Deep Inelastic Scattering through analogy with DGLAP and BFKL equations. He worked on extending this analysis to amplitudes with larger number of external legs.

6.2 Postdoc: Miguel Paulos

Paulos was a postdoc in Brown in 2012-2014. After Brown, he left to CERN as Marie-Curie fellow, and then accepted a faculty position in Cambridge University, UK. While in Brown he published papers [15], [16],[17], [18],[19], [5],[20], [21], [7], [22], [23], [8]. Two of them already have over 50 citations, and one over 100 citations. Paulos's research interests are very broad. While writing several papers with me on amplitudes, that I described above, he became a leader in the conformal bootstrap program.

Bootstrap program aims to understand conformal field theories using only their basic properties. Indeed, it is known that conformal field theories are highly constrained systems. However, while in 2d the infinite symmetry algebra leads to a wealth of exact solutions, in general dimensions there is only a finite conformal group, and up until recently this was believed to be insufficient to derive non-trivial consequences for conformal field theories. An important set of constraints arise from crossing symmetry, which is the statement that operator product expansions performed in different channels in a given correlation function have to match. Since the work of Rattazzi, Rychkov et al, a growing body of literature has shown that these constraints are in fact extremely powerful, in what has been termed the revived conformal bootstrap. These new methods have been used to derive bounds on generic 3d conformal field theories. Remarkably, the critical 3d Ising model seems not only to saturate the bound but to lie on a corner of parameter space (a kink). Since in the bootstrap approach the dimensionality is simply a parameter, it is straightforward to analytically continue. This is precisely what has been done in the work [21], repeating the analysis for various dimensions between 2 and 4. Starting from 4 one clearly sees the Wilson-Fisher fixed point, which again saturates the bound and lies on a kink. The results are in excellent agreement with (and improve on) Borel-resummed ε-expansion results.

In [16] Golden and Paulos considered the conformal bootstrap for dimension $1 < d < 2$ and determined bounds on operator dimensions. In [15] Paulos introduced `JuliBootS`, a package for numerical conformal bootstrap computations coded in `Julia`. Current supported features include conformal dimension bounds, OPE bounds, and bootstrap with or without global symmetries. The paper also gives a pedagogical introduction to the numerical bootstrap methods and several real-world applications.

The bootstrap methods can be used not only to derive universal bounds, but to explicitly reconstruct the spectrum of solutions to crossing symmetry. Using this it was possible to study the spectrum of the 3d Ising model in great detail [19], significantly improving on existing estimates for critical exponents. A concrete application in another context was made also recently [20]. In this work, they examined a recently proposed twist defect in the 3d Ising model. By directly bootstrapping the 1d defect CFT, they were able to make predictions and to confirm both ε-expansion calculations and Monte-Carlo simulations. This explicitly shows that the bootstrap methods are competitive and can surpass more traditional approaches to derive properties of critical points.

Paulos also contributed to the recent theoretical developments in entanglement entropy, which has become an important theoretical tool for probing quantum physics in diverse situations. In [17] Avery and Paulos derived bound on the time rate of change of geometric entanglement entropy for any relativistic quantum field theory. The bounds apply to both mixed and pure states, and may be extended to curved space.

6.3 Postdoc: Burkhard Schwab

Schwab started in 2013 as a postdoctoral researcher at Brown. He went on to to Harvard University in 2015 for a three year position as postdoctoral fellow. Schwab's work focused mainly on two areas. First of all, the soft behavior of scattering amplitudes in gauge, gravity, and string theories. Secondly, the description of on-shell graph BCFW recursion using cluster algebras. He published papers [4],[26],[18],[25].

Lately, there was a lot of interest in the soft behavior of gluon and graviton scattering amplitudes. Although the leading order was already understood by Weinberg more than 50 years ago, it was only recently discovered that there is a universal subleading soft term in gauge and gravity amplitudes at tree level. In gravity, this leading and subleading behavior has been linked to the existence of an extended asymptotic symmetry group of asymptotically flat space known as the extended BMS group. This group itself has been discovered a long time ago, essentially simultaneously with the original soft graviton theorem. But a connection between these two phenomena has only been formed recently by Storminger.

Schwab and I wrote one paper [4] on the subleading soft behavior of gauge and gravity amplitudes in arbitrary dimensions which I already described. He then turned to investigate the soft behavior of string theory amplitudes. He proved that there is universal subleading behavior for the massless modes (graviton and gluon states) in all string theories, type I, type II, and heterotic string theory for any compactification, any amount of supersymmetry and any number of external legs. On these topics he wrote two papers on his own [25, 26].

