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Leveraging Flexibility in Energy Minimization Algebraic Multigrid

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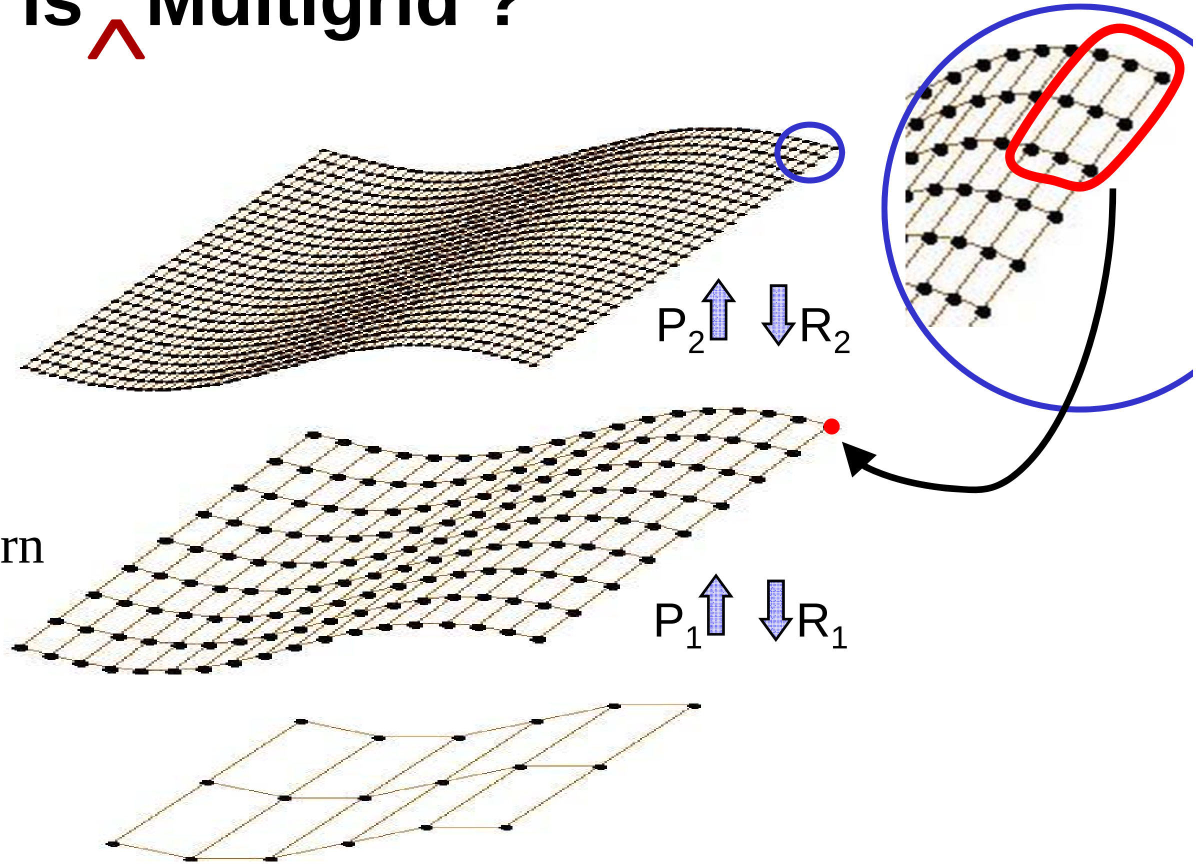
Outline

- Energy minimization AMG
  - Motivation: arbitrary coarsening, flexible coarse basis function support,
    accurate interpolation of arbitrary important modes,
    flexible choice of norm for minimization & search space
  - A ≠ A & Krylov Methods

- Leveraging flexibility of energy minimization AMG
  - weakly constrained & over-constrained
  - anisotropic elasticity
    - Flexible sparsity pattern & accurate rigid body mode interpolation
  - extended finite elements & fracture
    - Flexible pattern, flexible norm choice

What is Algebraic Multigrid ?

