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Coupling of Momentum Balance and Thickness Evolution Equations for Ice Sheet Modeling

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joint collaboration with

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Center for Computing Research

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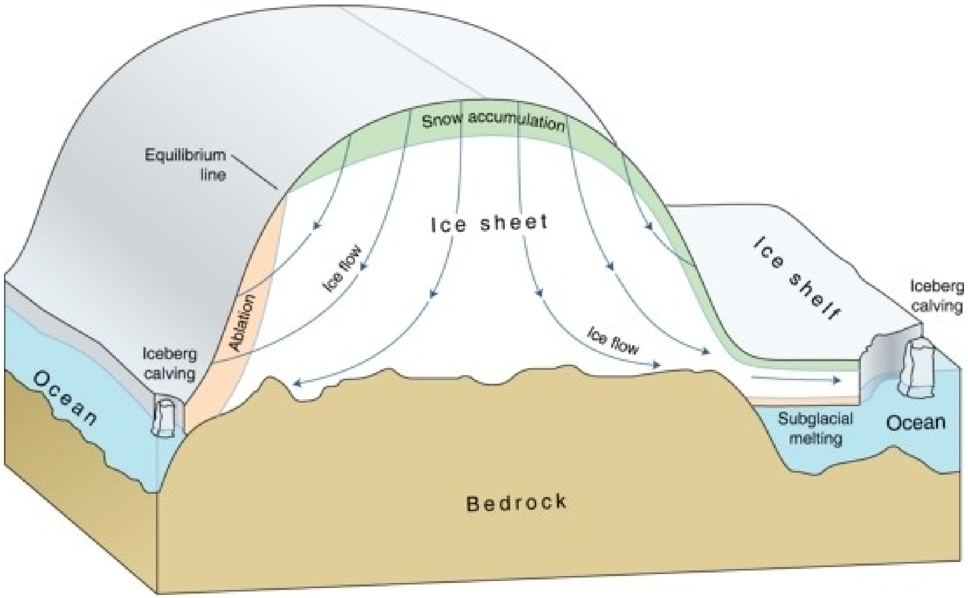
Brief introduction and motivation

- Modeling ice sheets (Greenland and Antarctica) dynamics is essential to provide estimates for sea level rise in next decades to centuries.
  Sea level rise predictions are important for policy makers.

- Ice behaves like a very viscous shear-thinning fluid (similar to lava flow).

- Different sources of nonlinearity including
  - nonlinear rheology
  - nonlinear basal boundary conditions
  - ice advance/retreat/calving
  - movement of grounding line
  - nonlinearity of of thickness evolution



from http://www.climate.be

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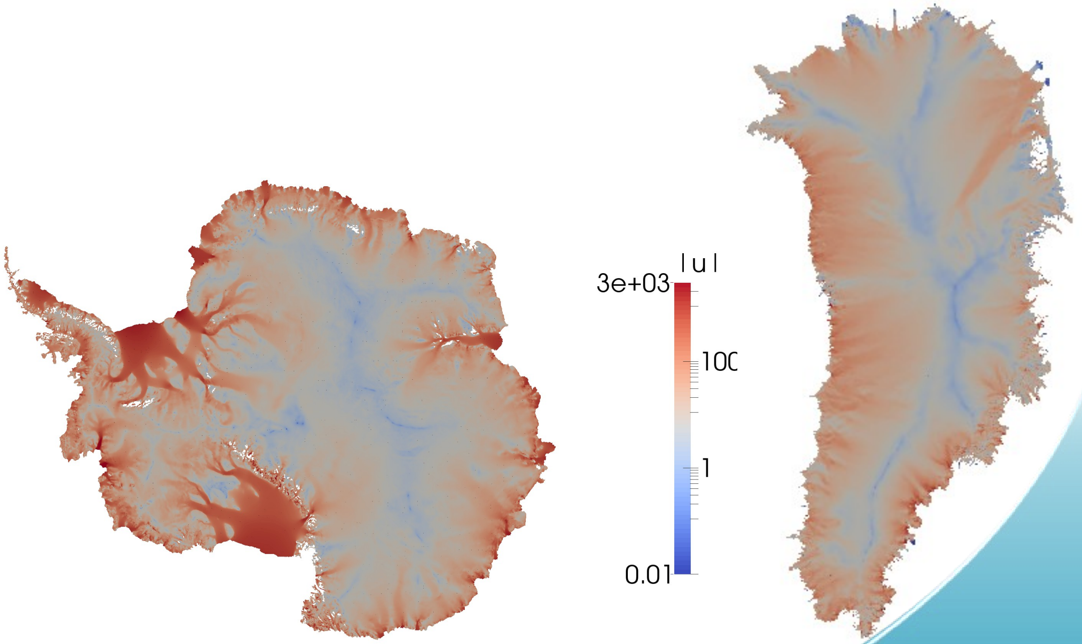
- Greenland and Antarctica ice sheets store most of the fresh water on hearth.
  They have a shallow geometry
  (thickness up to 3km,
  horizontal extensions of thousands of km).



