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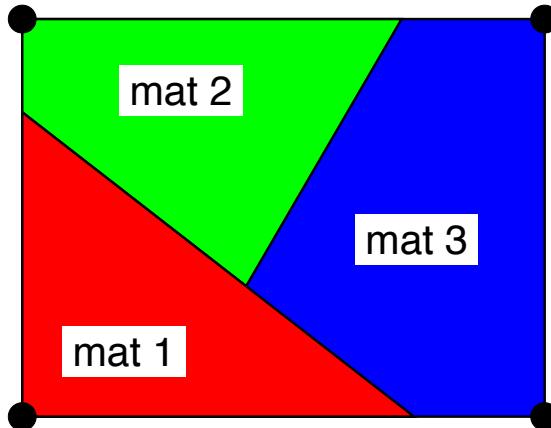


XFEM for Multi-Material Eulerian Solid/Hydrodynamics

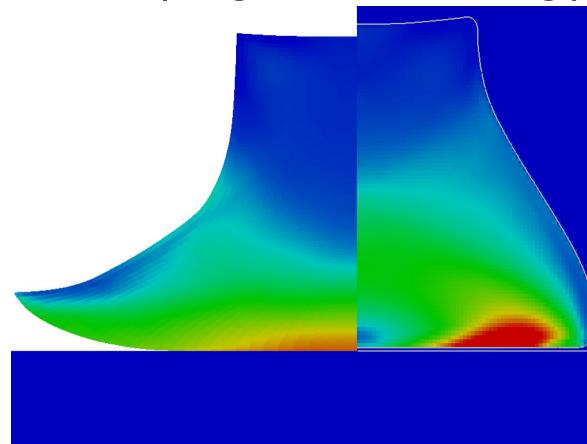
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Introduction:

- Our problems frequently require an Eulerian approach.
- Treatment of multi-material cells with **unmixed** materials are arguably deficient:
 - Single velocity/displacement field per element.
 - Lack of intra-element interfaces (e.g. for sliding).



A multi-material cell ...



... and it's effect with ad-hoc treatment.

- We are NOT attempting to address models for well mixed materials (e.g. gases).

Our problem:

- Solve model equations:

mass: $\frac{d}{dt}(\rho dv) = \frac{d}{dt}(dm) = 0$

momentum: $\rho \frac{d}{dt} \dot{\mathbf{u}} = \rho \ddot{\mathbf{u}} = \nabla \cdot \boldsymbol{\sigma} + \mathbf{f}$

energy: $\rho \frac{d}{dt} e = \boldsymbol{\sigma} : \mathbf{D}$

- Use traditional “Operator-Split” approach:
 - Lagrangian step solve of above equations.
 - Generate new mesh (Eulerian)
 - Perform remap (See Mosso et al. presentation)

Closure needed for mixed cells:

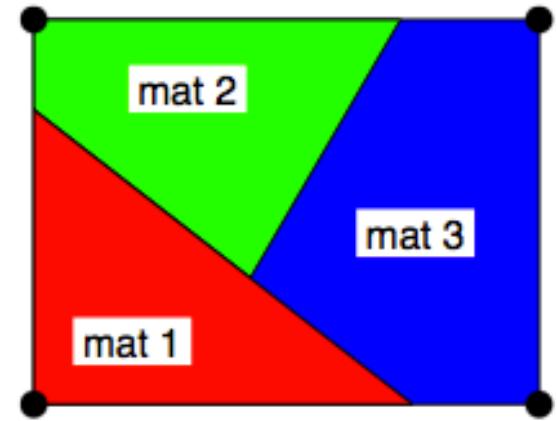
$$\mathbf{a}^n = \mathbf{M}^{-1} \left(\mathbf{f}_{ext}^n - \int \mathbf{B}^t \boldsymbol{\sigma}^n dV \right)$$

$$\mathbf{v}^{n+\frac{1}{2}} = \mathbf{v}^{n-\frac{1}{2}} + \Delta t \mathbf{a}^n$$

$$\mathbf{D}^{n+1/2} = \frac{1}{2} \left(\mathbf{L}^t + \mathbf{L} \right)^{n+\frac{1}{2}}$$

$$\boldsymbol{\sigma}_m^{n+1} = \mathcal{M}_m \left(\boldsymbol{\sigma}_m, \mathbf{D}_m^{n+\frac{1}{2}}, \dots \right)$$

$$e_m^{n+1} = e_m^n + \Delta t \int \boldsymbol{\sigma}_m^n : \mathbf{D}_m^{n+\frac{1}{2}} dV$$

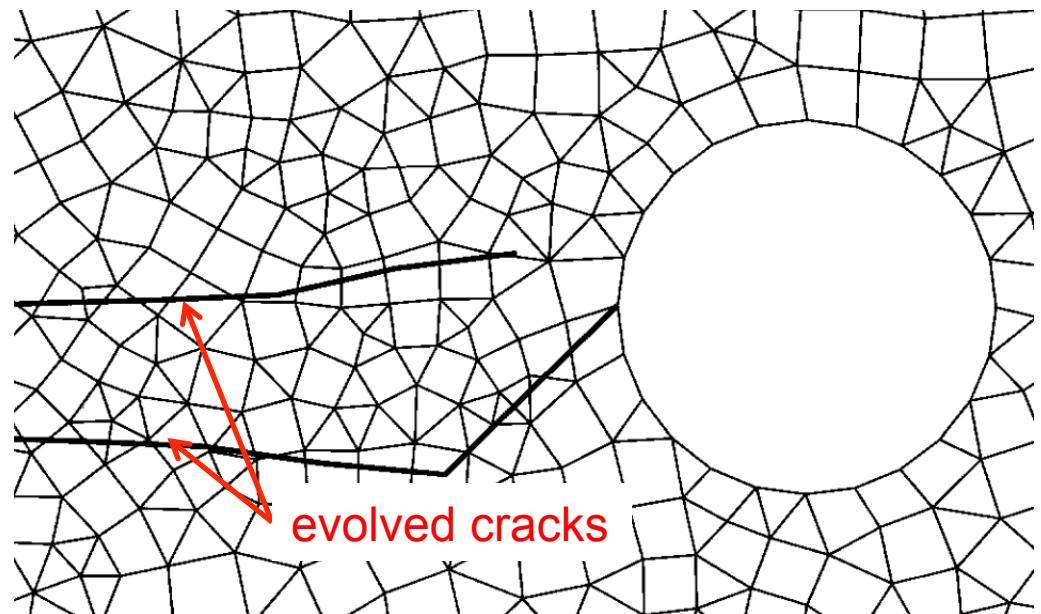
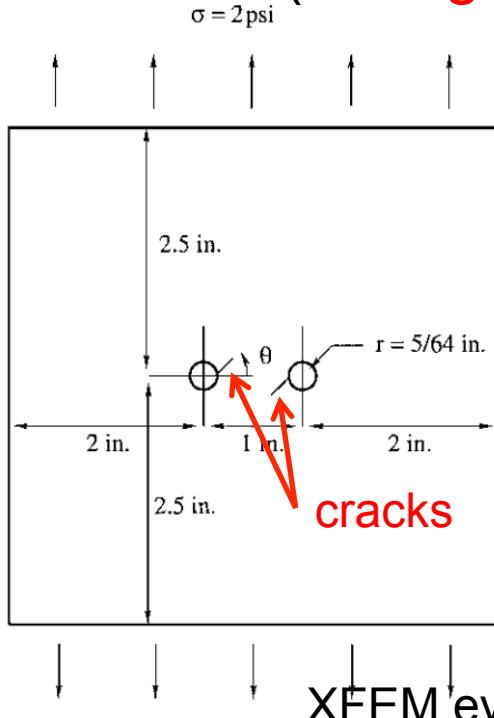


$$\mathbf{D}_m^e = \mathbf{D}^e$$

$$\boldsymbol{\sigma} = \sum_m \frac{V_m}{V} \boldsymbol{\sigma}_m$$

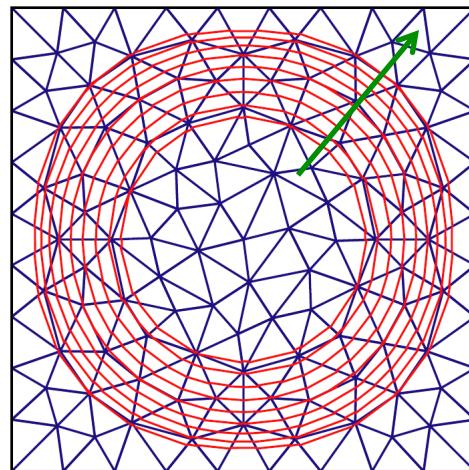
The XFEM:

- XFEM is the eXtended Finite Element Method
- Originated in the late 1990s at Northwestern University to model crack growth.
- Cracks are discontinuities in the displacement field variables (**strong discontinuity**)

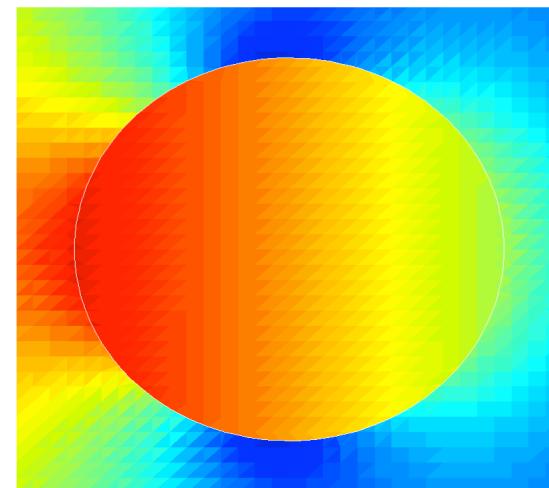


Extended to other physics:

- Further developed to model problems with important **evolving** “features.”
- Often characterized by discontinuities in field variable derivatives (temperature gradient; **weak discontinuity**).
- Less cumbersome than adaptively body-fitted mesh.



