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Multiscale Mortar Methods for Flow and Mechanics in Porous Media

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Outline

- Introduction
  - What are mortar methods?
  - Why do we care about flow and mechanics in porous media?
- Multiscale mortar mixed finite element methods
- Multiscale preconditioners
  - Nonintrusive stochastic methods
  - IMPES formulations of multiphase flow
  - Fully implicit formulation of two phase flow
- Mortar methods for coupled flow and mechanics
- Other applications using mortars
  - A posteriori error estimate for multiscale/multinumerics
  - Multilevel solvers based on mortars
- Conclusions

What Are Mortar Methods?

Mortar: a workable paste used to bind construction blocks together and fill the gaps between them.

-Wikipedia

Mortar methods: discretization methods for partial differential equations, which use interface variables to connect finite element discretizations on nonoverlapping subdomains.

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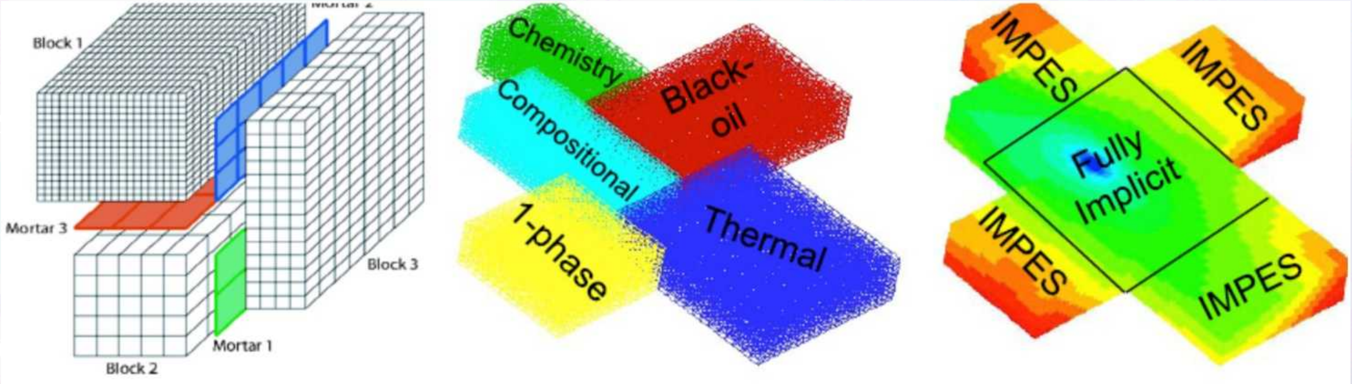
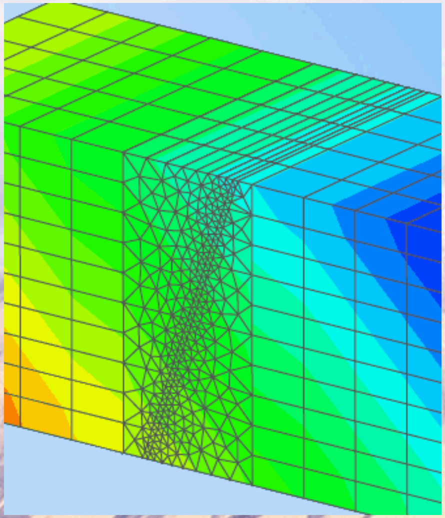
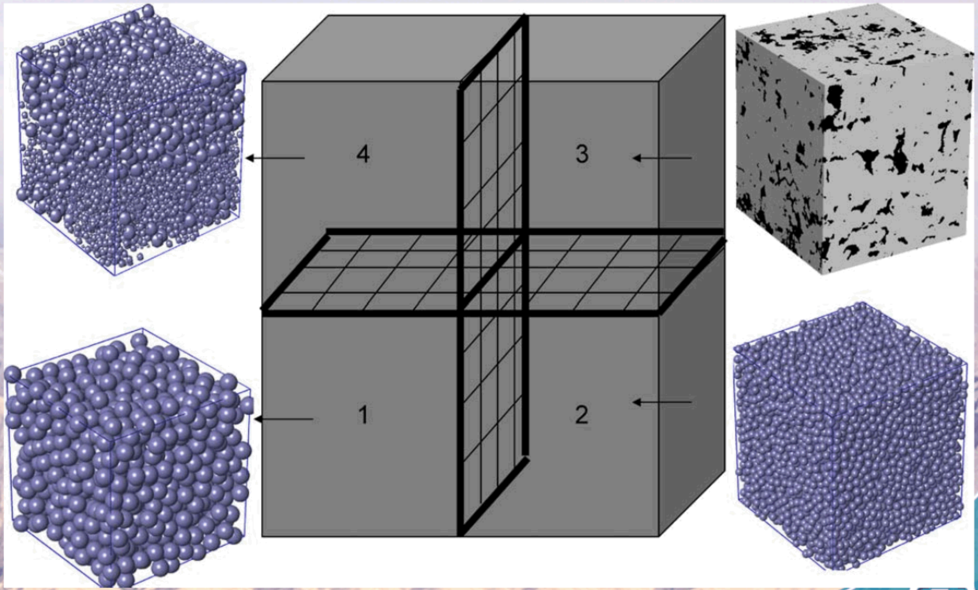
Advantages in Using Mortars

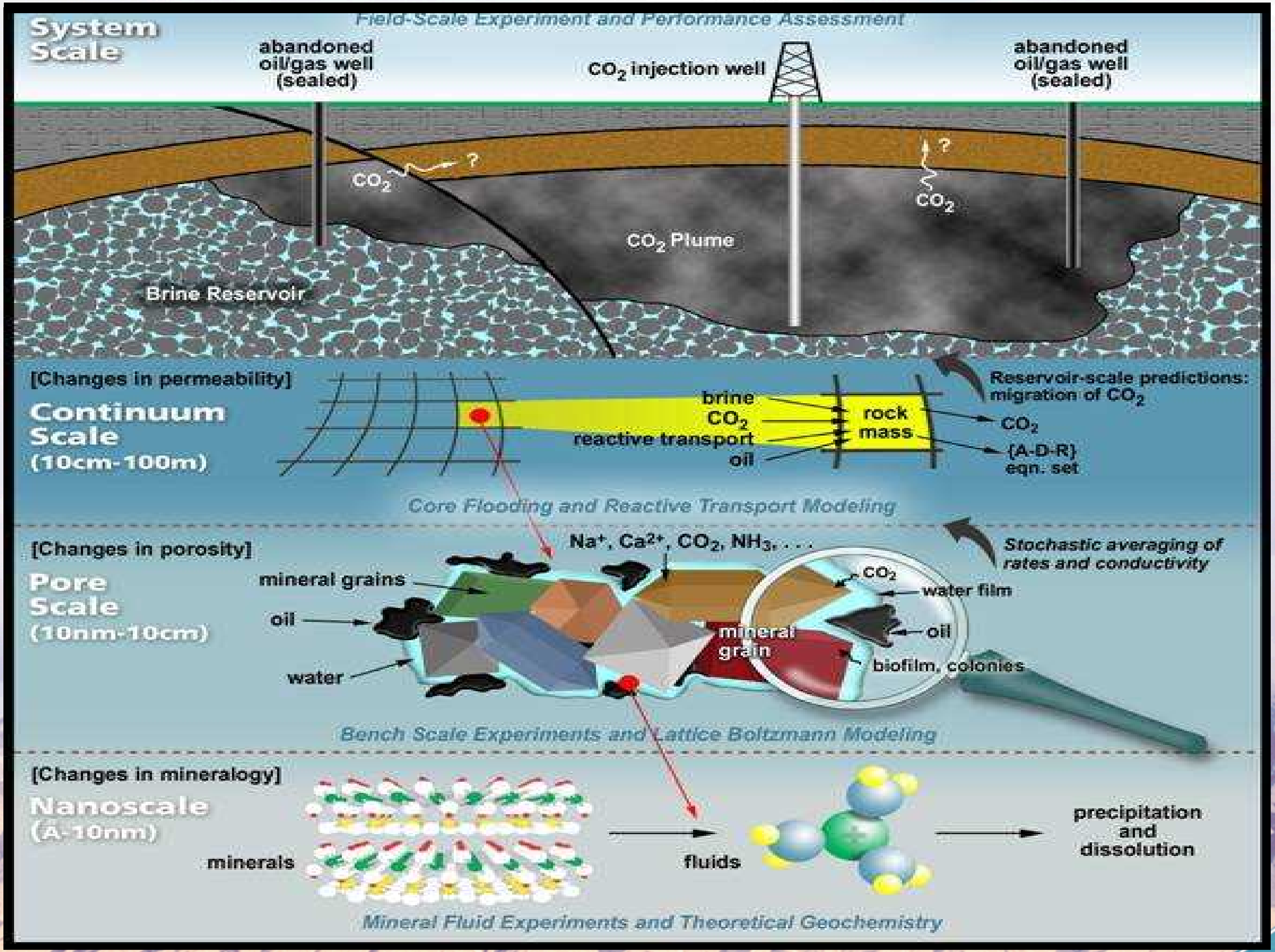






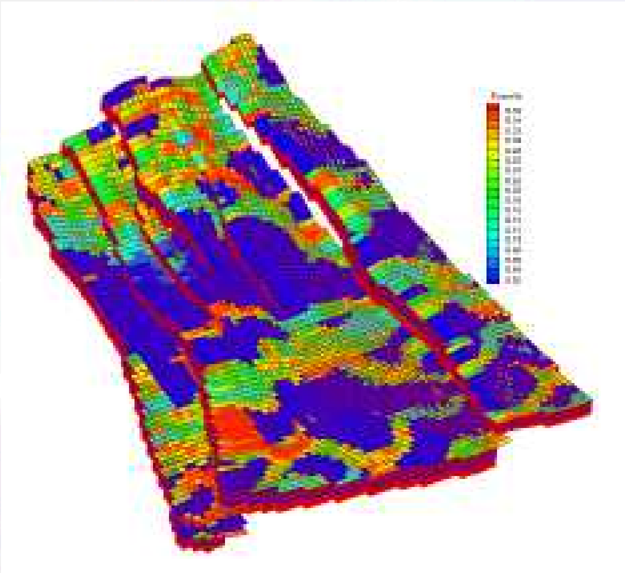
Courtesy of M. Balhoff (UT)

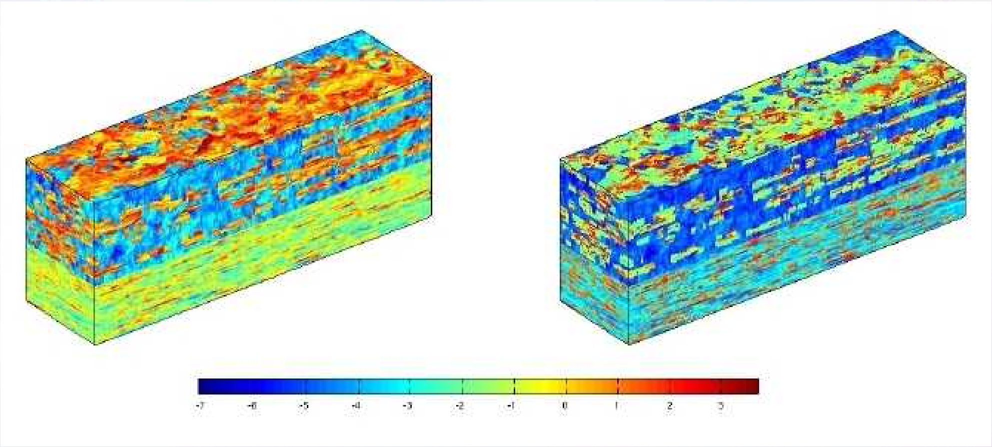
Scales in the Subsurface

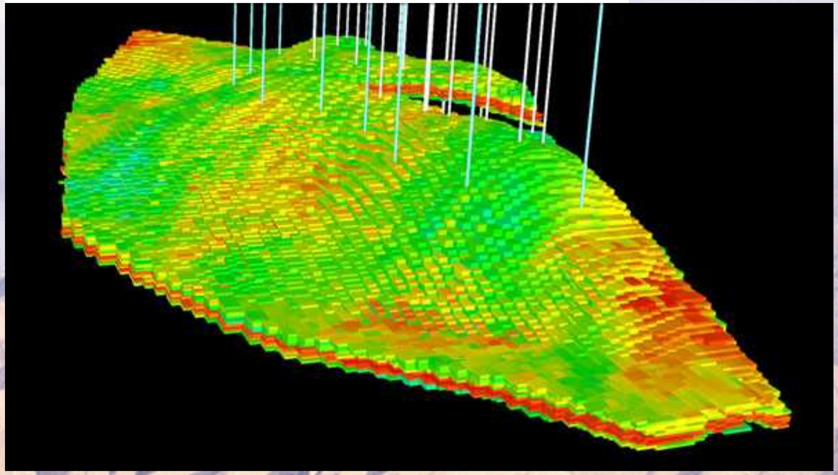


Courtesy of D. Zhang (LANL)

Typical Reservoir Models







Typical Dimensions

- 1000ft x 1000ft x 100ft
- Approx. 1 million elements
- Degrees of freedom 1-40 million

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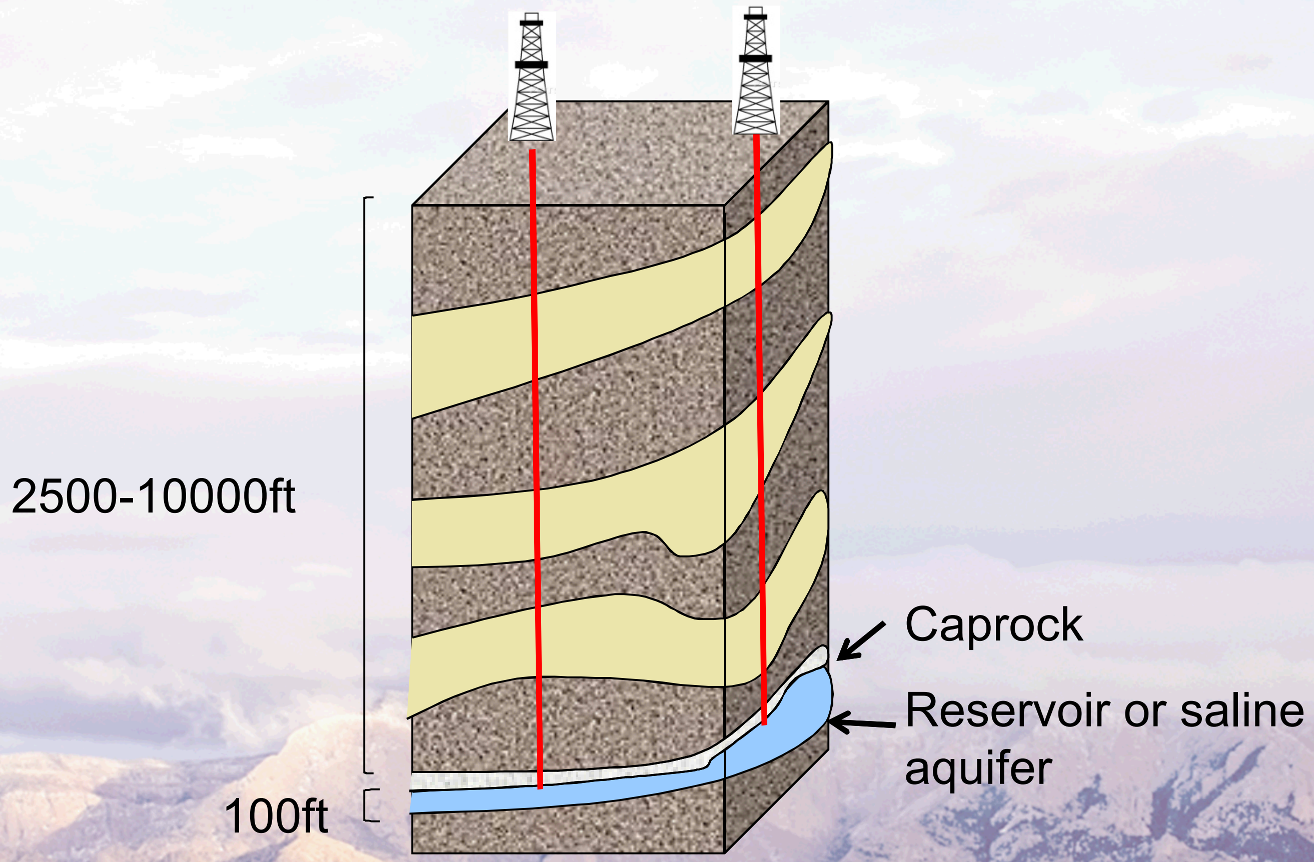
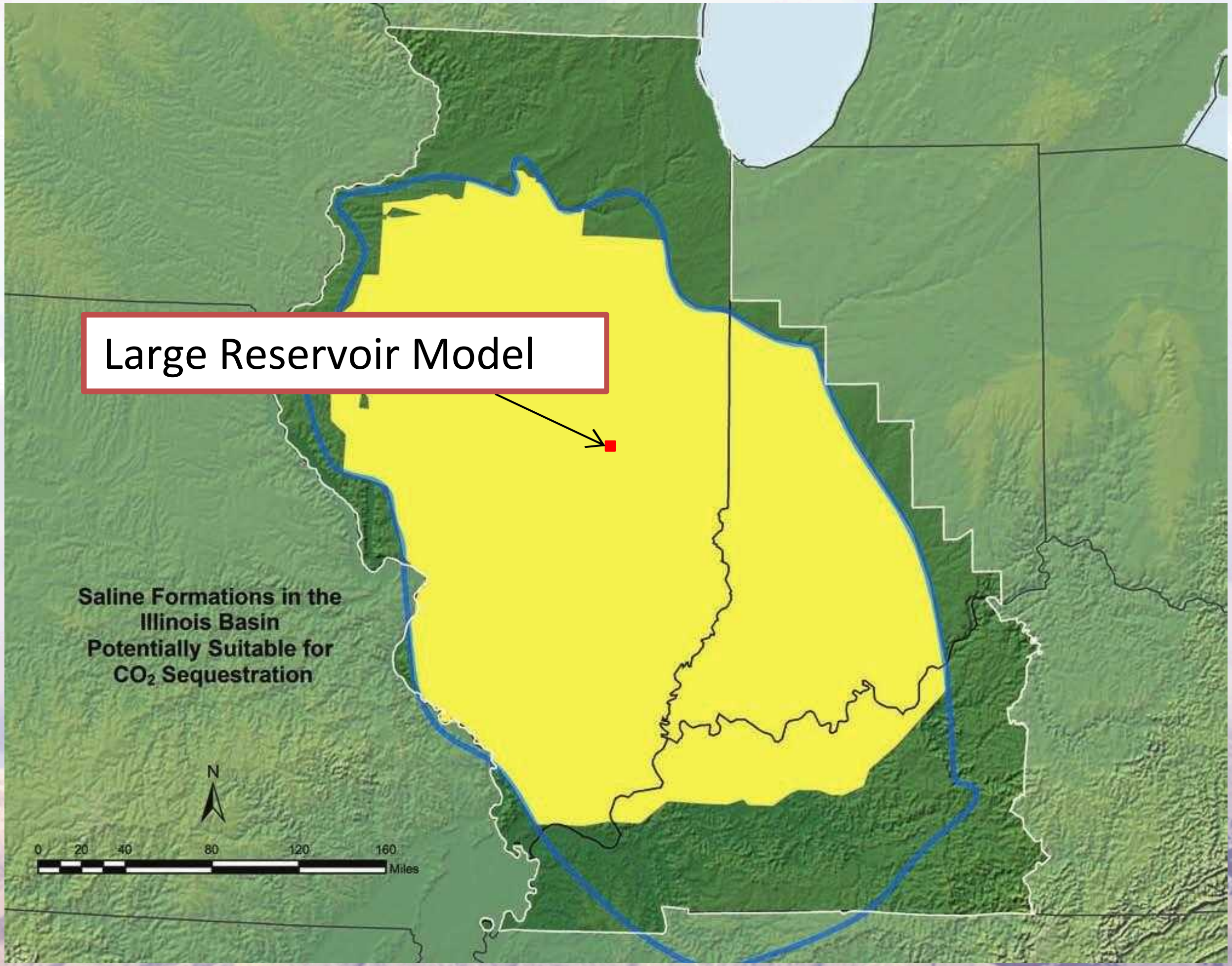
Vertical Scales



Areal Scales



Courtesy of Midwest Geological Consortium

Domain Decomposition and Multiscale Mortar Methods

Mixed Formulation

Model for flow in porous media:

$$\boldsymbol{u} = -K\nabla p, \qquad \text{(Darcy's Law)}$$
$$\nabla \cdot \boldsymbol{u} = f, \qquad \text{(Conservation of Mass)}$$

over Ω with $p = g$ on $\partial\Omega$.

Weak formulation: Find $\boldsymbol{u} \in H(\text{div}, \Omega)$ and $p \in L^2(\Omega)$ such that

$$\left(K^{-1}\boldsymbol{u}, \boldsymbol{v}\right) - (p, \nabla \cdot \boldsymbol{v}) = -\left\langle g, \boldsymbol{v} \cdot \boldsymbol{n} \right\rangle_{\partial\Omega}$$
$$(\nabla \cdot \boldsymbol{u}, w) = (f, w)$$

for all $\boldsymbol{v} \in H(\text{div}, \Omega)$ and $w \in L^2(\Omega)$.

Model Domain Decomposition Problem

Domain Decomposition: P nonoverlapping subdomains,

$$\Omega = \bigcup_{i=1}^{P} \Omega_i .$$

Interfaces: $\Gamma_{i,j} = \partial\Omega_i \cap \partial\Omega_j$ with $\Gamma = \bigcup_{1 \le i < j \le P} \Gamma_{i,j}$.

Weak Formulation: On each subdomain,

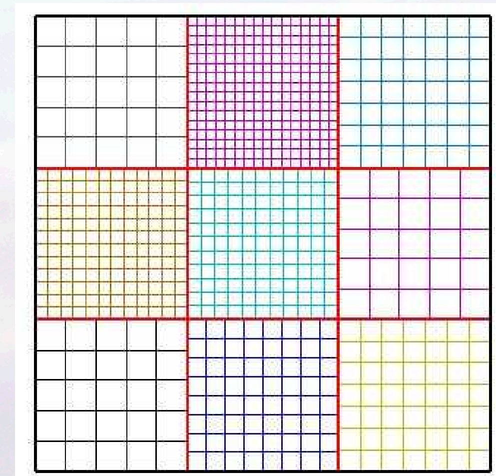
$$\begin{aligned}\left(K^{-1}\boldsymbol{u}, \boldsymbol{v}\right)_{\Omega_i} - (p, \nabla \cdot \boldsymbol{v})_{\Omega_i} + \langle \lambda, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\partial\Omega_i \cap \Gamma} &= - \langle g, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\partial\Omega_i \setminus \Gamma} \\ (\nabla \cdot \boldsymbol{u}, w)_{\Omega_i} &= (f, w)_{\Omega_i}\end{aligned}$$

Interface Conditions: Mass conservation,

$$\sum_{1 \le i < j \le P} \langle \boldsymbol{u}_i \cdot \boldsymbol{n}_i - \boldsymbol{u}_j \cdot \boldsymbol{n}_i, \mu \rangle_{\Gamma_{i,j}} .$$

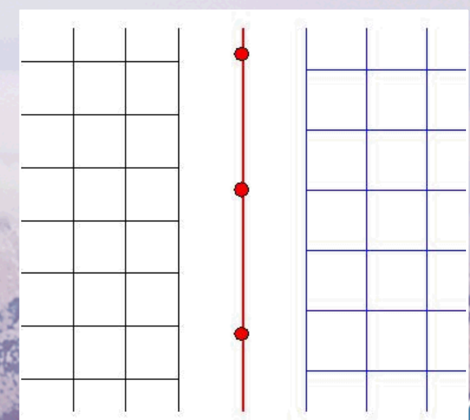
Multiscale Mortar Mixed Finite Element Method

Define a partition of each subdomain.