Computed surface velocity magnitude [m/yrs]

Ice Sheet Modeling

Main components of an ice model:

- Ice flow equations (momentum and mass balance)

$$\begin{cases} -\nabla \cdot \sigma = \rho \mathbf{g} \\ \nabla \cdot \mathbf{u} = 0 \end{cases}$$

with:

$$\sigma = 2\mu \mathbf{D} - \Phi I, \qquad \mathbf{D}_{ij}(\mathbf{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

Nonlinear viscosity:

$$\mu = \frac{1}{2}\alpha(T)\,|\mathbf{D}(\mathbf{u})|^{(p-2)}, \quad p \in (1,2] \quad (\text{tipically } p \simeq \frac{4}{3})$$

Viscosity is singular when ice is not deforming

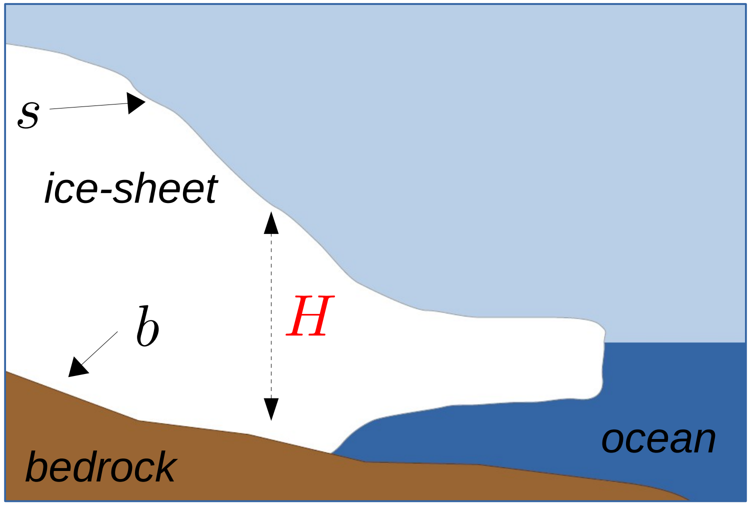
- Model for the evolution of the boundaries (thickness evolution equation)

$$\frac{\partial H}{\partial t} = -\nabla \cdot (\bar{\mathbf{u}}\, H) + \theta \qquad \text{with } \bar{\mathbf{u}} = \frac{1}{H}\int_z \mathbf{u}\, dz$$

Ice Sheet Modeling

Coupling

$$\begin{cases} -\nabla \cdot \sigma(\mathbf{u}) = \rho \mathbf{g} & \text{in } \Omega_H \\ \nabla \cdot \mathbf{u} = 0 & \text{in } \Omega_H \end{cases} \qquad \frac{\partial H}{\partial t} = -\nabla \cdot (\bar{\mathbf{u}}\, H) + \theta, \quad \text{in } \Sigma$$



For grounded ice:

$$\Omega_H := \{(x,y,z) \mid z = b(x,y) + H(x,y),\ (x,y) \in \Sigma\}$$

Ice Sheet Modeling

Coupling

$$\begin{cases} -\nabla \cdot \sigma(\mathbf{u}) = \rho \mathbf{g} & \text{in } \Omega_H \\ \nabla \cdot \mathbf{u} = 0 & \text{in } \Omega_H \end{cases} \qquad \frac{\partial H}{\partial t} = -\nabla \cdot (\bar{\mathbf{u}}\, H) + \theta, \quad \text{in } \Sigma$$

System typically coupled in a *sequential* way:

1. given H^n solve Stokes system for $\mathbf{u^n}$
2. compute $\bar{\mathbf{u}}^{\mathbf{n}}$ and solve thickness hyperbolic equation for H^{n+1}

Issue: stable only for tiny time steps.
Time steps satisfying CFL condition do NOT guarantee stability

Why? We need to simplify the equations in order to understand this.

Stokes approximations in different regimes

$$\text{Stokes}(\mathbf{u}, p) \text{ in } \Omega \in \mathbb{R}^3$$

Quasi-hydrostatic approximation

Scaling argument based on the fact that ice sheets are shallow

$$\mathbf{u} := (u,\, v,\, w)$$

$$\mathbf{D}(\mathbf{u}) = \begin{bmatrix} u_x & \frac{1}{2}(u_y + v_x) & \frac{1}{2}(u_z + \cancel{w_x}) \\ \frac{1}{2}(u_y + v_x) & v_y & \frac{1}{2}(v_z + \cancel{w_y}) \\ \cancel{\frac{1}{2}(u_z + w_x)} & \cancel{\frac{1}{2}(v_z + w_y)} & w_z \end{bmatrix}$$

$$p = \rho g(s - z) - 2\mu(u_x + v_y)$$

First Order* or Blatter-Pattyn model

$$\text{FO}(u, v) \text{ in } \Omega \in \mathbb{R}^3$$

$$-\nabla \cdot \left(2\mu\tilde{\mathbf{D}}\right) = -\rho g \nabla_{x,y}(H + b) \qquad \tilde{\mathbf{D}}(u, v) = \begin{bmatrix} 2u_x + v_y & \frac{1}{2}(u_y + v_x) & \frac{1}{2}u_z \\ \frac{1}{2}(u_y + v_x) & u_x + 2v_y & \frac{1}{2}v_z \end{bmatrix}$$