XFEM/VOF evolved phase-change interface (DMRV08).



XFEM for magnetics with edge-based elements (SBKV13).

Requires discontinuity location:

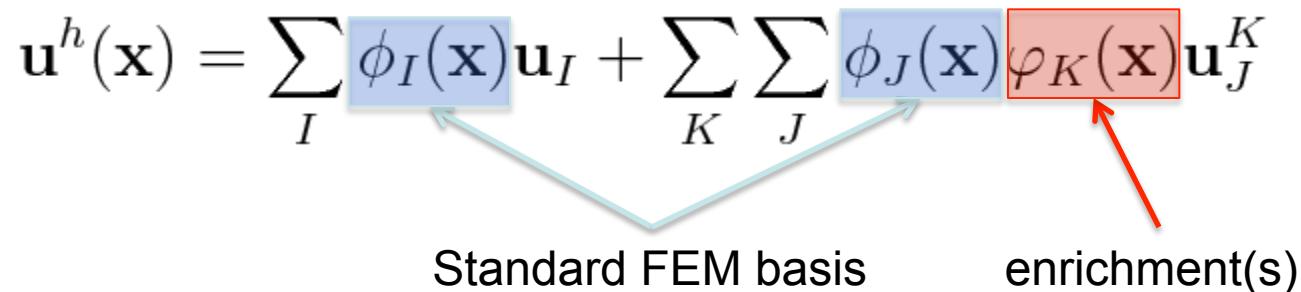


- Requires knowledge of the interface/discontinuity location (strong or weak).
- Traditionally uses Level-Set approach ...
 - Interface is located by evolving a level-set function.
 - Typically requires frequent fix-ups.
 - Confounded by complex interface intersections.
 - Doesn't conserve mass.
- We use Interface Reconstruction ...
 - Material volume fractions are advected with flow.
 - Interface is reconstructed from volume fraction field.
 - Allows evolving, complex interface intersections.
 - Conserves mass.

[See Kramer and Mosso's talk for details.](#)

Partition of Unity Framework:

- Partition-of-unity (POU) approach constructs basis functions as products of standard FEM bases and local, enriched bases.

$$\mathbf{u}^h(\mathbf{x}) = \sum_I \phi_I(\mathbf{x}) \mathbf{u}_I + \sum_K \sum_J \phi_J(\mathbf{x}) \varphi_K(\mathbf{x}) \mathbf{u}_J^K$$


Standard FEM basis enrichment(s)

- XFEM seeks to capture discontinuities, hence enrichment functions are generally strongly or weakly discontinuous.
- Aside: The Generalized Finite Element Method (GFEM) is essentially XFEM. Developed in parallel at different Universities.

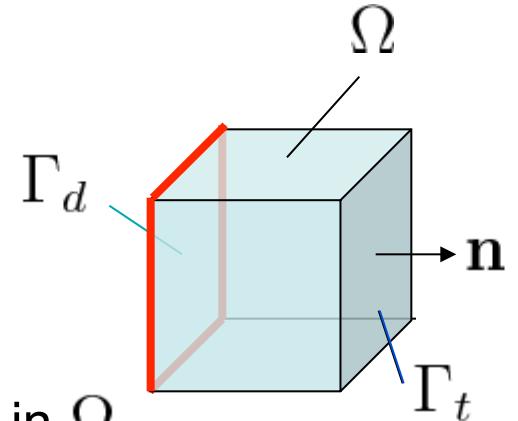
XFEM discretization:

Illustrate with the momentum balance equation:

PDE: $\rho \frac{d}{dt} \dot{\mathbf{u}} = \rho \ddot{\mathbf{u}} = \nabla \cdot \boldsymbol{\sigma} + \mathbf{f} \text{ in } \Omega$

ICs: $\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x})$ and $\dot{\mathbf{u}}(\mathbf{x}, 0) = \mathbf{v}_0(\mathbf{x})$ in Ω

BCs: traction: $\boldsymbol{\sigma} \cdot \mathbf{n} = \mathbf{t}$ on Γ_t
 dirichlet: $\mathbf{u} = \mathbf{u}_d$ on Γ_d



Or in weak form:

$$\int_{\Omega} [\rho \delta \mathbf{w} \cdot \ddot{\mathbf{u}} + \boldsymbol{\sigma} : \nabla \delta \mathbf{w}] d\Omega = \int_{\Omega} (\delta \mathbf{w} \cdot \mathbf{f}) d\Omega + \int_{\Gamma_t} (\delta \mathbf{w} \cdot \mathbf{t}) d\Gamma_t$$

consider
mass matrix

ICs: $\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x})$ and $\dot{\mathbf{u}}(\mathbf{x}, 0) = \mathbf{v}_0(\mathbf{x})$ in Ω

BC: $\mathbf{u} = \mathbf{u}_d$ on Γ_d

XFEM (consistent) mass matrix

For two materials we will want to use two “non-overlapping” Heaviside functions. One to “cover” the domain of each material.

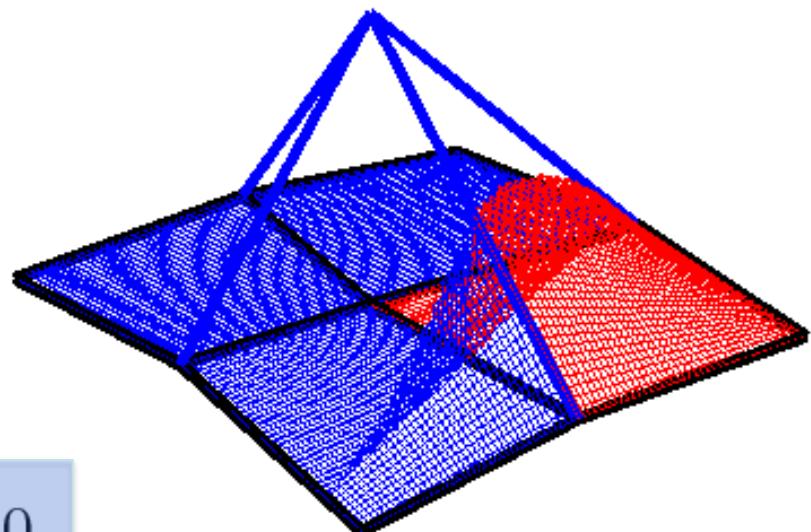
$$\mathbf{u}^h(\mathbf{x}) = \sum_I^{N_n} \left[\mathcal{H}^1(\mathbf{x})\phi_I(\mathbf{x})\mathbf{u}_I^1 + \mathcal{H}^2(\mathbf{x})\phi_I(\mathbf{x})\mathbf{u}_I^2 \right]$$

and the (consistent) Galerkin mass matrix is:

$$m_{I,J}^1 = \int_{\Omega} \rho \phi_I \mathcal{H}^1 \phi_J \mathcal{H}^1 d\Omega$$

$$m_{I,J}^2 = \int_{\Omega} \rho \phi_I \mathcal{H}^2 \phi_J \mathcal{H}^2 d\Omega$$

$$m_{I,J}^{12} = \int_{\Omega} \rho \phi_I \mathcal{H}^1 \phi_J \mathcal{H}^2 d\Omega = 0$$



No coupling between materials

Materials move independently:



Assume two materials for simplicity then the momentum equation is:

$$\begin{bmatrix} \mathbf{M}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_2 \end{bmatrix} \begin{bmatrix} \mathbf{v}_1^{new} \\ \mathbf{v}_2^{new} \end{bmatrix} = \Delta t \begin{bmatrix} \mathbf{f}_1^{int} + \mathbf{f}_1^{ext} \\ \mathbf{f}_2^{int} + \mathbf{f}_2^{ext} \end{bmatrix} + \begin{bmatrix} \mathbf{M}_1 \mathbf{v}_1^{old} \\ \mathbf{M}_2 \mathbf{v}_2^{old} \end{bmatrix}$$