Solve $A_3 u_3 = f_3$



- Construct Graph & Coarsen
- Determine P_i & R_i sparsity pattern
- Determine P_i & R_i's coefs
- Project: $A_i = R_i A_{i+1} P_i$

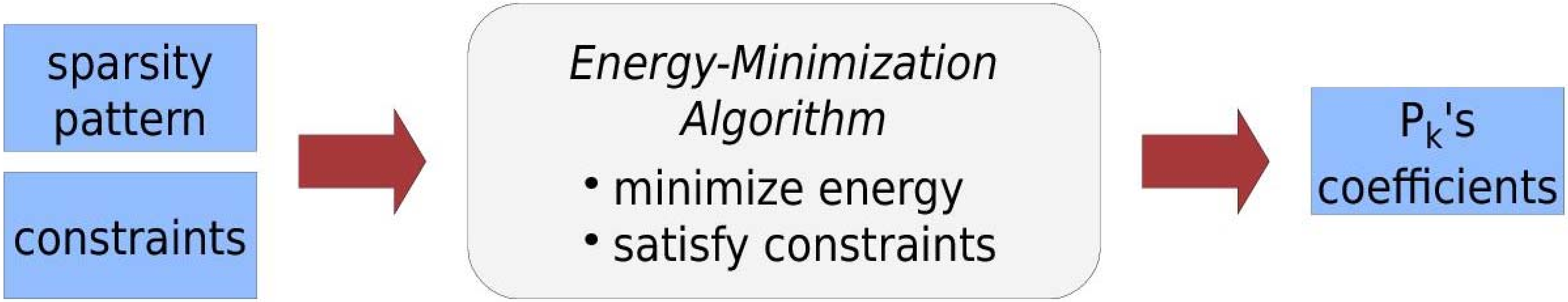
$$\min_{e_H} \| e - Pe_H \|_2^2 \le \beta \| e \|_A^2 \quad + \quad \| Se \|_A^2 \le \| e \|_A^2 - \alpha \| e \|_{A^2}^2 \quad \Rightarrow \quad \| ST_{2-level} \|_A \le \sqrt{1 - \frac{\alpha}{\beta}}$$

Algorithm

$min \sum \|p_i\|_e^2$ where p_i gives the i^{th} column of a prolongator

Idea: construct the grid transfer operator P by minimizing the energy of each column , while enforcing constraints (sparsity pattern and specified modes).

Input / output of energy-minimization algorithm:



sparsity pattern

constraints

Energy-Minimization Algorithm

- minimize energy
- satisfy constraints

P_k's coefficients

Advantages:

- Flexibility (input):
  - arbitrary coarsening
  - accept any sparsity pattern (arbitrary basis function support)
  - enforce constraints: important modes requiring accurate interpolation
  - choice of norm for minimization and search space
- Robustness

Energy-Minimization

Find $P = [\,p_1\ p_2 \cdots p_m\,]$ that

minimizes $\sum_i \|p_i\|_e$ in some space subject to $X \begin{pmatrix} p_1 \\ \vdots \\ p_m \end{pmatrix} = g$

if $e = A \quad \Rightarrow$

$$\begin{pmatrix} \underbrace{\begin{matrix} A & & & \\ & A & & \\ & & A & \\ & & & A \end{matrix}}_{\hat{A}} & X^T \\ X & \end{pmatrix} \begin{pmatrix} \underbrace{\begin{matrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{matrix}}_{\text{»}P\text{«}} \\ \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ \\ g \end{pmatrix}$$

Constraints enforce exact interpolation of null space, e.g. A B = 0 ➔ $B \in Range(P)$

“Solve” AP = 0
1) with minimization algorithm
2) in space satisfying constraints

$A\,(P_0 - \Delta P) = 0$ with $X\,\text{»}P_0\text{«} = g$

$X\,\text{»}\Delta P\text{«} = 0$

$Q\,\hat{A}\,\text{»}P_0\text{«},\ (Q\,\hat{A})^2\,\text{»}P_0\text{«},\ \ldots$

$Q = (\,I - X^T(X\,X^T)^{-1}\,X\,)$

CG minimization

Lemma: *Let A be SPD and apply CG to*

$$Q\,\hat{A}\,Q \;\gg\!\Delta P\!\ll \;=\; Q\,\hat{A}\,Q \gg\! P_0\!\ll$$

with 0 initial guess, then CG computes

$$\gg\!\Delta \mathrm{P}_{\mathrm{i}}\!\ll \;=\; \operatorname*{arg\,min}_{\gg \Delta \mathrm{P}_{\mathrm{i}} \ll \in \mathcal{K}_i} \left\| \gg\! \mathrm{P}_{\mathrm{i}}\! \ll \right\|_{\hat{A}}$$ *where* $P_i = P_0 - \Delta P_i$

Proof.