Let $\boldsymbol{V}_{h,i} \subset H(\text{div}, \Omega_i)$ and $W_{h,i} \subset L^2(\Omega_i)$ be mixed finite element spaces.

Define a coarse partition of each $\Gamma_{i,j}$.



Let $M_H \subset L^2(\Gamma)$ be space of (dis)continuous polynomials.

MSMMFEM: Find $\boldsymbol{u}_{h,i} \in \boldsymbol{V}_{h,i}$, $p_{h,i} \in W_{h,i}$ and $\lambda_H \in M_H$ such that weak formulation is satisfied for all $\boldsymbol{v} \in \boldsymbol{V}_{h,i}$, $w \in W_{h,i}$ and $\mu \in M_H$.

An Interface Operator

Let $\overline{\boldsymbol{u}}_{h,i} \in \boldsymbol{V}_{h,i}$ and $\overline{p}_{h,i} \in W_{h,i}$ solve,

$$\begin{aligned} \left(K^{-1}\overline{\boldsymbol{u}}_{h,i}, \boldsymbol{v}\right)_{\Omega_i} - \left(\overline{p}_{h,i}, \nabla \cdot \boldsymbol{v}\right)_{\Omega_i} &= -\left\langle g, \boldsymbol{v} \cdot \boldsymbol{n} \right\rangle_{\partial\Omega_i \setminus \Gamma} \\ \left(\nabla \cdot \overline{\boldsymbol{u}}_{h,i}, w\right)_{\Omega_i} &= \left(f, w\right)_{\Omega_i} \end{aligned}$$

Let $\boldsymbol{u}^*_{h,i} \in \boldsymbol{V}_{h,i}$ and $p^*_{h,i} \in W_{h,i}$ solve,

$$\begin{aligned} \left(K^{-1}\boldsymbol{u}^*_{h,i}, \boldsymbol{v}\right)_{\Omega_i} - \left(p^*_{h,i}, \nabla \cdot \boldsymbol{v}\right)_{\Omega_i} &= -\left\langle \lambda_H, \boldsymbol{v} \cdot \boldsymbol{n} \right\rangle_{\partial\Omega_i \cap \Gamma} \\ \left(\nabla \cdot \boldsymbol{u}^*_{h,i}, w\right)_{\Omega_i} &= 0 \end{aligned}$$

Superposition of solutions gives $\boldsymbol{u}_{h,i} = \overline{\boldsymbol{u}}_{h,i} + \boldsymbol{u}^*_{h,i}$ and $p_{h,i} = \overline{p}_{h,i} + p^*_{h,i}$.

An Interface Operator

Use $\overline{\boldsymbol{u}}_{h,i}$ to define a linear form,

$$g_H(\mu) = \sum_{1 \le i < j \le P} \langle \overline{\boldsymbol{u}}_{h,i} \cdot \boldsymbol{n}_i - \overline{\boldsymbol{u}}_{h,j} \cdot \boldsymbol{n}_i, \mu \rangle_{\Gamma_{i,j}}, \quad \forall \mu \in M_H.$$

Use $\boldsymbol{u}^*_{h,i}$ to define a bilinear form,

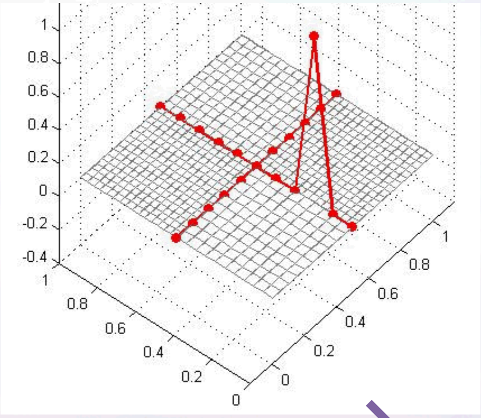
$$d_H(\lambda_H, \mu) = \sum_{1 \le i < j \le P} \left\langle \boldsymbol{u}^*_{h,i}(\lambda_H) \cdot \boldsymbol{n}_i - \boldsymbol{u}^*_{h,j}(\lambda_H) \cdot \boldsymbol{n}_i, \mu \right\rangle_{\Gamma_{i,j}}, \quad \forall \mu \in M_H.$$

Then $\lambda_H \in M_H$ satisfies,

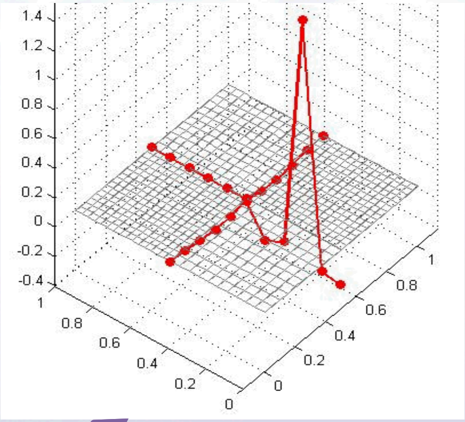
$$d_H(\lambda_h, \mu) = g_H(\mu), \qquad \forall \mu \in M_H.$$

Construction of a Multiscale Basis

For each mortar basis function



Project the flux into the mortar space and store.



Solve fine scale problem

$$\left(K^{-1}\boldsymbol{u}^*_{h,i},\boldsymbol{v}\right)_{\Omega_i} - \left(p^*_{h,i},\nabla\cdot\boldsymbol{v}\right)_{\Omega_i} = -\left\langle\lambda_H,\boldsymbol{v}\cdot\boldsymbol{n}\right\rangle_{\partial\Omega_i\cap\Gamma}$$

$$\left(\nabla\cdot\boldsymbol{u}^*_{h,i},w\right)_{\Omega_i} = 0$$

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Given the Mortar Multiscale Basis …

- Coarse scale equation is solved without solving further subdomain problems.
- Action of interface operator computed using a linear combination of the basis functions.
- Solved easily in parallel using an iterative method (CG, GMRES,…).
- For large problems, an acceleration may be required:
  - Block Jacobi: *Bramble, Pasciak, Schatz1986; Smith 1992; Achdou, Maday, Widlund 1999*
  - Balancing: *Mandel 1993; Cowsar, Mandel, Wheeler 1995; Pencheva, Yotov 2003*
  - Interface Multigrid: *Wheeler, Yotov 2000; Yotov 2001*
- Dominant cost is assembling the multiscale basis.

A Multiscale Mortar Preconditioner

Moving to More Complex Problems

- Consider a sequence of linear problems.
- With each linear problem, a coarse scale interface operator:

$$d_H^{\{m\}}(\lambda_H^{\{m\}}, \mu) = g_H^{\{m\}}(\mu).$$

- The algebraic problem,

$$\mathcal{D}_H^{\{m\}} \boldsymbol{\lambda}_H^{\{m\}} = \boldsymbol{g}_H^{\{m\}}.$$

- Must recompute the multiscale basis for each problem for mass conservation.
- Significant portion of the computational effort *[Ganis, Yotov 2009; Wheeler, W., Yotov 2009; : Lunati, Jenny 2006; Efendiev, Ginting, Hou, Ewing 2006]*

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A Multiscale Preconditioner

- Compute the multiscale basis for a training operator:

$$\bar{\mathcal{D}}_H \boldsymbol{\lambda}_H = \boldsymbol{g}_H.$$

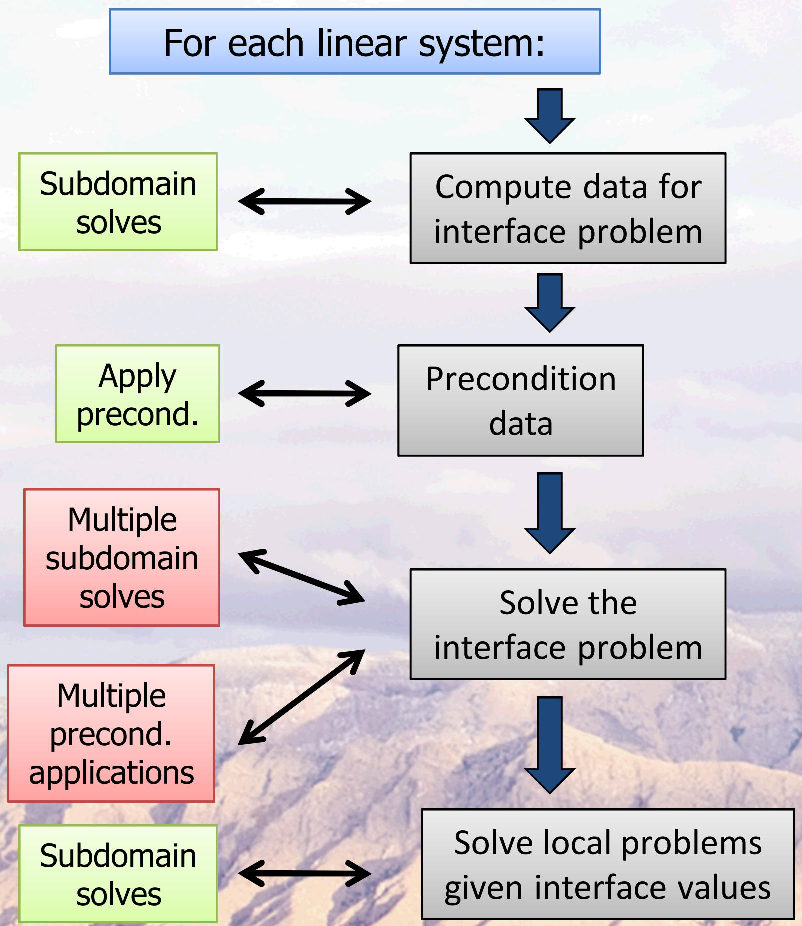
- Solve the preconditioned interface problem:

$$\bar{\mathcal{D}}_H^{-1} D_H^{\{m\}} \lambda_H = \bar{\mathcal{D}}_H^{-1} g_H^{\{m\}}.$$

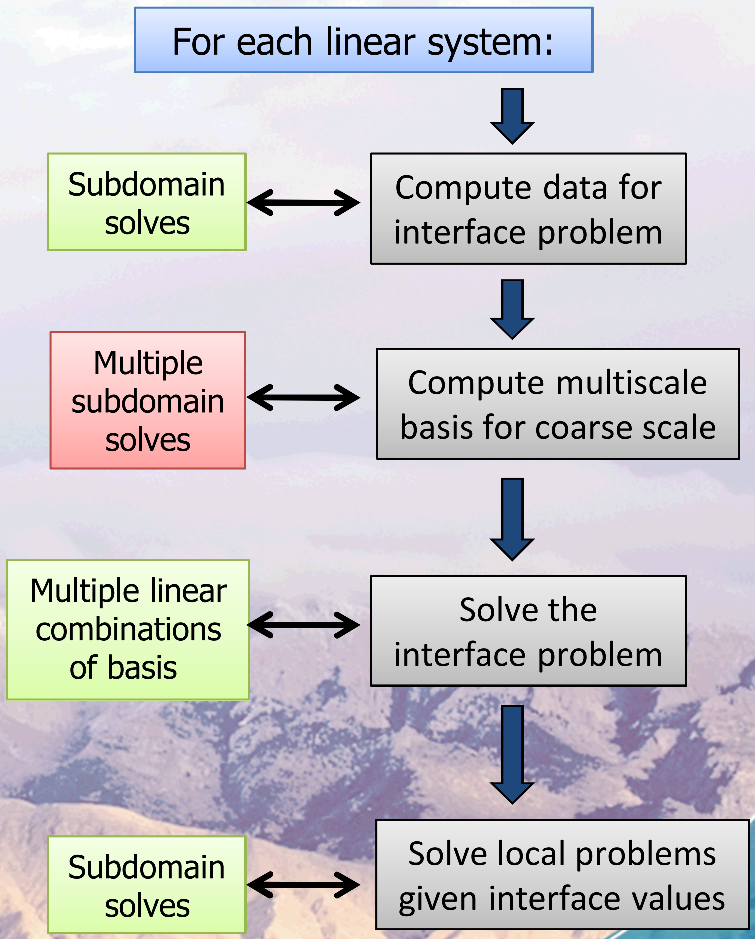
- Computing the action of $\mathcal{D}_H^{\{m\}}$ requires one solve per subdomain.
- Computing the action of $\mathcal{D}_H^{-1}$ requires one coarse scale solve using multiscale basis.

Domain Decomposition vs. Multiscale Basis

Domain Decomposition



Multiscale Basis



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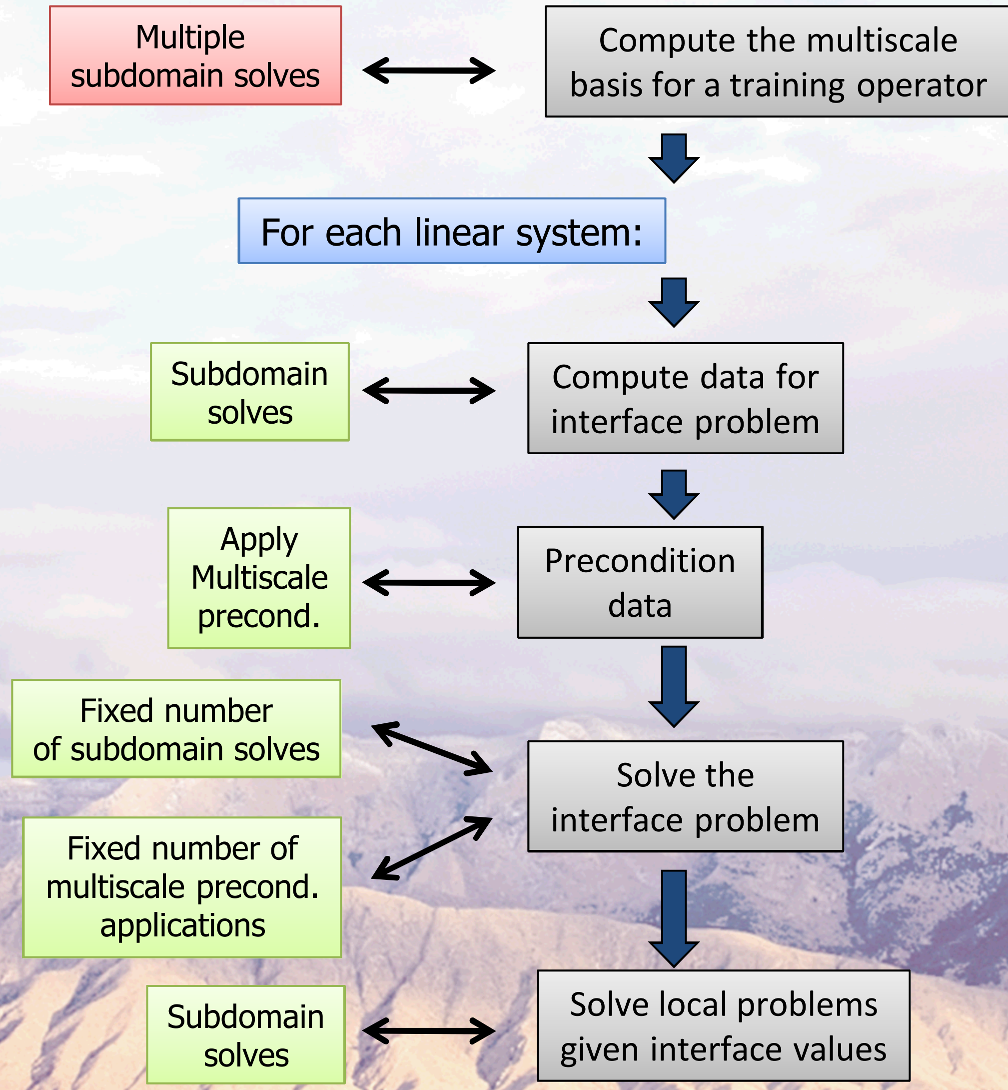
A Multiscale Preconditioner



Uniform Spectral Equivalence

Theorem

The coarse scale interface operators $\tilde{\mathcal{D}}_H$ and $\mathcal{D}_H^{\{m\}}$ are uniformly spectrally equivalent for $1 \leq m \leq M$. Moreover,

$$\operatorname{cond}\left(\tilde{\mathcal{D}}_H^{-1}\mathcal{D}_H^{\{m\}}\right) \leq C^{\{m\}},$$

where each constant is independent of the discretization, but does depend mildly on the choice of training permeability.

Example 1: Influence of the Discretization

- Define $\Omega = [0,1] \times [0,1]$, and consider,

$$\begin{cases} -\nabla \cdot (K(x,y)\nabla p) = f(x,y), & (x,y) \in \Omega, \\ p = 0, & (x,y) \in \partial\Omega, \end{cases}$$

with $f(x,y) = 8\pi^2 \sin(2\pi x)\sin(2\pi y)$

- True permeability,

$$K(x,y) = 1 - 0.5\sin(3\pi x)\sin(3\pi y)$$

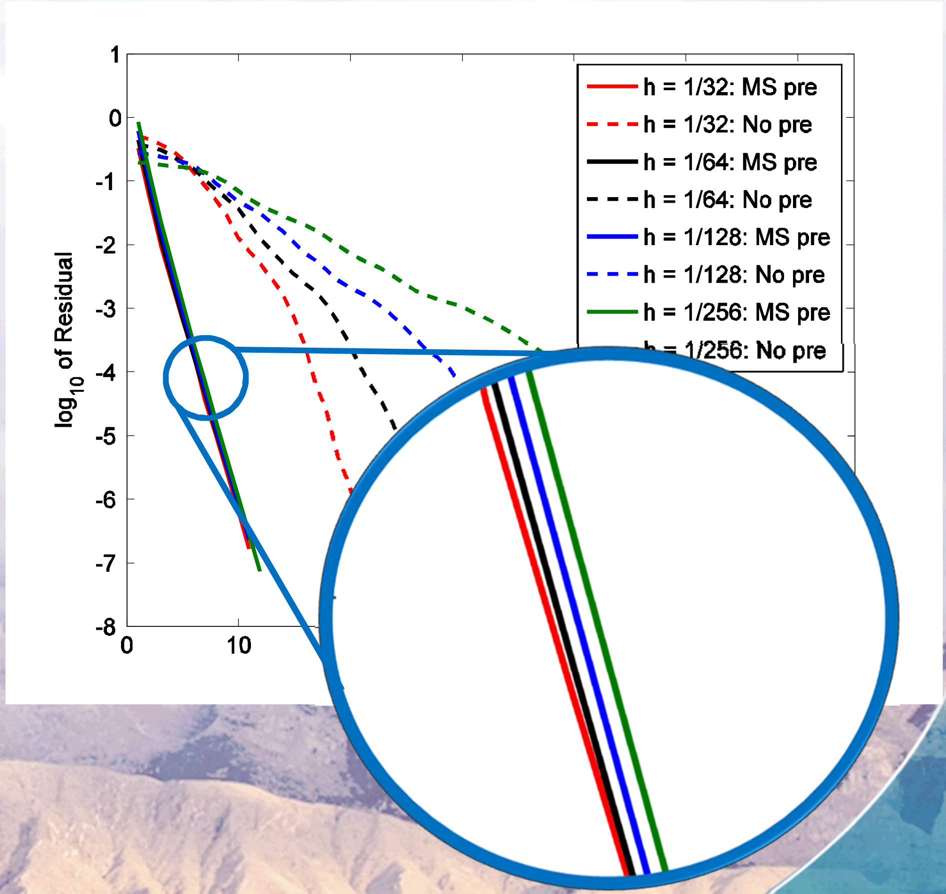
- Training permeability,

$$K(x,y) = 1$$

- We vary the mesh size, the number of subdomains, and the mortar approximation.