The element level versions of the above terms are:

$$\mathbf{f}_{1,e}^{int} = \int_{\Omega_e} \mathcal{H}_1 \mathbf{B}^T \boldsymbol{\sigma}_1 d\Omega_e$$

internal forces:

$$\mathbf{f}_{2,e}^{int} = \int_{\Omega_e} \mathcal{H}_2 \mathbf{B}^T \boldsymbol{\sigma}_2 d\Omega_e$$

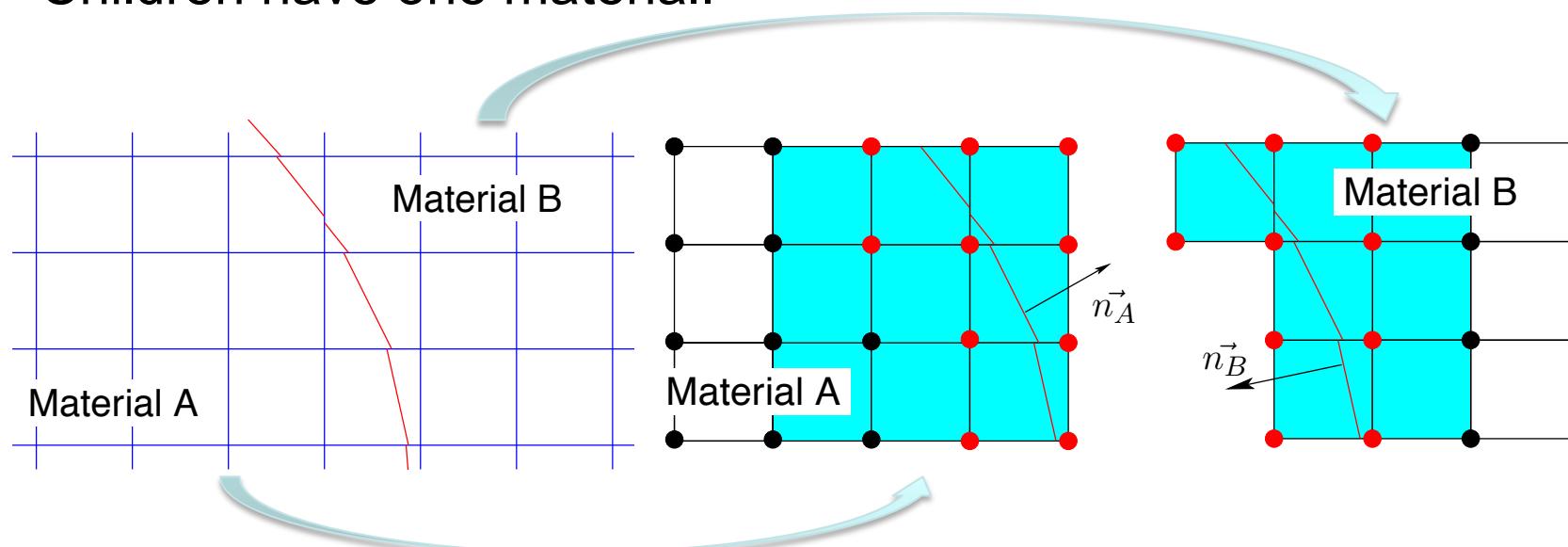
As stress for material 1 is only defined under Heaviside for material 1 (and vice-versa for material 2) we have:

$$\mathbf{f}_{12,e}^{int} = \int_{\Omega_e} \mathcal{H}_1 \mathbf{B}^T \boldsymbol{\sigma}_2 d\Omega_e = 0$$

No coupling
between materials

Construct hierarchy of entities:

- **Parents** are original elements / nodes.
- **Parent** elements are **enriched** (e.g. have **Children**) if they have more than one material (including void).
- Parent nodes are enriched if they are attached to an enriched element.
- Parents have as many Children as they do materials.
- Children have one material.



Recoupling material responses:



- Materials in a mixed cell now have unique velocity fields and hence deformation rates.
- Hence, individual material responses are decoupled from one-another.
- Without modification, materials move without regard to one another's deformation.
- Significant/active area of research to “recouple” material responses at their interfaces ([see C. Siefert's poster](#)).

Interaction enforced with LMs:

- Recouple materials via “Lagrange multipliers.”
- Lagrange multipliers applied to “constrain” materials such that they do not penetrate one another.
- We actually constrain materials to share normal component of velocity at shared interfaces.

$$\begin{bmatrix} \mathbf{M}_1 & \mathbf{0} & \mathbf{G}_1 \\ \mathbf{0} & \mathbf{M}_2 & \mathbf{G}_2 \\ \mathbf{G}_1^T & \mathbf{G}_2^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{v}_1^{new} \\ \mathbf{v}_2^{new} \end{bmatrix} = \begin{bmatrix} \Delta t (\mathbf{f}_1^{int} + \mathbf{f}_1^{ext}) + \mathbf{M}_1 \mathbf{v}_1^{old} \\ \Delta t (\mathbf{f}_2^{int} + \mathbf{f}_2^{ext}) + \mathbf{M}_2 \mathbf{v}_2^{old} \\ \mathbf{0} \end{bmatrix}$$

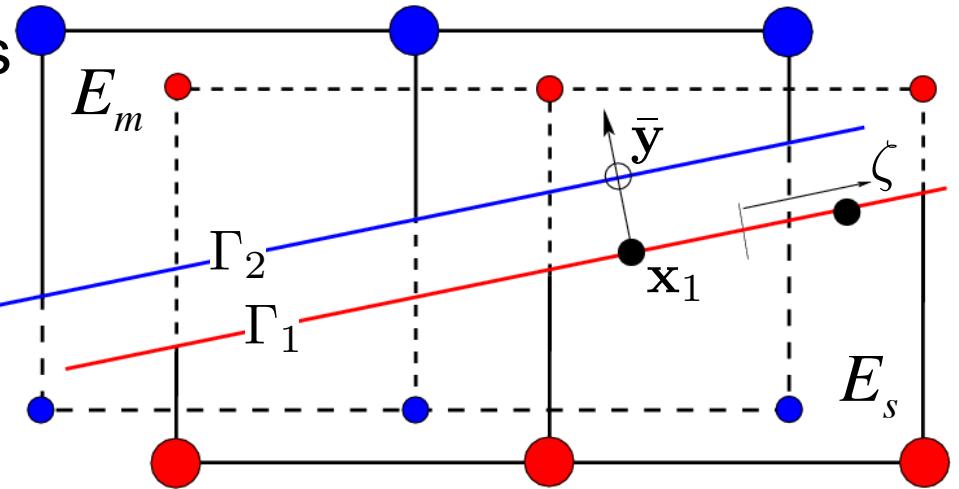
Diagram illustrating the components of the system matrix and right-hand side vector:

- Lagrange multipliers** are represented by the scalar λ in the middle column of the matrix.
- Constraint matrices** are represented by the matrices \mathbf{G}_1 and \mathbf{G}_2 in the third column of the matrix.
- Velocity constraint** is represented by the equation $\mathbf{G}_1^T \mathbf{v}_1^{new} + \mathbf{G}_2^T \mathbf{v}_2^{new} = \mathbf{0}$.

Lagrange Multipliers:

- Momentum residual is augmented with LM functional.
- Some essential components are:
 - Choice of basis functions
 - What to “enforce”
 - Finding interactions
 - Solving resultant system

$$\Pi_c = \int_{\Gamma_c} \dot{g} \lambda d\Gamma$$



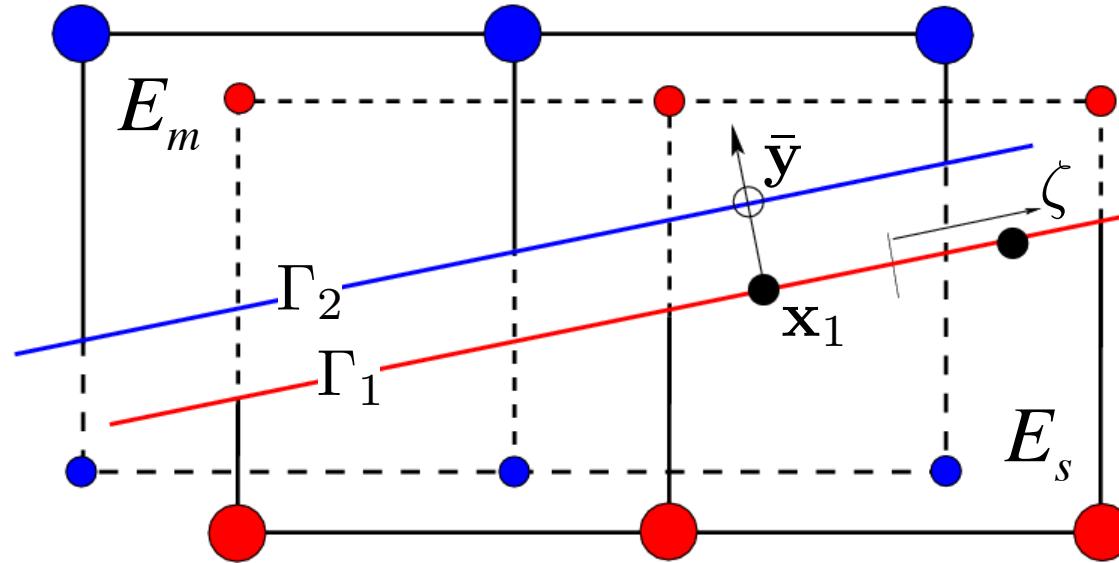
$$\delta \Pi_c = \int_{\Gamma_c} \delta \dot{g} \lambda d\Gamma + \int_{\Gamma_c} \dot{g} \delta \lambda d\Gamma = \delta \mathcal{P}_c + \delta \mathcal{C}$$