$$\begin{aligned}
\gg\!\Delta \mathrm{P}_{\mathrm{i}}\!\ll &= \operatorname*{arg\,min}_{\gg \Delta \mathrm{P}_{\mathrm{i}} \ll \in \mathcal{K}_i} \left\| \gg\! \mathrm{P}_{*}\! \ll - \gg\!\Delta \mathrm{P}_{\mathrm{i}}\!\ll \right\|^2_{Q\hat{A}Q} \\
&= \operatorname*{arg\,min}_{\gg \Delta \mathrm{P}_{\mathrm{i}} \ll \in \mathcal{K}_i} \left\| \gg\! \mathrm{P}_{*}\! \ll \right\|^2_{Q\hat{A}Q} + \left\| \gg\!\Delta \mathrm{P}_{\mathrm{i}}\!\ll \right\|^2_{Q\hat{A}Q} - 2\left\langle \gg\!\Delta \mathrm{P}_{\mathrm{i}}\!\ll , \gg\! \mathrm{P}_{0}\! \ll \right\rangle_{\hat{A}} \\
&= \operatorname*{arg\,min}_{\gg \Delta \mathrm{P}_{\mathrm{i}} \ll \in \mathcal{K}_i} \left\| \gg\! \mathrm{P}_{0}\! \ll \right\|^2_{Q\hat{A}Q} + \left\| \gg\!\Delta \mathrm{P}_{\mathrm{i}}\!\ll \right\|^2_{Q\hat{A}Q} - 2\left\langle \gg\!\Delta \mathrm{P}_{\mathrm{i}}\!\ll , \gg\! \mathrm{P}_{0}\! \ll \right\rangle_{\hat{A}} \\
&= \operatorname*{arg\,min}_{\gg \Delta \mathrm{P}_{\mathrm{i}} \ll \in \mathcal{K}_i} \left\| \gg\! \mathrm{P}_{0}\! \ll - \gg\!\Delta \mathrm{P}_{\mathrm{i}}\!\ll \right\|^2_{\hat{A}}
\end{aligned}$$

Corollary: CG solution is unique

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GMRES minimization

Lemma: *Let A be nonsingular and apply GMRES to*

$$Q\hat{A}\;\text{»}\Delta P\text{«} = Q\hat{A}\;\text{»}P_0\text{«} \qquad (*)$$

with 0 initial guess, then GMRES computes

$$\text{»}\Delta P_i\text{«} = \arg\min_{\text{»}\Delta P_i\text{«}\in\mathcal{K}_i} \left\|\text{»}P_i\text{«}\right\|_{\hat{A}^T Q\hat{A}} \quad \textit{where } P_i = P_0 - \Delta P_i$$

which is unique.

$$||\;\text{»}AP_i\text{«}\;||_Q$$

Proof.

Define S and $S_\perp$ *such that*

$$QS = 0,\quad S_\perp S_\perp^T = Q,\quad S_\perp^T S_\perp = I,\quad S^T S = I,\quad S^T S_\perp = I$$

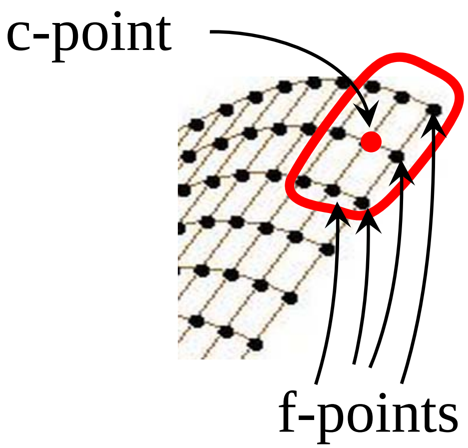
Use properties of GMRES applied to nonsingular system

$$S_\perp^T \hat{A} S_\perp y = S_\perp^T \hat{A}\,\text{»}P_0\text{«} \qquad (**)$$

Pre-multiplication of (**) & associated Kyrlov space by $S_\perp$ *reveal equivalence with (*).*

$A \neq A^T$ (weakly constrained)

- Use GMRES to “solve” $AP = 0$ with only constraint $P = [P_f \;\; I_c]^T$
- Use GMRES to “solve” $RA = 0$ with only constraint $R = [R_f \;\; I_c]$



Consider

$$\begin{pmatrix} A_{ff} & A_{fc} \\ A_{cf} & A_{cc} \end{pmatrix}\begin{pmatrix} u_f \\ u_c \end{pmatrix} = \begin{pmatrix} b_f \\ b_c \end{pmatrix}$$

Then

$$\underbrace{\begin{pmatrix} I_f & 0 \\ & R \end{pmatrix}\begin{pmatrix} A_{ff} & A_{fc} \\ A_{cf} & A_{cc} \end{pmatrix}\begin{pmatrix} I_f & P \\ 0 & \end{pmatrix}}_{\begin{pmatrix} A_{ff} & E_{fc} \\ E_{cf} & A_H \end{pmatrix}}\begin{pmatrix} y_f \\ y_c \end{pmatrix} = \begin{pmatrix} b_f \\ Rb \end{pmatrix}$$

Decouples if $E_{fc} = \begin{pmatrix} A_{ff} & A_{fc} \end{pmatrix} P$ & $E_{cf} = R\begin{pmatrix} A_{ff} \\ A_{cf} \end{pmatrix}$ are 0.

