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CHIRP-Like Signals: Estimation, Detection and Processing A Sequential Model-Based Approach

J. V. Candy

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I INTRODUCTION

Chirp signals have evolved primarily from radar/sonar signal processing applications specifically attempting to estimate the location of a target in surveillance/tracking volume. The chirp, which is essentially a sinusoidal signal whose phase changes instantaneously at each time sample, has an interesting property in that its correlation approximates an impulse function. It is well-known that a matched-filter detector in radar/sonar estimates the target range by cross-correlating a replicant of the transmitted chirp with the measurement data reflected from the target back to the radar/sonar receiver yielding a maximum peak corresponding to the echo time and therefore enabling the desired range estimate.

In this application, we perform the same operation as a radar or sonar system, that is, we transmit a "chirp-like pulse" into the target medium and attempt to first detect its presence and second estimate its location or range. Our problem is complicated by the presence of disturbance signals from surrounding broadcast stations as well as extraneous sources of interference in our frequency bands and of course the ever present random noise from instrumentation.

First, we discuss the chirp signal itself and illustrate its inherent properties and then develop a model-based processing scheme enabling both the detection and estimation of the signal from noisy measurement data.

II CHIRP-Like SIGNALS

II.1 LINEAR CHIRP

A *chirp signal* is a sinusoidal signal characterized by its amplitude α and instantaneous phase ϕ, that is,

$$s(t) = \alpha \times \cos\Big(\phi(t)\Big) \tag{1}$$

where the phase is defined by

$$\phi(t) = 2\pi f(t) + \phi(t_o) \tag{2}$$

for $f(t)$ the corresponding *instantaneous frequency.* The rate of change of the phase is defined by its derivative

$$\frac{d}{dt}\phi(t) = 2\pi f(t) =: \omega_i(t) \tag{3}$$

where ω_i is defined as the instantaneous *angular radian frequency.*

The instantaneous frequency can be characterized by a number of different relations one of which is the "linear" relation

$$f(t) = \beta \cdot t + f_o \tag{4}$$

where f_o is the initial (sweep) frequency at time t_o and β is the rate given by

$$\beta = \frac{f_f - f_o}{t_f} \tag{5}$$

where f_f is the final (sweep) frequency in the time window bounded by the final (sweep) time t_f.

For the linear chirp, the corresponding phase is found by integrating Eq. 3

$$\phi(t) = \phi(t_o) + 2\pi \int_0^t f(\alpha)d\alpha = \phi(t_o) + 2\pi\Big(f_o\ t + \frac{\beta\ t^2}{2}\Big) \tag{6}$$

Thus, the corresponding *sinusoidal chirp* signal is then given by (see Fig. 1)

$$s(t) = \alpha\ \cos\Big(2\pi(f_o \cdot t + \frac{\beta}{2} \cdot t^2)\Big) \tag{7}$$

Unfortunately for our problem, we have disturbances and noise that contaminate the measurement, that is, assuming a sampled-data representation ($t \to t_k$) we have

$$y(t_k) = s(t_k) + d(t_k) + e(t_k) + v(t_k) \tag{8}$$

where s is the transmitted chirp, d is potential broadcast disturbances, e is an extraneous disturbances and v is a zero-mean, Gaussian random noise process with variance $R_{vv}(t_k)$. A typical raw data output and spectrum are shown in Fig. 2 where we see the noisy sequence with the disturbances. Is the chirp present?

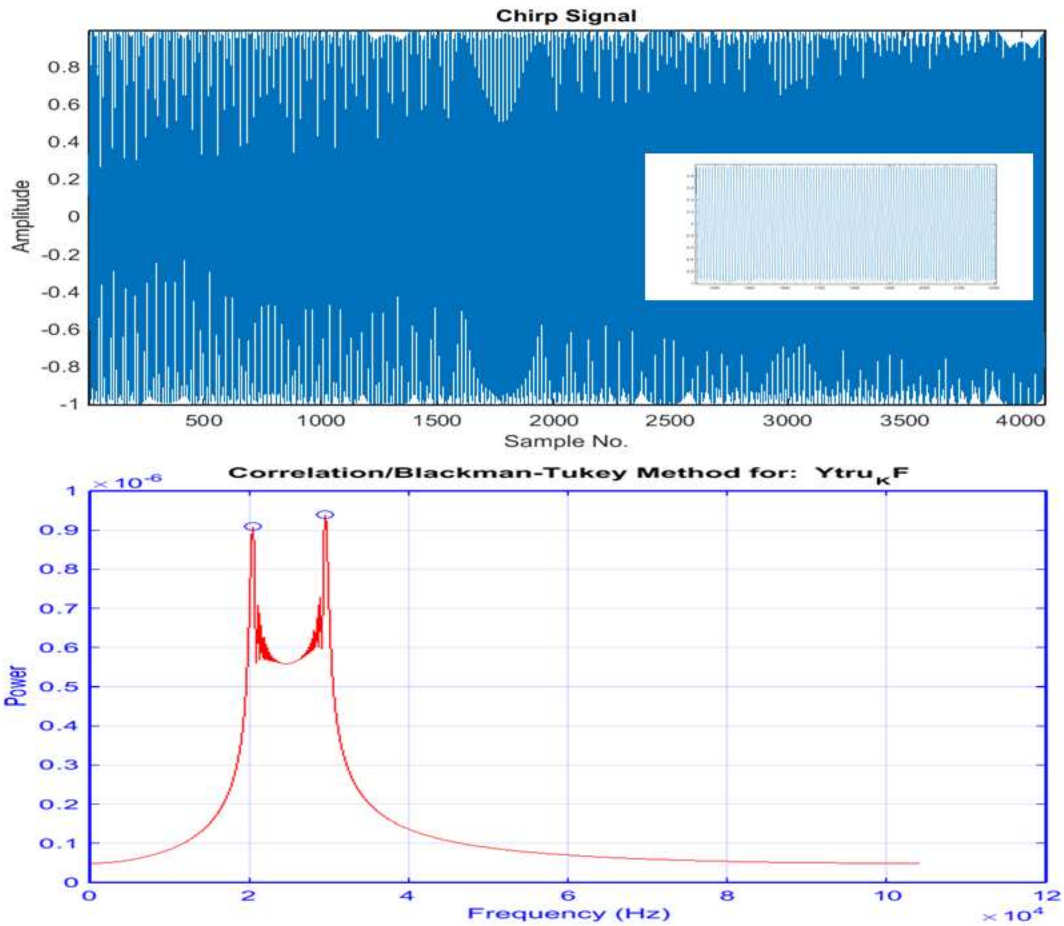

Figure 1: Chirp Signal: (a) Time series. (b) Power spectrum (Welch method).

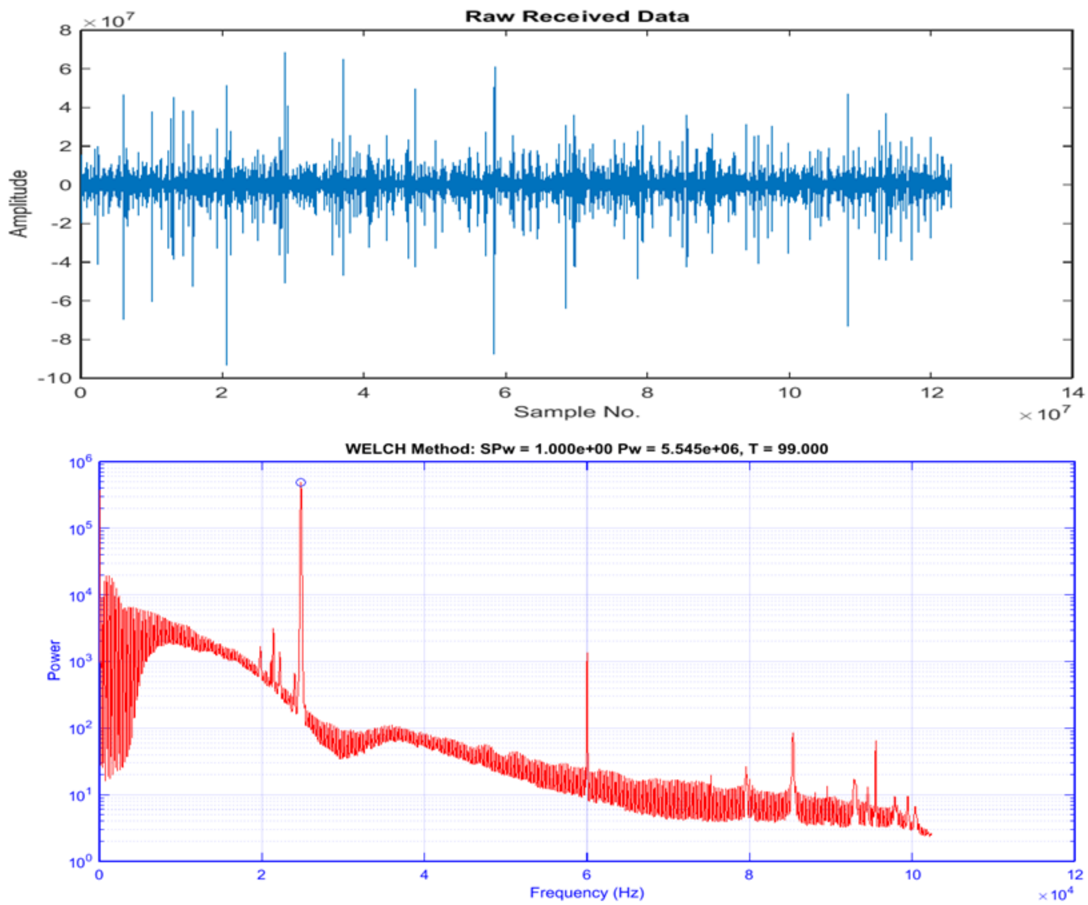

Figure 2: Raw Chirp Data: (a) Time series. (b) Power spectrum (Welch method) with extraneous and broadcast disturbances indicated.

II.2 Frequency-Shift Keying (*FSK*) Signal

An other alternative to the linear frequency modulation (*FM*) chirp is the frequency-shift key (*FSK*) modulation deemed the simplest form of *FM*.[15] In the *M-ary* case, M different frequencies are used to transmit a coded information sequence. The choice of *frequency separation* $\triangle f$ is defined by the *FSK-signal*

$$s_m(t_k) = \sqrt{\frac{2\mathcal{E}_s}{\mathcal{T}}} \cos 2\pi\Big(f_o + m\triangle f\Big)t_k; \quad m = 0, \cdots, M-1 \tag{9}$$

where $\mathcal{E}_s = m \times \mathcal{E}_b$ is defined as the *energy/symbol*, $\mathcal{T} = m \times \mathcal{T}_b$ is the *symbol interval* and $\triangle f = f_m - f_{m-1}$ with $\mathcal{E}_b$ is the signal *energy/bit* and $\mathcal{T}_b$ is the duration of the *bit interval*. Usually, *M-ary FSK* is used to transmit a block of $n = \log_2 M$ *bits/signal*. Here the *M-FSK* signals have *equal* energy $\mathcal{E}_s$.