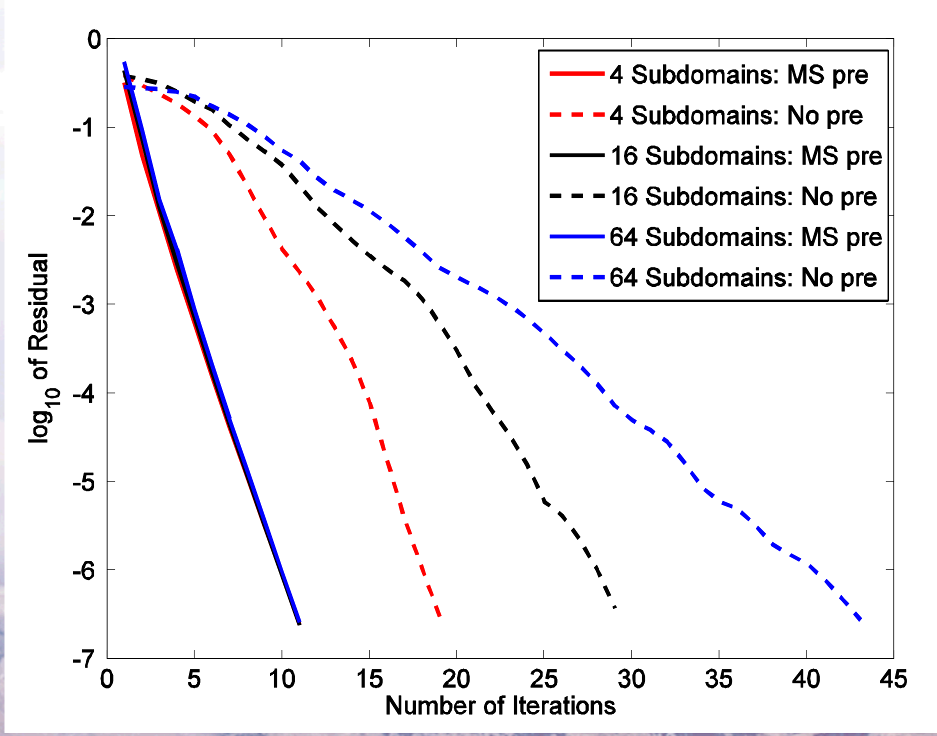
Example 1: Decreasing the Mesh Size

- 16 equal-sized subdomains
- Matching grids
- Continuous linear mortars
- Vary the subdomain mesh size
  - h = 1/64, 1/128, 1/256, 1/512
- GMRES for the outer iterations
- CG for the preconditioner



Example 1: Increasing the Number of Subdomains

- Matching grids
- Continuous linear mortars
- h = 1/128
- Vary the number of subdomains
  - P = 4, 16, 64
- GMRES for the outer iterations
- CG for the preconditioner



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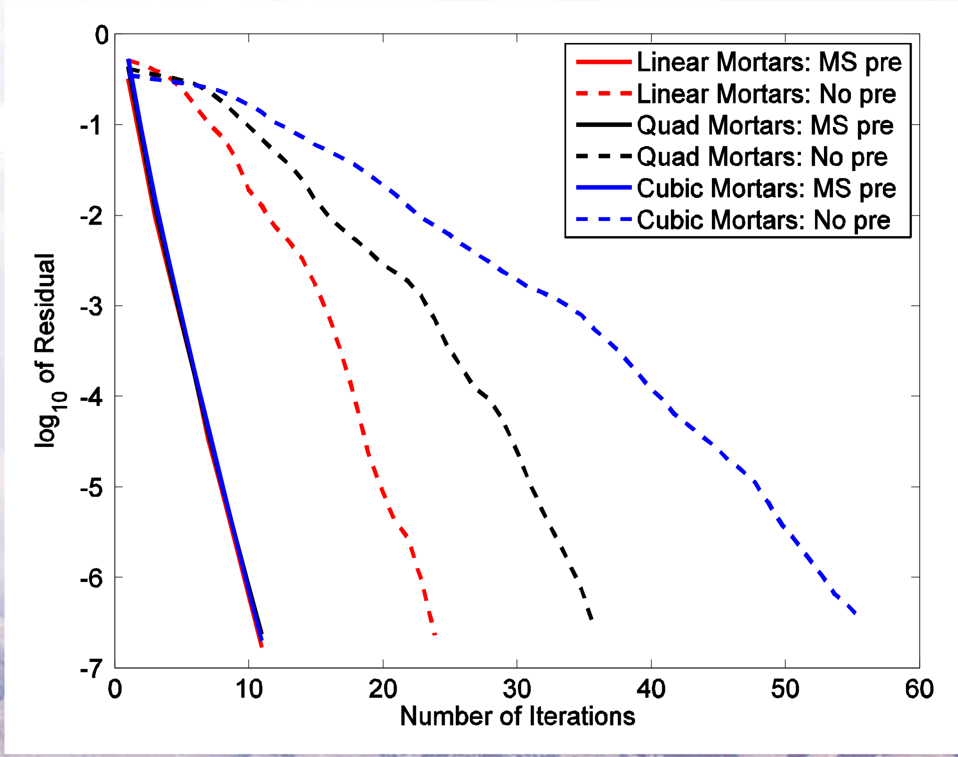
Example 1: Changing the Mortar Degree

- Matching grids
- h = 1/128
- Continuous mortars
- Vary the degree of the mortars
  - p = 1, 2, 3
- GMRES for the outer iterations
- CG for the preconditioner



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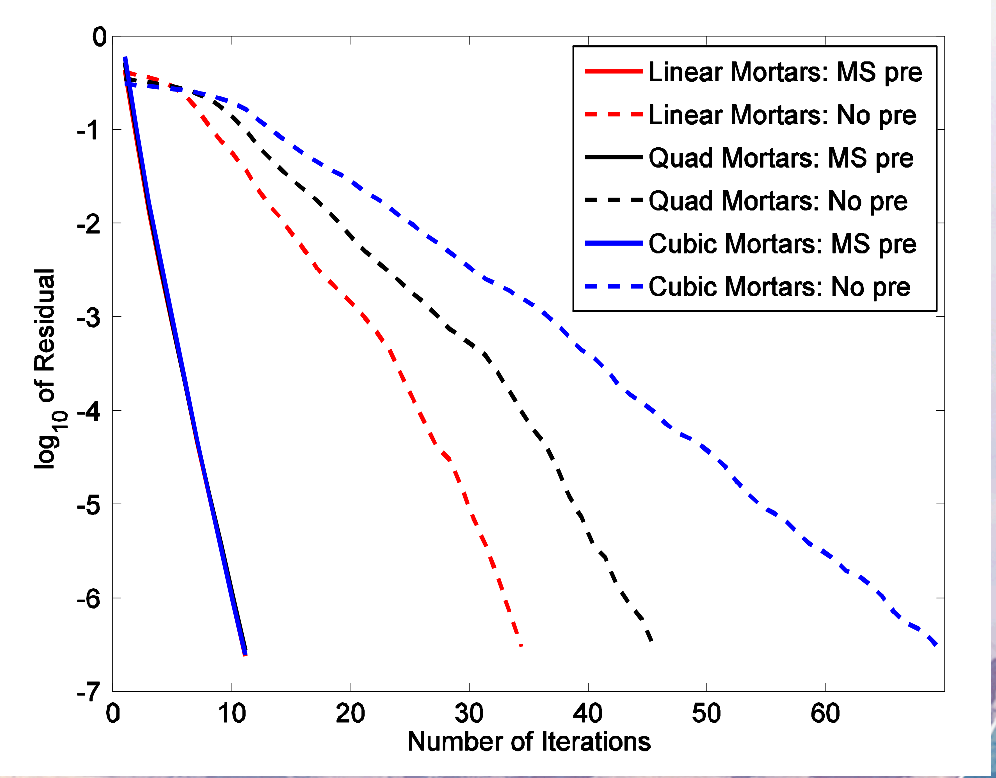
Example 1: Changing the Mortar Degree

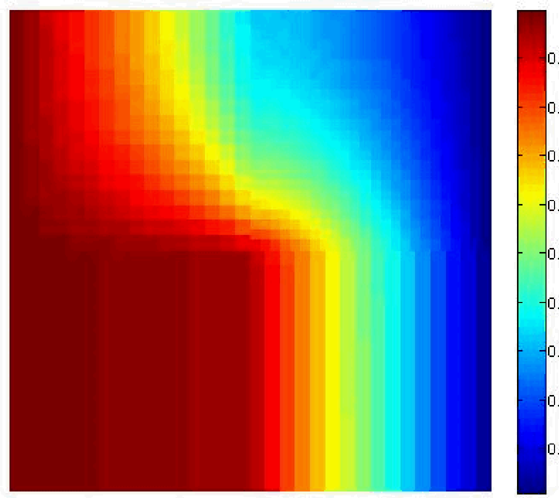
- Matching grids
- h = 1/128
- Discontinuous mortars
- Vary the degree of the mortars
  - p = 1, 2, 3
- GMRES for the outer iterations
- CG for the preconditioner

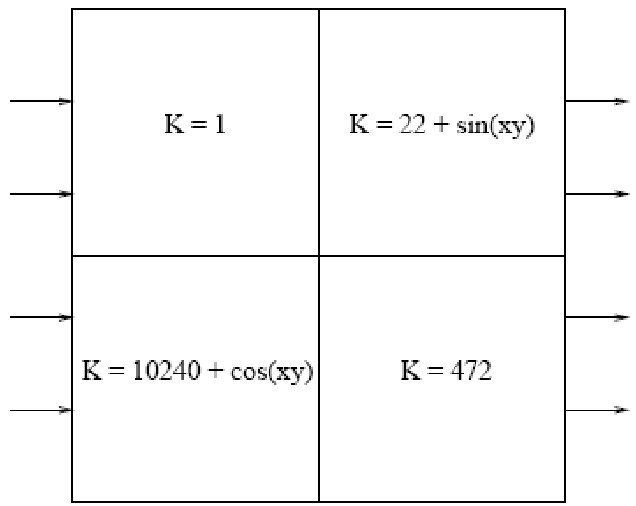


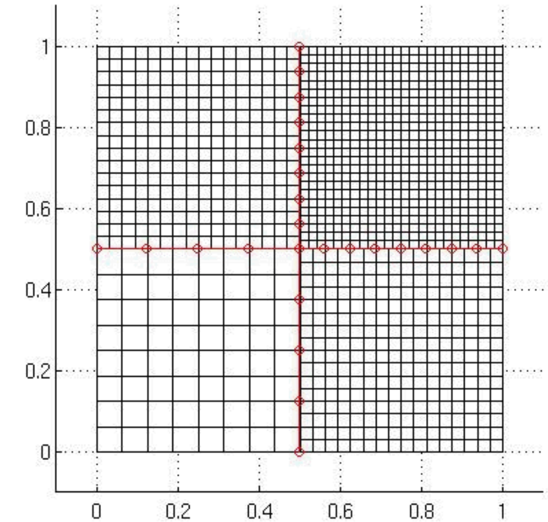
Example 2: Changing the Training Permeability

- 4 equal-sized subdomains
- Nonmatching grids
- Continuous linear mortars

- GMRES for outer iterations
- CG to apply the preconditioner
- Vary the quality of the training permeability







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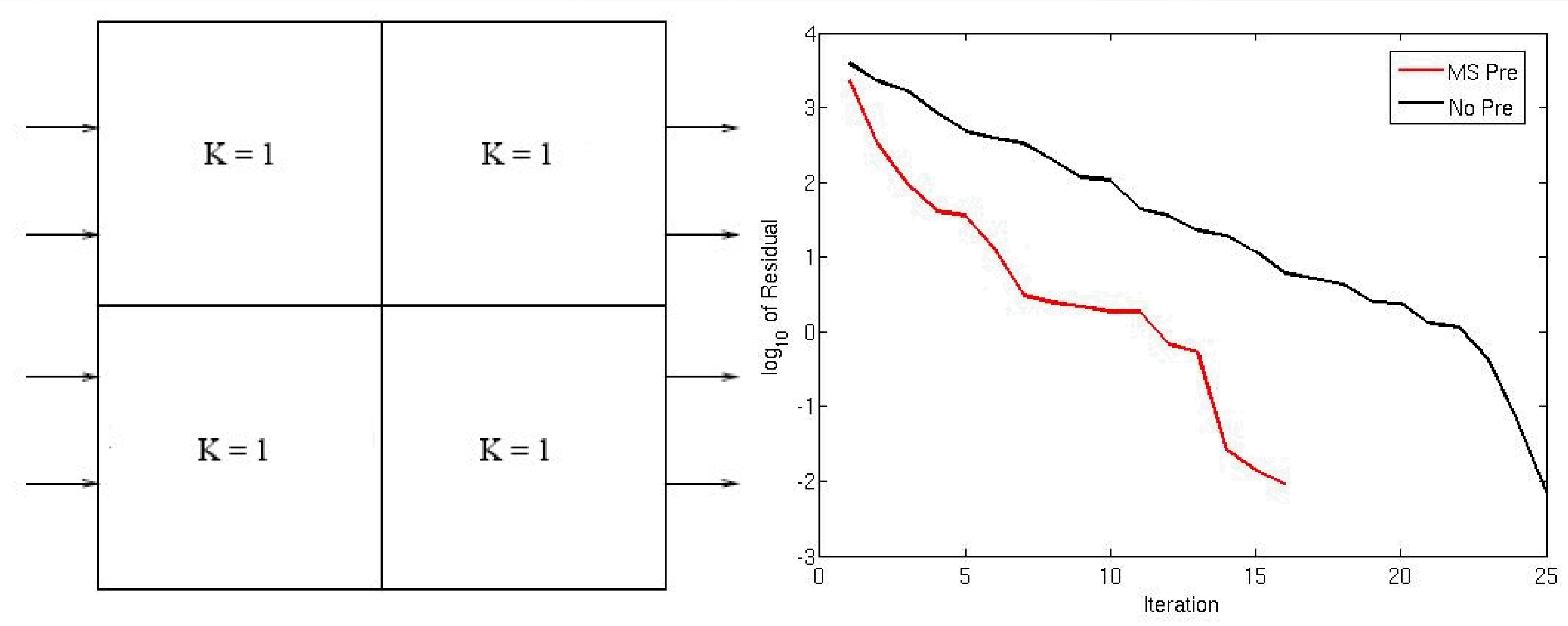

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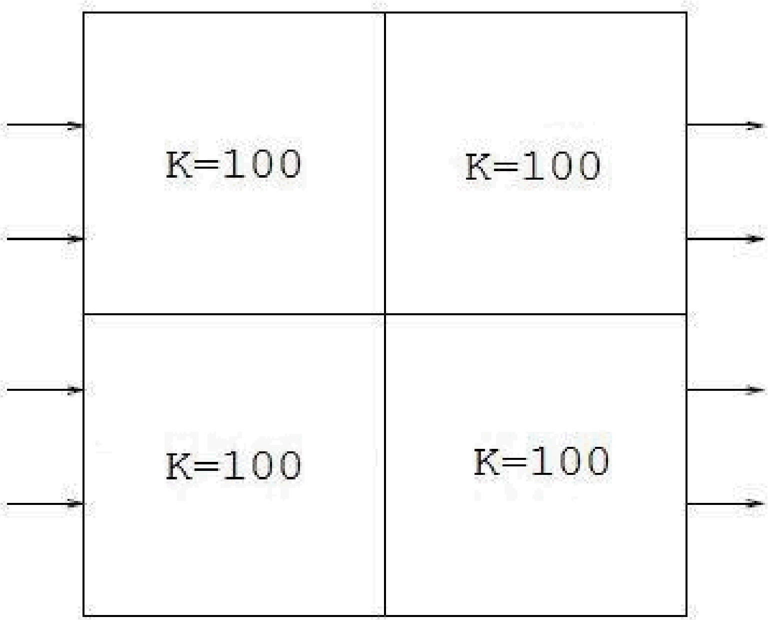

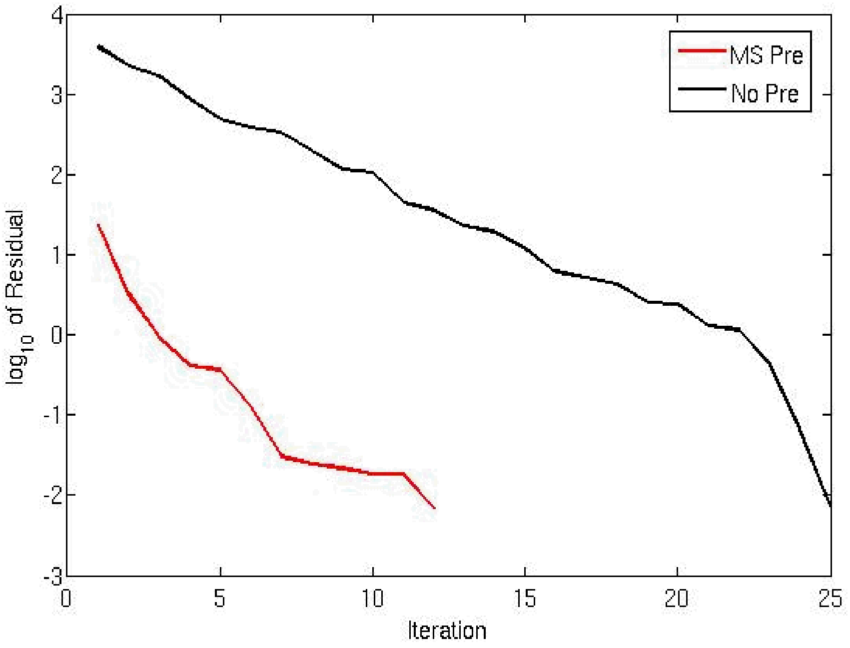

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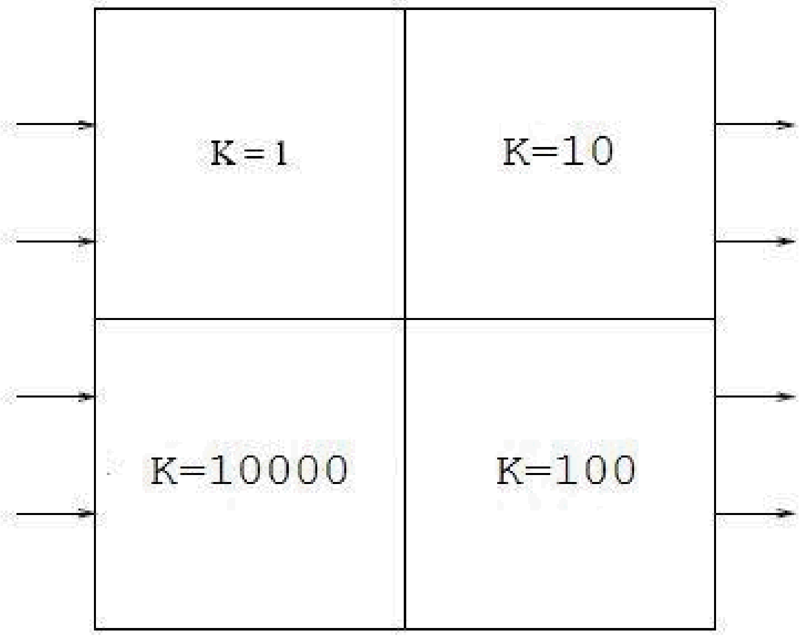
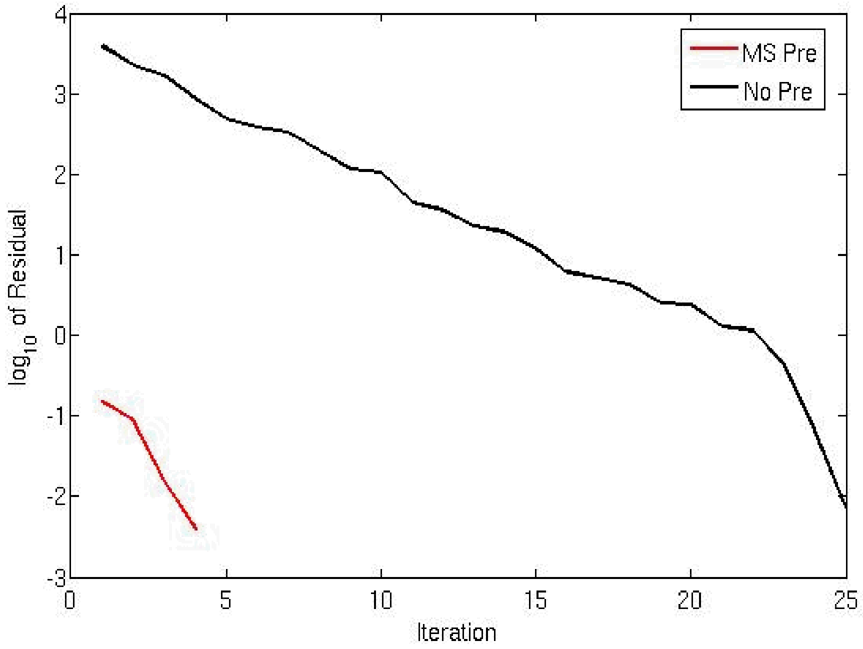
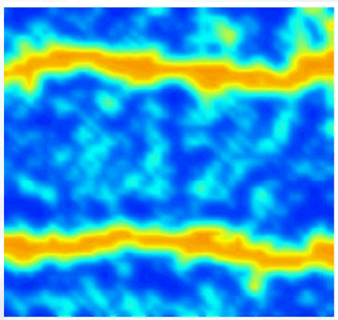




Applications to Stochastic Flow in Porous Media

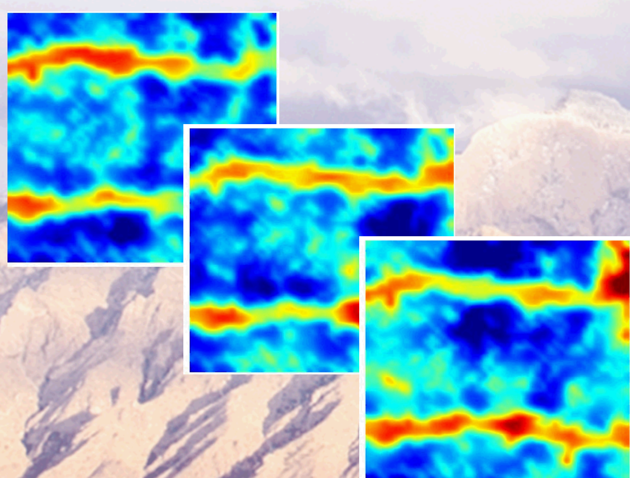
A Multiscale Preconditioner

Select a training permeability



Compute multiscale basis from the training permeability

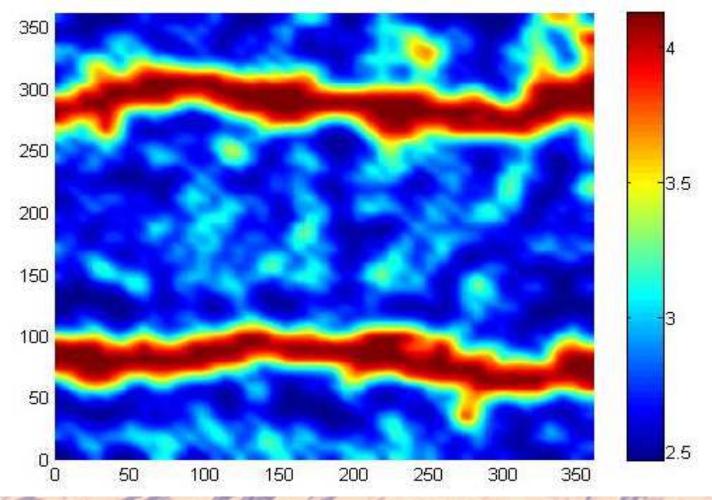
Generate stochastic realizations

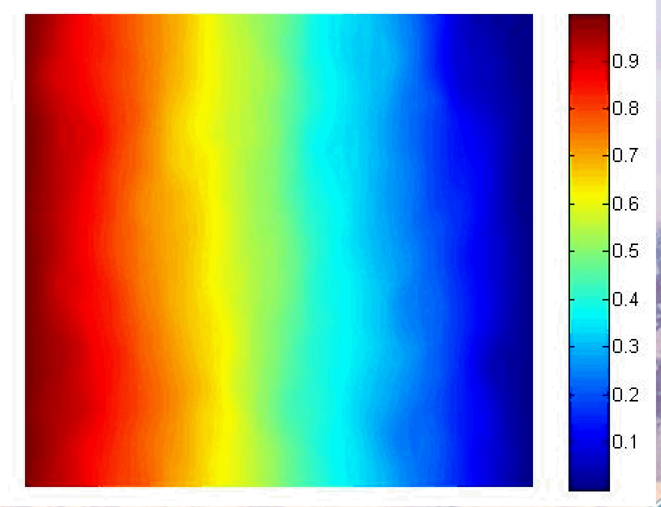


Use the multiscale basis to construct a preconditioner for each realization.

