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WKB Approximation to the Thin Shell Equations

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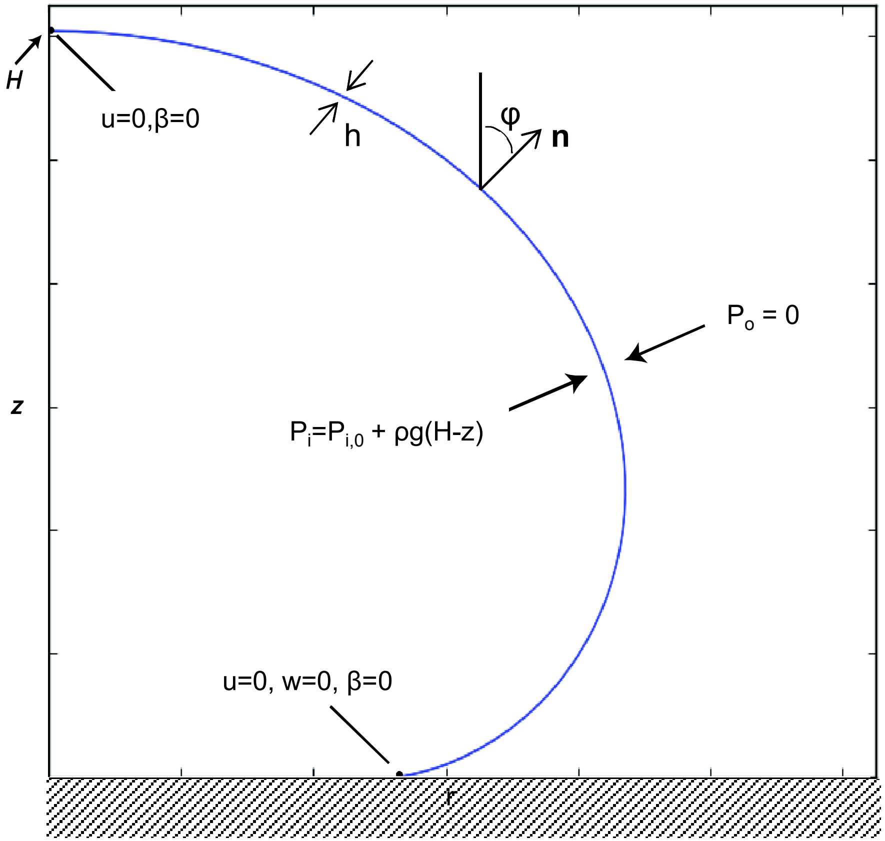
Problem Statement

- Consider a liquid drop (e.g. a molten metal) covered by an ultra-thin shell (e.g. oxide layer) resting on a solid surface



u – tangential extension of shell

w – normal displacement of shell

β – local rotation of shell

φ – "turning angle" angle between the normal to the shell and the z axis

r_φ , r_θ – radii of curvature such that $ds^2 = r_\varphi^2 d\varphi^2 + r_\theta^2 d\theta^2$

P_i,P_o – hydrostatic pressure inside and outside drop

$P_{i,0}$ – hydrostatic pressure inside drop at z = H

E, *v* – Young's modulus, Poisson's ratio

H – height of drop

h – thickness of the shell

Problem Statement

- In the limit of $\frac{h}{r} \to 0$, thin-shell equations with bending become extremely difficult to solve numerically
- Previous asymptotic solutions
  - Assumption of a constant-curvature shell (flat, spherical, cylindrical)
    - Geckeler approximation
    - Asymptotic integration
  - Boundary-layer analysis
    - Leading-order term for the inner bending region, can't match to an outer solution

Governing Differential Equations – Linear Elastic Deformation of Shell

$$h^3 \sum_{i=1}^{4} a_i(\varphi)\beta^{(i)} + \left(h a_{0,K}(\varphi) + h^3 a_{0,D}(\varphi)\right)\beta = g \qquad (1)$$

$$u - \frac{dw}{d\varphi} = r_\varphi \beta \qquad (2)$$

$$u = \sin(\varphi)\int \frac{1}{\sin(\varphi)}\left(\frac{(\nu r_\theta + r_\varphi)(-f_1 + \frac{\cot(\varphi) Q r_\theta}{r_\theta})}{Eh} - \frac{(r_\theta + \nu r_\varphi)(-q_n r_\theta + \frac{f_1 r_\theta}{r_\varphi} + \frac{(Q r_\theta)'}{r_\varphi})}{Eh} \right) d\varphi \qquad (3)$$