The frequency separation $\triangle f$ determines the degree of discrimination possible of the M transmitted signals. Each of these signals can be represented as *orthogonal* unit-vectors $\mathbf{u}_m$ scaled by $\sqrt{\mathcal{E}_s}$, that is, $\mathbf{s}_m := \sqrt{\mathcal{E}_s}\mathbf{u}_m$ where the basis functions are defined $\mathbf{b}_m(t_k) = \sqrt{\frac{2}{\mathcal{T}}} \cos 2\pi(f_o + m\triangle f)t_k$. The minimum distance between the pairs of signal vectors is given by $d = \sqrt{\mathcal{E}_s}$. We will incorporate this *FSK* representation to develop our model-based estimators and detectors to follow.[15]

In order to represent the *FSK* signal in the time domain, we must introduce the *gate* function $G_{t_k}(\triangle\tau)$ defined

$$G_{t_k}(\triangle\tau) := \mu(t_k - m\triangle\tau) - \mu(t_k - (m+1)\triangle\tau); \quad m = 0, \cdots, M-1 \tag{10}$$

for $\mu(\cdot)$ a unit-step function and $\triangle\tau = \frac{1}{\triangle f}$ the *reciprocal* of the frequency separation or equivalently $\mathcal{T}_b$ the bit interval duration. With this operator available, we can now represent the temporal *FSK* signal as

$$s(t_k) = \sum_{m=0}^{M-1} s_m(t_k) \times G_{t_k}(\mathcal{T}_b) = \sqrt{\frac{2\mathcal{E}_s}{\mathcal{T}}} \sum_{m=0}^{M-1} \cos 2\pi\Big(f_o + m\triangle f\Big)t_k \Big[\mu(t_k - m\mathcal{T}_b) - \mu(t_k - (m+1)\mathcal{T}_b)\Big] \tag{11}$$

A typical *FSK* signal is shown in Fig. 3 where we see the time series, the instantaneous frequency (step-frequencies) and time-frequency spectrogram. We will use this model subsequently for both signal estimation and detection.

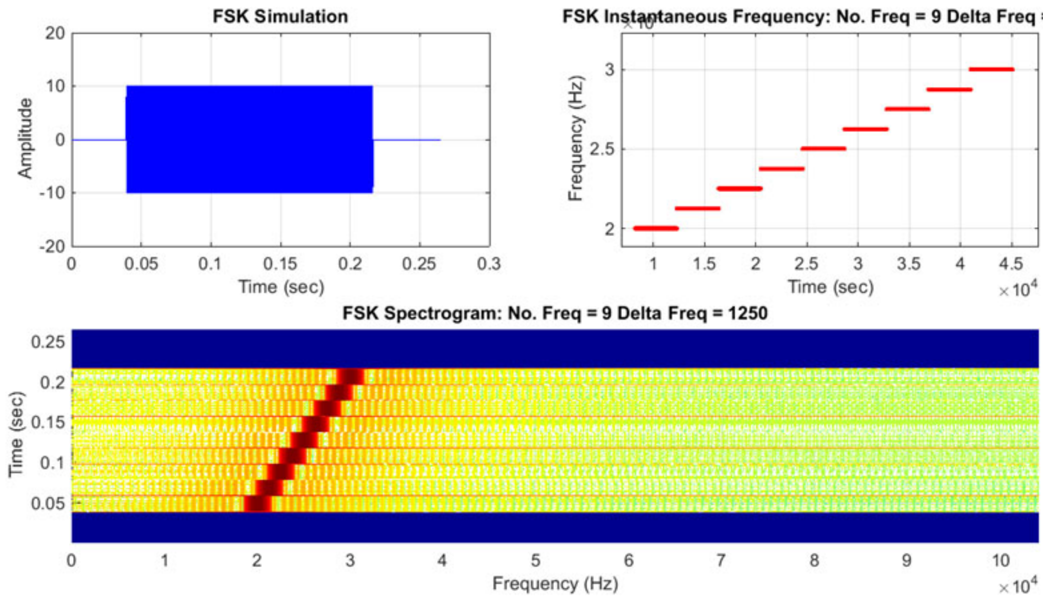

Figure 3: True *FSK* Signal: (a) Time series. (b) Step-Frequencies $(20.00, 21.25, 22.5, 23.75, 25.00, 26.25, 27.5, 28.75, 30.00\ KHz)$. (c) Time-Frequency Spectrogram.

III Model-Based Signal Estimation: Linear Chirp

Thus, the model-based linear *chirp estimation* problem can simply be stated as

GIVEN a set of noisy measurement data $\{y(t_k)\}, k = 0, 1, \cdots, K$ contaminated with broadcast disturbances $\{d(t_k)\}$, extraneous disturbances $\{e(t_k)\}$ and Gaussian noise $\{v(t_k)\}$, FIND the best estimate of the chirp signal $\hat{s}(t_k)$ or equivalently the parameters $\{\hat{\alpha}, \hat{f}_o, \hat{f}_f\}$

The estimated chirp is obtained directly from its parameter estimates as

$$\hat{s}(t_k) = \hat{\alpha} \cos\left(\hat{f}_o \cdot t + 2\pi(\frac{\hat{\beta}}{2} \cdot t^2)\right) \tag{12}$$

for

$$\hat{\beta} = \frac{\hat{f}_f - \hat{f}_o}{t_f} \tag{13}$$

Thus, the solution to estimating the linear chirp in the midst of these disturbances reduces to a problem of parameter estimation.

III.1 Linear Chirp: Gauss-Markov State-Space Model

We choose to apply the unscented Kalman filter (*UKF*) to this nonlinear problem by modeling the parameter variations as a *random walk*, that is, in state-space form we have that

$$\begin{aligned}\mathbf{f}(t_k) &= \mathbf{f}(t_{k-1}) + \mathbf{w}_f(t_{k-1}) \\ \hat{\alpha}(t_k) &= \alpha(t_{k-1}) + w_\alpha(t_{k-1})\end{aligned} \tag{14}$$

where $\mathbf{f} \in \mathcal{R}^{N_f \times 1}$ is the vector of frequency parameters, α is the scalar amplitude and $\mathbf{w} \in \mathcal{R}^{N_f \times 1}$ a zero-mean, multivariate Gaussian vector process with covariance $\mathbf{R}_{ww}(t_k)$. Here $\mathbf{f} = [f_o \ f_f]^T$ and $\mathbf{w}_f = [w_o \ w_f]^T$. The corresponding measurement model is given in Eq. 12 above.

Generally, the *UKF* algorithm provides the update signal (parameter) estimates as:[1]

$$\begin{aligned}\hat{\mathbf{f}}(t_k|t_k) &= \hat{\mathbf{f}}(t_k|t_{k-1}) + \mathbf{K}_f(t_k)\epsilon(t_k|t_{k-1}) \\ \hat{\alpha}(t_k|t_k) &= \hat{\alpha}(t_k|t_{k-1}) + \mathbf{K}_\alpha(t_k)\epsilon(t_k|t_{k-1}) \\ \epsilon(t_k|t_{k-1}) &= y(t_k) - \hat{y}(t_k|t_{k-1}) - d(t_k) - e(t_k) \\ \hat{y}(t_k|t_{k-1}) &= \hat{s}(t_k|t_k) = \hat{\alpha}(t_k|t_{k-1}) \cos\left(2\pi(\hat{f}_o(t_k|t_{k-1}) \cdot t_k + \frac{\hat{\beta}(t_k|t_{k-1})}{2} \cdot t_k^2)\right) \\ \mathbf{K}(t_k) &= \mathbf{R}_{f\epsilon}(t_k|t_{k-1}) \ \mathbf{R}_{\epsilon\epsilon}^{-1}(t_k|t_{k-1})\end{aligned} \tag{15}$$

with the instantaneous frequency (linear chirp) and amplitude $\hat{\alpha}(t_k|t_k)$ given by

$$\begin{aligned}\hat{f}(t_k) &:= \hat{f}_o(t_k|t_k) \cdot t_k + \frac{\hat{\beta}(t_k|t_k)}{2} \cdot t_k^2 \\ \hat{\beta}(t_k|t_k) &= \frac{\hat{f}_f(t_k|t_k) - \hat{f}_o(t_k|t_k)}{t_f}\end{aligned} \tag{16}$$

Since we are transmitting the chirp signal, we have a good estimate of the starting values for α, f_o, f_f, t_f that can be used in the signal estimation/detection problem. An interesting property of the *UKF* processor is that the innovations (residual errors) should be approximately zero-mean and white (uncorrelated) indicating that the model "matches" the data that can be used in "detecting" the presence of the transmitted chirp. That is, the processor is tuned when the innovations are zero-mean/white and not when these statistical conditions are not met. Therefore, the processor can be used as an *anomaly* detector when these conditions are not satisfied. We discuss this property subsequently.

[1] The notation $\hat{\theta}(t_k|t_{k-1})$ means the estimate of θ at time t_k based on all the data up to time t_{k-1}, $Y_{k-1}, \{y(t_k)\}; k = 0, \cdots, K$

Thus, the received data can be considered to consist of extraneous and broadcast disturbances along with random measurement noise. We developed a set of synthesized measurement data by filtering (low pass) a snip-it of raw measurements to extract the extraneous disturbance with its power spectrum shown in Fig. 5. Next we synthesized a unit amplitude ($\alpha = 1$) chirp and broadcast disturbances. That is, we assume a chirp signal was swept over the entire record length sweeping up from $f_o = 20kHz$ to $f_f = 30kHz$ for a period of $t_f = 19.7msec$ at a sampling interval of $\triangle t = 4.8\mu sec$. From the power spectrum of Fig. 1, we see that the extraneous disturbance is in the frequency range of $0 - 19kHz$ (Fig. 5) with the broadcast (sinusoid) disturbance at $2.48kHz$. For our simulation we chose amplitudes of 0.2 and 0.5, respectively. The simulated data and spectrum (without instrumentation noise) are shown in Fig. 14. Note that all of the amplitude data was scaled down from actual measurements. For the simulation we chose a measurement model of

$$y(t_k) = 1.0 \cos\Big(2\pi(2\times 10^4 t_k + 3\times 10^4 t_k^2\Big) + 0.5 \sin\Big(2\pi\ 2.48\times 10^4 t_k\Big) + 0.25 e(t_k) + v(t_k) \quad (17)$$

for $v \sim \mathcal{N}(0, 1\times 10^{-1})$. Finally we show the simulated noisy measurement and spectrum in Fig. 14 with instrumentation noise included obscuring the chirp signal completely.