Example: Uncertainty Quantification

- 360x360 grid
- 25 (5x5) subdomains of equal size
- 129,600 degrees of freedom on fine scale
- Continuous quadratic mortars
- Matching grids on the interfaces
- 380 degrees of freedom on the coarse scale
- GMRES for the outer iterations
- PCG to apply the preconditioner
- Vary the number of terms in the KL expansion
- Vary the experimental covariance
- Second order stochastic collocation or Monte Carlo sampling





Computational Results

Standard deviation	0.25
Terms in KL expansion	4
Number of realizations	16

Method	Total	Average	Minimum	Maximum
Standard DD	2825	176.6	165	186
Recomputing the multiscale basis	1088	68	68	68
Multiscale basis preconditioner	64	4	4	4

Computational Results

Standard deviation	1.0
Terms in KL expansion	4
Number of realizations	16

Method	Total	Average	Minimum	Maximum
Standard DD	2960	185.0	169	195
Recomputing the multiscale basis	1088	68	68	68
Multiscale basis preconditioner	84	5.25	4	6

Computational Results

Standard deviation	10.0
Terms in KL expansion	4
Number of realizations	16

Method	Total	Average	Minimum	Maximum
Standard DD	3827	239.2	195	285
Recomputing the multiscale basis	1088	68	68	68
Multiscale basis preconditioner	216	13.3	7	19

Computational Results

Standard deviation	1.0
Terms in KL expansion	9
Number of realizations	512

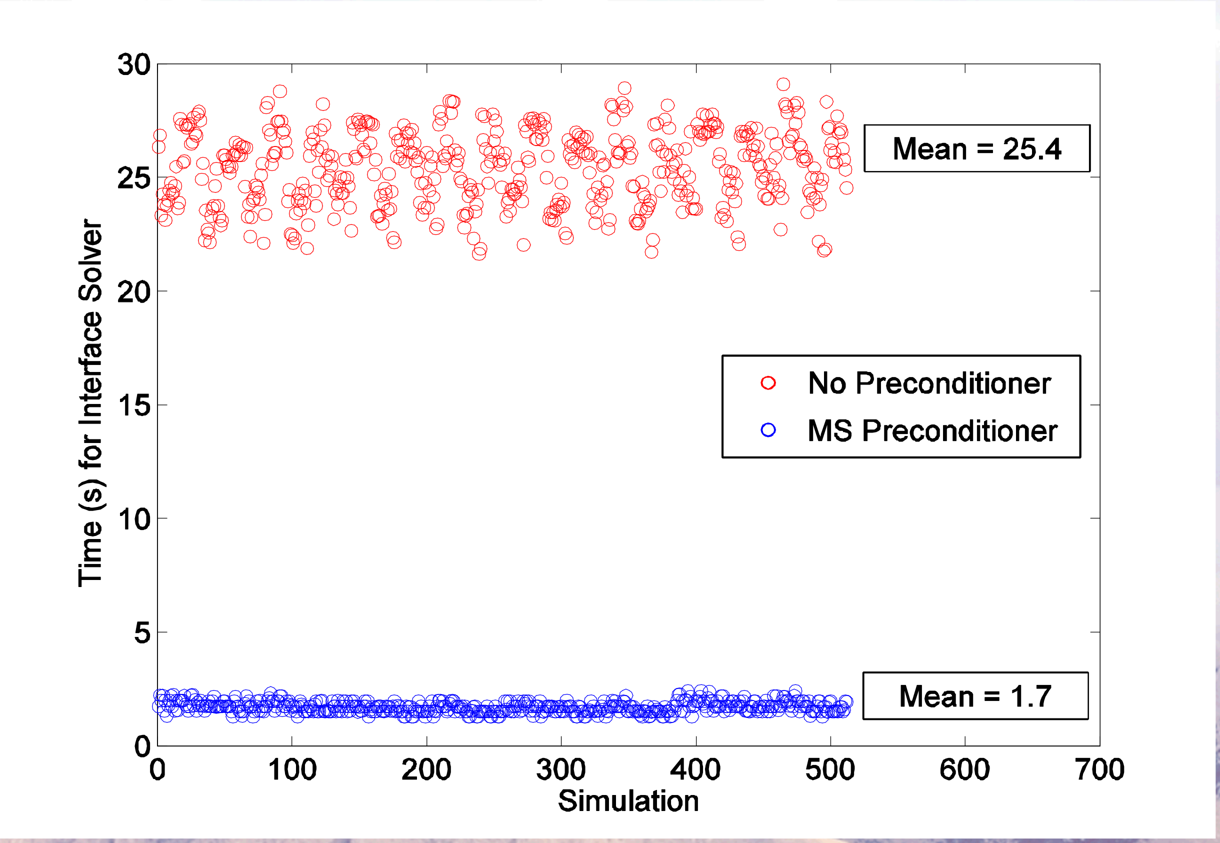
Method	Total	Average	Minimum	Maximum
Standard DD	98544	192.5	160	225
Recomputing the multiscale basis	34816	68	68	68
Multiscale basis preconditioner	3463	6.8	5	10

Computational Results

Standard deviation	1.0
Terms in KL expansion	100
Number of realizations	1000

Method	Total	Average	Minimum	Maximum
Standard DD	203587	203.6	144	283
Recomputing the multiscale basis	68000	68	68	68
Multiscale basis preconditioner	11262	11.3	6	33

Computational Results



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Application to an IMPES Formulation for Two Phase Flow

Equations Two Phase Flow

- Mass balance equation for wetting and non-wetting phase:

$$\frac{\partial}{\partial t}(\phi \rho_\alpha S_\alpha) + \nabla \cdot (\rho_\alpha \boldsymbol{u}_\alpha) = q_\alpha, \qquad \alpha = w, n$$

- Darcy law for each phase:

$$\boldsymbol{u}_\alpha = -K \frac{k_{r,\alpha}(S_w)}{\mu_\alpha} \left(\nabla P_\alpha - g \rho_\alpha \nabla D\right), \qquad \alpha = w, n$$

- Saturation constraint: $\sum_\alpha S_\alpha = 1,$

- Capillary pressure: $P_c(S_w) = P_n - P_w,$

IMPES Formulation

- Define mobilities:

$$\lambda_w(S_w) = \frac{k_{rw}(S_w)}{\mu_w}, \quad \lambda_n(S_w) = \frac{k_{rn}(S_w)}{\mu_n}, \quad \lambda_t(S_w) = \lambda_w(S_w) + \lambda_n(S_w)$$

- Primary velocity and capillary velocity:

$$\boldsymbol{u}_a = -\lambda_t K \left(\nabla P_w - \rho_w g \nabla D\right),$$
$$\boldsymbol{u}_c = -\lambda_n K \left(\nabla P_c - (\rho_n - \rho_w) g \nabla D\right),$$

- Implicit equation for primary velocity:

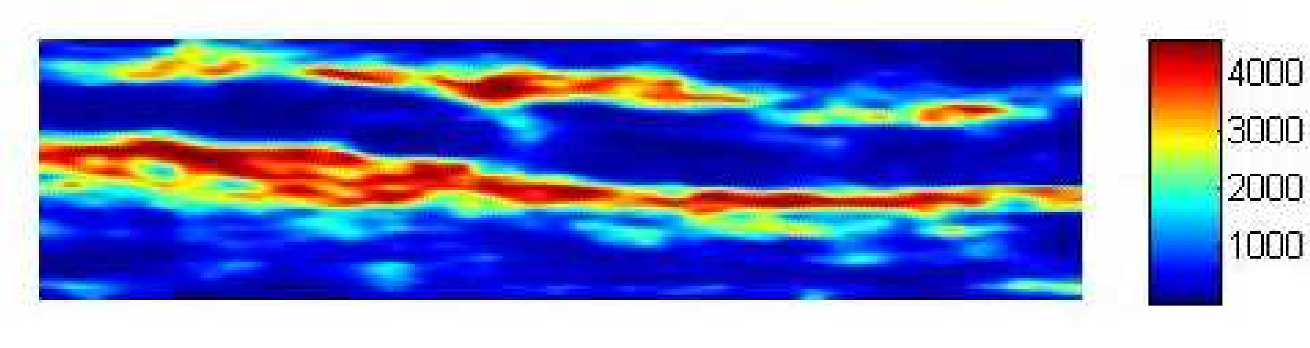
$$\nabla \cdot (\boldsymbol{u}_a + \boldsymbol{u}_c) = q_a + q_c,$$

- Explicit method for saturation:

$$\frac{\partial}{\partial t}(\phi S_w) + \nabla \cdot \left(\frac{\lambda_w}{\lambda_t} \boldsymbol{u}_a\right) = q_w,$$

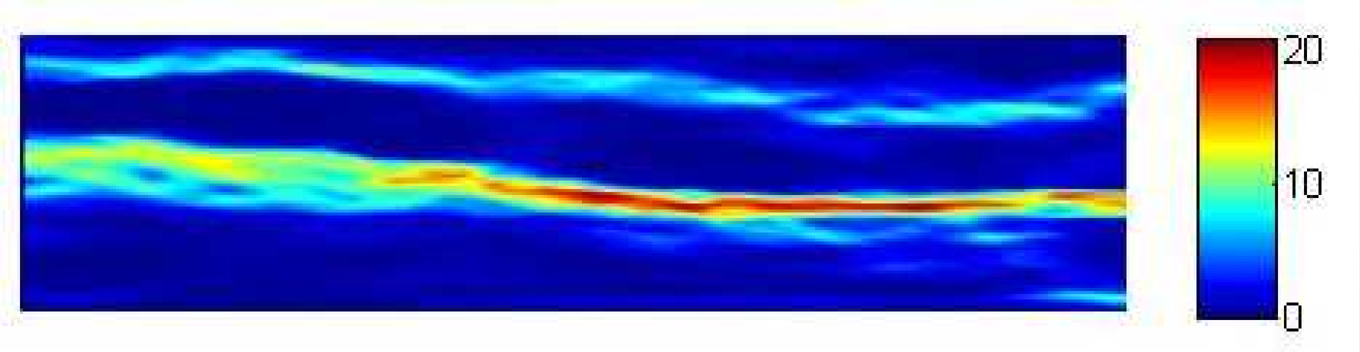
Example: Water Flood

- Compare multiscale mortar solution with fine scale solution.
- Fine scale: 180x180 → 32,400 fine scale degrees of freedom
- Coarse scale: 4x4 quadratic mortars → 456 degrees of freedom on coarse scale
- Channelized permeability.
- Inject at one end, produce at the other.
- Neglect gravity.
- Improved IMPES:100 saturation time steps per pressure time step

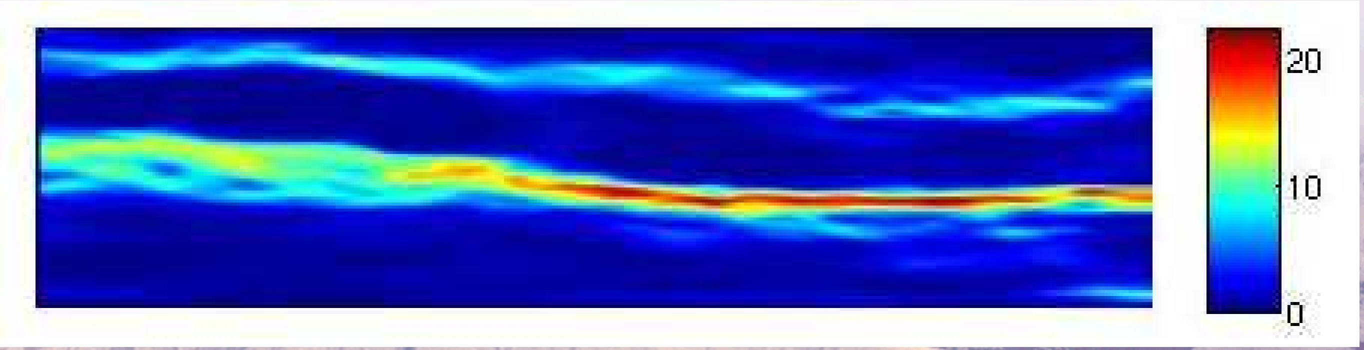


Fine scale permeability

Fine Scale and Multiscale Velocities

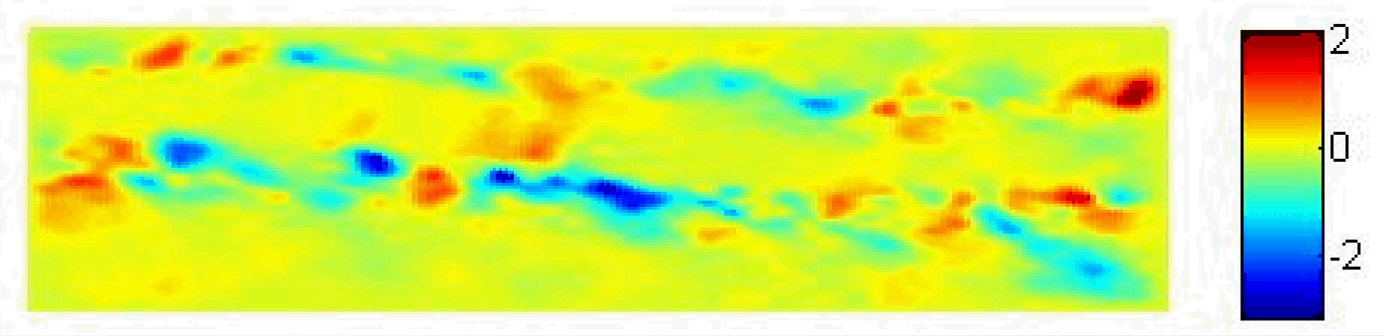


x-velocity from fine scale solution:

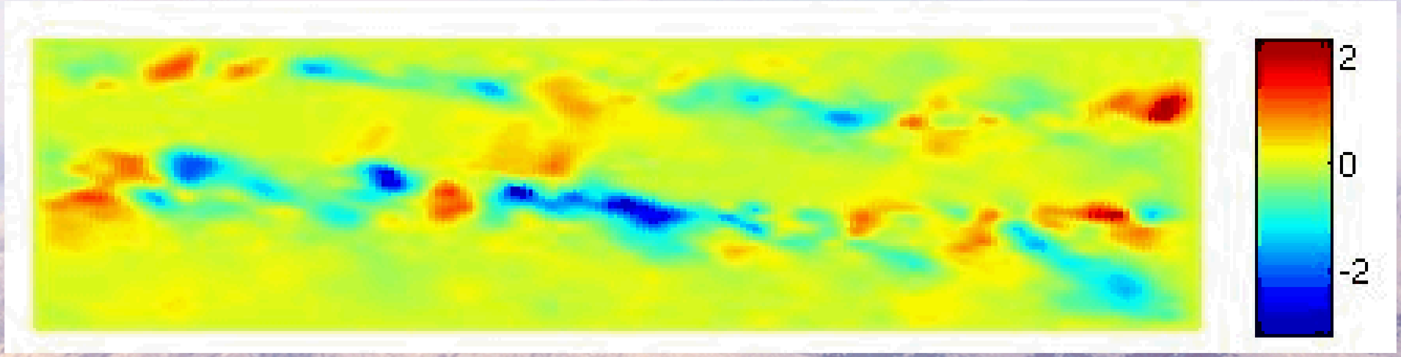


x-velocity from multiscale solution:

Fine Scale and Multiscale Velocities

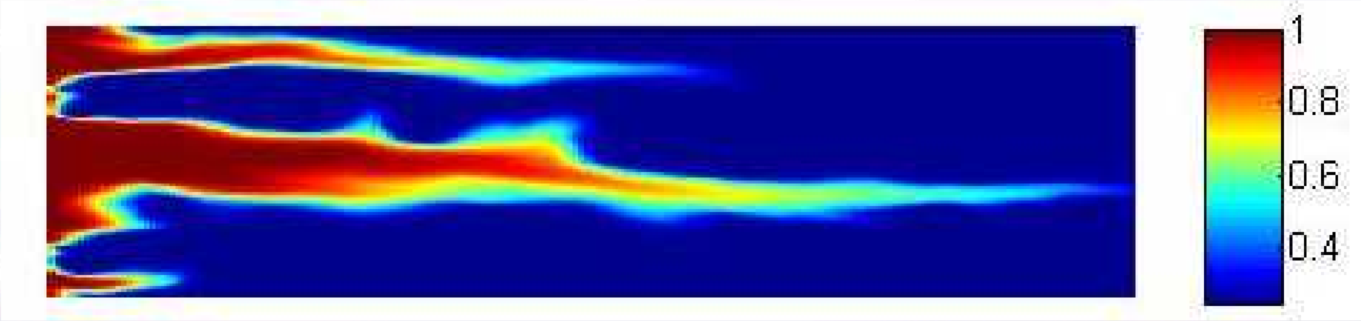


y-velocity from fine scale solution:

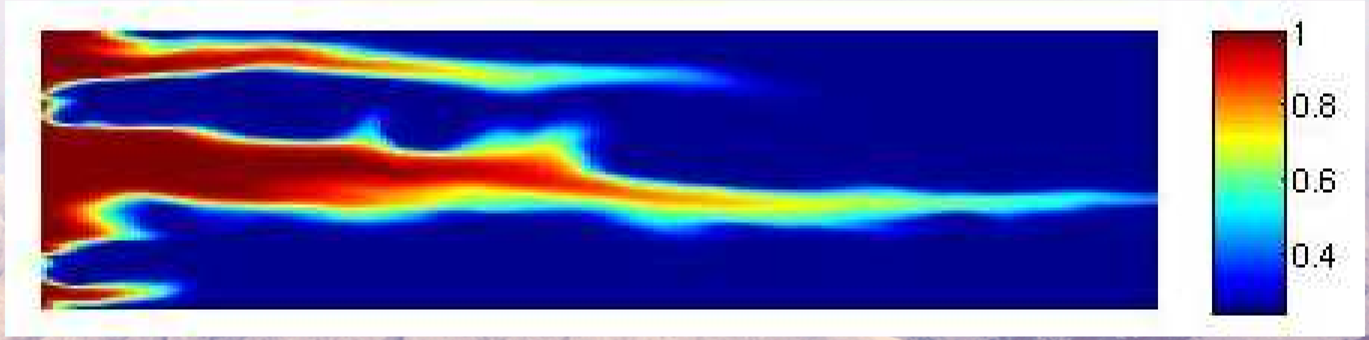


y-velocity from multiscale solution:

Fine Scale and Multiscale Saturations



Water saturation from fine scale solution:



Water saturation from multiscale solution:

Efficiency of the Multiscale Preconditioner

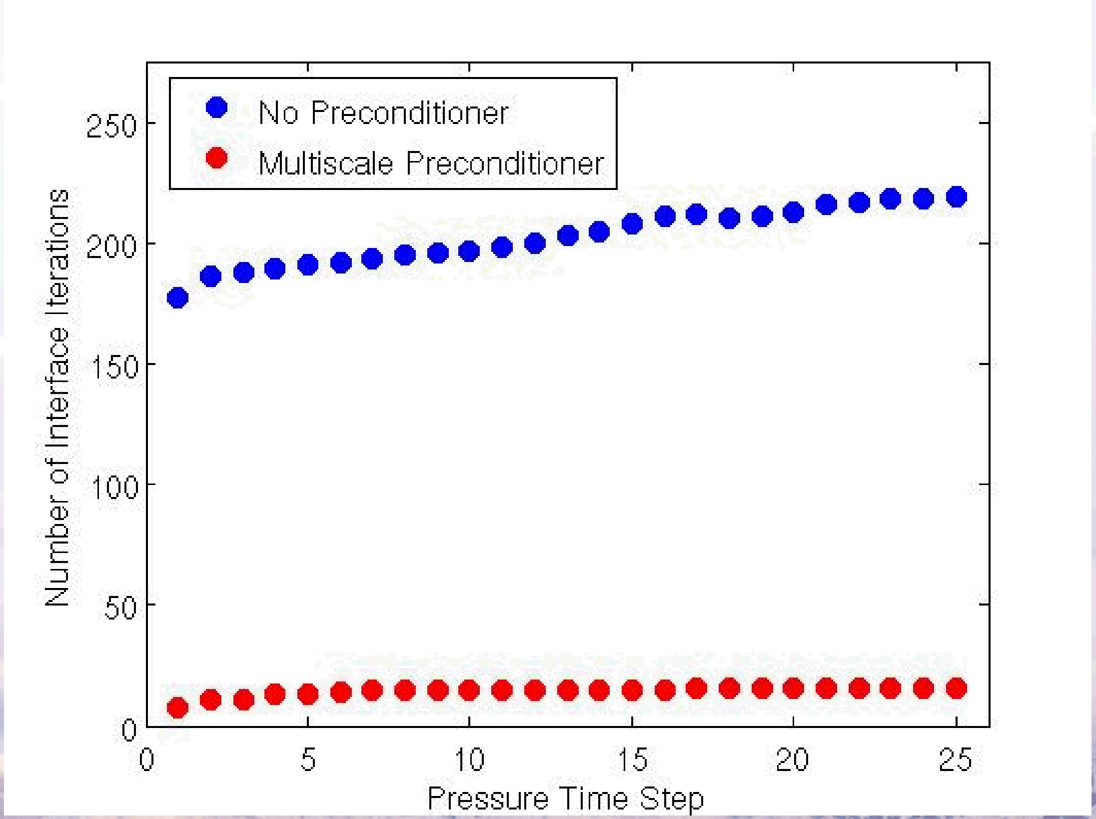


Figure: Number of interface iterations without preconditioner and with multiscale preconditioner based on initial conditions.