Coercive system for the horizontal components of the velocity

**Dukowicz, Price and Lipscomb, 2010. J. Glaciol*

Stokes approximations in different regimes

$$\text{FO}(u,v) \text{ in } \Omega \in \mathbb{R}^3$$

Ice regime:
grounded ice with frozen bed

$$\mathbf{D} = \begin{bmatrix} 0 & 0 & \frac{1}{2}u_z \\ 0 & 0 & \frac{1}{2}v_z \\ 0 & 0 & w_z \end{bmatrix}$$

$$p = \rho g(s-z)$$

Ice regime:
shelves or fast sliding grounded ice

$$\mathbf{D} = \begin{bmatrix} u_x & \frac{1}{2}(u_y+v_x) & 0 \\ \frac{1}{2}(u_y+v_x) & v_y & 0 \\ 0 & 0 & w_z \end{bmatrix}$$

$$p = \rho g(s-z) - 2\mu(u_x+v_y)$$

$$\text{SIA}(u,v) \text{ in } \Omega \in \mathbb{R}^3$$

Shallow Ice Approximation

$$\text{SSA}(u,v) \text{ in } \Sigma \in \mathbb{R}^2$$

Shallow Shelf Approximation

$$\text{Hybrid models, } \simeq SIA + SSA$$

SIA coupled with thickness evolution

It is possible to compute the SIA solution in closed form. For constant flow rate we get

$$\begin{bmatrix} u \\ v \end{bmatrix} = C\left((s-z)^{n+1} - H^{n+1}\right)|\nabla s|^{n-1}\nabla s, \qquad (s = H + b)$$

Substituting the expression of the velocity into the thickness evolution equation*

$$\frac{\partial H}{\partial t} + \mathrm{div}\big(\eta\,\nabla H\big) = \theta - \mathrm{div}\big(\eta\,\nabla b\big), \qquad \text{with } \eta = C_1 H^2 - C_2 H^{n+2}|\nabla s|^{n-1}$$

Which is a nonlinear ***parabolic*** equation H.

In the limit case of shallow ice on frozen bedrock, the thickness evolution equation is not hyperbolic but parabolic.
We have a *diffusive CFL* condition**: $\Delta t \le \mathrm{CFL}_{\mathrm{diff}}(\Delta x)^2$

Note: coupling the thickness evolution equation with SSA we obtain an integro-differential equation that does not feature a diffusive term.

*Fowler, ice sheets and glaciers, 1997
**Bueler and Brown, JGR, 2009

Possible strategies to couple momentum and thickness evolution equations

1. Sequential coupling. Possibly use adaptive time steps that relies on notion of diffusive CFL, as computed using SIA approximation. **Requires many time steps.**

2. Operator splitting*: try to identify "diffusive" and "advective" parts of operator and solve the evolution equation with a IMEX scheme.

$$\mathbf{u} = \alpha \mathbf{u}_{\mathrm{SIA}} + (1-\alpha)\,\mathbf{u}_{\mathrm{SSA}}$$

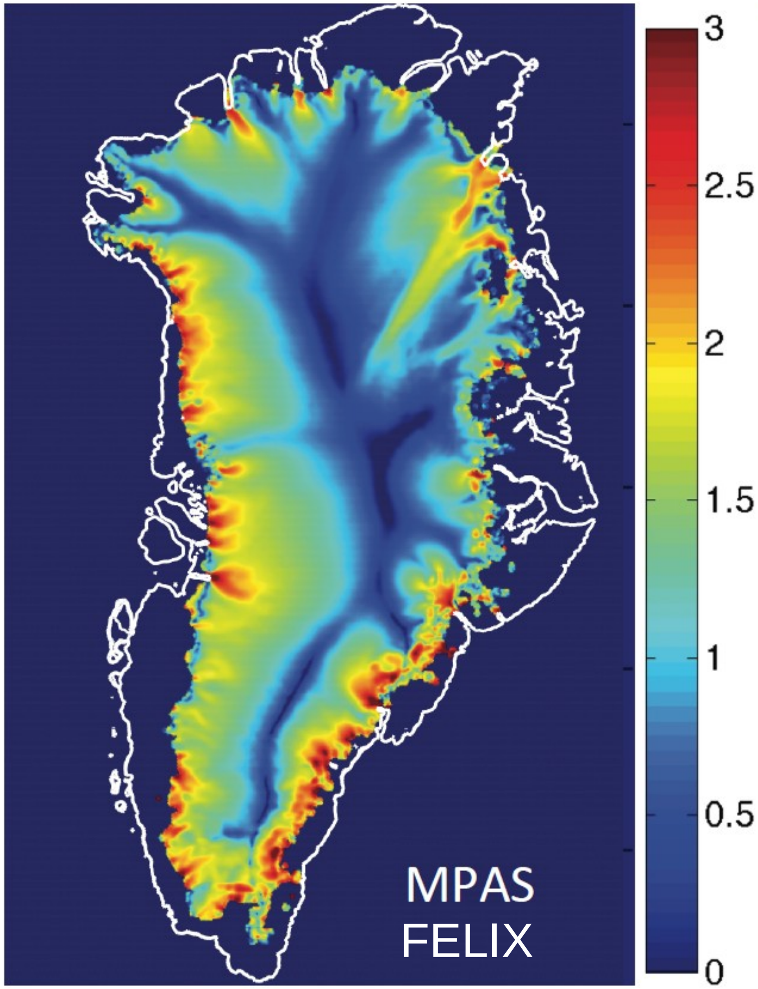
Hard to identify "diffusive" part in Stokes and FO models.