Next we applied the *UKF* processor to the data estimate the instantaneous frequency and compared it in steady state ($\tilde{f} = E\{\hat{f}(t_k|t_k)\}$) and dynamically at each instant as shown in Fig. 3. Here we see that the model can track the parameters as indicated by its estimate of the instantaneous frequency and the resulting chirp with the corresponding error. The *UKF* processor is "tuned", since the innovations statistics are zero-mean/white using both a statistical zero-mean/whiteness test [] as well as the weighted-sum squared residual (*WSSR*) test indicating a reasonably tuned processor. The *WSSR* processor performs a dynamic whiteness test by sliding a finite length (parameter) window through the data and tests for whiteness as long as the statistic lies *below* the threshold.

IV MODEL-BASED SIGNAL ESTIMATION: *FSK* SIGNAL

In this section we develop the model-based estimator for the *FSK* signal discussed earlier in Sec. II.2. Recall that the *FSK* signal can be parameterized as:

$$s_m(t_k) = \alpha \cos 2\pi\Big(f_o + m\triangle f\Big)t_k; \quad m = 0, \cdots, M-1 \quad (18)$$

where the amplitude is $\alpha = \sqrt{\frac{2\mathcal{E}_s}{\mathcal{T}}}$, the carrier frequency is f_o and the frequency separation is $\triangle f$. From the signal estimation perspective we need only estimate the amplitude and separation, since the carrier is well-known as well as the number of transmitted frequencies M. Therefore the model-based *FSK estimation* problem including the disturbances (as before) can be stated as:

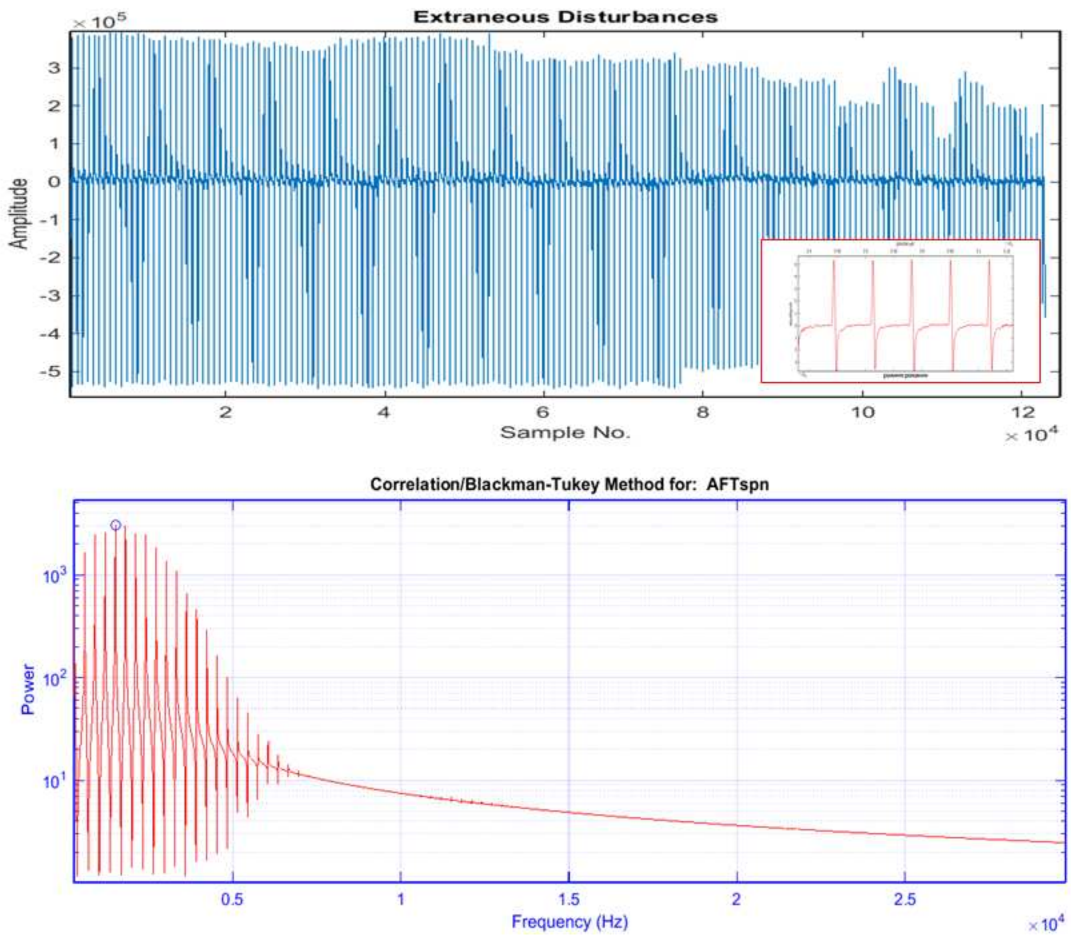

Figure 4: Synthesized Extraneous Disturbance Data: (a) Time series. (b) Power spectrum (Correlation method).

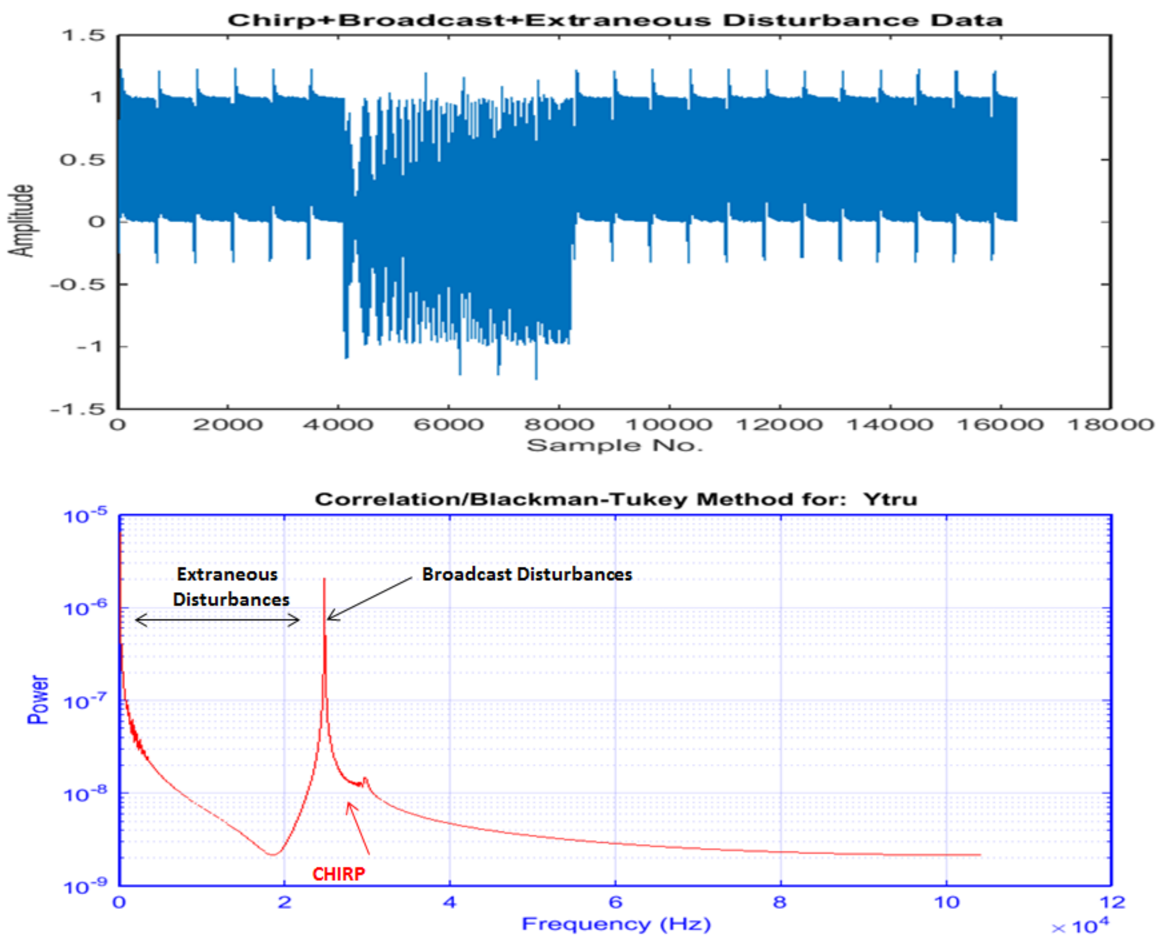

Figure 5: Synthesized Data without random instrumentation noise: (a) Time series. (b) Power spectrum (Correlation method).

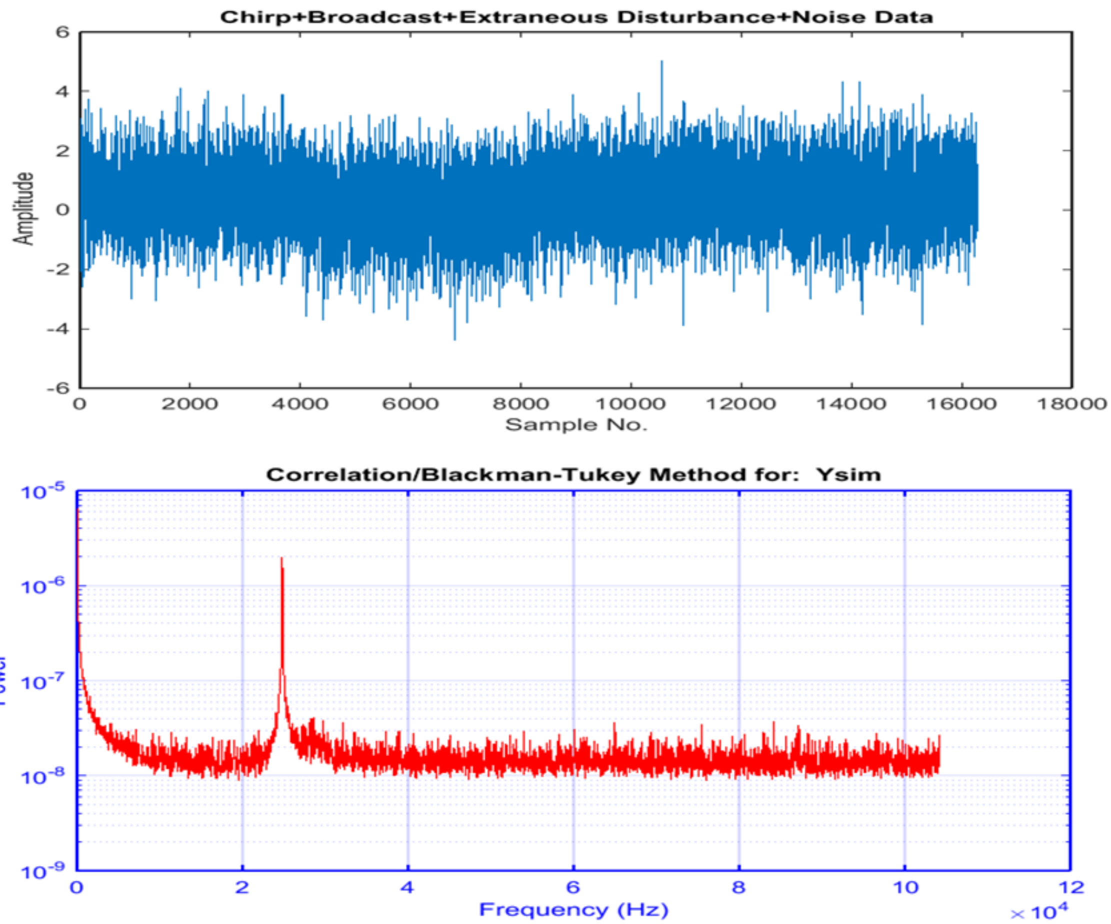

Figure 6: Synthesized Data with random instrumentation noise: (a) Time series. (b) Power spectrum (Correlation method).