A Nonlinear Interface Operator

- Consider the nonlinear interface equation: $B(\lambda) = 0$

$$\langle B(\lambda), \mu \rangle = b(\lambda, \mu) = \sum_i \langle [\boldsymbol{u}_h(\lambda) \cdot \boldsymbol{n}_i], \mu \rangle_{\Gamma_i}$$

- Computing $[\boldsymbol{u}_h(\lambda) \cdot \boldsymbol{n}]$ requires nonlinear subdomain solves.
- Inexact Newton-GMRES algorithm for interface problem.
- Each Newton correction is computed by solving

$$D_\delta B(\lambda) s = -B(\lambda)$$

- Finite difference approximation of Jacobian

$$D_\delta B(\lambda, \mu) = \begin{cases} 0, & \mu = 0, \\ \|\mu\| \frac{B(\lambda + \delta \|\lambda\| \mu / \|\mu\|) - B(\lambda)}{\delta \|\lambda\|}, & \mu \neq 0, \lambda \neq 0, \\ \|\mu\| \frac{B(\delta \mu / \|\mu\|) - B(\lambda)}{\delta}, & \mu \neq 0, \lambda = 0, \end{cases}$$

A Nonlinear Interface Operator

- May be implemented in a matrix-free algorithm.
- Each action of the Jacobian requires subdomain solves.
- Computing the Jacobian is similar to computing a multiscale mortar basis.
- Compute the Jacobian based on a particular state.

$$M = D_\delta B(\overline{\lambda})$$

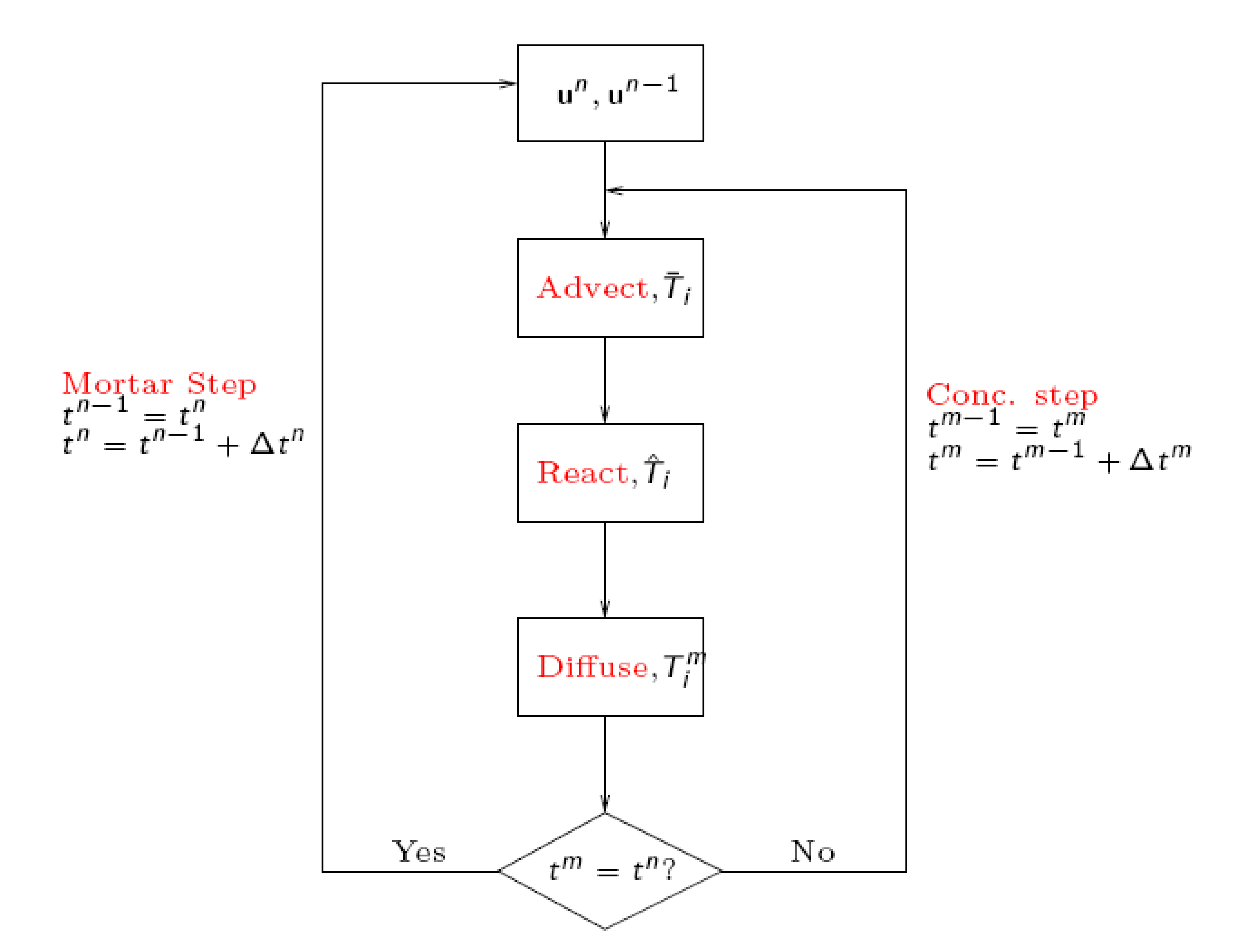
- Used as a preconditioner for subsequent Jacobians.

$$M^{-1} D_\delta B(\lambda) s = -M^{-1} B(\lambda)$$

- Recomputed if necessary.

Application to Slightly Compressible Single Phase Flow with Reactive Transport

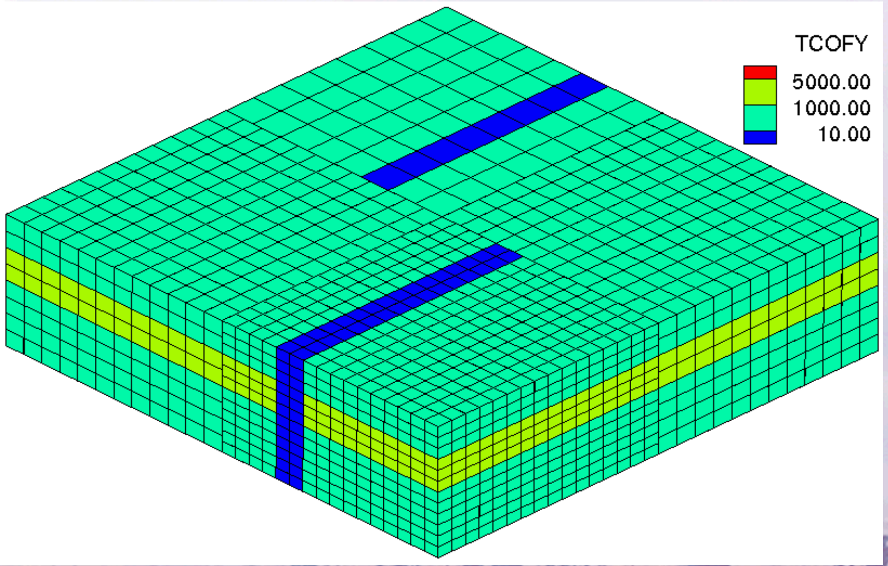
Split Solution Algorithm for Transport



Numerical Results: Flow Around a Barrier

- 24 ft x 400ft x 400ft
- Grid sizes
  - 2ft x 12.5ft x 12.5ft (12x16x16)
  - 3ft x 16.6ft x 16.6ft (8x12x12)
  - 3ft x 16.6ft x 16.6ft (8x12x12)
  - 4ft x 25ft x 25ft (6x8x8)
- Injection pressure: increase linearly
  - 505 [psi] (initially)
  - 1000 [psi] (50 days)
- Production pressure: decrease linearly
  - 480 [psi] (initially)
  - 350 [psi] (30 days)
- Initial concentrations:
  - 100 [M/cu-ft] in first grid cell
  - 0 everywhere else
- Continuous linear mortars
- High order Godunov for advection
- Van Leer slope limiting with parameter 0.85
- Molecular diffusivity: 1.0 [sq ft / day]
- Physical (Diffusion)-Dispersion:
  - Longitudinal: 1.0 [ft]
  - Transverse: 0.2 [ft]

Production



Injection

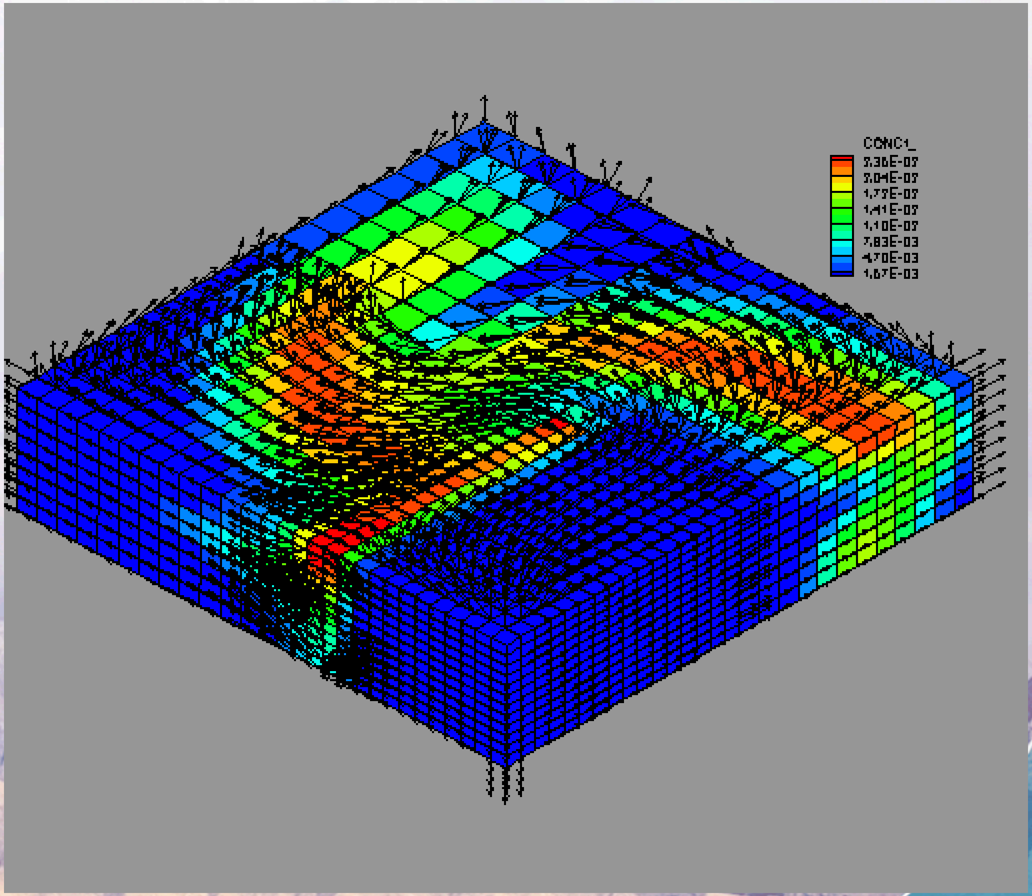
Numerical Results: Flow Around a Barrier

Avg. number of interface iterations

- No preconditioner: **27.5**
- Multiscale preconditioner: **4.0**

Total solve time

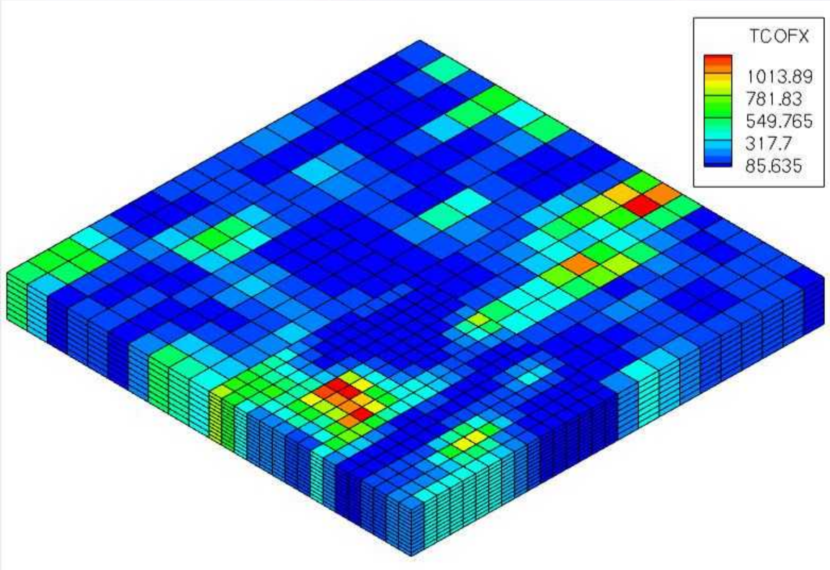
- No preconditioner: **7 hr**
- Multiscale preconditioner: **3 hr**



Numerical Results: Heterogeneous Permeability

- 24 ft x 400ft x 400ft
- Grid sizes
  - 2ft x 12.5ft x 12.5ft (12x16x16)
  - 3ft x 16.6ft x 16.6ft (8x12x12)
  - 3ft x 16.6ft x 16.6ft (8x12x12)
  - 4ft x 25ft x 25ft (6x8x8)
- Injection pressure: increase linearly
  - 505 [psi] (initially)
  - 1000 [psi] (50 days)
- Production pressure: decrease linearly
  - 480 [psi] (initially)
  - 350 [psi] (30 days)
- One species, initial concentrations:
  - 100 [M/cu-ft] in first grid cell
  - 0 everywhere else
- Continuous linear mortars
- High order Godunov for advection
- Van Leer slope limiting with parameter 0.85
- Molecular diffusivity: 1.0 [sq ft / day]
- Physical (Diffusion)-Dispersion:
  - Longitudinal: 1.0 [ft]
  - Transverse: 0.2 [ft]

Production



Injection

Numerical Results: Heterogeneous Permeability

Avg. number of interface iterations

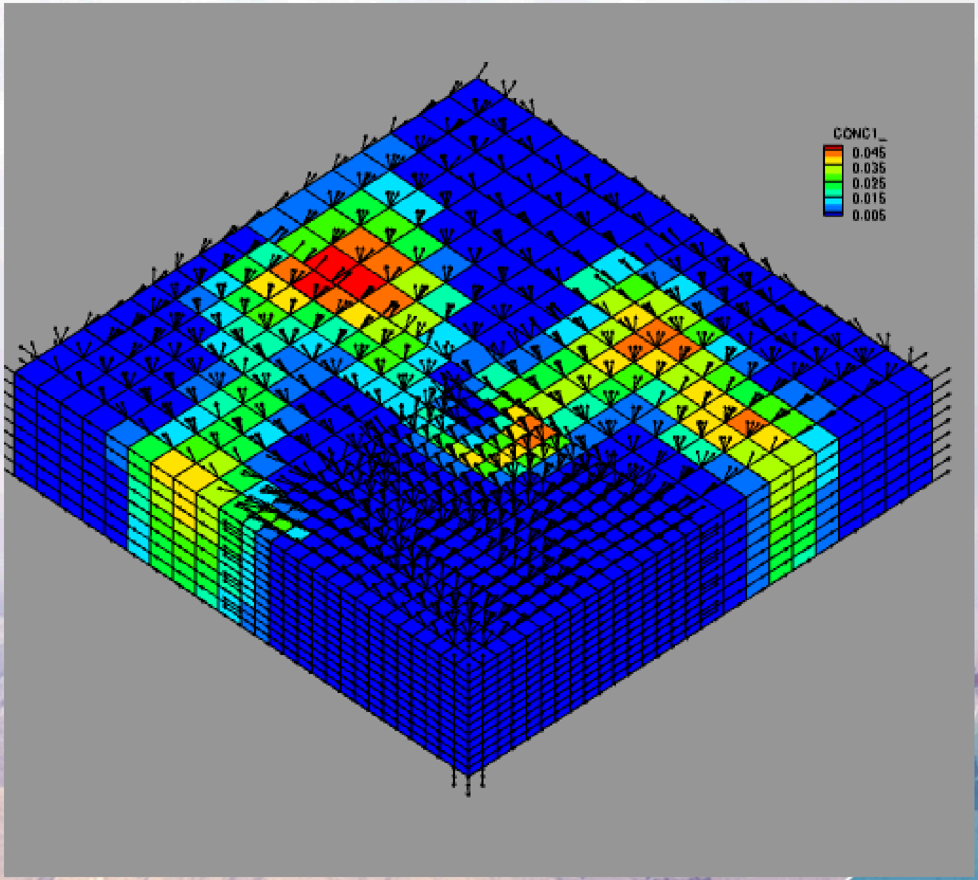
- No preconditioner: **56.4**
- Multiscale preconditioner: **3.5**

Total solve time

- No preconditioner: **66 min**
- Multiscale preconditioner: **21 min**

Time solving subdomain problems

- No preconditioner: **50 min**
- Multiscale preconditioner: **6 min**



Application to Fully Implicit Formulations for Multiphase Flow

Equations for Two Phase Flow

- Mass balance equation and Darcy Law:

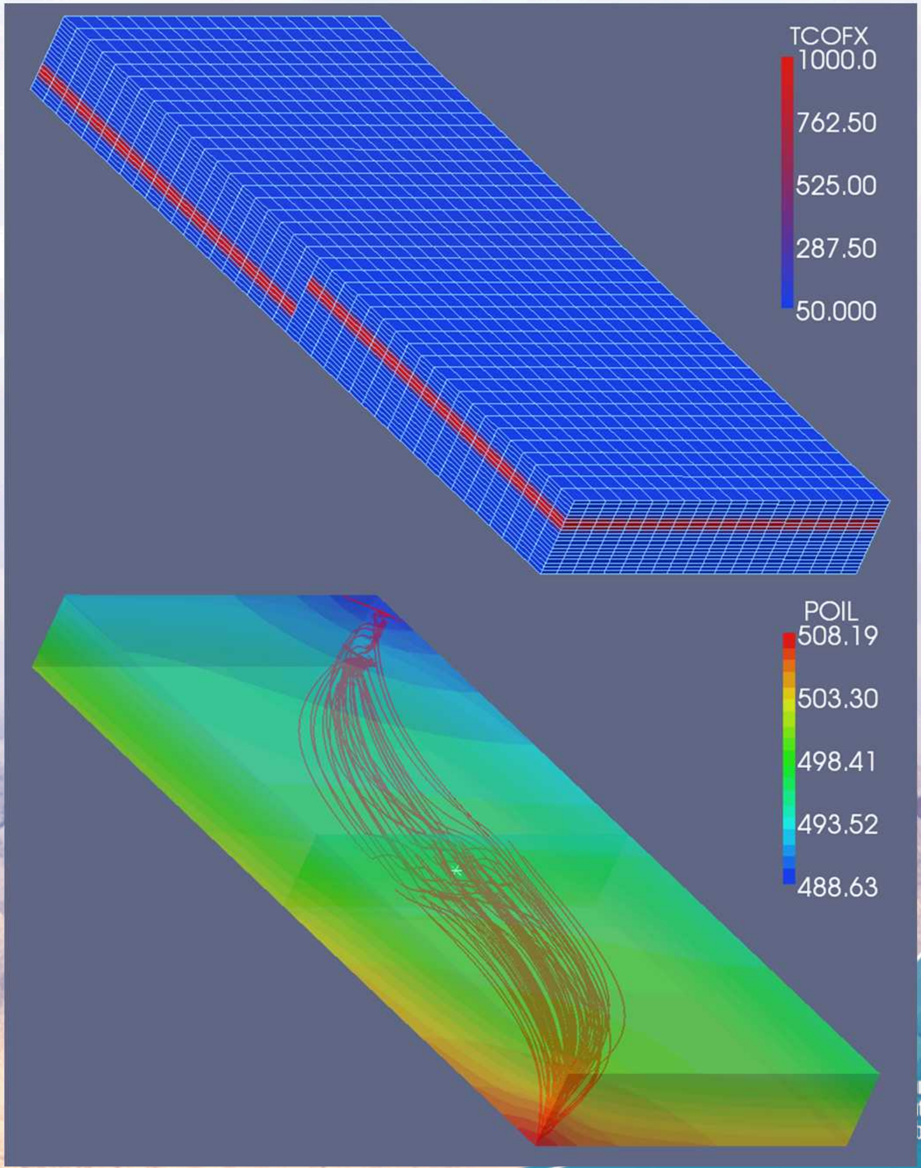
$$\frac{\partial}{\partial t}(\phi \rho_\alpha S_\alpha) + \nabla \cdot (\rho_\alpha \boldsymbol{u}_\alpha) = q_\alpha, \qquad \boldsymbol{u}_\alpha = -K \frac{k_{r,\alpha}}{\mu_\alpha} (\nabla P_\alpha - g \rho_\alpha \nabla D),$$

- Solve for a pressure and a concentration on subdomains
- Choice of interface variables
  - Both phase pressures: $\lambda_1 = P_w, \quad \lambda_2 = P_o$
  - One phase pressure and one concentration: $\lambda_1 = P_w, \quad \lambda_2 = N_o = \rho_o S_o$
- Match fluxes: $[\boldsymbol{u}_w] = 0, \; [\boldsymbol{u}_o] = 0.$
- The interface Jacobian has 2x2 block structure.

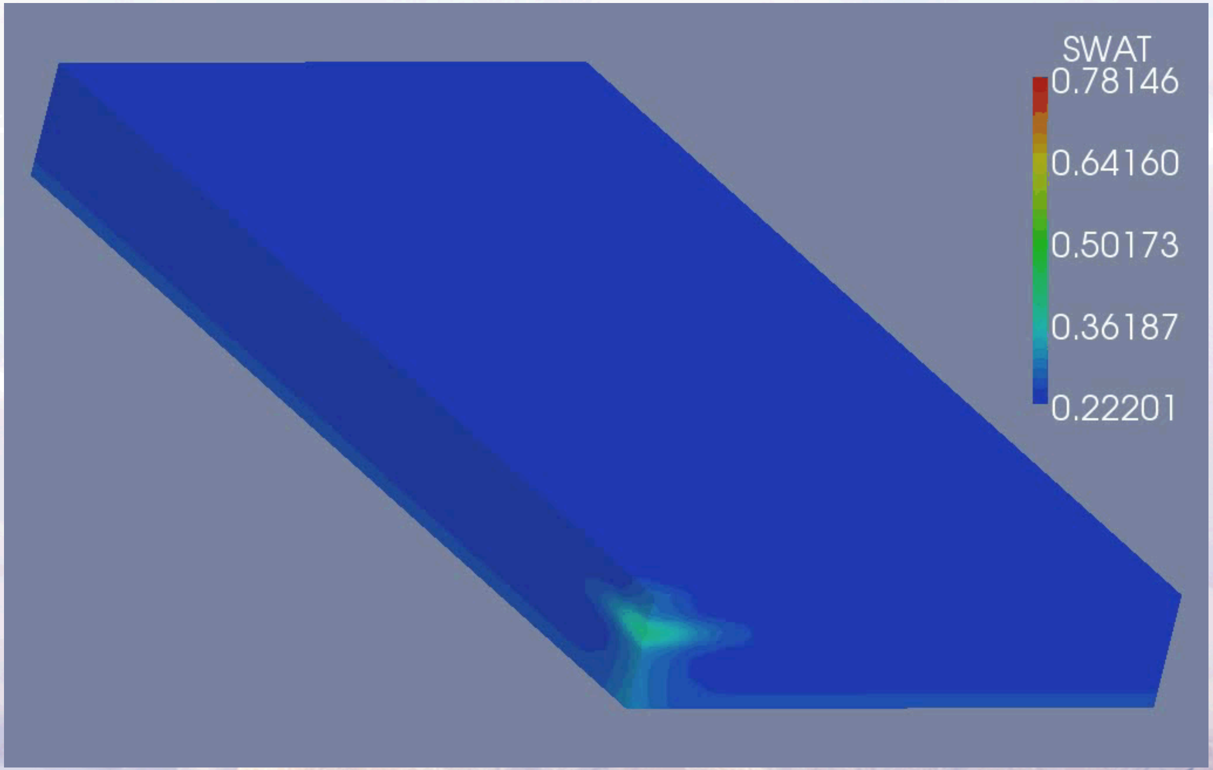
$$D_\delta B(\lambda) = \begin{pmatrix} \frac{\partial B_1}{\partial \lambda_1} & \frac{\partial B_1}{\partial \lambda_2} \\ \frac{\partial B_2}{\partial \lambda_1} & \frac{\partial B_2}{\partial \lambda_2} \end{pmatrix}$$

Numerical Results: Two Phase Flow

- 20 [ft] x 100 [ft] x 200 [ft]
- First grid (coarser)
  - 2 [ft] x 10 [ft] x 10 [ft] (10x10x10)
  - 2 [ft] x 10 [ft] x 10 [ft] (10x10x10)
- Second grid (finer)
  - 1 [ft] x 5 [ft] x 5 [ft] (20x20x20)
  - 1 [ft] x 5 [ft] x 5 [ft] (20x20x20)
- Layered permeability
- Initial pressure: 500 [psi]
- Initial water saturation: 0.22
- Injection pressure: 505 [psi]
- Production pressure: 495 [psi]
- Includes gravity and capillary pressure
- Discontinuous constant mortars
- Construct preconditioner from initial Jacobian.