3. Solve implicitly the coupling between momentum and thickness equations.

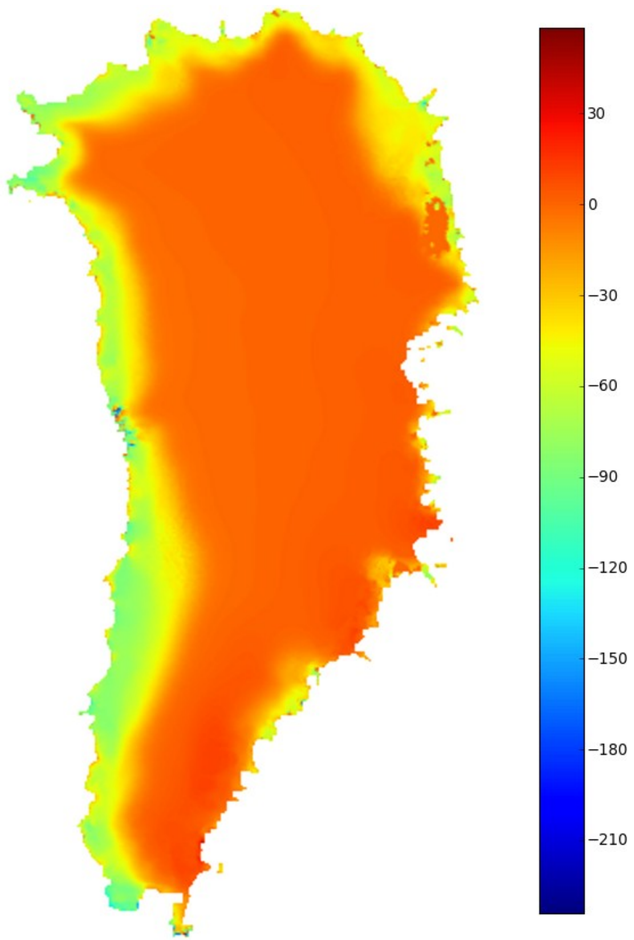
*PISM, Parallel Ice Sheet Model.

Results using sequential approach

(Ice2Sea experiment *A.J. Payne et al, PNAS 2013)*



Surface velocity magnitude [km/yr]



Thickness [m] change after 100 yr

FO eq. solved using the finite element implementation in Albany [SNL].

Evolution eq. solved using the finite volume implementation in MPAS [LANL]

Sequential approach works fine for relatively coarse meshes (here is 5km resolution). However our goal is to use resolution of 1km, 500m and in this case the approach becomes prohibitive.

Implicit coupling of Stokes and thickness evolution equation

$$\begin{cases} -\nabla \cdot \sigma(\mathbf{u}^{(n+1)}) = \rho \mathbf{g} & \text{in } \Omega_{H^{(n+1)}} \\ \nabla \cdot \mathbf{u}^{(n+1)} = 0 & \text{in } \Omega_{H^{(n+1)}} \end{cases} \qquad \frac{H^{(n+1)} - H^n}{\Delta t} + \nabla \cdot \left(\bar{\mathbf{u}}^{(n+1)} H^{(n+1)} \right) = \theta^n$$

This implicit discretization should mitigate the stability issues but is very expensive because the geometry is changing during the iterations.

Using Newton method we need to compute shape derivatives.

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Using Newton method we need to compute shape derivatives.

Idea: when using FO the thickness is exposed in the momentum equation and we may not need to to change the domain.

$$-\nabla\cdot\left(\mu\tilde{\mathbf{D}}\left(\mathbf{u}^{(n+1)}\right)\right) = -\rho g\nabla\left(b + H^{(n+1)}\right) \ \text{ in } \Omega_{H^n} \qquad \frac{H^{(n+1)} - H^n}{\Delta t} + \nabla\cdot\left(\bar{\mathbf{u}}^{(n+1)}\, H^{(n+1)}\right) = \theta^n$$

Working with external code limitations

MPAS (climate library, LANL)

unstructured **explicit** finite volume on Voronoi grids.

solves for thickness (upwind method).

H →

← $\mathbf{u}$

Albany-FELIX (finite element, SNL)

unstructured finite element method

solves FO for velocity

Need to improve overall solution acting only on the velocity solver.