$$\dot{g}^h = \sum_I M_I(\zeta) g_I$$

$$\lambda^h = \sum_J Q_J(\zeta) \Lambda_J$$

$$\dot{g}^h(\mathbf{x}) = (\mathbf{v}_m(\bar{y}) - \mathbf{v}_s(\mathbf{x})) \cdot \mathbf{n}$$

“Face-Face” strategy ...



Quadrature and Lagrange Multiplier:

$$\mathcal{C} = \sum_{e \in \Gamma_1} \Lambda^e \sum_{q \in e} W_q j_q (\mathbf{v}_m(\bar{\mathbf{y}}_q) - \mathbf{v}_s(\mathbf{x}_q)) \cdot \mathbf{n}$$

$$\delta \mathcal{C}_e = (\mathbf{G}^t)_e \mathbf{v} = \sum_{q \in e} W_q j_q (\mathbf{v}_m(\bar{\mathbf{y}}_q) - \mathbf{v}_s(\mathbf{x}_q)) \cdot \mathbf{n} = \mathbf{0}$$

Enforcement phase:

- Solve resulting system for Lagrange multipliers.
- Compute contact forces.
- Update “new” velocities.

$$\begin{bmatrix} \mathbf{M} & \mathbf{G} \\ \mathbf{G}^T & \mathbf{M} \end{bmatrix} \begin{bmatrix} \mathbf{v}^{new} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \Delta t \mathbf{f}^{int} + \mathbf{M} \mathbf{v}^{old} \\ \mathbf{0} \end{bmatrix}$$

$$\mathbf{G}^T \mathbf{v}^{new} = \mathbf{G}^T \mathbf{M}^{-1} \mathbf{G} \boldsymbol{\lambda} - \mathbf{G}^T (\mathbf{v}^{old} + \mathbf{M}^{-1} \mathbf{f}^{int}) = \mathbf{0}$$

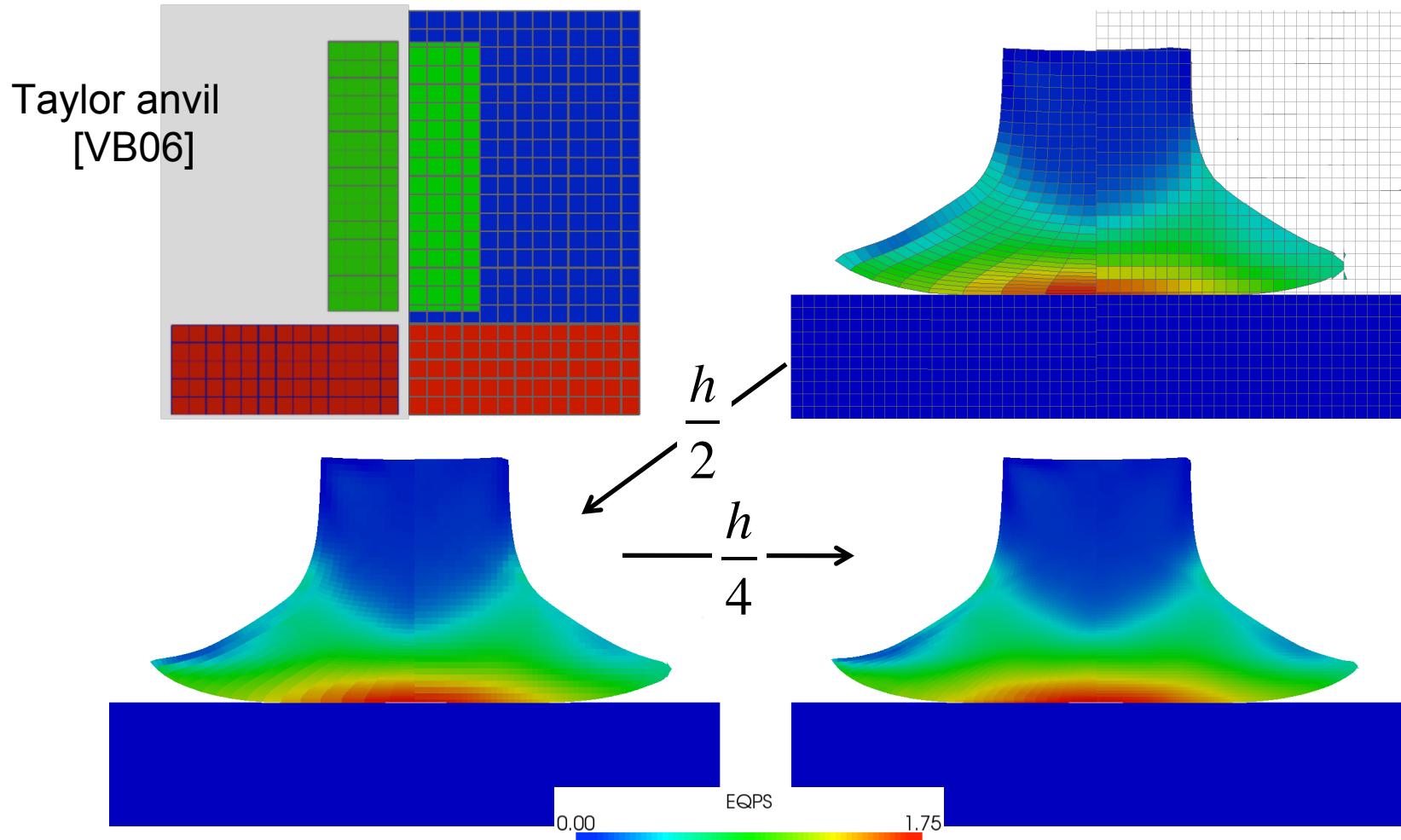
$\mathbf{G}^T \mathbf{v}^{new} \quad \mathbf{G}^T \mathbf{M}^{-1} \mathbf{G} \boldsymbol{\lambda} - \mathbf{G}^T (\mathbf{v}^{old} + \mathbf{M}^{-1} \mathbf{f}^{int}) = \mathbf{0}$