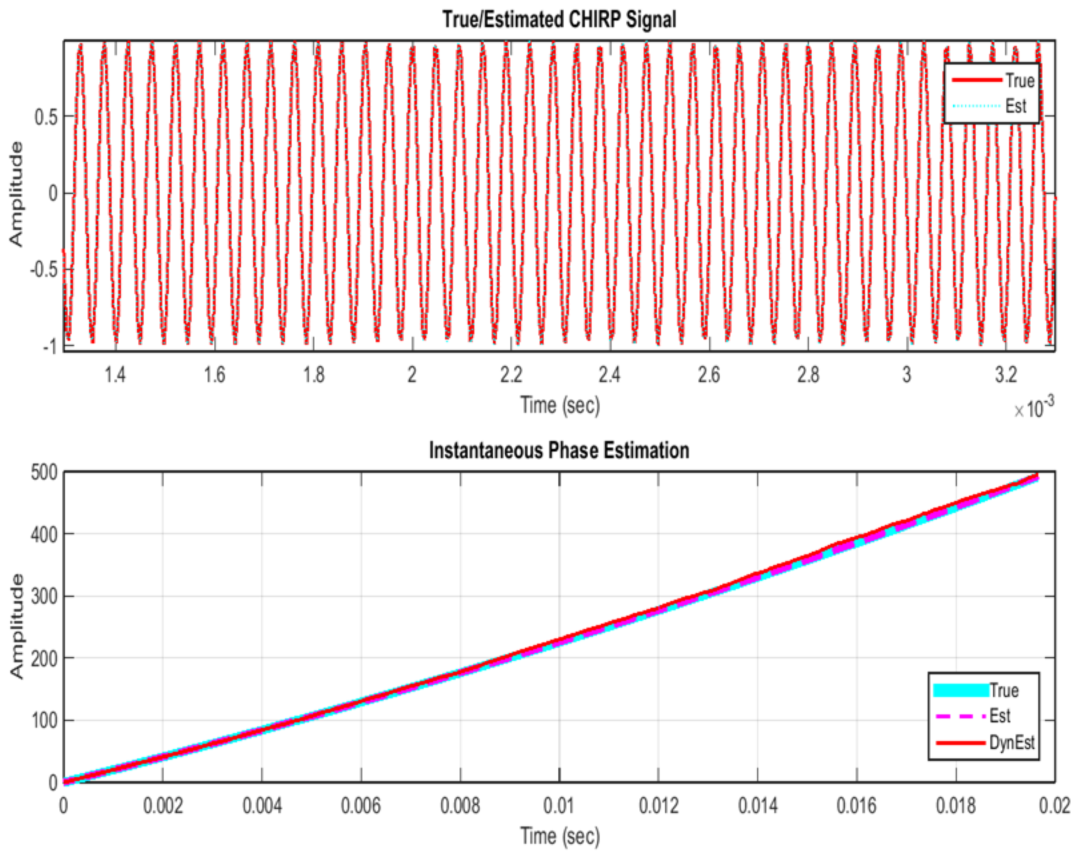

Figure 7: Model-Based CHIRP Estimation: (a) Estimated Chirp. (b) Linear chirp: True, Steady-State, Dynamic).

GIVEN a set of noisy measurement data $\{y(t_k)\}, k = 0, 1, \cdots, K$ contaminated with broadcast disturbances $\{d(t_k)\}$, extraneous disturbances $\{e(t_k)\}$ and Gaussian noise $\{v(t_k)\}$, FIND the best estimate of the *FSK* signal $\hat{s}(t_k)$ or equivalently the parameters $\{\hat{\alpha}, \triangle\hat{f}\}$

For this case the measurement model is

$$y(t_k) = s(t_k) + d(t_k) + e(t_k) + v(t_k) \tag{19}$$

and the estimated *FSK* signal is obtained directly from its parameter estimates as

$$\hat{s}(t_k) = \hat{\alpha} \sum_{m=0}^{M-1} \cos 2\pi \Big(f_o + m\triangle\hat{f}\Big) t_k \times G_{t_k}(\mathcal{T}_b) \tag{20}$$

Thus, the solution to estimating the *FSK* in the midst of these disturbances again reduces to a problem of parameter estimation as before in the linear chirp case.

IV.1 *FSK*: Gauss-Markov State-Space Model

We choose to apply the unscented Kalman filter (*UKF*) to this nonlinear problem by modeling the parameter variations as a *random walk*, that is, in state-space form we have that

$$\begin{aligned} \triangle f(t_k) &= \triangle f(t_{k-1}) + w_f(t_{k-1}) \\ \alpha(t_k) &= \alpha(t_{k-1}) + w_\alpha(t_{k-1}) \end{aligned} \tag{21}$$

where $\triangle f$ is the frequency separation parameter, α is the scalar amplitude and $\mathbf{w} \in \mathcal{R}^{2\times 1}$ a zero-mean, multivariate Gaussian vector process with covariance $\mathbf{R}_{ww}(t_k)$. Here $\mathbf{w} = [w_f \ w_\alpha]^T$. The corresponding measurement model is given in Eq. 19 above.

Generally, the *UKF* algorithm provides the update signal (parameter) estimates as:

$$\begin{aligned} \triangle\hat{f}(t_k|t_k) &= \triangle\hat{f}(t_k|t_{k-1}) + K_f(t_k)\epsilon(t_k|t_{k-1}) \\ \hat{\alpha}(t_k|t_k) &= \hat{\alpha}(t_k|t_{k-1}) + K_\alpha(t_k)\epsilon(t_k|t_{k-1}) \\ \epsilon(t_k|t_{k-1}) &= y(t_k) - \hat{y}(t_k|t_{k-1}) - d(t_k) - e(t_k) \\ \hat{y}(t_k|t_{k-1}) &= \hat{s}(t_k|t_k) = \hat{\alpha}(t_k|t_{k-1}) \sum_{m=0}^{M-1} \cos 2\pi\Big(f_o + m\triangle\hat{f}(t_k|t_{k-1})\Big) t_k \times G_{t_k}(\mathcal{T}_b) \\ \mathbf{K}(t_k) &= \mathbf{R}_{f\epsilon}(t_k|t_{k-1})\ \mathbf{R}_{\epsilon\epsilon}^{-1}(t_k|t_{k-1}) \end{aligned} \tag{22}$$

Since we are transmitting the *FSK* signal, we have a good estimate of the starting values for α, f_o, M that can be used in the signal estimation/detection problem. Recall that the processor is tuned when the innovations are zero-mean/white and not when these statistical conditions are not met. Therefore, the processor can be used as an *anomaly* detector (as before) when these conditions are not satisfied. We discuss this property subsequently.

Thus, the received data can be considered to consist of extraneous and broadcast disturbances along with random measurement noise. We developed a set of synthesized measurement data (as before for the linear chirp) by filtering (low pass) a snip-it of raw measurements to extract the extraneous disturbance with its power spectrum shown in Fig. 5. Next we synthesized a unit amplitude ($\alpha = 1$) *FSK* measurement and broadcast disturbances. That is, we assume a *FSK* pulse signal was swept over the entire record length sweeping up from $f_o = 20KHz$ to $f_f = 30KHz$ for a period of $t_f = 0.177sec$ at a sampling interval of $\triangle t = 4.8\mu sec$. From the power spectrum of Fig. 1, we see that the extraneous disturbance is in the frequency range of $0 - 19KHz$ (Fig. 8) with the broadcast (sinusoid) disturbance at $2.48KHz$.

For the first simulation, we generated a pulsed *FSK* signal contaminated in uncorrelated Gaussian noise at a $-40dB$ *SNR* as shown in Fig. 8. Here we see the application of the model-based processor (*MBP*) to extract the known *FSK* signal from the noisy data in Fig. 8a and observe its corresponding estimated power spectrum in (b). Note that each of the step-frequencies correspond to our true model simulation with small errors. The performance of the *MBP* is illustrated in Fig. 9 where we see the zero-mean/whiteness statistics ($0.09 > 0.011/1.5\%out$) and the *WSSR* test below the threshold indicating a reasonably tuned processor with a slight bias. When the disturbances are added to the simulation the true (noise-free) power spectrum appears in Fig. 10 where we locate the disturbance and signal spectral lines. For our second simulation we added less noise ($-22dB$ *SNR*)so that the disturbances could be observed in the spectral data as shown in Fig. 11. For our signal/disturbance simulation we chose amplitudes of 2.5 and 0.5, respectively. Thus,for the simulation we chose a measurement model of

$$y(t_k) = \sum_{m=0}^{M-1} \cos 2\pi \Big(2\times 10^4 + 0.125\times 10^4 m\Big) t_k \, G_{t_k}(1.97\times 10^{-3}) + 0.5 \sin\Big(2\pi \; 2.48\times 10^4\Big) t_k + 2.5 e(t_k) + v(t_k) \tag{23}$$

for $v \sim \mathcal{N}(0, 8 \times 10^1)$ for a *SNR*=$-22dB$.

The *MBP* result is shown in Fig. 12 indicating a reasonably tuned processor with the predicted measurement and estimated (average) spectrum using the parameterized signal estimates in Fig. 12b. we note the ability of the processor to extract the desired step-frequencies with little error. This completes the section of the signal estimation problem. Next we consider the detection problem in detail.