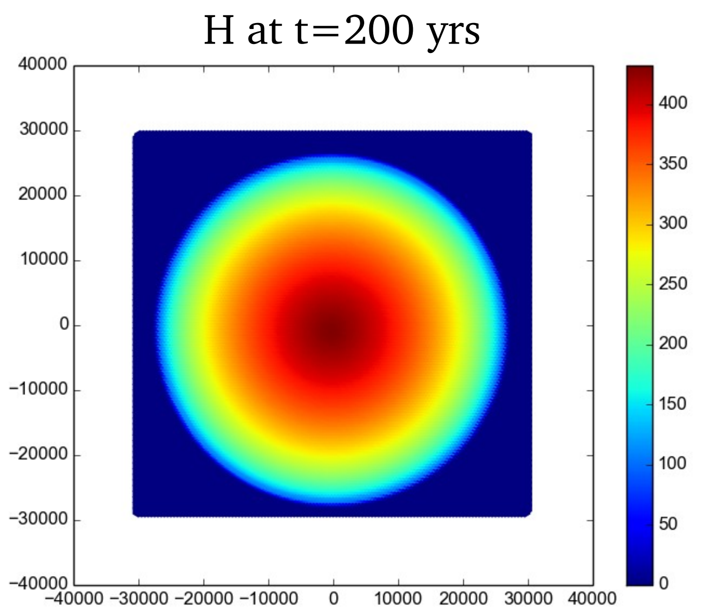
Albany-FELIX

$$-\nabla \cdot \left(\mu \tilde{\mathbf{D}}\left(\mathbf{u}\right)\right) = -\rho g \nabla \left(b + H\right) \;\; \text{in } \Omega_{H^n} \qquad \frac{H - H^n}{\Delta t} + \nabla \cdot \left(\bar{\mathbf{u}}\, H^n\right) = \theta^n$$

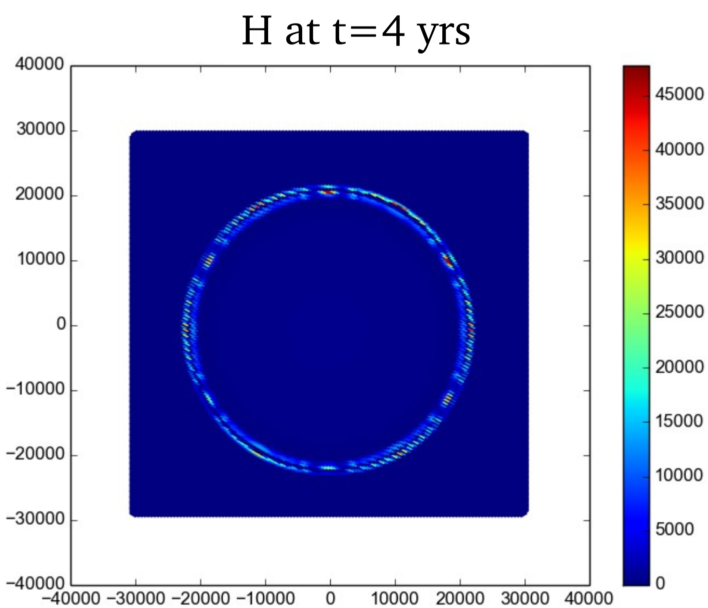
MPAS

$$\frac{H^{n+1} - H^n}{\Delta t} + \sum_i F_i\left(\mathbf{u},\, H^n\right) = \theta^n$$

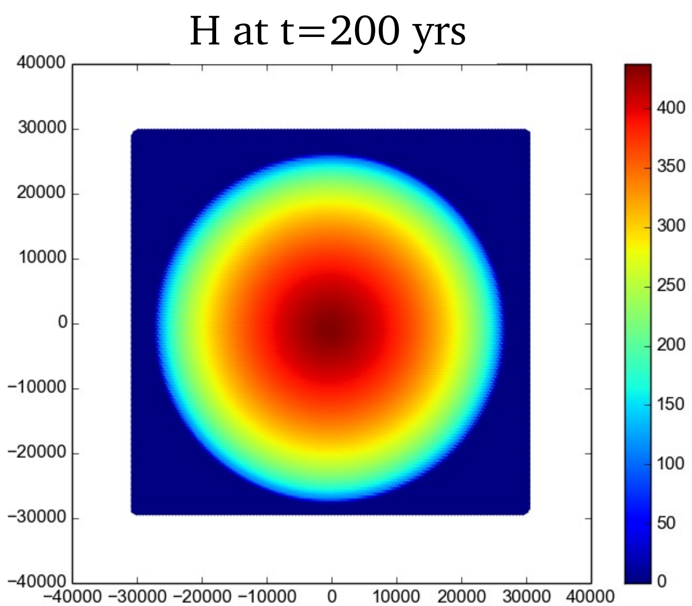
Preliminary results on dome benchmark



Reference solution computed with sequential approach and time step of 5 months.