$\mathbf{A} \boldsymbol{\lambda} = \mathbf{b}$

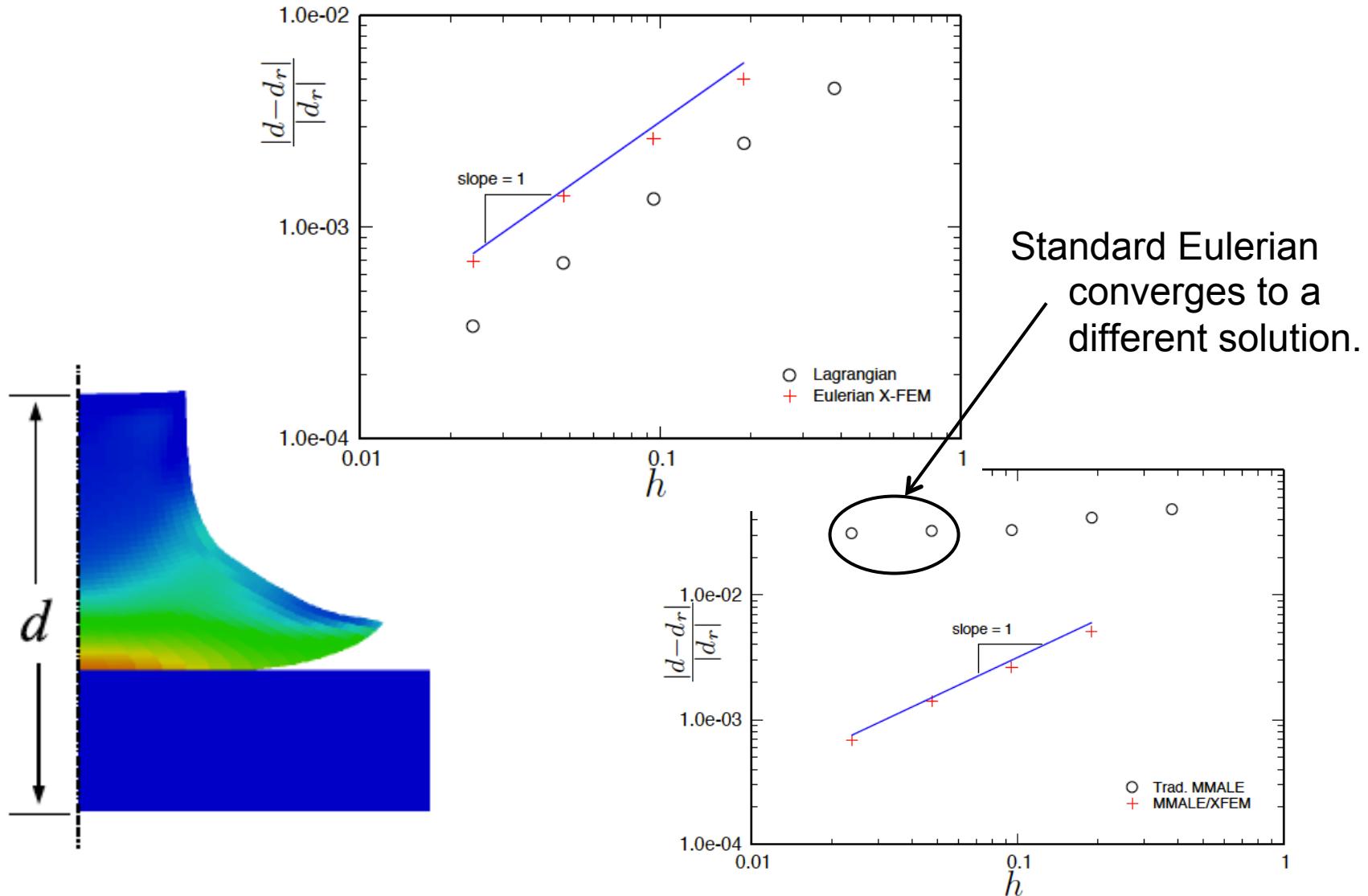
$$\mathbf{v}^{new} = \mathbf{M}^{-1} \mathbf{G} \boldsymbol{\lambda} - (\mathbf{v}^{old} + \mathbf{M}^{-1} \mathbf{f}^{int})$$

contact forces

Lagrangian quality results:

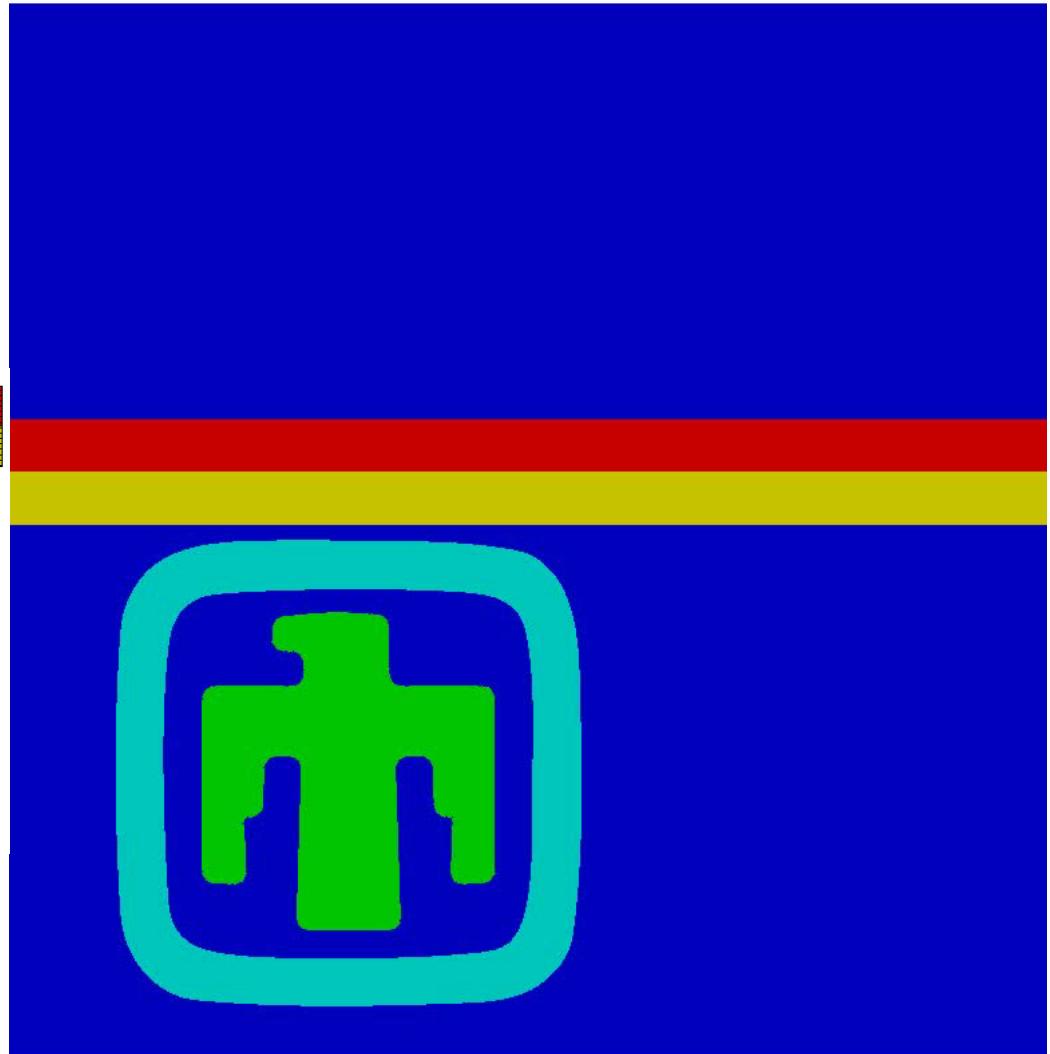
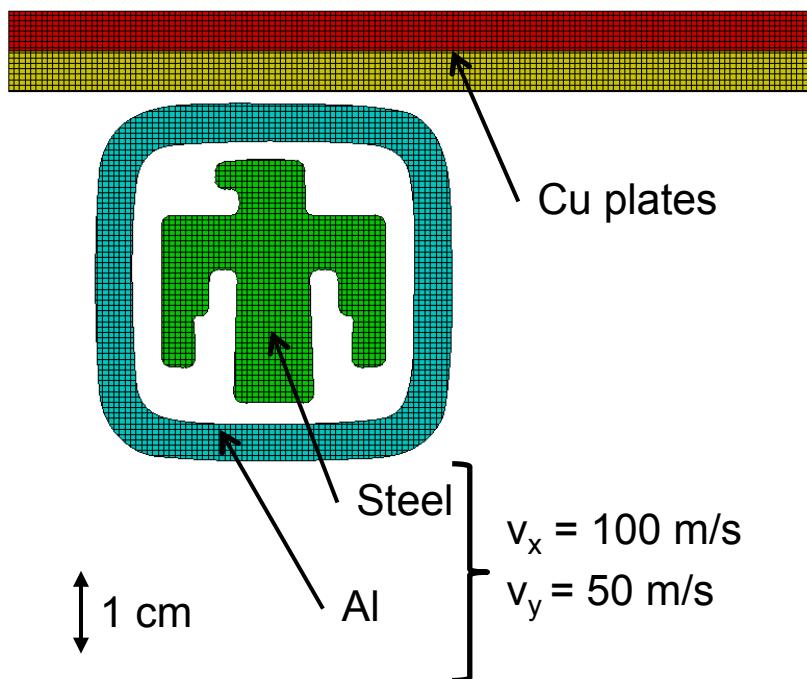


Eulerian gets wrong answer:



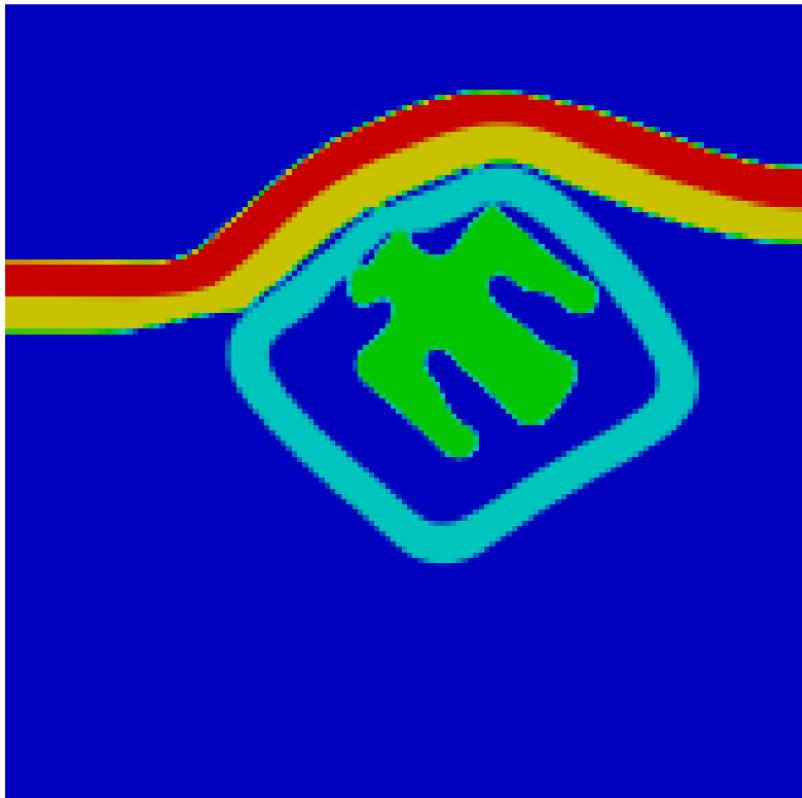
A more whimsical example:

Logo with initial velocity impacts
stationary, layered copper plates.