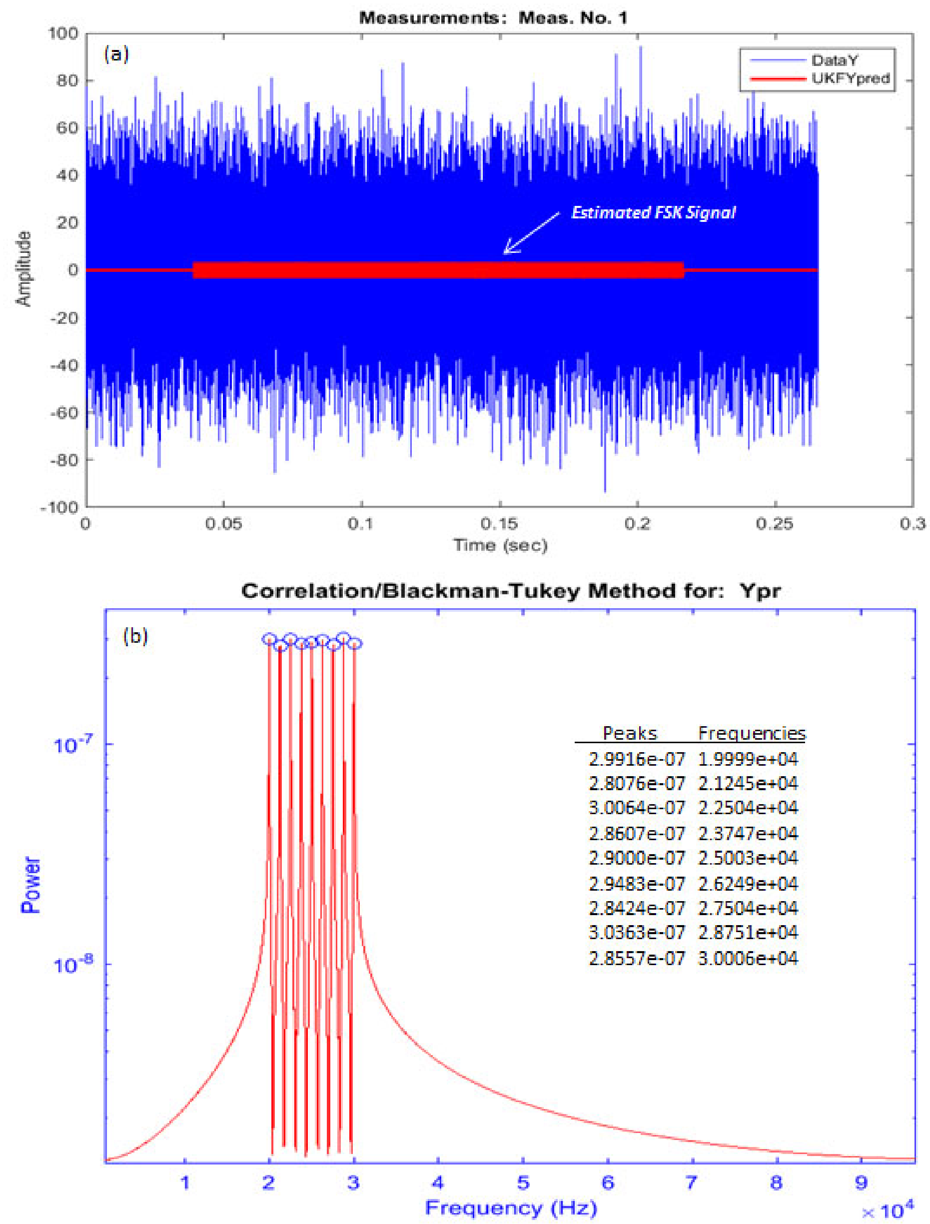

Figure 8: Model-Based *FSK* Signal Estimation: (a) Raw data ($-40\ dB\ SNR$) and estimated *FSK* signal. (b) Estimated power spectrum of estimated *FSK* signal showing frequency lines.

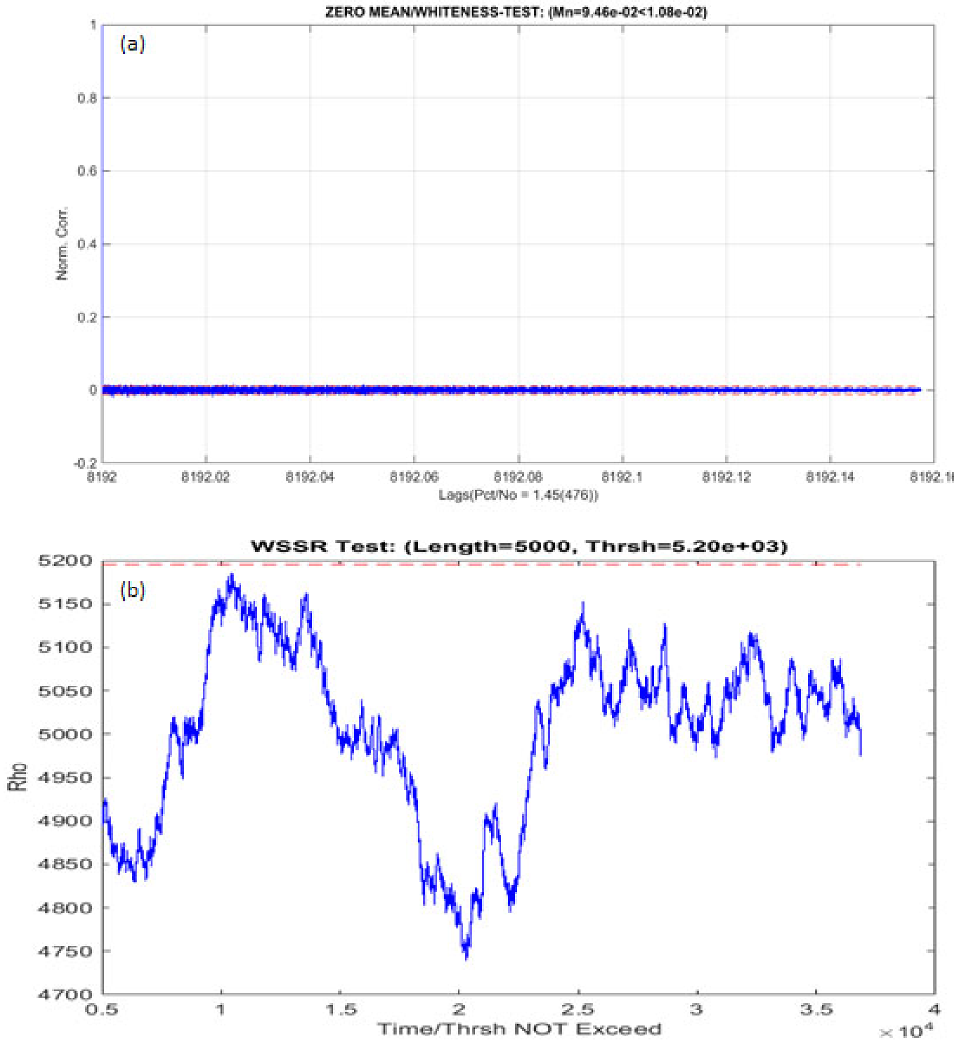

Figure 9: Model-Based Estimation Performance Metrics: (a) Zero-Mean/Whiteness tests $(Z-M: 0.095 > 0.011, WT: 1.5\%)$. (b) *WSSR* test (below threshold).

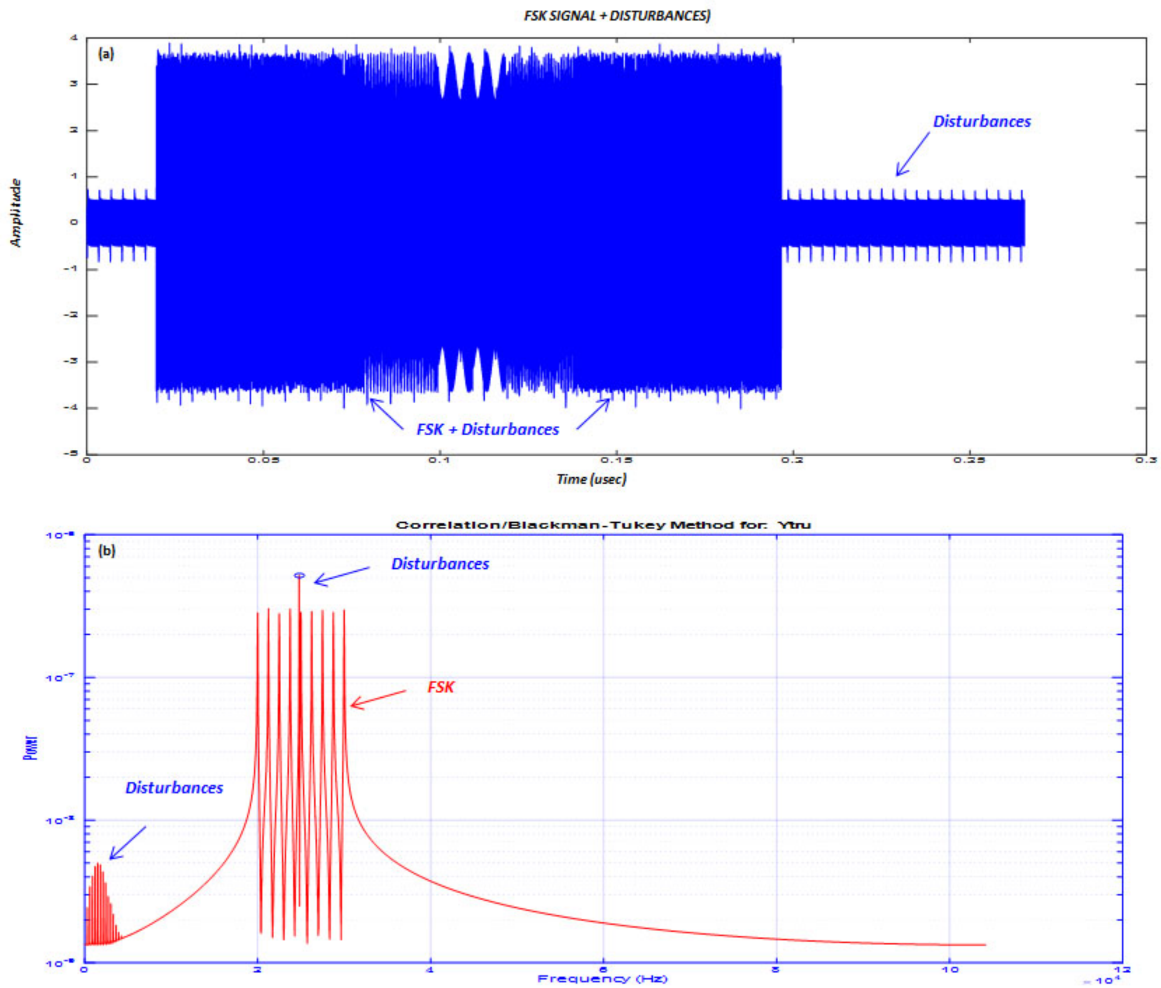

Figure 10: True *FSK* with Disturbances: (a) Time series. (b) Power spectrum.