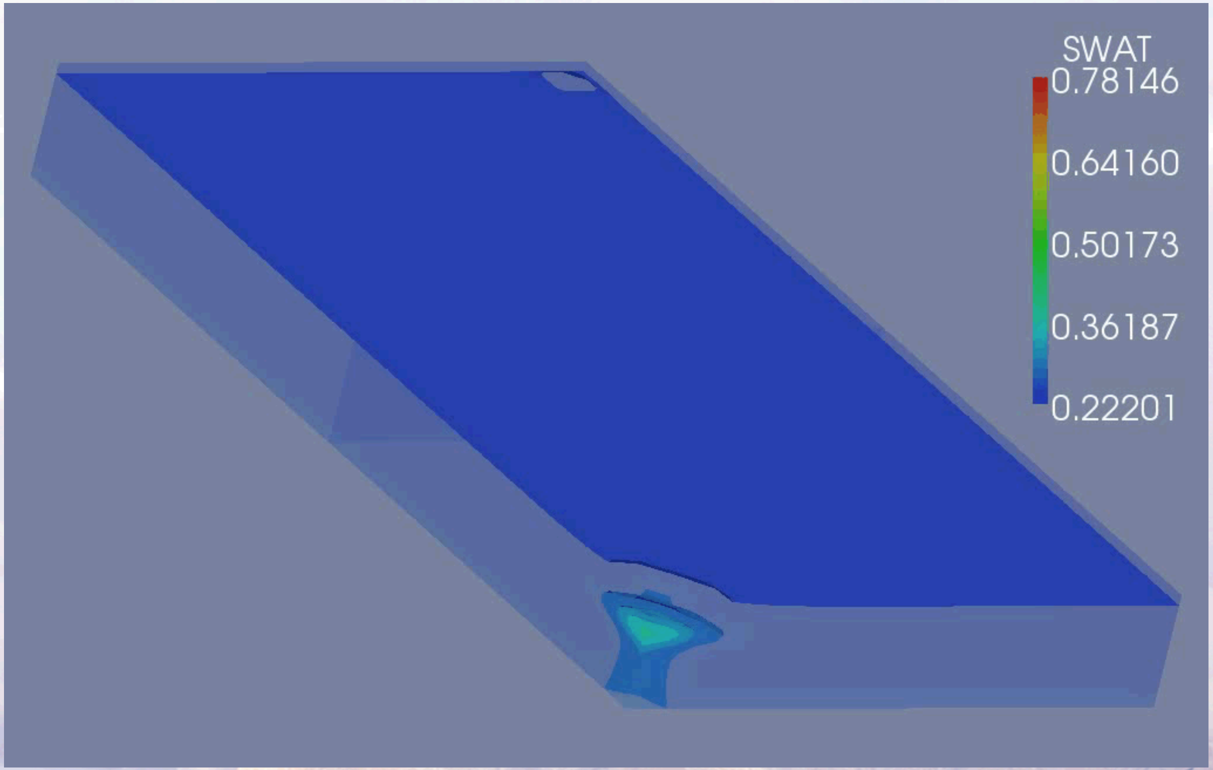


Numerical Results: Two Phase Flow



Coarser Grid	Avg. Interface Iterations	Time Per Newton Step (min)	Multiscale Precond. Assembly Time (min)	Total Time (min)
No Precond.	81.5	0.45	-	147.0
Multiscale Precond.	8.6	0.0037	0.42	12.2

Numerical Results: Two Phase Flow



Finer Grid	Avg. Interface Iterations	Time Per Newton Step (min)	Multiscale Precond. Assembly Time (min)	Total Time (min)
No Precond.	157.7	7.86	-	1730.3
Multiscale Precond.	7.2	0.41	15.7	105.7

Mortar Coupling for Elasticity and Poroelasticity

Relevant Work

Poromechanics

Biot	General theory of three-dimensional consolidation	1941
Showalter	Diffusion in poro-elastic media	2000
von Terzaghi	Theoretical Soil Mechanics	1943

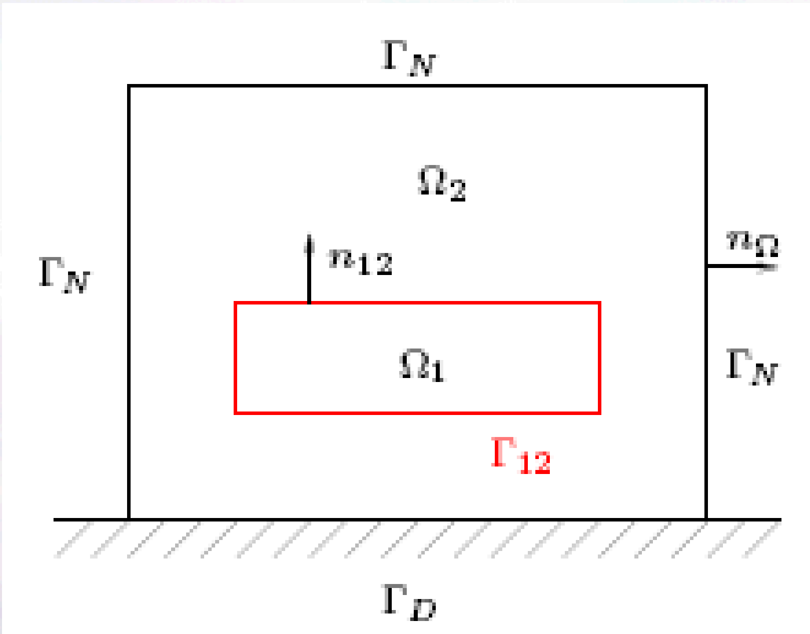
Coupled Geomechanical Models

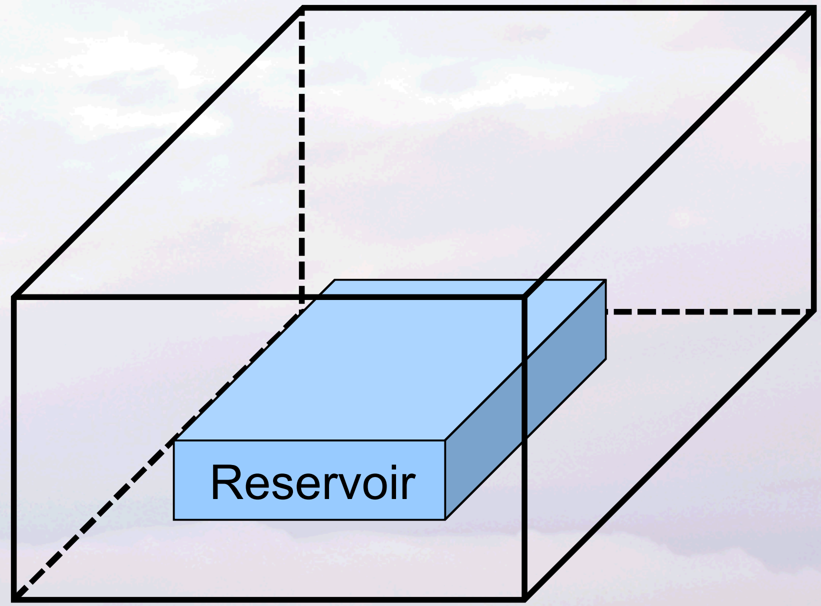
Chin et al	Fully coupled geomechanics and fluid flow ...	1998
Chin et al	Iteratively coupled analysis of ...	2002
Phillips, Wheeler	A coupling of mixed and continuous ...	2007-08
Settari, Mourits	Coupling of geomechanics and reservoir ...	1994

Domain Decomposition with Mortars

Girault et al	Coupling discontinuous Galerkin and mixed ...	2008
Girault et al	Domain decomposition for linear elasticity ...	2009

Domain Decomposition





Ω_1	Reservoir (pay-zone)
Ω_2	Nonpay-zone
Ω	$\Omega_1 \cup \Omega_2$
Γ_{12}	Interface between Ω_1 and Ω_2

Pay-zone Model: Poroelasticity

- Equation for Cauchy stress tensor: $\tilde{\boldsymbol{\sigma}} = \boldsymbol{\sigma} - \alpha p \boldsymbol{I}$

$\boldsymbol{u}$	displacement
p	fluid pressure
$\boldsymbol{\sigma}(\boldsymbol{u}) = \lambda(\operatorname{div} \boldsymbol{u})\boldsymbol{I} + 2\mu\,\epsilon(\boldsymbol{u})$	effective stress tensor
$\epsilon(\boldsymbol{u}) = \frac{1}{2}\left(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T\right)$	strain tensor
$\boldsymbol{I}$	identity tensor
$\lambda > 0,\ \mu > 0$	Lame coefficients
$\alpha > 0,$	Biot-Willis constant

- Darcy's Law: $\boldsymbol{v}_f = -\dfrac{\boldsymbol{K}}{\mu_f}\left(\nabla p - \rho_f \boldsymbol{g}\right)$

$\boldsymbol{v}_f$	fluid flux
$\boldsymbol{K}$	permeability tensor (SPD)
$\mu_f > 0,\ \rho_f > 0$	fluid viscocity and density
$\boldsymbol{g}$	gravitational force

Pay-zone Model: Poroelasticity

- Equation for mass conservation: $\dfrac{\partial \eta}{\partial t} = -\operatorname{div} \boldsymbol{v}_f + q$

$$\begin{array}{c|l} \eta & \text{fluid content of medium} \\ q & \text{volumetric fluid source term} \end{array}$$

- Fluid content is related to pressure and material volume

$$\eta = c_0\, p + \alpha \operatorname{div} \boldsymbol{u}$$

$$\begin{array}{c|l} c_0 & \text{constrained specific storage coefficient} \end{array}$$

- Balance of linear momentum under quasi-static assumption

$$-\operatorname{div} \tilde{\boldsymbol{\sigma}} = \boldsymbol{f}_1$$

$$\begin{array}{c|l} \boldsymbol{f}_1 & \text{body force in } \Omega_1 \end{array}$$

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Pay-zone Model: Poroelasticity

- Combining gives the system in $\Omega_1 \times (0, T]$:

$$\frac{\partial}{\partial t}(c_0 p + \alpha \operatorname{div} \boldsymbol{u}) - \frac{\boldsymbol{K}}{\mu_f}(\nabla p - \rho_f \boldsymbol{g}) = q$$
$$-(\lambda + \mu)\nabla(\operatorname{div} \boldsymbol{u}) - \mu \Delta \boldsymbol{u} + \alpha \nabla p = \boldsymbol{f}_1$$

- Requires initial condition on pressure: $p(0) = p_0$

- Initial condition for displacement must be compatible

$$-(\lambda + \mu)\nabla(\operatorname{div} \boldsymbol{u}_0) - \mu \Delta \boldsymbol{u}_0 + \alpha \nabla p_0 = \boldsymbol{f}_1(0)$$

- Plus adequate boundary or interface conditions

Nonpay-zone Model: Elasticity

- Linear elastic model in nonpay-zone
- Equations are steady but depend on transmission conditions
- In $\Omega_2 \times (0, T]$:

$$-(\lambda + \mu)\nabla(\operatorname{div} \boldsymbol{u}) - \mu \Delta \boldsymbol{u} = \boldsymbol{f}_2$$

$$\begin{array}{r|l} \boldsymbol{f}_2 & \text{body force in } \Omega_2 \\ \boldsymbol{u} = \boldsymbol{0} & \text{Displacement on } \Gamma_D \\ \boldsymbol{\sigma}(\boldsymbol{u})\, \boldsymbol{n}_\Omega = \boldsymbol{t}_N & \text{Traction on } \Gamma_N \end{array}$$

- Need to define interface conditions

A Coupled Model

- Define the jump and average along interface

$$[v] = (v|_{\Omega_1} - v|_{\Omega_2})\,|_{\Gamma_{12}}$$

$$\{v\} = \frac{1}{2}(v|_{\Omega_1} + v|_{\Omega_2})\,|_{\Gamma_{12}}$$

- Prescribe the following transmission conditions:

$$\begin{array}{r|l} [\boldsymbol{u}] = \mathbf{0} & \text{continuity of the medium} \\ [\boldsymbol{\sigma}(\boldsymbol{u})]\,\boldsymbol{n}_{12} = \alpha p\,\boldsymbol{n}_{12} & \text{continuity of normal stresses} \\ -\frac{\boldsymbol{K}}{\mu_f}(\nabla p - \rho_f \boldsymbol{g}) \cdot \boldsymbol{n}_{12} = 0 & \text{no flow on the interface} \end{array}$$

Variational Formulation

Define the space: $H^1_{0D}(\Omega) = \left\{ v \in H^1(\Omega) \, : \, v|_{\Gamma_D} = 0 \right\}$

Find $\boldsymbol{u} \in L^\infty\left(0, T; H^1_{0D}(\Omega)^d\right)$ and $p \in L^\infty\left(0, T; L^2(\Omega_1)\right) \cap L^2\left(0, T; H^1(\Omega_1)\right)$ such that

$$\int_\Omega \boldsymbol{\sigma}(\boldsymbol{u}) \, : \, \epsilon(\boldsymbol{v}) \; d\boldsymbol{x} - \alpha \int_{\Omega_1} p \, \mathrm{div} \, \boldsymbol{v} \; d\boldsymbol{x} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \; d\boldsymbol{x} + \int_{\Gamma_N} \boldsymbol{t}_N(s) \cdot \boldsymbol{v}(s) \; ds$$

$$\int_{\Omega_1} \frac{\partial}{\partial t}(c_0 p + \alpha \, \mathrm{div} \, \boldsymbol{u})\theta \; d\boldsymbol{x} + \int_{\Omega_1} \frac{\boldsymbol{K}}{\mu_f} \nabla p \cdot \nabla \theta \; d\boldsymbol{x} = \int_{\Omega_1} q \; \theta \; d\boldsymbol{x} + \int_{\Omega_1} \frac{\boldsymbol{K}}{\mu_f} \rho_f \boldsymbol{g} \cdot \nabla \theta \; d\boldsymbol{x}$$

for all $\boldsymbol{v} \in H^1_{0D}(\Omega)^d$ and $\theta \in H^1(\Omega_1)$

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Existence and Uniqueness

Theorem

Let $p_o \in H^1(\Omega)$, $\boldsymbol{f} \in H^1(0,T;L^2(\Omega)^d)$, $q \in L^2(\Omega_1 \times (0,T))$, and $\boldsymbol{t}_N \in H^1(0,T;L^2(\Gamma_N)^2)$. Then there exists a unique solution to the variational formulation.

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Discontinuous Galerkin on Interface

- For simplicity, define

$$\tilde{\alpha} = \begin{cases} \alpha, & \text{in}\, \Omega_1 \\ 0, & \text{in}\, \Omega_2 \end{cases}$$

- Transmission condition becomes: $[\boldsymbol{\sigma}(\boldsymbol{u}) - \tilde{\alpha} p \boldsymbol{I}]\, \boldsymbol{n}_{12} = \boldsymbol{0}$
- Allow test function to be discontinuous along interface.
- Interface term becomes

$$-\int_{\Gamma_{12}} \{\boldsymbol{\sigma}(\boldsymbol{u}) - \tilde{\alpha} p \boldsymbol{I}\}\, \boldsymbol{n}_{12} \cdot [\boldsymbol{v}]\; ds$$

- Since displacements are continuous, we can add

$$\int_{\Gamma_{12}} \{\boldsymbol{\sigma}(\boldsymbol{v})\}\, \boldsymbol{n}_{12} \cdot [\boldsymbol{u}]\; ds$$

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Discontinuous Galerkin on Interface

- Let $\Gamma_{h,i}$ be a conforming triangulation of Γ_{12} with elements of size h_{γ_i}
- Add the stabilization terms

$$(\lambda + 2\mu) \sum_{\gamma_i \in \Gamma_{h,i}} \frac{\sigma_{\gamma_i}}{h_{\gamma_i}} \int_{\gamma_i} [\boldsymbol{u}] \cdot [\boldsymbol{v}] \; ds$$

- To prove stability, we need the additional stabilization term

$$(\lambda + 2\mu) \sum_{\gamma_i \in \Gamma_{h,i}} \frac{\bar{\sigma}_{\gamma_i}}{h_{\gamma_i}} \int_{\gamma_i} [\boldsymbol{u}'] \cdot [\boldsymbol{v}] \; ds$$

- Similar terms are added to enforce the Dirichlet boundary condition.

DG with Lagrange Multipliers

Introduce the Lagrange multiplier $\boldsymbol{\lambda}$ on Γ_{12}.
Elasticity and poroelasticity equations become:

$$\sum_{i=1}^{2} \int_{\Omega_i} \boldsymbol{\sigma}(\boldsymbol{u}_i) : \epsilon(\boldsymbol{v}_i)\, d\boldsymbol{x} - \alpha \int_{\Omega_1} p \,\text{div}\, \boldsymbol{v}_1 \, d\boldsymbol{x} +$$

$$\sum_{i=1}^{2} \Big[- \int_{\Gamma_{12}} \boldsymbol{\sigma}(\boldsymbol{u}_i)\boldsymbol{n}_i \cdot \boldsymbol{v}_i \, ds + \int_{\Gamma_{12}} \boldsymbol{\sigma}(\boldsymbol{v}_i)\boldsymbol{n}_i \cdot \boldsymbol{u}_i \, ds \Big] + \alpha \int_{\Gamma_{12}} p \, \boldsymbol{n}_{12} \cdot \boldsymbol{v}_i \, ds$$

$$\sum_{i=1}^{2} \Big[(\lambda + 2\mu) \sum_{\gamma_i \in \Gamma_{h,i}} \Big(\frac{\sigma_{\gamma_i}}{h_{\gamma_i}} \int_{\gamma_i} \boldsymbol{u}_i \cdot \boldsymbol{v}_i \, ds + \frac{\bar{\sigma}_{\gamma_i}}{h_{\gamma_i}} \int_{\gamma_i} \boldsymbol{u}_i' \cdot \boldsymbol{v}_i \, ds \Big) \Big]$$

$$- \int_{\Gamma_D} \boldsymbol{\sigma}(\boldsymbol{u}_2)\boldsymbol{n}_\Omega \cdot \boldsymbol{v}_2 \, ds - \int_{\Gamma_D} \boldsymbol{\sigma}(\boldsymbol{v}_2)\boldsymbol{n}_\Omega \cdot \boldsymbol{u}_2 \, ds + \sum_{\gamma \in \Gamma_{h,D}} (\lambda + 2\mu) \frac{\sigma_\gamma}{h_\gamma} \int_\gamma \boldsymbol{u}_2 \cdot \boldsymbol{v}_2 \, ds$$

$$- \sum_{i=1}^{2} \Big[\int_{\Gamma_{12}} \boldsymbol{\sigma}(\boldsymbol{v}_i)\boldsymbol{n}_i \cdot \boldsymbol{\lambda} \, ds - (\lambda + 2\mu) \sum_{\gamma_i \in \Gamma_{h,i}} \Big(\frac{\sigma_{\gamma_i}}{h_{\gamma_i}} \int_{\gamma_i} \boldsymbol{\lambda} \cdot \boldsymbol{v}_i \, ds + \frac{\bar{\sigma}_{\gamma_i}}{h_{\gamma_i}} \int_{\gamma_i} \boldsymbol{\lambda}' \cdot \boldsymbol{v}_i \, ds \Big) \Big]$$

$$= \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \, d\boldsymbol{x} + \int_{\Gamma_N} \boldsymbol{t}_N(s) \cdot \boldsymbol{v}(s) \, ds$$

Interface term based on DG method (NIPG)

Interface terms from integration by parts

Usual interior terms

Penalty terms on interface

Lagrange multiplier terms

Analogous terms on Dirichlet boundary

Usual source terms

DG with Lagrange Multipliers

Interface equation:

$$-(\lambda + 2\mu) \sum_{i=1}^{2} \sum_{\gamma_i \in \Gamma_{h,i}} \Big(\frac{\sigma_{\gamma_i}}{h_{\gamma_i}} \int_{\gamma_i} (\boldsymbol{u}_i - \boldsymbol{\lambda}) \cdot \boldsymbol{\mu} \; ds + \frac{\bar{\sigma}_{\gamma_i}}{h_{\gamma_i}} \int_{\gamma_i} (\boldsymbol{u}_i' - \boldsymbol{\lambda}') \cdot \boldsymbol{\mu} \; ds \Big) \Big]$$

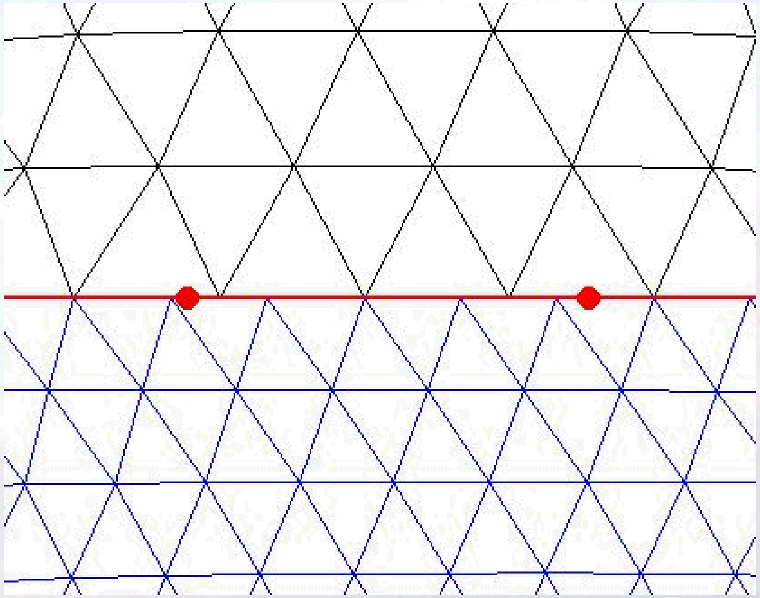
$$+ \int_{\Gamma_{12}} ([\boldsymbol{\sigma}(\boldsymbol{u})] - \alpha \; p) \, \boldsymbol{n}_{12} \cdot \boldsymbol{\mu} \; ds = 0$$

Pressure equation is unchanged:

$$\int_{\Omega_1} \frac{\partial}{\partial t}(c_0 p + \alpha \text{ div } \boldsymbol{u})\theta \; d\boldsymbol{x} + \int_{\Omega_1} \frac{\boldsymbol{K}}{\mu_f} \nabla p \cdot \nabla \theta \; d\boldsymbol{x} =$$

$$\int_{\Omega_1} q \; \theta \; d\boldsymbol{x} + \int_{\Omega_1} \frac{\boldsymbol{K}}{\mu_f} \rho_f \boldsymbol{g} \cdot \nabla \theta \; d\boldsymbol{x}$$

Discretizations



$\mathcal{T}_{h,i}, i = 1, 2$	two independent regular triangulations of size h_i
Γ_H	triangulation of Γ_{12} of size H
f^n	$f(t^n)$
$\boldsymbol{u}_{h_i}^n \in \boldsymbol{X}_{h,i}$	continuous in Ω_1 and Ω_2, piecewise $\mathbb{P}_k^d$, $k \geq 1$
$p_h^n \in M_{h,1}$	continuous in Ω_1, piecewise $\mathbb{P}_m$, $m \geq 1$
$\boldsymbol{\lambda}_H^n \in \boldsymbol{\Lambda}_H$	continuous in Γ_{12}, piecewise $\mathbb{P}_l^d$, $l \geq 1$
Time Stepping	Backward Euler with fixed Δt