Solution obtained with sequential coupling, **dt =1 yr**



Solution obtained with semi-implicit coupling, **dt =5 yrs**

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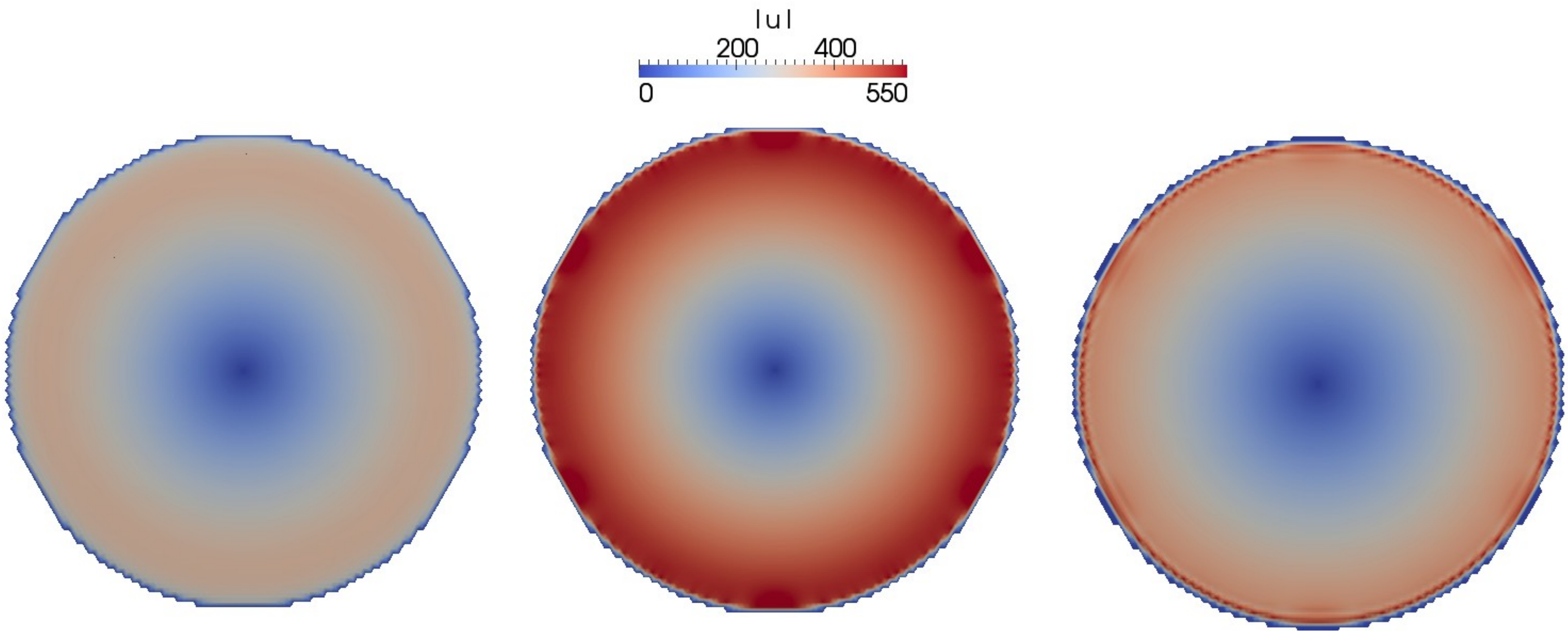
How the coupling is affecting the computed velocity?



Semi-implicit Coupling, t=0, dt = 1yr

Sequential Coupling, t=0, dt = 4 months

Sequential Coupling, t=1yr, dt = 4 months

The velocity computed with the coupled scheme is anticipating the effect of a change in thickness and in general results “smoother” than the one computed with the explicit scheme

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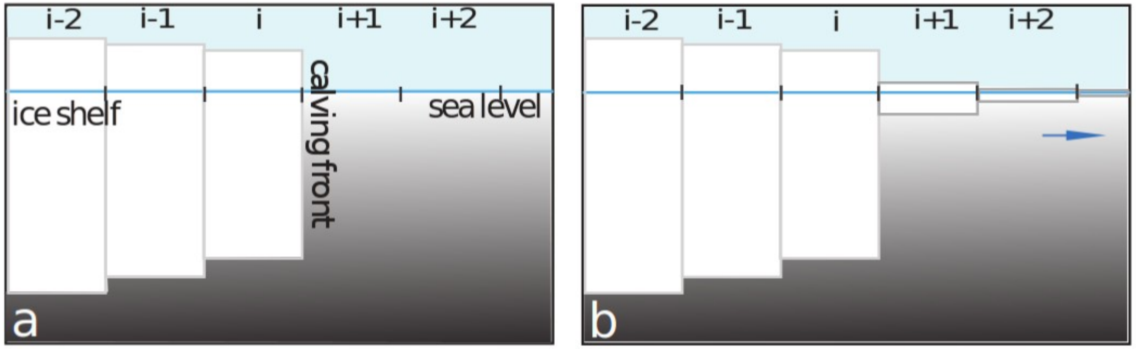
Other nonlienarities and sources of instability

Lateral boundary advance/retreat:

- Grounded boundary. Explicit method might show instability if the time-step is not small enough. Possible solution: reinterpret the thickness evolution equation as an obstacle problem*,**(requires solution of a variational inequality).
- Calving fronts are typically treated explicitly and might in principle be a source of instability