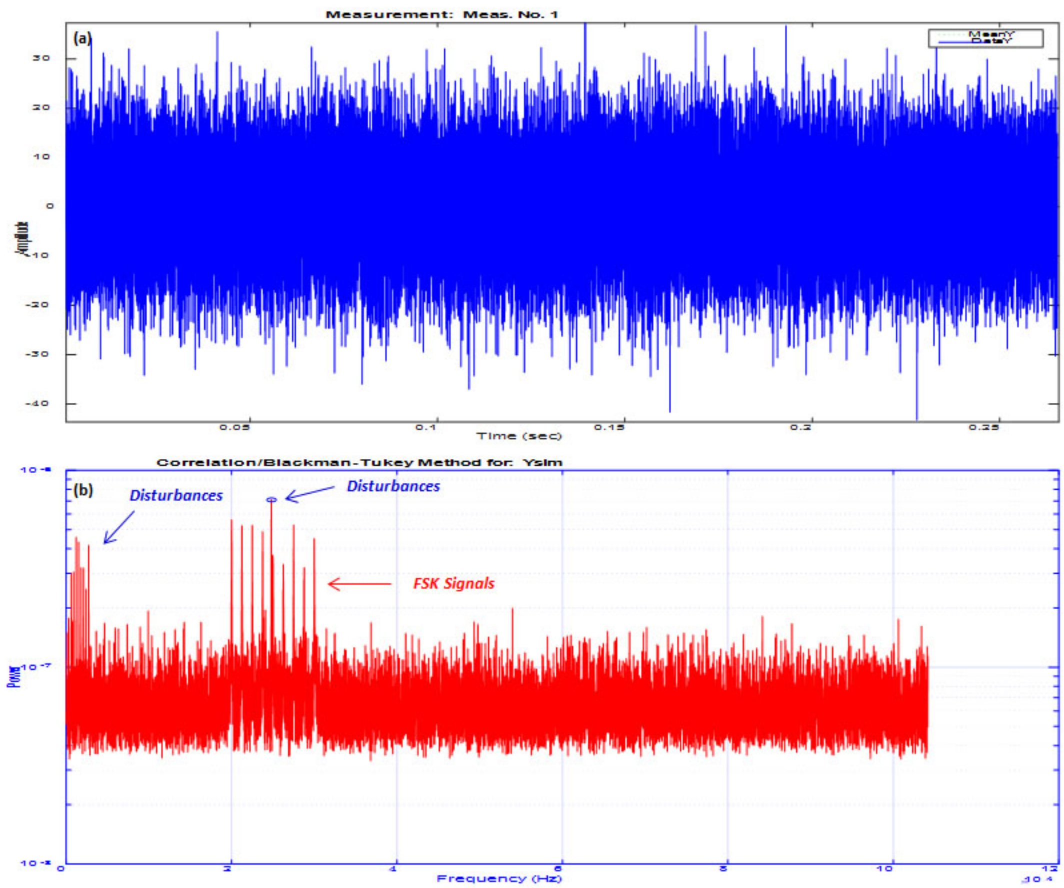

Figure 11: Model-Based *FSK* Signal Estimation with Disturbances: (a) Raw data (-22 dB SNR) and estimated *FSK* signal. (b) Estimated power spectrum of estimated *FSK* signal showing frequency lines.

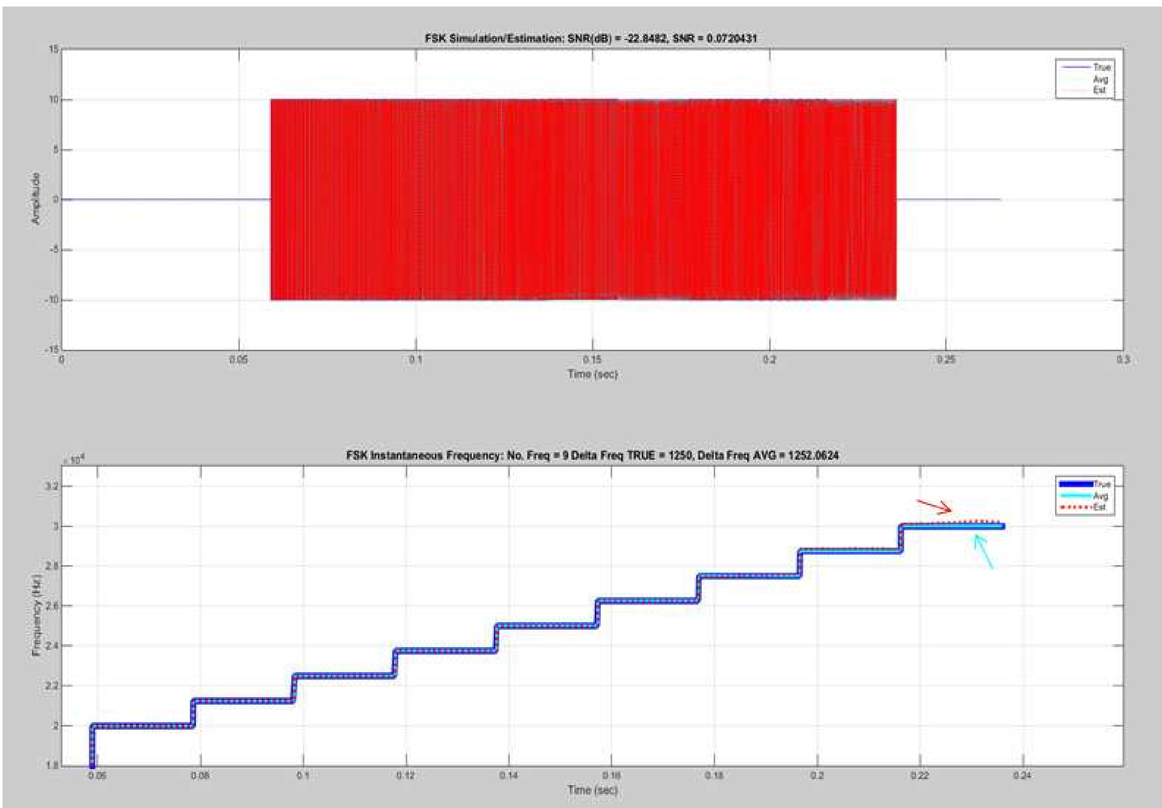

Figure 12: Model-Based *FSK* Signal Estimation with Disturbances: (a) True/Estimated *FSK* data (-22 dB SNR). (b) Estimated *FSK* step-frequency (0.1252 KHz) estimates.

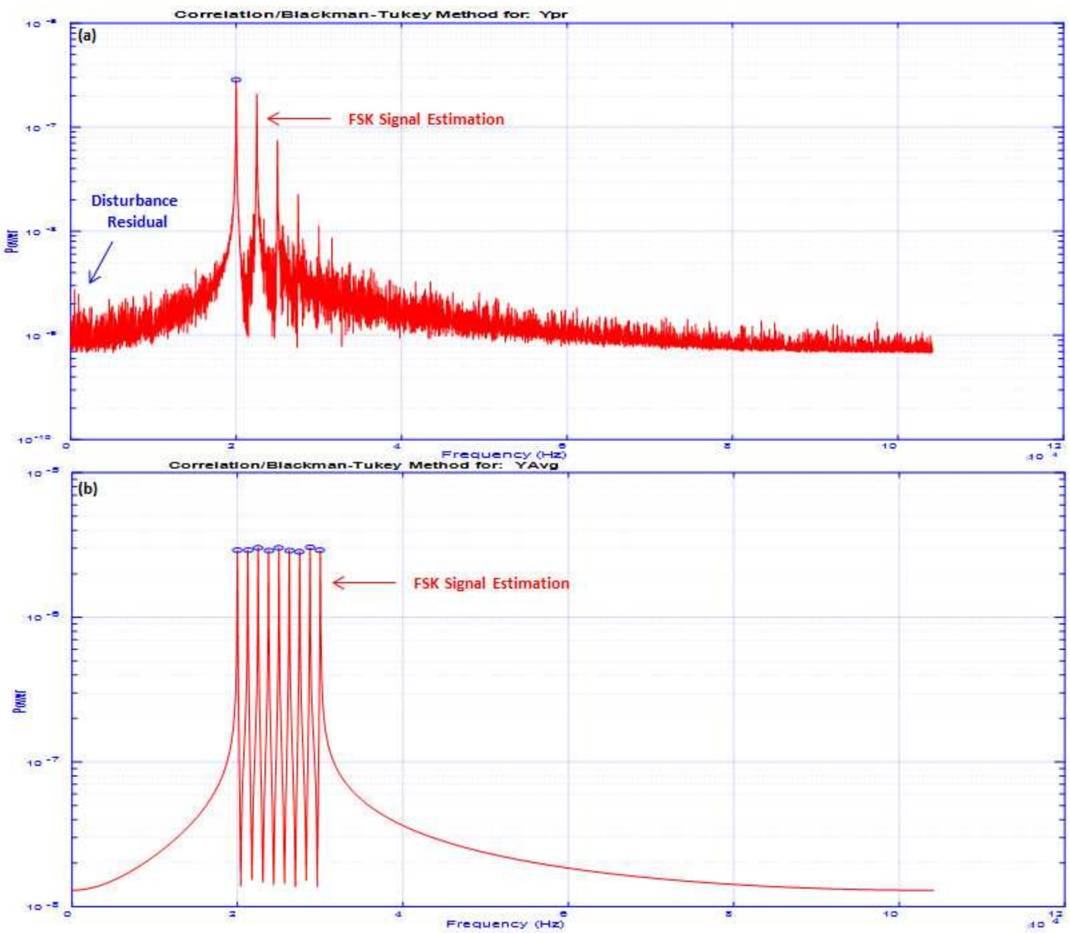

Figure 13: Model-Based *FSK* Signal Estimation with Disturbances: (a) Predicted *FSK* power spectrum. (b) Estimated *FSK* power spectrum showing step-frequencies.

V SEQUENTIAL DETECTION OF CHIRP PROCESSES

In order to develop a sequential processor [], we must test the binary hypothesis that whether the measured data have evolved from the transmitted chirp or not. The basic decision problem is simply stated as:

GIVEN a set of uncertain measurements $\{y(t_k)\}; k = 0, 1, \cdots, K$ from a receiver, DECIDE whether or not the signal is the transmitted chirp . If so, "extract" its characteristic parameters, $\Theta := \{\hat{f}_f, \hat{f}_o, \hat{\alpha}\}$ and "classify" its type.

We are to test the hypothesis that the set of measurements Y_K contain our transmitted chirp. Therefore, we specify the hypothesis test by

$$
\begin{aligned}
\mathcal{H}_0: \; y(t_k) &= d(t_k) + e(t_k) + v(t_k) && \text{[DISTURBANCE/NOISE]} \\
\mathcal{H}_1: \; y(t_k) &= s(t_k; \Theta) + d(t_k) + e(t_k) + v(t_k) && \text{[CHIRP SIGNAL]}
\end{aligned}
\tag{24}
$$

where s is the chirp signal, Θ is the set of known chirp parameters, d is the broadcast disturbances, e is the extraneous disturbances and v is the measurement noise (as before).