Simplified Notation: (Poro)Elasticity

$$\sum_{i=1}^{2}\int_{\Omega_i}\boldsymbol{\sigma}(\boldsymbol{u}_{h_i}^n) : \epsilon(\boldsymbol{v}_i)\,d\boldsymbol{x} - \alpha\int_{\Omega_1} p_h^n \operatorname{div}\boldsymbol{v}_1\,d\boldsymbol{x} +$$

$$\sum_{i=1}^{2}\Big[-\int_{\Gamma_{12}}\boldsymbol{\sigma}(\boldsymbol{u}_{h_i}^n)\boldsymbol{n}_i\cdot\boldsymbol{v}_i\,ds + \int_{\Gamma_{12}}\boldsymbol{\sigma}(\boldsymbol{v}_i)\boldsymbol{n}_i\cdot\boldsymbol{u}_{h_i}^n\,ds\Big] + \alpha\int_{\Gamma_{12}} p_h^n\,\boldsymbol{n}_{12}\cdot\boldsymbol{v}_i\,ds$$

$$\sum_{i=1}^{2}\Big[(\lambda+2\mu)\sum_{\gamma_i\in\Gamma_{h,i}}\Big(\frac{\sigma_{\gamma_i}}{h_{\gamma_i}}\int_{\gamma_i}\boldsymbol{u}_{h_i}^n\cdot\boldsymbol{v}_i\,ds + \frac{\bar{\sigma}_{\gamma_i}}{h_{\gamma_i}}\frac{1}{\Delta t}\int_{\gamma_i}\boldsymbol{u}_{h_i}^n\cdot\boldsymbol{v}_i\,ds\Big)\Big]$$

$$-\int_{\Gamma_D}\boldsymbol{\sigma}(\boldsymbol{u}_{h_2}^n)\boldsymbol{n}_\Omega\cdot\boldsymbol{v}_2\,ds - \int_{\Gamma_D}\boldsymbol{\sigma}(\boldsymbol{v}_2)\boldsymbol{n}_\Omega\cdot\boldsymbol{u}_{h_2}^n\,ds + \sum_{\gamma\in\Gamma_{h,D}}(\lambda+2\mu)\frac{\sigma_\gamma}{h_\gamma}\int_\gamma\boldsymbol{u}_{h_2}^n\cdot\boldsymbol{v}_2\,ds$$

$$-\sum_{i=1}^{2}\Big[\int_{\Gamma_{12}}\boldsymbol{\sigma}(\boldsymbol{v}_i)\boldsymbol{n}_i\cdot\boldsymbol{\lambda}_H^n\,ds - (\lambda+2\mu)\sum_{\gamma_i\in\Gamma_{h,i}}\Big(\frac{\sigma_{\gamma_i}}{h_{\gamma_i}}\int_{\gamma_i}\boldsymbol{\lambda}_H^n\cdot\boldsymbol{v}_i\,ds + \frac{\bar{\sigma}_{\gamma_i}}{h_{\gamma_i}}\int_{\gamma_i}\frac{1}{\Delta t}\boldsymbol{\lambda}_H^n\cdot\boldsymbol{v}_i\,ds\Big)\Big]$$

$$= \int_\Omega \boldsymbol{f}^n\cdot\boldsymbol{v}\,d\boldsymbol{x} + \int_{\Gamma_N}\boldsymbol{t}_N^n(s)\cdot\boldsymbol{v}(s)\,ds$$

$$+\sum_{i=1}^{2}\sum_{\gamma_h\in\Gamma_{h,i}}(\lambda+2\mu)\frac{\bar{\sigma}_{\gamma_i}}{h_{\gamma_i}}\frac{1}{\Delta t}\int_{\gamma_h}(\boldsymbol{u}_{h_i}^{n-1} - \boldsymbol{\lambda}_H^{n-1})\cdot\boldsymbol{v}_i\,ds$$

$$B(\boldsymbol{u}_h^n,\boldsymbol{v}) + D(\boldsymbol{\lambda}_H^n,\boldsymbol{v}) + C(p_h^n,\boldsymbol{v}) = l^n(\boldsymbol{v})$$

Simplified Notation: Interface

$$\int_{\Gamma_{12}} [\boldsymbol{\sigma}(\boldsymbol{u}_{h_i}^n)]\boldsymbol{n}_{12} \cdot \boldsymbol{\mu}\ ds - \int_{\Gamma_{12}} \alpha\ p_h^n \boldsymbol{n}_{12} \cdot \boldsymbol{\mu}\ ds$$

$$- (\lambda + 2\mu) \sum_{i=1}^{2} \sum_{\gamma_h \in \Gamma_{h,i}} \left(\frac{\sigma_{\gamma_i}}{h_{\gamma_i}} + \frac{\bar{\sigma}_{\gamma_i}}{h_{\gamma_i}} \frac{1}{\Delta t} \right) \int_{\gamma_h} (\boldsymbol{u}_{h_i}^n - \boldsymbol{\lambda}_H^n) \cdot \boldsymbol{\mu}\ ds$$

$$= -(\lambda + 2\mu) \sum_{i=1}^{2} \sum_{\gamma_h \in \Gamma_{h,i}} \frac{\bar{\sigma}_{\gamma_i}}{h_{\gamma_i}} \frac{1}{\Delta t} \int_{\gamma_h} (\boldsymbol{u}_{h_i}^{n-1} - \boldsymbol{\lambda}_H^{n-1}) \cdot \boldsymbol{\mu}\ ds$$

$$G(\boldsymbol{u}_h^n, \boldsymbol{\mu}) + M(\boldsymbol{\lambda}_H^n, \boldsymbol{\mu}) + H(p_h^n, \boldsymbol{\mu}) = j^n(\boldsymbol{\mu})$$

Simplified Notation: Flow

$$\frac{\alpha}{\Delta t}\int_{\Omega_1} \text{div}\ \boldsymbol{u}_{h_1}^n\ \theta\ d\boldsymbol{x} + \frac{c_0}{\Delta t}\int_{\Omega_1} p_h^n\ \theta\ d\boldsymbol{x} + \frac{1}{\mu_f}\int_{\Omega_1} \boldsymbol{K}\nabla p_h^n \cdot \nabla\theta\ d\boldsymbol{x}$$

$$= \frac{\rho_f}{\mu_f}\int_{\Omega_1} \boldsymbol{K}\boldsymbol{g}\cdot\nabla\theta\ d\boldsymbol{x} + \int_{\Omega_1} q^n\ \theta\ d\boldsymbol{x} + \frac{\alpha}{\Delta t}\int_{\Omega_1} \text{div}\ \boldsymbol{u}_{h_1}^{n-1}\ \theta\ d\boldsymbol{x} + \frac{c_0}{\Delta t}\int_{\Omega_1} p_h^{n-1}\ \theta\ d\boldsymbol{x}$$

$$T(\boldsymbol{u}_h^n, \theta) + A(p_h^n, \theta) = q^n(\theta) + T(\boldsymbol{u}_h^{n-1}, \theta)$$

Simplified Notation

$$
\begin{array}{r|l}
\text{(Poro)Elasticity} & B(\boldsymbol{u}_h^n, \boldsymbol{v}) + D(\boldsymbol{\lambda}_H^n, \boldsymbol{v}) + C(p_h^n, \boldsymbol{v}) = l^n(\boldsymbol{v}) \\
\text{Interface} & G(\boldsymbol{u}_h^n, \boldsymbol{\mu}) + M(\boldsymbol{\lambda}_H^n, \boldsymbol{\mu}) + H(p_h^n, \boldsymbol{\mu}) = j^n(\boldsymbol{\mu}) \\
\text{Flow} & T(\boldsymbol{u}_h^n, \theta) \qquad\qquad + A(p_h^n, \theta) = q^n(\theta) + T(\boldsymbol{u}_h^{n-1}, \theta)
\end{array}
$$

Time Stepping Algorithm

For *n = 0* ➡ Set $p_h^0 = \pi_h p_0$, $\boldsymbol{u}_{h_i}^0 = I_{h_i}\boldsymbol{u}_0$, $\boldsymbol{\lambda}_H^0 = \rho_H \boldsymbol{\lambda}_0 = \rho_H \boldsymbol{u}_0|_{\Gamma_{12}}$

For *n = 1* ➡ ?

For *n > 1* ➡ Given p_h^{n-1}, $\boldsymbol{u}_{h_i}^{n-2}$, $\boldsymbol{u}_{h_i}^{n-1}$, $\boldsymbol{\lambda}_H^{n-1}$,

- Compute p_h^n by solving the reaction-diffusion equation:

$$A(p_h^n, \theta) = q^n(\theta) + T(\boldsymbol{u}_h^{n-2}, \theta) - T(\boldsymbol{u}_h^{n-1}, \theta)$$

- Compute $\boldsymbol{u}_{h_i}^n$ and $\boldsymbol{\lambda}_H^n$ by solving the elasticity equation:

$$B(\boldsymbol{u}_h^n, \boldsymbol{v}) + D(\boldsymbol{\lambda}_H^n, \boldsymbol{v}) = l^n(\boldsymbol{v}) - C(p_h^n, \boldsymbol{v}),$$
$$G(\boldsymbol{u}_h^n, \boldsymbol{\mu}) + M(\boldsymbol{\lambda}_H^n, \boldsymbol{\mu}) = j^n(\boldsymbol{\mu}) - H(p_h^n, \boldsymbol{\mu}),$$

Time Stepping Algorithm

For n = 1 ➡

Given p_h^0, $\boldsymbol{u}_{h_i}^0$, $\boldsymbol{\lambda}_H^0$,

- Compute $\boldsymbol{u}_{h_i}^1$ and $\boldsymbol{\lambda}_H^1$ by solving the elasticity equation:

$$B(\boldsymbol{u}_h^1, \boldsymbol{v}) + D(\boldsymbol{\lambda}_H^1, \boldsymbol{v}) = l^1(\boldsymbol{v}) - C(p_h^0, \boldsymbol{v}),$$
$$G(\boldsymbol{u}_h^1, \boldsymbol{\mu}) + M(\boldsymbol{\lambda}_H^1, \boldsymbol{\mu}) = j^1(\boldsymbol{\mu}) - H(p_h^0, \boldsymbol{\mu}),$$

- Compute p_h^1 by solving the reaction-diffusion equation:

$$A(p_h^1, \theta) = q^1(\theta) + T(\boldsymbol{u}_h^0, \theta) - T(\boldsymbol{u}_h^1, \theta)$$

The Interface Equation

- For each time step we solve the elasticity equation:

$$
\begin{aligned}
B\boldsymbol{u}^n + D\boldsymbol{\lambda}^n &= l^n - Cp^n, \\
G\boldsymbol{u}^n + M\boldsymbol{\lambda}^n &= j^n - Hp^n,
\end{aligned}
$$

- Relatively few degrees of freedom for Lagrange multipliers
- Define the Schur complement (Steklov-Poincare operator):

$$S = M - GB^{-1}D$$

- Solve the interface equation:

$$S\boldsymbol{\lambda} = j^n - Hp^n - GB^{-1}(l^n - Cp^n)$$

- S is positive definite, but not symmetric.
- S is time invariant if elasticity parameters are static and constant time steps.

Error Estimates

Let δ denote the discrete difference in time operator, e.g.,

$$\delta \boldsymbol{u}_{h_i}^n = \boldsymbol{u}_{h_i}^n - \boldsymbol{u}_{h_i}^{n-1}$$

Define the semi-norm,

$$|\boldsymbol{v}|_{\mathbf{X}_h}^2 = \sum_{i=1}^{2} \left(\lambda \|\text{div } \boldsymbol{v}_i\|_{L^2(\Omega_i)}^2 + 2\mu \|\boldsymbol{\epsilon}(\boldsymbol{v})\|_{L^2(\Omega_i)}^2 \right)$$

Assume the exact solution has the following regularity:

$$\boldsymbol{u}_i \in H^1(0,T; H^{s_{\boldsymbol{u}}}(\Omega_i)^3), \quad \text{div } \boldsymbol{u}_1'' \in L^2(\Omega_1 \times (0,T))$$

$$p \in H^1(0,T; H^{s_p}(\Omega_1)), \quad p'' \in L^2(\Omega_1 \times (0,T))$$

with $s_{\boldsymbol{u}} > 3/2$ and $s_p > 1/2$.

Error Estimates

Assumption

There exists a constant C independent of h_1, h_2, and H such that for any $\tau \in \Gamma_H$ and any $\gamma_i \in \Gamma_{h,i}$ that intersects τ,

$$\frac{H_\tau}{h_{\gamma_i}} \leq C \frac{H}{h_i}$$

This assumption is needed to prove the interpolation result,

$$\sum_{\gamma_i \in \Gamma_{h,i}} \frac{\sigma_{\gamma_i}}{h_{\gamma_i}} \|\boldsymbol{\lambda}(t^n) - \rho_H \boldsymbol{\lambda}(t^n)\|^2_{L^2(\gamma_i)} \leq C \left(\frac{H}{h_i}\right) H^{2r_{\boldsymbol{\lambda}}-1} \|\boldsymbol{\lambda}\|^2_{L^\infty(0,t^n;H^{r_{\boldsymbol{\lambda}}}(\Gamma_{12})}$$

where $r_{\boldsymbol{\lambda}} = \min(l+1, s_{\boldsymbol{u}} - 1/2)$.

Error Estimates

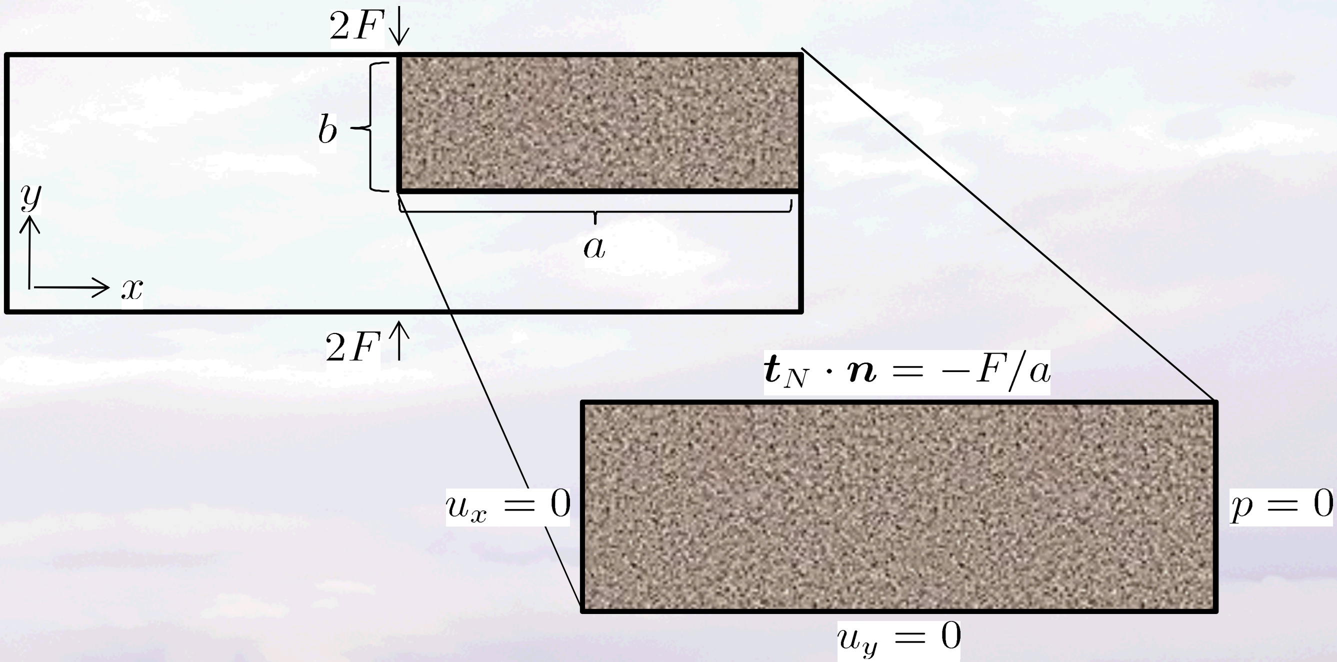
Theorem

Assume the triangulations are regular in the sense of Ciarlet and satisfy Assumption 1. Then there exist constants C_1 and C_2 independent of $h_1, h_2, H, \Delta t$ and n such that for $n \geq 2$,

$$E_{\boldsymbol{u}}^n + E_p^n + E_{\Gamma_{12}}^n + E_{\Gamma_D}^n \leq C_1 \exp(C_2 t^n) \Big(\Delta t^2 + h_1^{2(r_{\boldsymbol{u}}-1)} + h_2^{2(r_{\boldsymbol{u}}-1)} + h_1^{2(r_p-1)} + \left(\frac{H}{h_1} + \frac{H}{h_2} \right) H^{2r_{\boldsymbol{\lambda}}-1} \Big)$$

where $r_{\boldsymbol{u}} = \min(k+1, s_{\boldsymbol{u}})$, $r_p = \min(m+1, s_p)$, $r_{\boldsymbol{\lambda}} = \min(l+1, s_{\boldsymbol{u}} - 1/2)$.

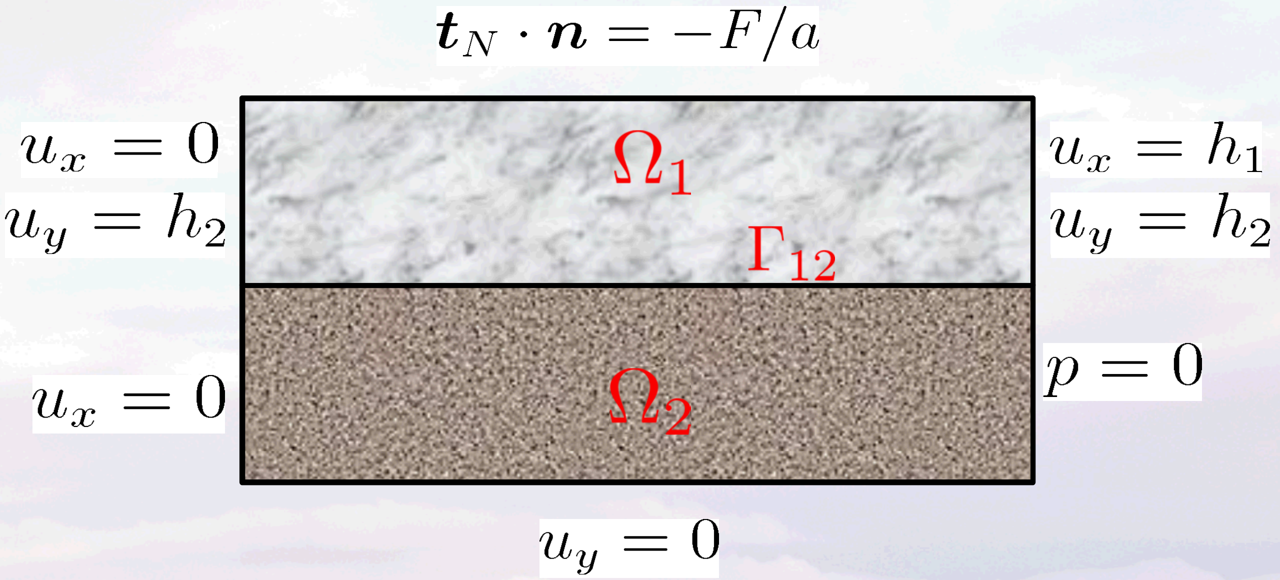
Mandel's Problem



- Infinite domain subjected to a loading of 2F from above and below
- Symmetry allows domain to be reduced to 2D upper right quadrant
- Analytical solution for pressure and displacements[1]
- Solution demonstrates the Mandel-Creyer effect.

[1] *Mandel 1953, Abouslieman et al 1996*

Extension of Mandel's Problem



α	1
E	1×10^4
ν	0.2
K	100
F	2000
c_0	0.1
Δt	1×10^{-8}

- Extend Mandel's problem to include an elastic domain

$$u_x^{\text{nonpay}} = u_x^{\text{pay}} \qquad u_y^{\text{nonpay}} = u_y^{\text{pay}} - \frac{\alpha}{\lambda + 2\mu}(y-b)p$$

- Piecewise linear finite elements and continuous linear mortars with H = h
- Small time step to isolate spatial discretization error

Extension of Mandel's Problem

Total Degrees of Freedom	Error in Energy Norm	Predicted Rate	Observed Rate
1,197	1.07E-2		
4,387	4.91E-3	1	1.09
16,767	2.34E-3	1	1.05
65,527	1.14E-3	1	1.03
259,047	5.65E-4	1	1.01

Table: Numerical error after 100 time steps

A Posteriori Error Bounds for Multiscale and Multinumerics and Mortar Coupling

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What is/should be an a posteriori error estimate?

Usual form

- $\|p - p_h\|^2 \lesssim \sum_{T \in \mathcal{T}_h} \eta_T(p_h)^2$
- Can be used to determine mesh elements with large error.
- We can then refine these elements: mesh adaptivity.