Albrecht et al, The Cryosphere, 2011

Grounding line. Parametrization might mitigate the nonlinearity

*Ed Bueler, submitted, 2015

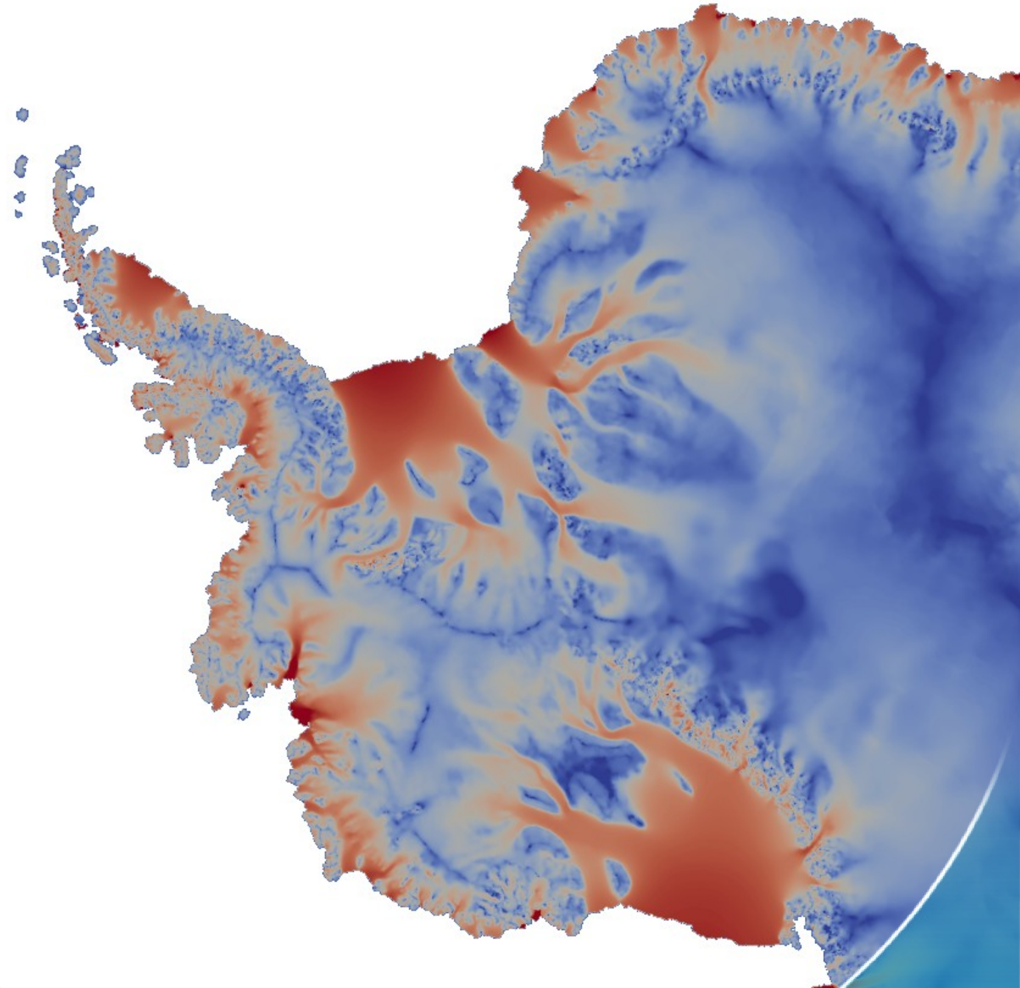
** Jouvet and Bueler, Siam J. Appl. Math, 2012

Results on Antarctica
(preliminary results using semi-implicit scheme)

We compared the solution times for solving Antarctica ice sheet using the **semi-implicit** scheme and with adaptive time step based on the **advective CFL** condition vs the **explicit scheme** based on **diffusive CFL** condition.

The cost per time-step of the semi-implicit scheme larger, because of the increased dimension of the nonlinear system (more expensive assembly and solve).

Using the semi-implicit scheme we had a **speedup of 4.5** times (over an unstructured mesh with **max resolution = 3km**).



Geometry: Bedmap2 (Fretwell et al. The Cryosphere, 2013), massaged by D. Martin and X. Asay-Davis.

Basal friction obtained with Inversion

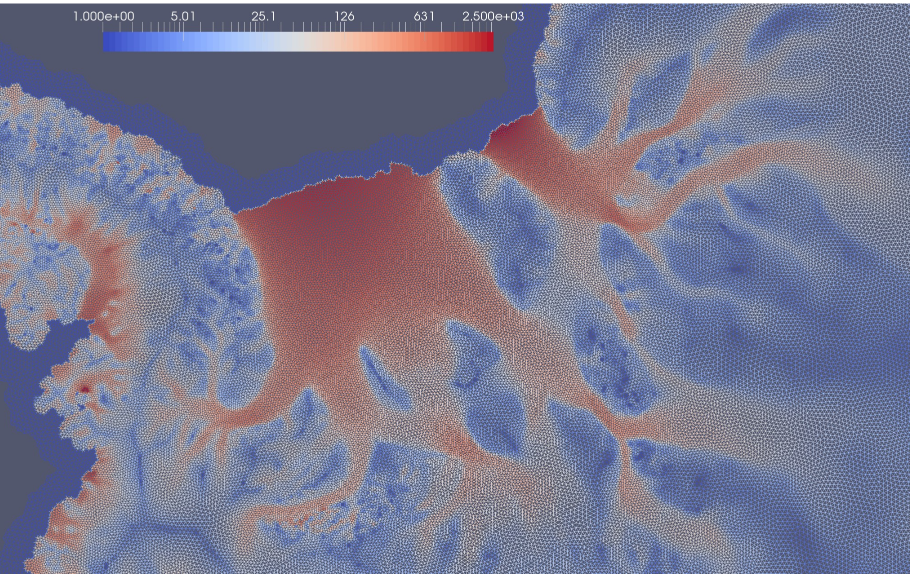
Temperature (Cornford, Martin et al., 2014; Pattyn et al., 2010)