The fundamental approach of classical detection theory to solving this binary decision problem is to apply the Neyman-Pearson criterion of maximizing the detection probability for a specified false alarm rate [] with the chirp parameters *known*. The result leads to a *likelihood-ratio* decision function defined by []

$$
\mathcal{L}(Y_K; \Theta) := \frac{\Pr[Y_K|\Theta; \mathcal{H}_1]}{\Pr[Y_K|\Theta; \mathcal{H}_0]} \quad \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \; \mathcal{T} \tag{25}
$$

with threshold $\mathcal{T}$. This expression implies a "batch" decision, that is, we gather the K samples Y_K, calculate the likelihood (Eq. 25) over the entire batch of data and compare it to the threshold $\mathcal{T}$ to make the decision.

V.1 Sequential Processor

An alternative to the batch approach is the sequential method which can be developed by expanding the likelihood ratio for each inter-arrival to obtain

$$
\mathcal{L}(Y_K; \Theta) = \frac{\Pr[Y_K|\Theta; \mathcal{H}_1]}{\Pr[Y_K|\Theta; \mathcal{H}_0]} = \frac{\Pr[y(t_0), y(t_1), \cdots, y(t_K)|\Theta; \mathcal{H}_1]}{\Pr[y(t_0), y(t_1), \cdots, y(t_K)|\Theta; \mathcal{H}_0]} \tag{26}
$$

From the chain rule of probability and Bayes' rule [?] for $\ell = 0, 1$, we have that

$$\Pr[Y_K|\Theta;\mathcal{H}_\ell] \;=\; \Pr[y(t_K), Y_{k-1}|\Theta;\mathcal{H}_\ell] = \Pr[y(t_K)|Y_{k-1},\Theta;\mathcal{H}_\ell] \times \Pr[Y_{k-1}|\Theta;\mathcal{H}_\ell] \quad (27)$$

Substituting these expressions into the likelihood ratio above, replacing $k \to K$ and grouping, we obtain

$$\mathcal{L}(Y_k;\Theta) = \left[\frac{\Pr[Y_{k-1}|\Theta;\mathcal{H}_1]}{\Pr[Y_{k-1}|\Theta;\mathcal{H}_0]}\right] \times \frac{\Pr[y(t_k)|Y_{k-1},\Theta;\mathcal{H}_1]}{\Pr[y(t_k)|Y_{k-1},\Theta;\mathcal{H}_0]} \quad (28)$$

and the recursion or equivalently *sequential likelihood ratio* for the k-th arrival follows as

$$\mathcal{L}(Y_k;\Theta) = \mathcal{L}(Y_{k-1};\Theta) \times \frac{\Pr[y(t_k)|Y_{k-1},\Theta;\mathcal{H}_1]}{\Pr[y(t_k)|Y_{k-1},\Theta;\mathcal{H}_0]}; k = 0,\cdots,K \quad (29)$$

with $\Pr[y(t_0)|Y_{-1},\Theta;\mathcal{H}_\ell] = \Pr[y(t_0)|\Theta;\mathcal{H}_\ell]$, the *prior* under each hypothesis.

Therefore, the Wald *sequential probability-ratio test* is [?], [?]

$$\begin{array}{cccc} & \mathcal{L}(Y_k;\Theta) & > \mathcal{T}_1(k) & \text{Accept} \quad \mathcal{H}_1 \\ \mathcal{T}_0(k) & \leq \mathcal{L}(Y_k;\Theta) & \leq \mathcal{T}_1(k) & \text{Continue} \\ & \mathcal{L}(Y_k;\Theta) & < \mathcal{T}_0(k) & \text{Accept} \quad \mathcal{H}_0 \end{array} \quad (30)$$

where the thresholds are specified in terms of the false alarm (P_{FA}) and miss (P_M) probabilities as

$$\mathcal{T}_0(k) = \frac{\mathrm{P}_M(k)}{1-\mathrm{P}_{FA}(k)} \qquad \mathcal{T}_1(k) = \frac{1-\mathrm{P}_M(k)}{\mathrm{P}_{FA}(k)} \quad (31)$$

These thresholds are determined from a receiver operating characteristic (*ROC*) curve (detection versus false alarm probabilities) obtained by simulation or a controlled experiment to calculate the decision function. That is, an operating point is selected from the *ROC* corresponding to specific detection (or equivalently miss) and false-alarm probabilities specifying the required thresholds which are calculated according to Eq. 31 for each parameter update.

A reasonable approach to this problem of making a reliable decision with high confidence in a timely manner is to develop a sequential detection processor as illustrated in Fig. 14. At each sample arrival (at $y(t_k)$), we *sequentially update* the decision function and compare it to the thresholds to perform the detection—"sample-by-sample". Here as each sample is monitored producing the arrival sequence, the processor takes each arrival measurement and attempts to "decide" whether or not it evolves from a chirp or non-chirp. For each arrival, the decision function is "sequentially" updated and compared to the detection thresholds obtained from the *ROC* curve operating point enabling a rapid decision. Once the threshold is crossed, the decision (chirp or non-chirp) is made and the arrival is processed; however, if not enough data is available to make the decision, then another measurement is obtained.

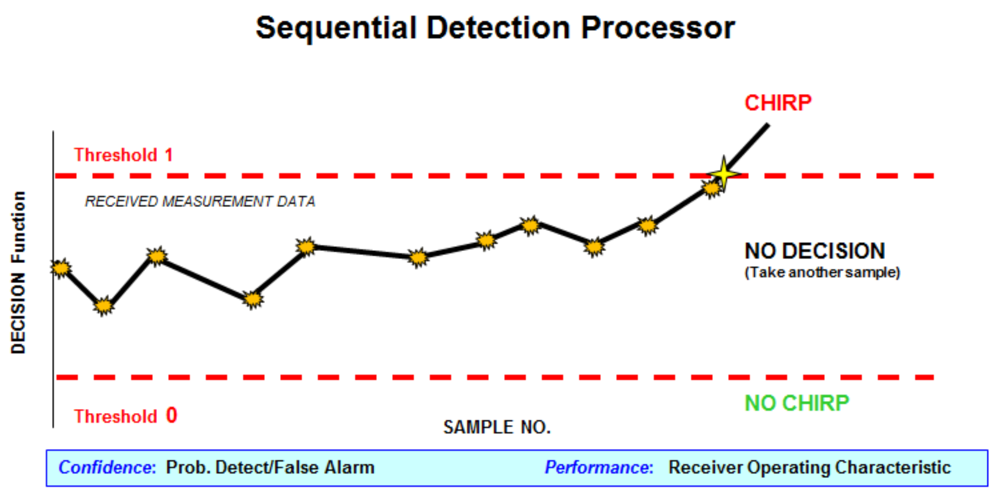

Figure 14: As each individual data sample is digitized, it is discriminated, estimated, the decision function calculated and compared to thresholds to DECIDE if the chirp is detected. Quantitative performance and sequential thresholds are determined from estimated *ROC* curve and the selected operating point (detection/false alarm probability).

V.2 Generalized Likelihood-Ratio Test

For our problem, we typically have information about the background, disturbance and noise parameters, but we rarely have the source information. Therefore, we still can make a decision, but require estimates of the unknown parameters, that is, $\hat{\Theta} \to \Theta$. In this case, we must construct a *composite* or *generalized* likelihood-ratio test (*GLRT*).

Θ can be considered to be deterministic but unknown, since we are transmitted the chirp. Here the approach is to estimate the unknown parameter vector $\hat{\Theta} \to \Theta$ and proceed with the simple testing. A *maximum likelihood estimate* $\hat{\Theta}_{ML}$, can be used to create the *GLRT* such that

$$\mathcal{L}(Y_k;\Theta) = \frac{\max\limits_{\Theta_1} \Pr[Y_k|\Theta_1;\mathcal{H}_1]}{\max\limits_{\Theta_0} \Pr[Y_k|\mathcal{H}_0]} \tag{32}$$

This is the approach we employ *initially*. The batch solution for the *GLRT* can also be extended to the sequential case as before giving the solution by simply replacing $\hat{\Theta}_{ML} \to \Theta$, that is,

$$\mathcal{L}(Y_k;\hat{\Theta}) \quad = \quad \mathcal{L}(Y_{k-1};\hat{\Theta}) \times \frac{\Pr[y(t_k)|Y_{k-1},\hat{\Theta}_1;\mathcal{H}_1]}{\Pr[y(t_k)|Y_{k-1},\hat{\Theta}_0;\mathcal{H}_0]}; k = 0, 1, \cdots, K \tag{33}$$

Anticipating Gaussian models (exponential family [**?**]) for our unknown parameters, we develop the logarithmic form of the sequential likelihood decision function. Simply taking the natural logarithm of Eq. 33, that is, $\Lambda(Y_K;\Theta) := \ln \mathcal{L}(Y_K;\Theta)$ we obtain the *log-likelihood* sequential decision function as

$$\Lambda(Y_k;\hat{\Theta}) = \Lambda(Y_{k-1};\hat{\Theta}) + \ln \Pr[y(t_k)|Y_{k-1},\hat{\Theta}_1;\mathcal{H}_1] - \ln \Pr[y(t_k)|Y_{k-1},\hat{\Theta}_0;\mathcal{H}_0] \tag{34}$$

Using these formulations, we develop the detection algorithm for our problem next. We should note that we only consider the "chirp detection problem" in this paper.

V.3 Sequential Chirp Detection

Here we start with the results of the previous section and incorporate the chirp and disturbance processes. For chirp detection, we start with the simple model of Eq. **??** at time $y(t_k)$ leading to the subsequent (sequential) hypothesis test:

$$\begin{aligned}
\mathcal{H}_0:\ y(t_k) &= d(t_k) + e(t_k) + v(t_k) && \text{[DISTURBANCE/NOISE]} \\
\mathcal{H}_1:\ y(t_k;\Theta) &= s(t_k;\Theta) + d(t_k) + e(t_k) + v(t_k) && \text{[CHIRP/DISTURBANCE]}
\end{aligned} \tag{35}$$

The sequential detection solution (as before) for this problem with unknown chirp parameters follows directly from the *GLRT* results of Eq. 33 to yield

$$\mathcal{L}(Y_k;\hat{\Theta}) = \mathcal{L}(Y_{k-1};\hat{\Theta}) \times \frac{\Pr[y(t_k)|Y_{k-1},\hat{\Theta};\mathcal{H}_1]}{\Pr[y(t_k)|Y_{k-1};\mathcal{H}_0]} \tag{36}$$

To implement the processor, we must first determine the required conditional probabilities in order to specify the decision function, that is,

$$\Pr[y(t_k)|Y_{k-1},\hat{\Theta};\mathcal{H}_1] = \frac{1}{\sqrt{2\pi R_{vv}}} \exp\Big\{-\frac{1}{2R_{vv}}\Big(y(t_k) - \hat{y}(t_k|t_{k-1};\hat{\Theta})\Big)^2\Big\} \tag{37}$$

and under the null hypothesis

$$\Pr[y(t_k)|Y_{k-1};\mathcal{H}_0] = \frac{1}{\sqrt{2\pi R_{vv}}} \exp\Big\{-\frac{1}{2R_{vv}} y^2(t_k)\Big\} \tag{38}$$

where the Gaussian noise is distributed as $v \sim \mathcal{N}(0, R_{vv})$.