Reliability

- $\|p - p_h\|^2 \leq C \sum_{T \in \mathcal{T}_h} \eta_T(p_h)^2$

Guaranteed upper bound (evaluation of the constants)

- $\|p - p_h\|^2 \leq \sum_{T \in \mathcal{T}_h} \eta_T(p_h)^2$

Local efficiency

- $\eta_T(p_h)^2 \leq C_{\mathrm{eff},T}^2 \sum_{K \text{ close to } T} \|p - p_h\|_K^2$

Asymptotic exactness

- $\sum_{T \in \mathcal{T}_h} \eta_T(p_h)^2 / \|p - p_h\|^2 \to 1$

Robustness

- Independence of the data variation or mesh properties

Negligible evaluation cost

- Estimators which can be evaluated locally

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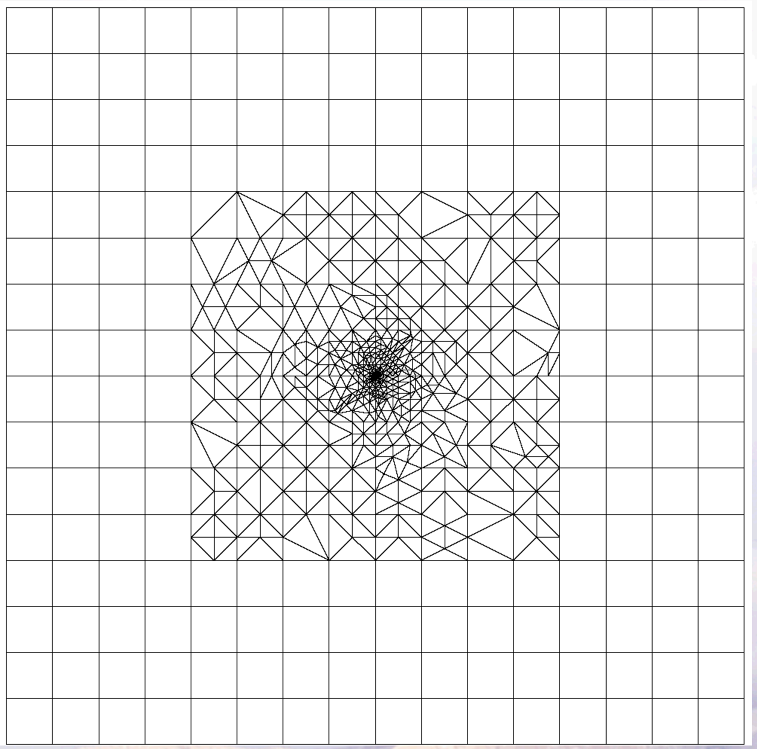
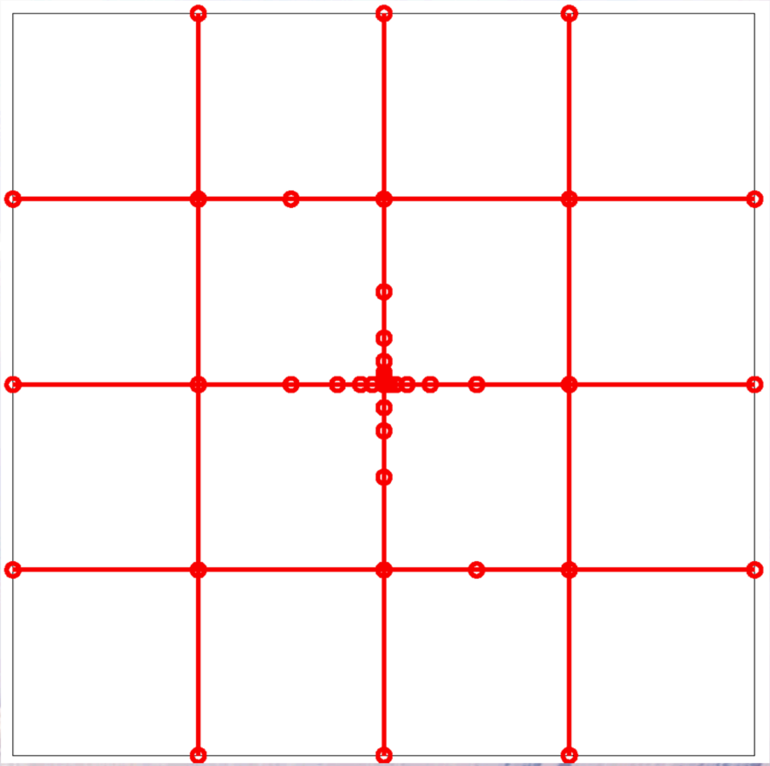
Adaptive Meshes



Adaptive subdomain mesh



Adaptive mortar mesh

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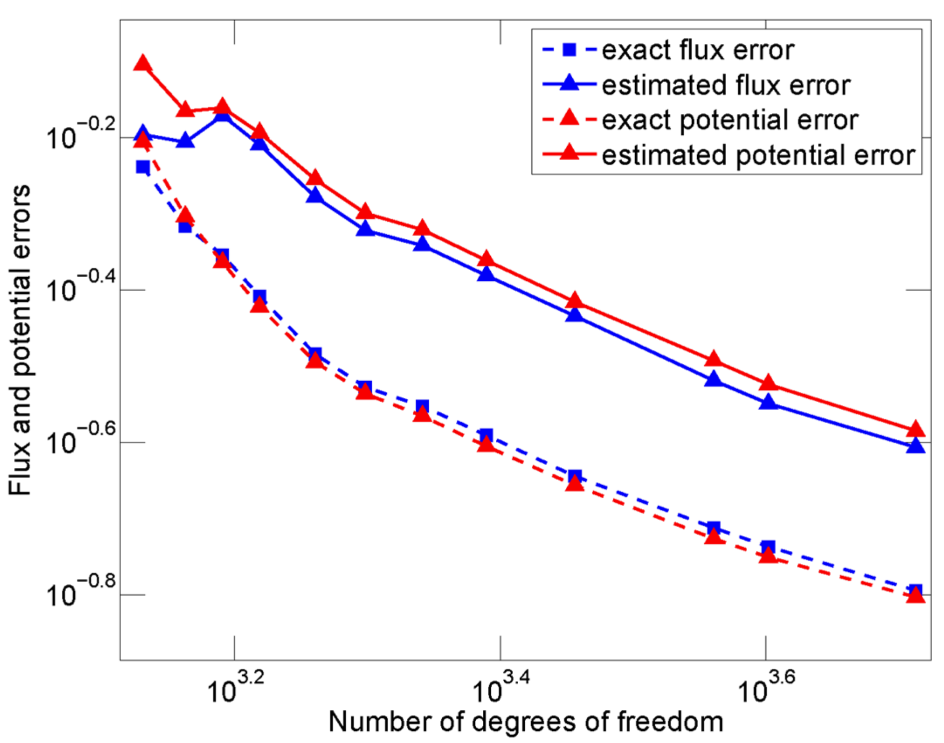
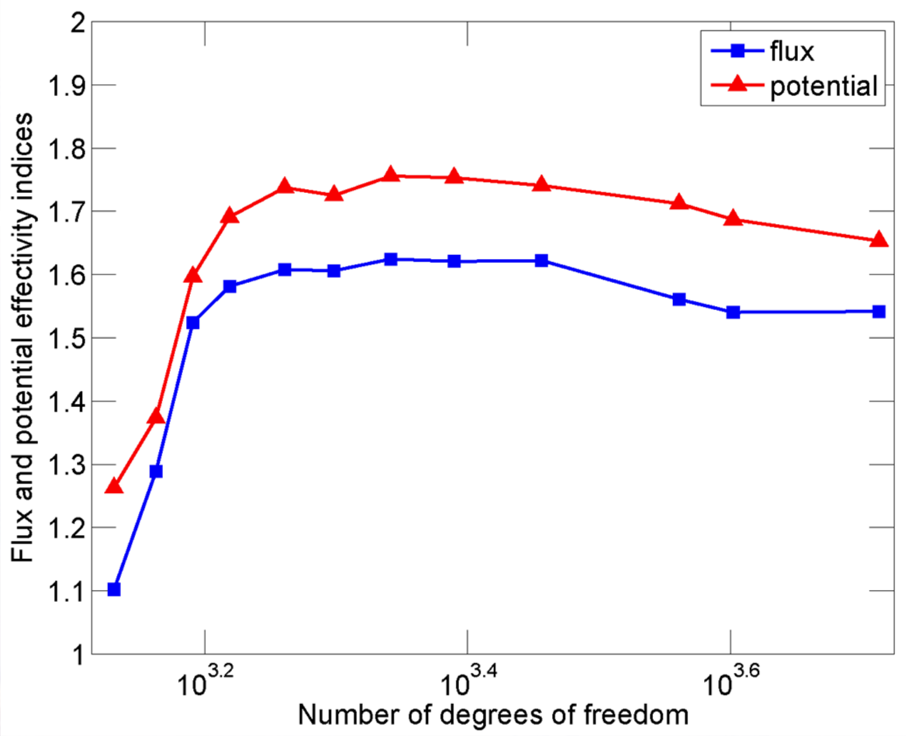
Flux and Potential Errors and Effectivity Indices

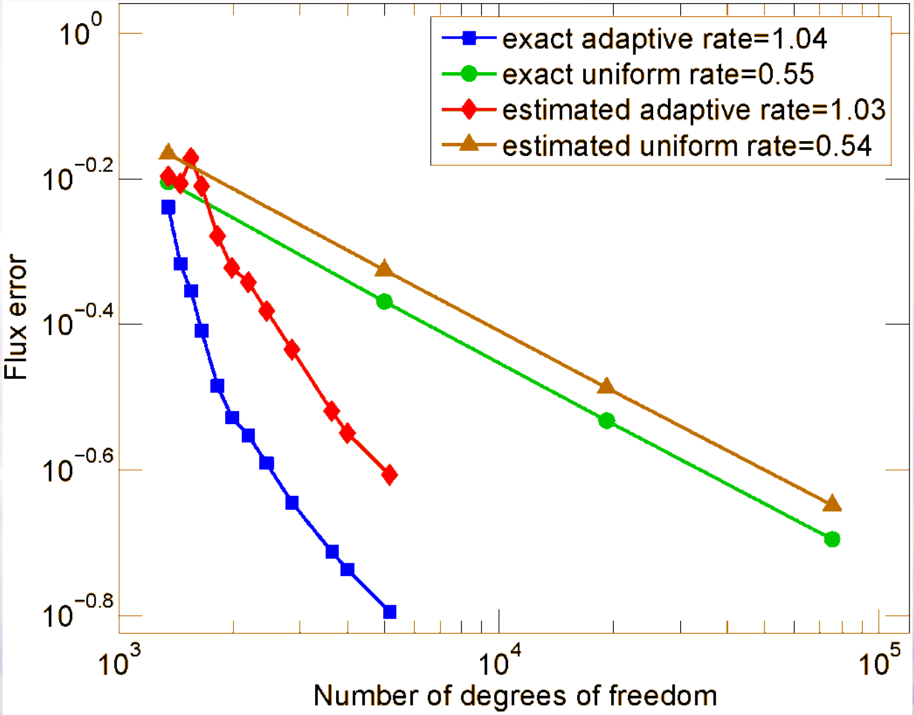


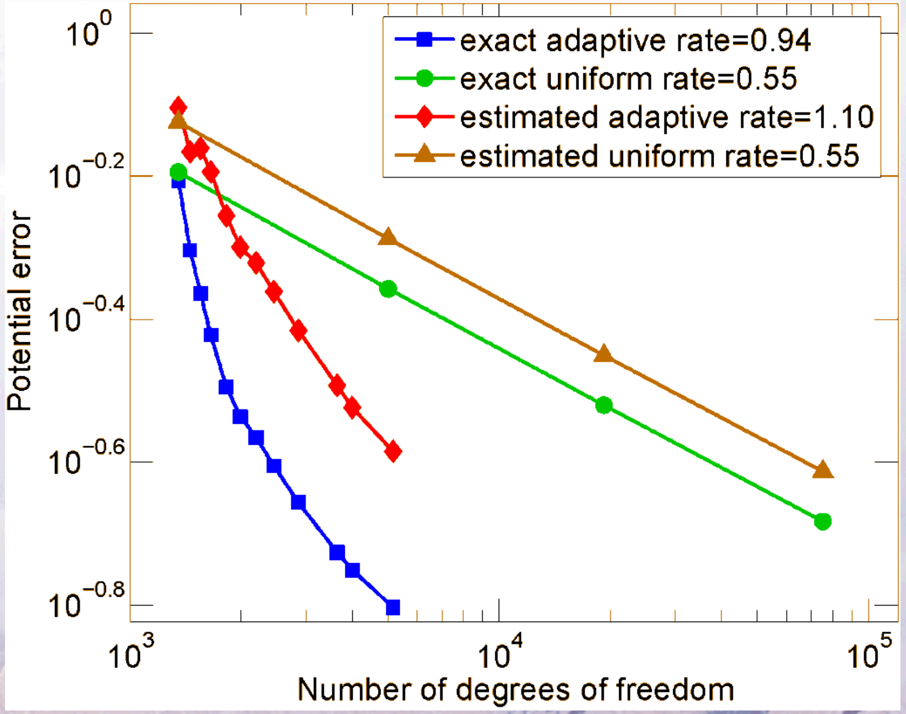


Estimated and actual flux error and potential error

Effectivity indices

Flux and Potential Errors





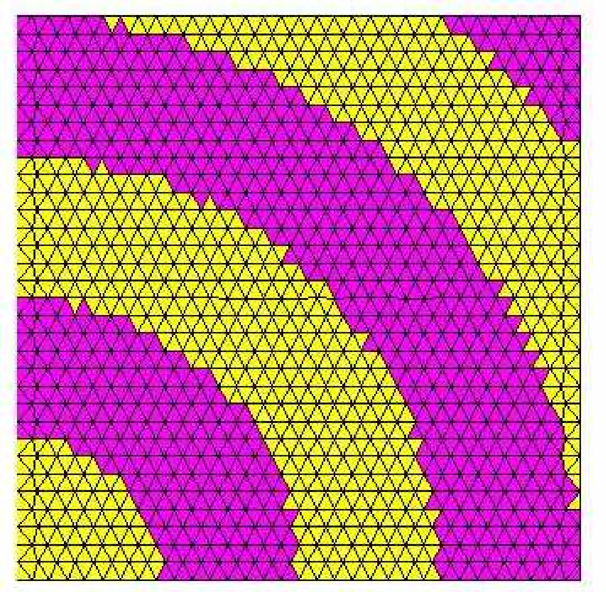
Estimated and actual flux error

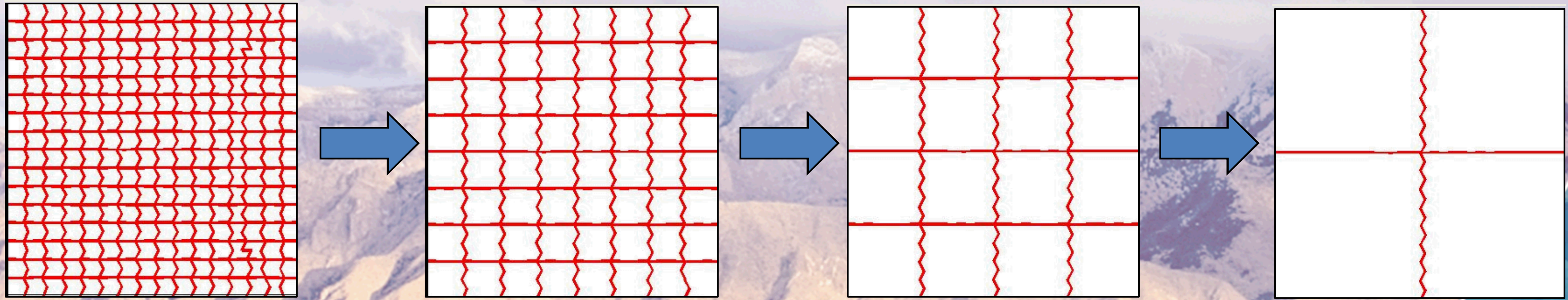
Potential error

Multilevel Solvers for Hybridized Formulations

Laplace Equation – Multinumerics

Model	Laplace equation
Domain	$(0,1) \times (0,1)$
Method	DG-NIPG$_1$ (Yellow)
Method	Mixed-RT$_1$ (Pink)
Mesh	Triangles
True solution	$p = xye^{x^2 y^3}$
Tolerance	1×10^{-6}





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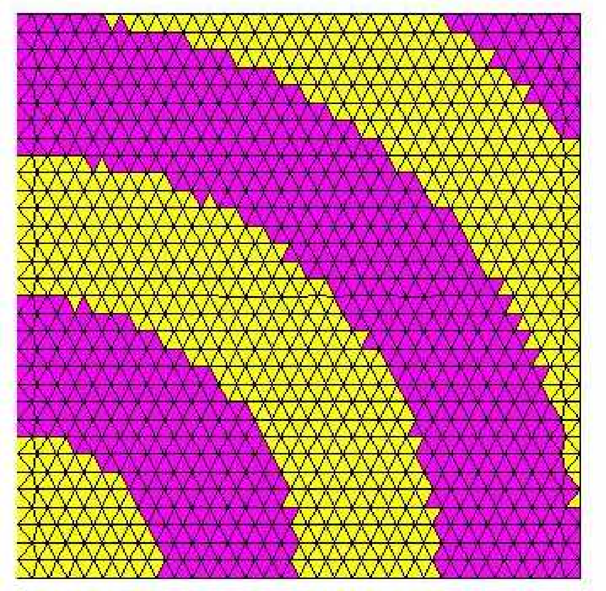
Laplace Equation – Multinumerics

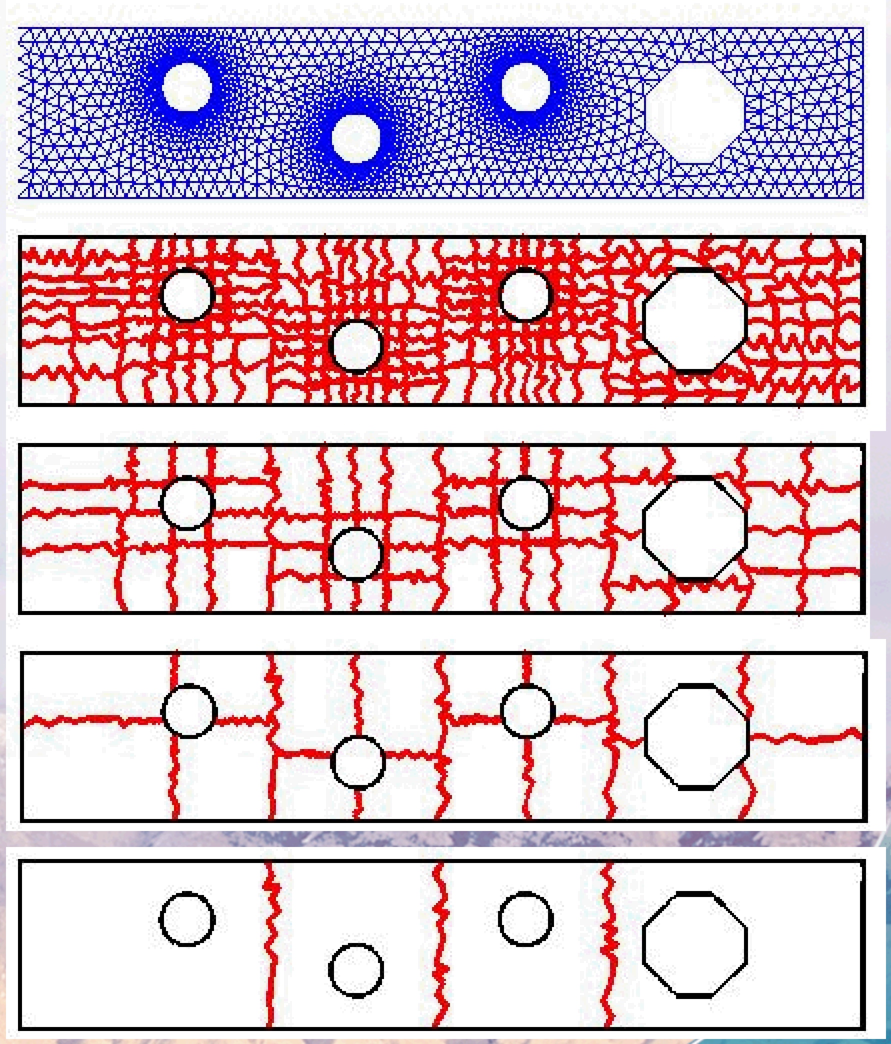
Model	Laplace equation
Domain	$(0,1)\times(0,1)$
Method	DG-NIPG$_1$ (Yellow)
Method	Mixed-RT$_1$ (Pink)
Mesh	Triangles
True solution	$p = xye^{x^2y^3}$
Tolerance	1×10^{-6}



Levels	DOF	V-cycles	MG Factor
3	224	8	0.19
4	960	8	0.19
5	3968	8	0.20
6	16128	8	0.20

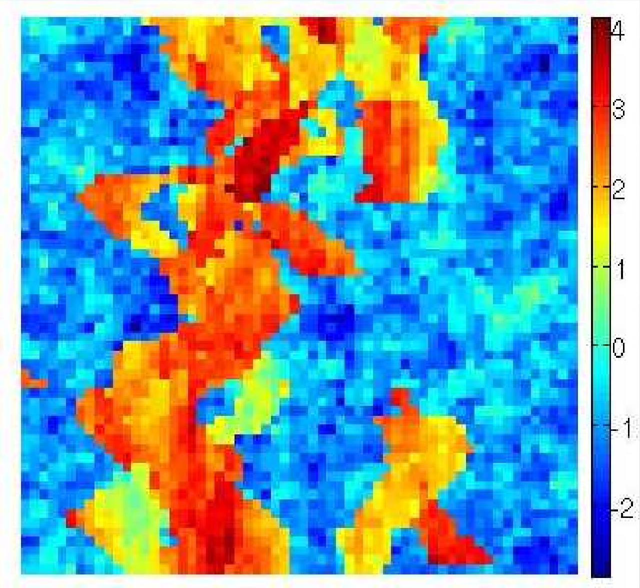
Poisson Equation – Unstructured Mesh

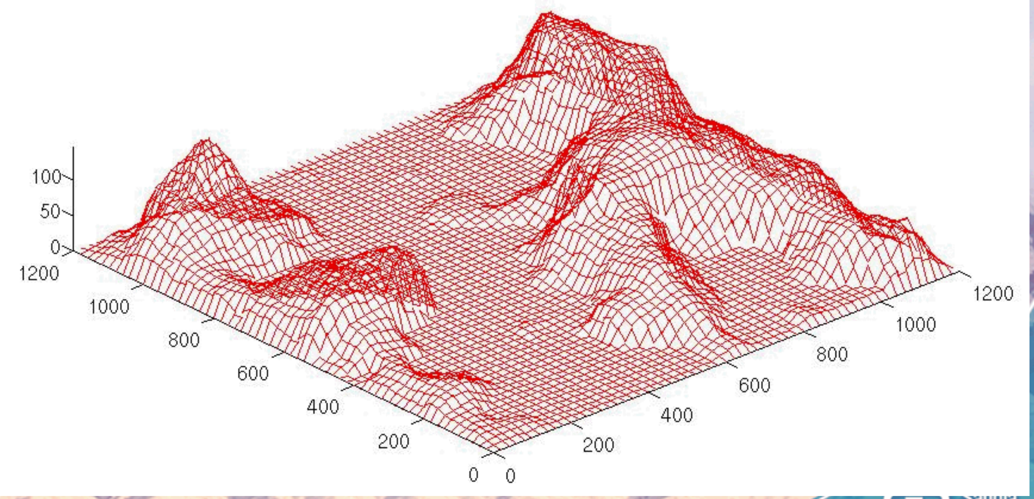
Model	Poisson equation
Permeability	$5.5e^{\sin(2\pi x)}$
Method	Mixed RT_1
Tolerance	1×10^{-6}
Degrees of freedom	11492
Levels	5
V-cycles	29
Convergence rate	0.51
PCG Iters	11



Single Phase Flow with Heterogeneities

Model	Single phase - incompr.
Domain	$(0, 1200) \times (0, 1200)$
Method	Mixed RT_1
Mesh	Quadrilaterals
Permeability	Layer 75 from SPE10
Tolerance	1×10^{-6}
Degrees of freedom	52240
Levels	8
V-cycles	27
Convergence rate	0.57
PGMRES Iters.	8





Conclusions

Conclusions

- Mortars provide flexibility using weak coupling.
- Ideal for multiscale/multiphysics/multinerics coupling.

- Construction of a multiscale mortar basis is sometimes useful.
- Construction of a multiscale mortar preconditioner is much more efficient.
- Applications include stochastic operators, IMPES formulations, and nonlinear interface operators.

- Coupled elastic/poroelastic model has been formulated and analyzed.
- Theoretical convergence has been established.
- Verification using a modified analytical solution.

- Robust a posteriori bounds for the energy error may be obtained.
- Optimal linear solvers can be developed based on mortars.

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Thank you for your attention!
Questions?

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