Unstructured Delaunay mesh refined based on gradient of surface velocity (MPAS planar Voronoi grid generator by M. Duda, NCAR)

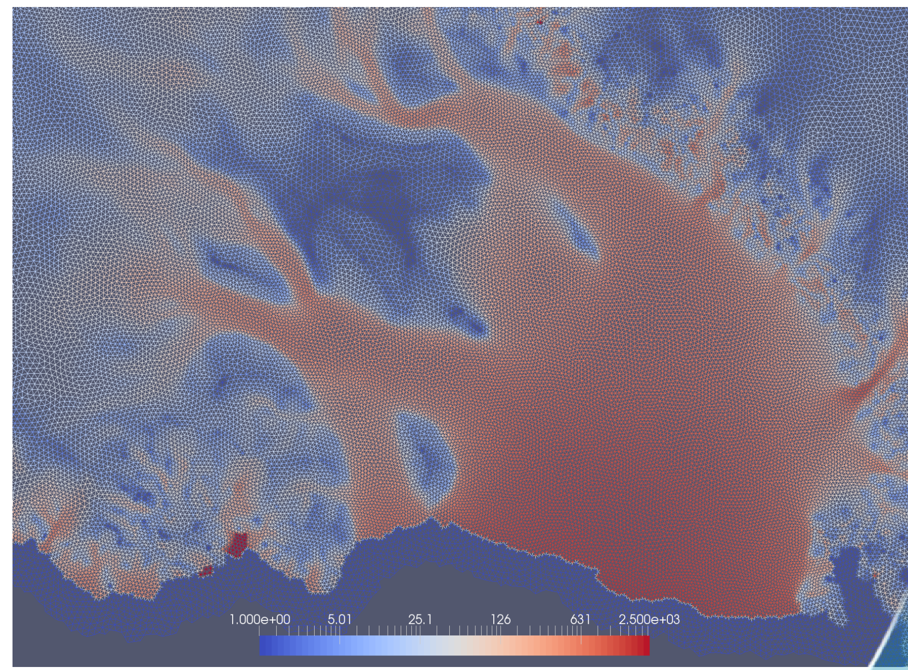
Simulation performed using MPAS land ice module, that relies on FELIX-Albany for the solution of the FO momentum equations.

Results on Antarctica
(surface velocity magnitude m/yr)

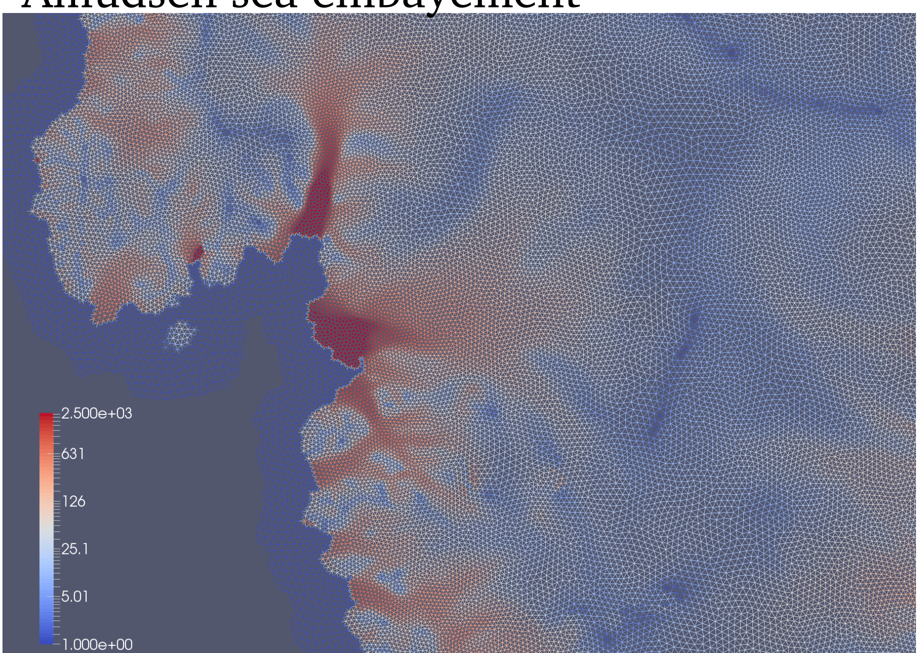
Flicher-Ronne Ice Shelf



Ross sea embayement



Amudsen sea embayement



Future development

- Continue investigating robustness/efficiency/accuracy of the implicit method on synthetic test cases and realistic problems.

- Fully implicit scheme: Solve thickness evolution as an obstacle problem* (variational inequality) in order to avoid negative thickness and solve more accurately the margin of ice sheets.
  Would other nonlinearities limit time-steps' size?

*Ed Bueler, submitted, 2015

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