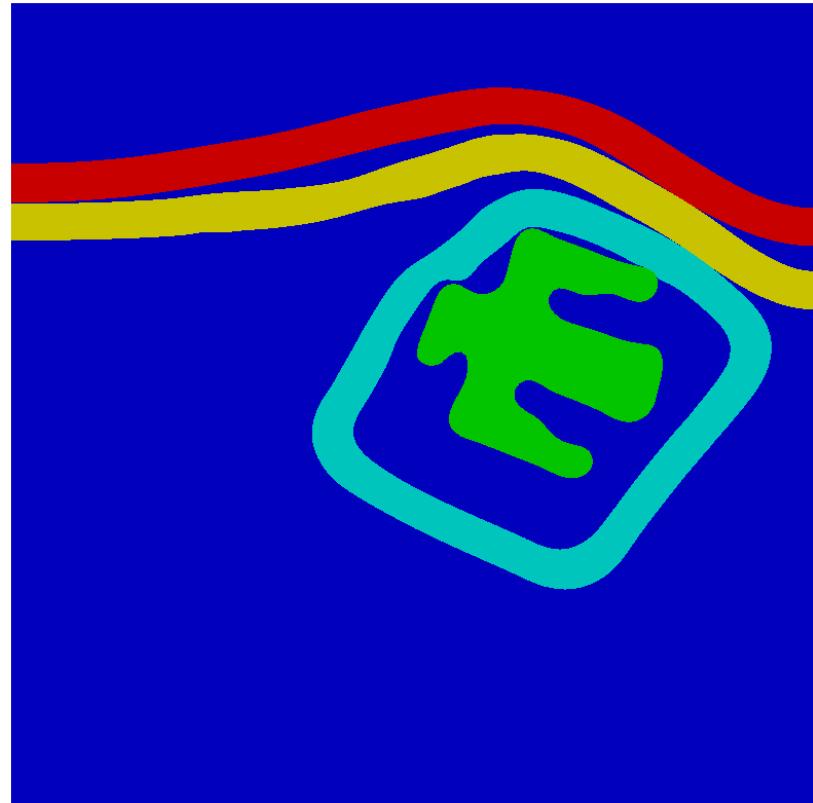


... and a comparison:

Standard approach



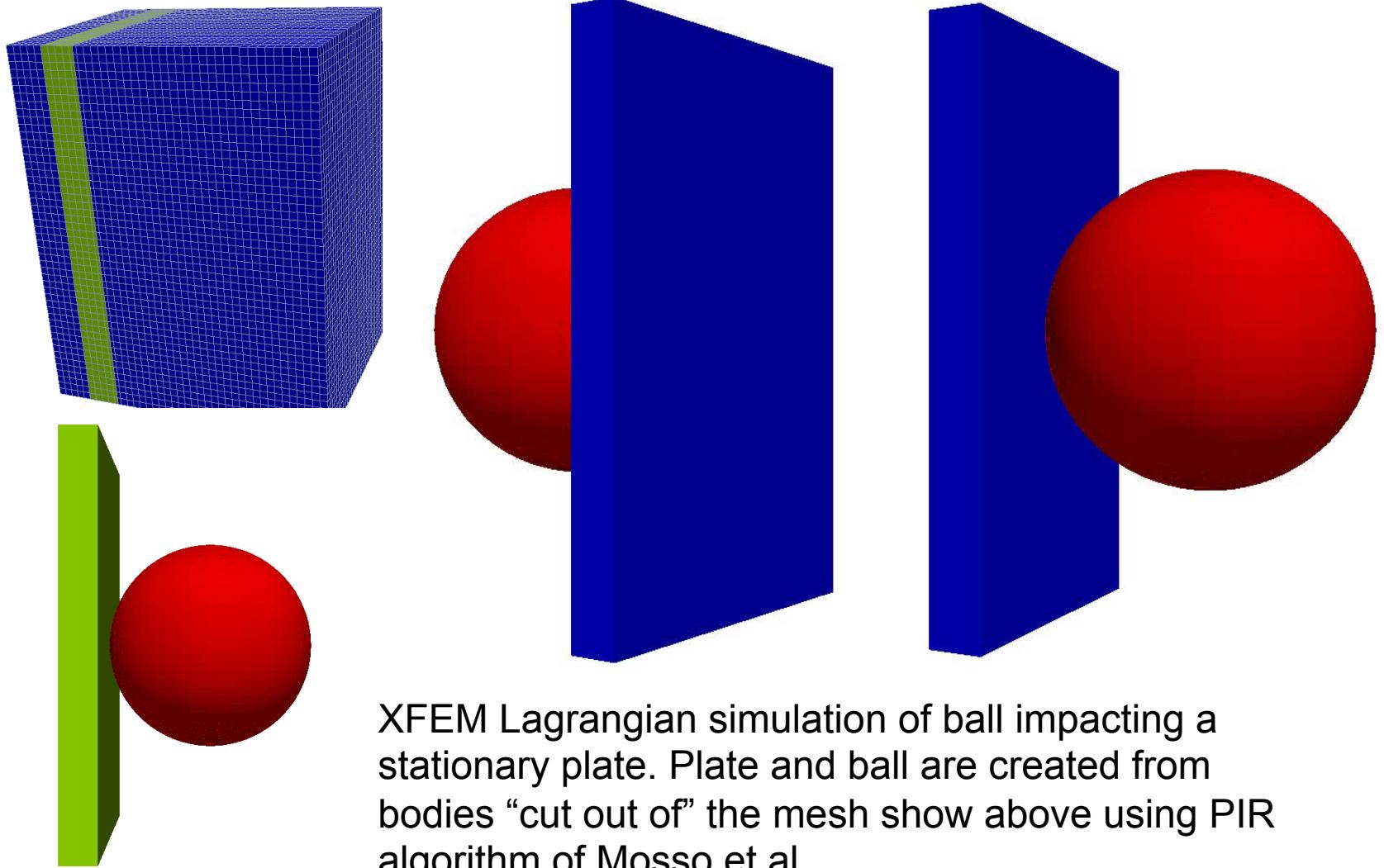
XFEM approach



Geometry at time = 1.0e-2 s.

As expected, standard approach with shared velocity fields between materials shows bonding while **XFEM allows material impact and separation.**

3D work is ongoing:



XFEM Lagrangian simulation of ball impacting a stationary plate. Plate and ball are created from bodies “cut out of” the mesh show above using PIR algorithm of Mosso et al.

Conclusions:



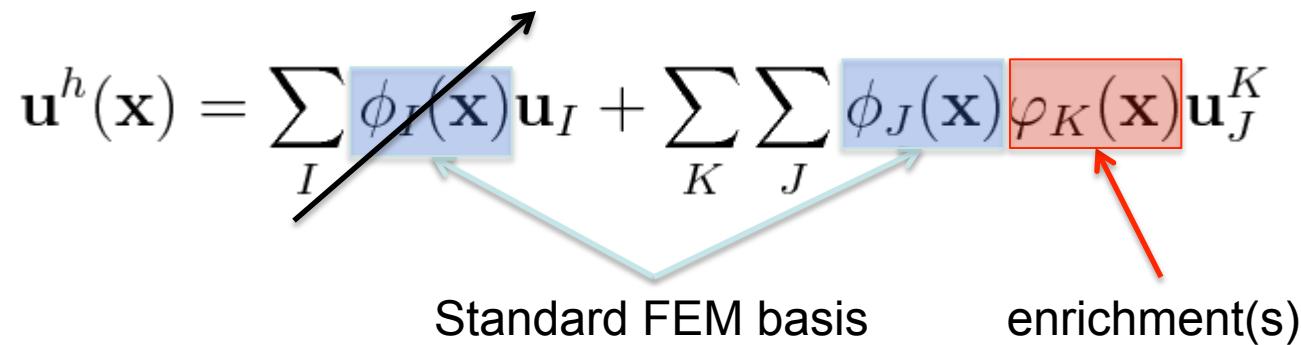
- Developing capability to more accurately treat multi-material cells in an “operator-split” ALE context.
- Capability builds on existing ALE infrastructure.
- Uses X-FEM ideas to provide unique kinematics for each material in a cell.
- Uses interface reconstruction rather than level-set ideas to address conservation and complex interface intersections.
- Demonstrates good convergence/accuracy for problems investigated here.