$$Q r_\theta = \frac{Eh^3}{12(1-\nu^2)}\left(-\frac{\cot(\varphi)^2 \beta}{r_\theta} + \frac{-\nu\beta + \cot(\varphi)\beta'}{r_\varphi} + \frac{r_\theta(-r_\varphi{}'\beta' + r_\varphi \beta'')}{r_\varphi{}^3} \right) \qquad (4)$$

$$a_4(\varphi) = -\frac{E r_\theta{}^2}{12 r_\varphi (1-\nu^2)}, \quad a_3(\varphi) = \frac{E}{12(1-\nu^2)}\left(6\frac{r_\theta{}^2 r_\varphi{}'}{r_\varphi^2} + 2\frac{r_\theta^2 \cot(\varphi)}{r_\varphi} - 4 r_\theta \cot(\varphi) \right), \quad a_{0,K}(\varphi) = -E r_\varphi^3, \qquad (5)$$

g, f_1 contain the driving force acting on the shell (e.g. external pressure, tangential stress)

Boundary-layer type analysis

- Can solve for an asymptotic series in the "inner region"

$$\beta(\varphi) = h^{-\frac{1}{2}}\beta_{inner,0}(\varphi_c + h^{\frac{1}{2}}x) + \varepsilon^0\beta_{inner,1}(\varphi_c + h^{\frac{1}{2}}x) + h^{\frac{1}{2}}\beta_{inner,1}(\varphi_c + h^{\frac{1}{2}}x) + \ldots$$

  - φ_c - location where bending is occurring
  - x is the inner variable
- "Outer region" of equation (1), where β*(φ) ~ O(1)*

$$h a_{0,K}(\varphi)\beta_{outer}(\varphi) = g(\varphi)$$

  - Can't match to inner solution – limits the utility of the approximation
  - Indicates length scales are not linearly separable

WKB Approximation

- WKB global approximation

$$\beta(\varphi) = \exp\left(h^{-\frac{1}{2}} s_0(\varphi) + h^0 s_1(\varphi) + \ldots \right)$$

$$\left(s_0'(\varphi) \right)^4 = -\frac{a_{0,K}(\varphi)}{a_4(\varphi)}$$

$$s_1'(\varphi) = -\frac{a_3(\varphi) s_0'(\varphi) - 6 a_4(\varphi) s_0''(\varphi)}{4 a_4(\varphi) s_0'(\varphi)}$$

$\Longrightarrow$ $$\beta(\varphi) = \exp\left(-m(\varphi)\right)\left[\exp\left(q(\varphi)\right)\left[C_1 \sin\left(q(\varphi)\right) + C_2 \sin\left(q(\varphi)\right)\right] + \exp\left(-q(\varphi)\right)\left[C_3 \sin\left(q(\varphi)\right) + C_4 \sin\left(q(\varphi)\right)\right]\right],$$

$$q(\varphi) = \frac{3^{1/4}(1-\nu^2)^{1/4}}{\sqrt{h}} \int \frac{r_\varphi}{\sqrt{r_\theta}} d\varphi, \qquad m(\varphi) = \int \frac{\cot(\varphi)(r_\theta + r_\varphi)}{4 r_\theta} d\varphi$$

spherical shell: $r_\theta = r_\varphi = r$ $\Longrightarrow$ Obtain classical asymptotic integration result consistent with Kraus and Timoshenko

$$\beta(\varphi) = \sin(\varphi)^{-1/2}\left[\exp\left(q_s \varphi\right)\left[C_1 \sin\left(q_s \varphi\right) + C_2 \sin\left(q_s \varphi\right)\right] + \exp\left(-q_s \varphi\right)\left[C_3 \sin\left(q_s \varphi\right) + C_4 \sin\left(q_s \varphi\right)\right]\right], \qquad q_s = \frac{3^{1/4}(1-\nu^2)^{1/4}}{\sqrt{h}} \sqrt{r}$$

Conclusion

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- WKB expansion doesn't require assumption of constant curvature, can be extended (with some effort) to higher-order terms
- Should be possible to include various types of equations-of-state, non-linearity
- Can provide a boundary condition for numerical simulations
- We have not been able to find this approximation in the literature or in textbooks on shell mechanics.

Questions?