Therefore, the likelihood ratio becomes (simply)

$$\mathcal{L}(Y_k;\Theta) = \mathcal{L}(Y_{k-1};\Theta) \times \exp\Big\{ -\frac{1}{2R_{vv}}\Big[\Big(y(t_k) - \hat{y}(t_k|t_{k-1};\hat{\Theta})\Big)^2 + y^2(t_k)\Big]\Big\} \tag{39}$$

which completes the problem with chirp parameters Θ, unknown constants to be estimated independently.

The equivalent *log-likelihood ratio* is given by

$$\Lambda(Y_k;\Theta) = \Lambda(Y_{k-1};\Theta) - \frac{1}{2R_{vv}}\Big[\Big(y(t_k) - \hat{y}(t_k|t_{k-1};\hat{\Theta})\Big)^2 + y^2(t_k)\Big] \tag{40}$$

or simplifying

$$\Lambda(Y_k;\Theta) = \Lambda(Y_{k-1};\Theta) + \frac{1}{2R_{vv}}\Big[\Big(2y(t_k) - \hat{y}(t_k|t_{k-1};\hat{\Theta})\Big))\Big] \times \hat{y}(t_k|t_{k-1};\hat{\Theta})\Big) \tag{41}$$

We synthesized an ensemble of contaminated chirp data and ran the sequential log-likelihood ratio detector. The decision function is shown in Fig. 15 where we see each of the realizations as well as the "jump" create by the windowed chirp signal. A check of its onset and termination times coincide with our transmit and sweep times. Note that the detection threshold must be calculated and that implies a *ROC* curve simulation is required to establish the thresholds. This completes the development of the sequential Bayesian detection approach for chirp processes.

VI Summary

We have developed a model-based approach to processing, estimating and detecting a windowed chirp in noisy simulated data.

VII Acknowledgments

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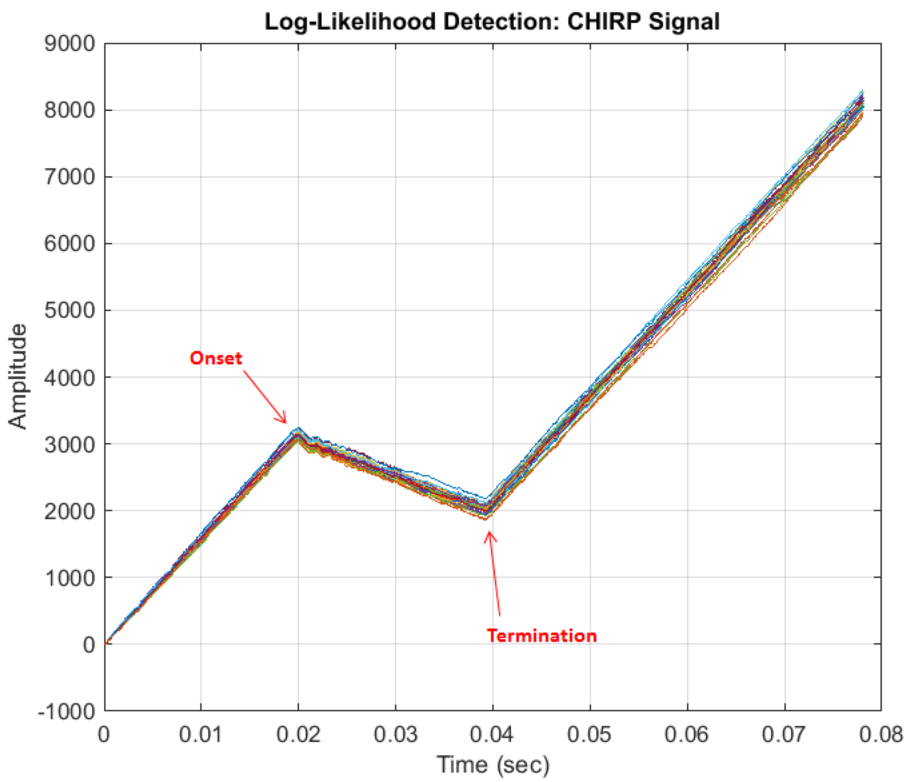

Figure 15: Model-Based CHIRP Detection over an Ensemble of 30 Log-Likelihood Realizations: Note the onset(initial sweep time) and termination (sweep final time) jumps.

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APPENDIX

A Unscented Kalman Filter Processor

In this section we briefly discuss the processors for our problem with details available in Refr. [13]. The *UKF* is an alternative to the nonlinear or extended Kalman filter processor applied successfully in many of the model-based ocean acoustic applications.[3–7] Like the *EKF* it is still restricted to a unimodal distribution (single peak), but that distribution need not be Gaussian. It also performs a linearization (statistical), but not of the system dynamical model, but of an inherent nonlinear vector transformation requiring "sigma points" which deterministically characterize the underlying unimodal distribution. These points have been pre-calculated for the Gaussian case.[13] If we place the *EKF*/*UKF* into the Bayesian framework, then we see that the underlying posterior distribution has already been decided to be approximately multivariate Gaussian with the objective to extract the corresponding conditional mean and covariance as accurately as possible. Therefore, we see that the *UKF* provides the multivariate posterior solution

$$\hat{\Pr}\Big[\mathbf{\Phi}(z_\ell;\mathbf{\Theta})|P_\ell\Big] = (2\pi)^{-N_\Phi/2}|R_{\Phi\Phi}(z_\ell)|^{-N_\Phi/2} \times \\ \exp\Big\{-\frac{1}{2}(\mathbf{\Phi}(z_\ell;\mathbf{\Theta}) - \hat{\mathbf{\Phi}}(z_\ell;\mathbf{\Theta}))^T R_{\Phi\Phi}^{-1}(z_\ell)(\mathbf{\Phi}(z_\ell;\mathbf{\Theta}) - \hat{\mathbf{\Phi}}(z_\ell;\mathbf{\Theta}))\Big\} \tag{42}$$

where $\hat{\mathbf{\Phi}}(z_\ell;\mathbf{\Theta})$ is the augmented conditional (modal/parameter) mean at depth z_ℓ and $R_{\Phi\Phi}(z_\ell)$ is the conditional modal covariance based on pressure-field measurements up to depth z_ℓ.

A detailed flow diagram of the *UKF* is shown in Fig. 16 where we note the basic predictor/update structure. Much of the algorithm is devoted to the statistical linearization in which regression estimators are used to perform the transformation while the usual Kalman filtering equations are used to perform the updates.

We refer the interested reader to the current texts or basic papers for more details.[21–25] Thus, we see that there exists a fundamental philosophical difference between the *UKF* (Kalman) processor. Their implementations are completely different as well: one based on approximating the required distribution through statistical linearization and one through an empirical *PMF* estimator. This completes the Appendix.

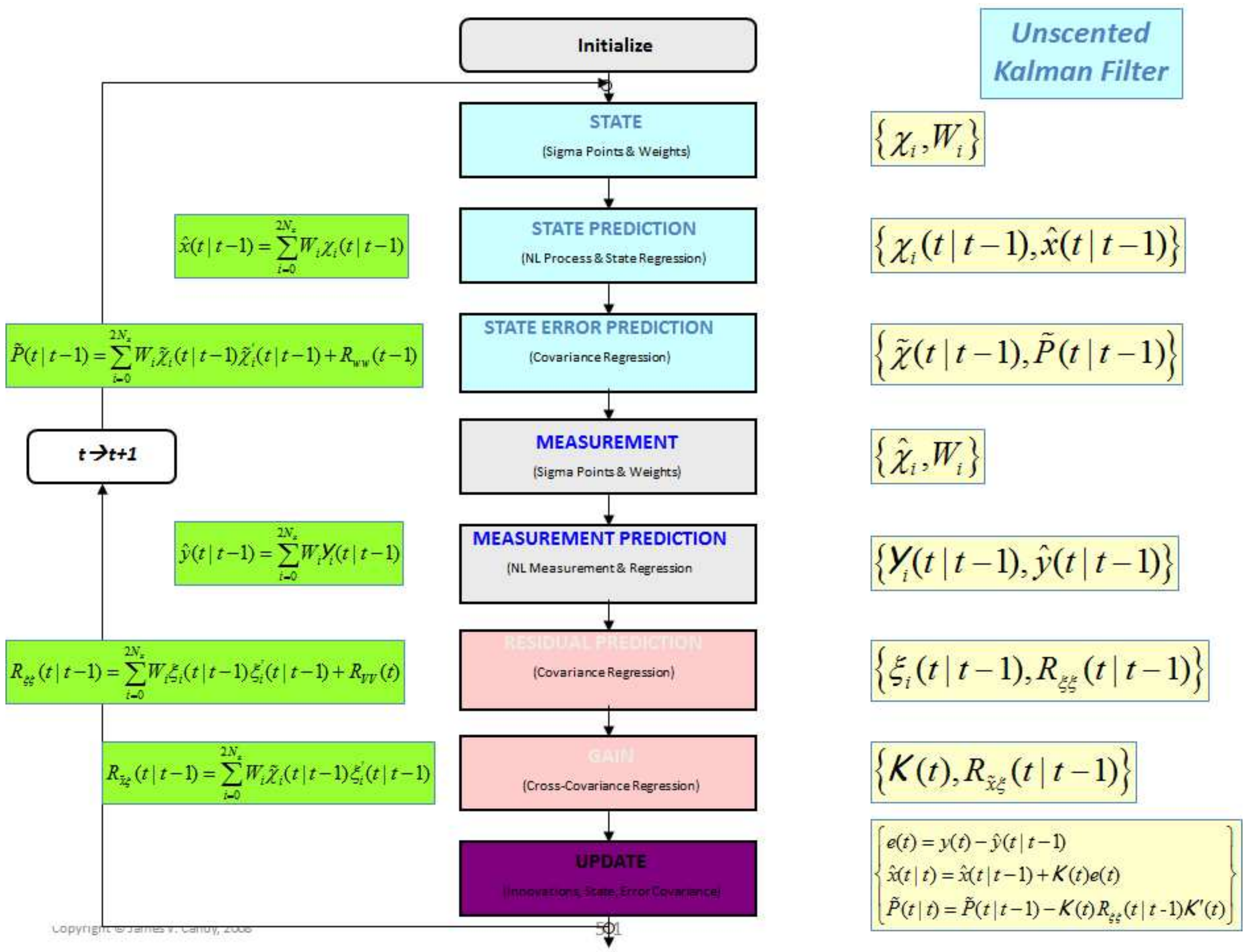

Figure 16: Unscented Kalman filter algorithm flow diagram: initialization, prediction, update and innovation with $z_\ell \rightarrow t$ as the index variable.