Hold Back Slides



XFEM Heaviside enrichments:

- Enrichment term of the Heaviside enriched XFEM basis contains the space of the classical term (can represent a constant and linear). Hence the classical term is dropped.

$$\mathbf{u}^h(\mathbf{x}) = \sum_I \phi_I(\mathbf{x}) \mathbf{u}_I + \sum_K \sum_J \phi_J(\mathbf{x}) \varphi_K(\mathbf{x}) \mathbf{u}_J^K$$


Standard FEM basis enrichment(s)

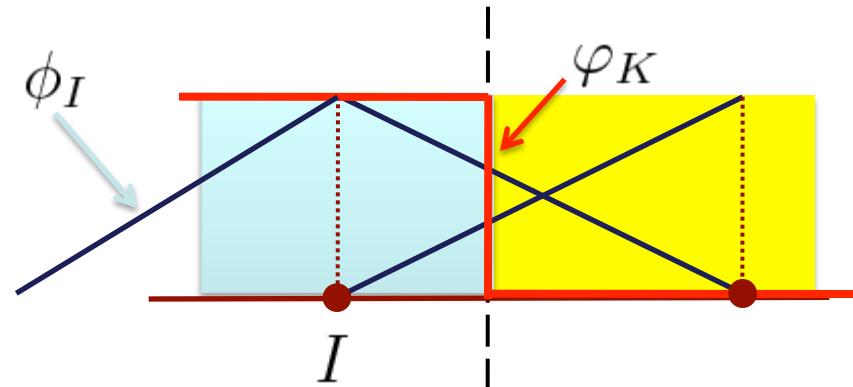
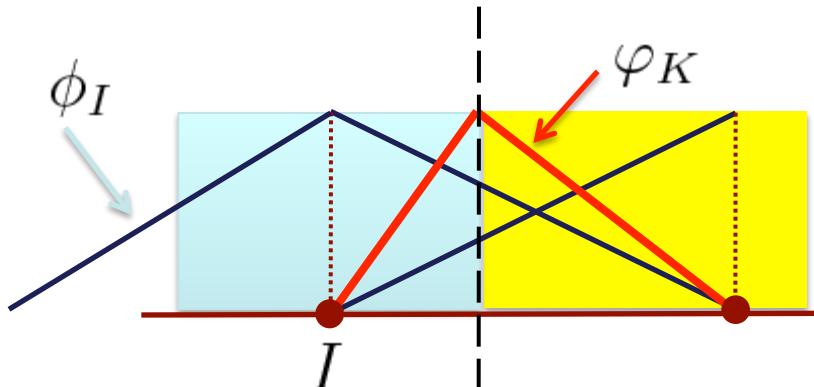
- This reduced basis is an important component of our ALEGRA implementation as we will see later.
- From this point on XFEM implies Heaviside XFEM.

XFEM enrichments:

$$\mathbf{u}^h(\mathbf{x}) = \sum_I \phi_I(\mathbf{x}) \mathbf{u}_I + \sum_K \sum_J \phi_J(\mathbf{x}) \varphi_K(\mathbf{x}) \mathbf{u}_J^K$$

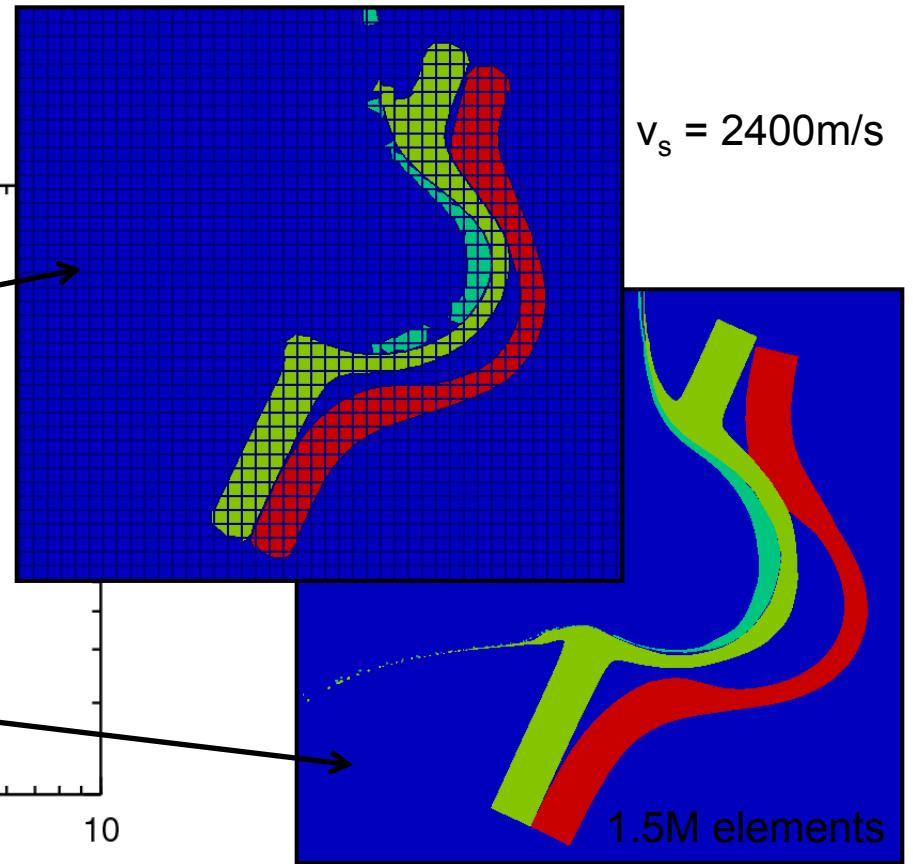
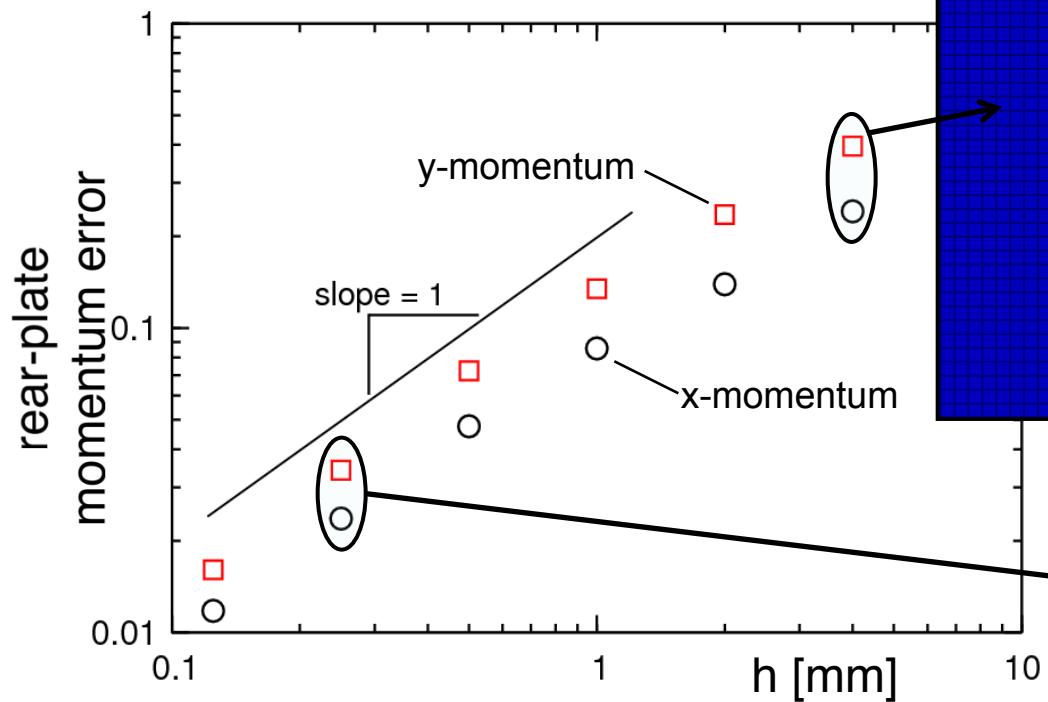
Standard FEM basis enrichment(s)

- Ridge (weak)
 - parasitic high order terms
 - complex multi-interface treatment
 - Some forms require blending
- Heaviside (weak & strong)
 - re-tie weak discontinuities
 - simple multi-interface treatment
 - nice implementation “tricks”



Whipple shield example:

Whipple Shield used in satellite protection.