Together with Paulos, Schwab also worked on the cluster algebra structure of tree level scattering amplitudes in the maximally supersymmetric Yang-Mills theory in four dimensions in the planar limit[18]. Specifically, they investigated the Grassmannian on-shell amplitude formulation proposed by Arkani-Hamed and collaborators. Paulos and Schwab showed that a rigorous understanding of the so called "boundary measurement" process is possible from the point of view of cluster algebras. In this formulation, a boundary measurement corresponds to the "freezing" of a variable and the subsequent elimination of the equations associated with this variable – called "deletion" – in cluster algebras. By doing so, an on-shell diagram can be systematically taken apart.

6.4 Postdoc: Tristan Dennen

Dennen joined Brown in 2015 as a postdoc, and left to become a software engineer in Google in 2016. He wrote a paper on Landau singularities with me which I described above [1].

Dennen has spent most of his research career working on scattering amplitudes in gauge and gravity theories, and he has been transporting his experience towards precision calculation and Standard Model phenomenology. His scientific interests were in the analytic calculation of multiloop scattering amplitudes, including the BCJ color-kinematics duality, integral evaluation through the use of IBP identities and Mellin-Barnes techniques, and, most importantly, the management of large analytic expressions. Also he has been working on analytic methods for the calculation of $2 \to 4$ processes at NLO for collider phenomenology, with particular emphasis on controlling the analytic structure of the resulting expressions. These will be important ingredients in the next generation of NNLO precision calculations.

6.5 Graduate student: Dhritiman Nandan

Nandan defended his Ph.D in 2013, and went to Humbolt University of Berlin as a postdoc.

In addition to the papers described above with me, he also wrote [27] with Wen on generating all tree amplitudes in $\mathcal{N} = 4$ Yang-Mills by inverse soft limit. The idea of adding particles to construct amplitudes has been utilized in various ways in exploring the structure of scattering amplitudes. This idea is often called Inverse Soft Limit, namely it is the reverse mechanism of taking particles to be soft. They applied the Inverse Soft Limit to the tree-level amplitudes in $\mathcal{N} = 4$ super Yang-Mills theory, which allowed them to generate full tree-level superamplitudes by adding "soft" particles in a certain way. With the help from BCFW recursion relations, a systematic and concrete way of adding particles was determined recursively. The amplitudes constructed solely by adding particles not only have manifest Yangian symmetry, but also make the soft limit transparent. The method of generating amplitudes by Inverse Soft Limit can also be generalized for constructing form factors.

6.6 Graduate student: John Golden

Golden defended his Ph.D in 2015, and went to University of Michigan as a postdoc.

In addition to the papers described above with me, in [31] Golden and Spradlin investigated collinear and soft limits of multi-loop integrands in $\mathcal{N} = 4$ Yang-Mills with Spradlin. In [30] Golden and Spradlin computed the differential of all two-loop MHV amplitudes in $\mathcal{N} = 4$ Yang-Mills. In [29] they described a general algorithm which builds on several pieces of data available in the literature to construct explicit analytic formulas for two-loop MHV amplitudes in $\mathcal{N} = 4$ super-Yang-Mills theory. They presented an explicit formula for the seven-point amplitude as a sample application. In [28], they applied a bootstrap procedure to two-loop MHV amplitudes in planar

$\mathcal{N} = 4$ super-Yang-Mills theory. They argued that the mathematically most complicated part of the n-particle amplitude is uniquely determined by a simple cluster algebra property together with a few physical constraints such as dihedral symmetry, analytic structure, supersymmetry, and well-defined collinear limits. They presented a concise, closed-form expression which manifested these properties for all n.

6.7 Graduate student: Michael Zlotnikov

Zlotnikov has been working with me since summer 2014. In addition to a paper with me and Wen that I described above, he also published [32], [33] under my supervision. In [33] Zlotnikov used CHY formula to prove the sub-sub-leading soft- graviton theorem in any dimension. In [32] he developed a polynomial reduction procedure that transforms any gauge fixed CHY amplitude integrand for n scattering particles into a σ-moduli multivariate polynomial of what he called the standard form. He showed that a standard form polynomial must have a specific ladder type monomial structure, which has finite size at any n, with highest multivariate degree given by $(n-3)(n-4)/2$. This set of monomials spans a complete basis for polynomials with rational coefficients in kinematic data on the support of scattering equations. Subsequently, at tree and one-loop level, he employed the global residue theorem to derive a prescription that evaluates any CHY amplitude by means of collecting simple residues at infinity only. The prescription was then applied explicitly to some tree and one-loop amplitude examples.