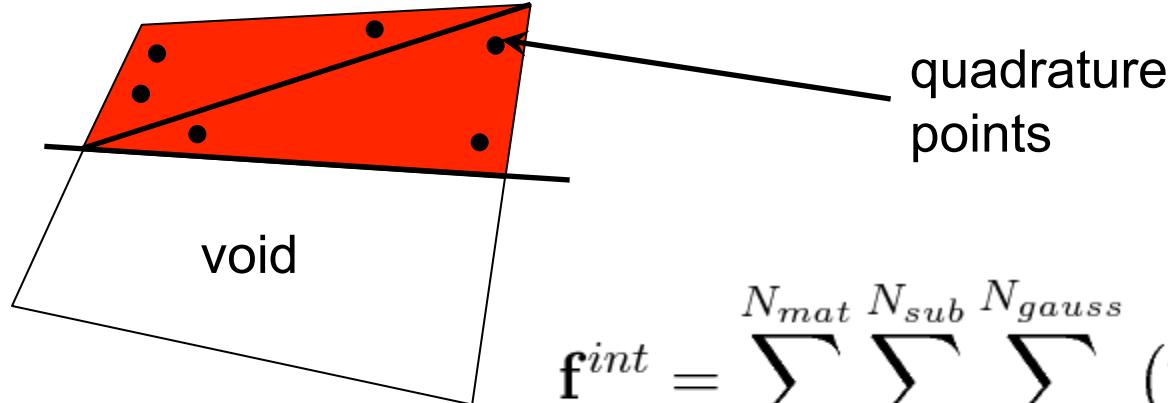
High-velocity impact difficult for Lagrangian and unrealistic for Eulerian are possible with X-FEM.

We use XFEM to ...

Develop an Eulerian capability to treat intra-element interface behavior incorporating sound, realistic physics:

- as a true surface (not volumetric) phenomenon,
- with distinct velocity/displacement fields for each material in an element,
- while maintaining the advantages of ALEGRA's explicit-dynamics code-base,
- and capitalizing on our existing infrastructure.

Subdomain quadrature:

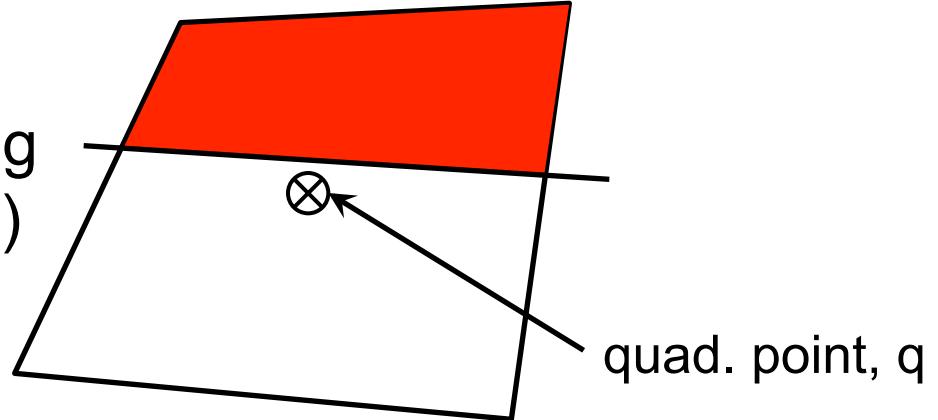


$$\mathbf{f}^{int} = \sum_{m=1}^{N_{mat}} \sum_{s=1}^{N_{sub}} \sum_{l=1}^{N_{gauss}} (\mathcal{H}_m \mathbf{B}^T \boldsymbol{\sigma}_m |J|)_l \cdot w_l$$

- Frequently used in literature but...
 - Overly stiff (subject to volumetric locking)
 - Requires tracking of state at many more points.
 - Many more material model evaluations (expensive).
 - So, not practical in “real” code.

Using mean quadrature:

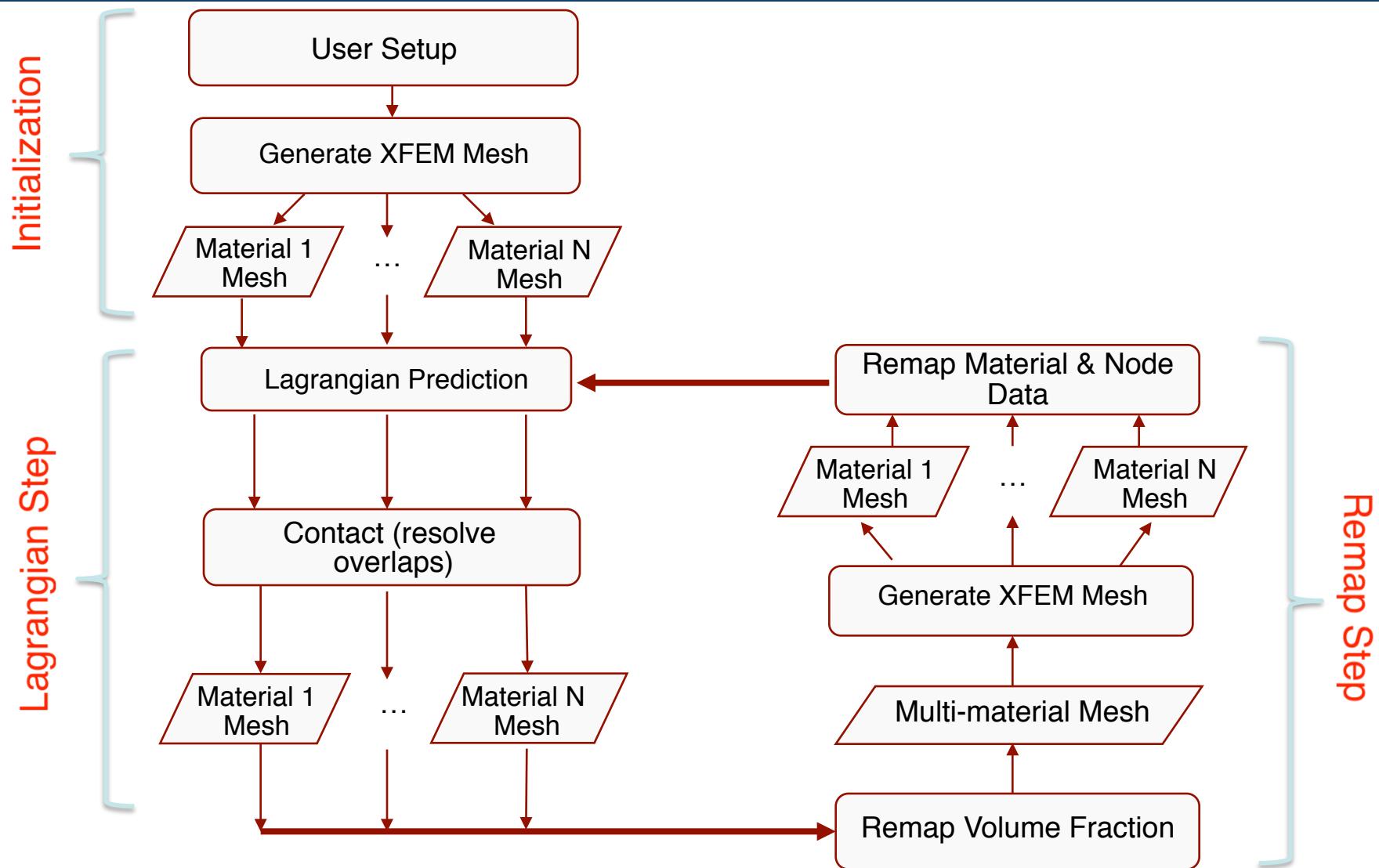
- Approach follows Song and Belytschko (2006)



- Compute single-point result as if fully filled and scale by material volume fraction in cell/element:

$$\begin{aligned}
 \mathbf{f}_{1,e}^{int} &= \int_{\Omega_e} \mathcal{H}_1 \mathbf{B}^T \boldsymbol{\sigma}_1 d\Omega_e \\
 &\approx \frac{A_1}{A_e} \int_{\Omega^e} \mathbf{B}^T \boldsymbol{\sigma}_1 d\Omega^e \\
 &\approx \frac{A_1}{A_e} (\mathbf{B}^T \boldsymbol{\sigma}_1 |J|)_q \cdot w_q
 \end{aligned}$$

Summary of algorithm flow:



Remap Step:

- Transfer volume fractions from end of previous Lagrangian step to start of next step.
- Construct new parent-child hierarchy.
- Transfer velocities and material state from mesh at end of Lagrangian step to start of next step.
 - Conserve mass, momentum and internal energy.
- Construct interfaces for next Lagrangian step.

[See S. J. Mosso's talk for details.](#)

Search phase:

- Search phase finds interactions between materials.
- Key to computing constraint matrices.
- Begins by solving unconstrained momentum equations to get predicted, t^* , configuration.