= Q »AP« = »RA«Q

Exploiting Flexibility

Coarsening, sparsity pattern, p_{ij} choice are often tied together within many **AMG methods**

Example: smoothed aggregation

1) **Aggregate:** $\mathcal{A}_i \cup \mathcal{A}_j = \{1, \ldots , |V|\}, \mathcal{A}_i \cap \mathcal{A}_j = \emptyset$, $diam(\mathcal{A}_i) \approx 3$

2) $P_0 = BlkDiag(\mathcal{R}_i \; B) \quad = \begin{bmatrix} \mathcal{B}_1 & & \\ & \mathcal{B}_2 & \\ & & \ddots \end{bmatrix}$

3) $P = P_0 + \omega D^{-1} \tilde{A} P_0$

Graph of $\tilde{A}$

- ColDim(B) = ColDim($\mathcal{B}_\mathrm{i}$)
- sparsity pattern is $|\tilde{A}| \, P_0$
- Should have $\tilde{A} \, P_0 \, B_c = 0$ where B_c is coarse representation of B (modes requiring accurate interpolation)

$$\Rightarrow \tilde{A} \, B = 0 \text{ (as } P_0 \, B_c = B)$$

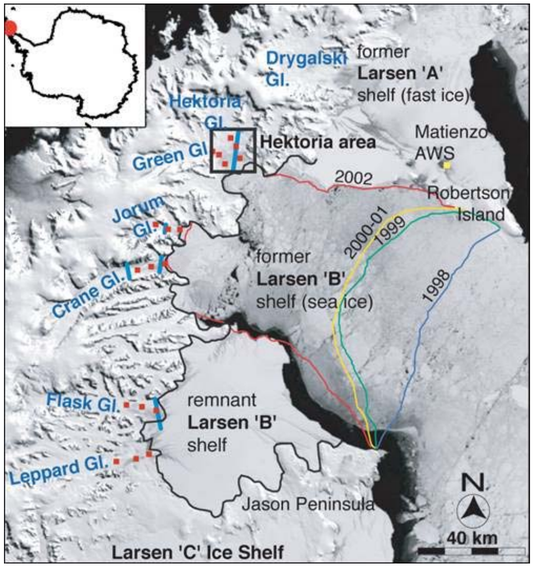
We will also exploit ability to change norm.

Motivation: Importance of Ice Fracture

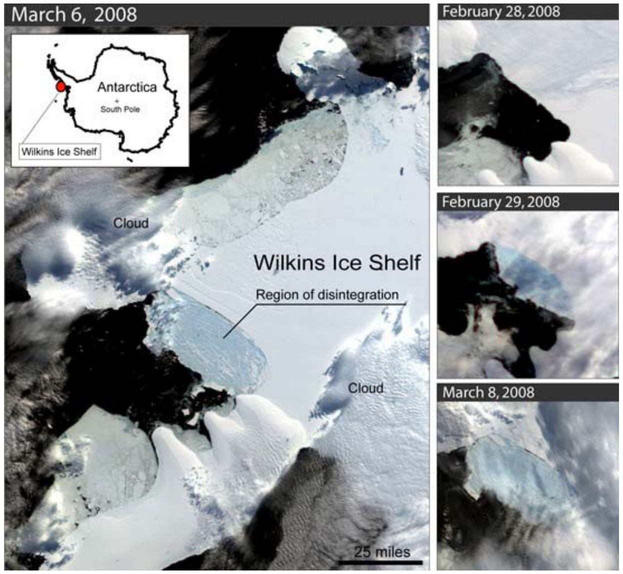
Objective: Employ parallel computers to study the fracture of land ice to better understand how it affects global climate change. In particular

- the collapse mechanism of ice shelves,
- the calving of large icebergs, and
- the role of fracture in the delivery of water to the bed of ice sheets.

ice shelves in Antarctica

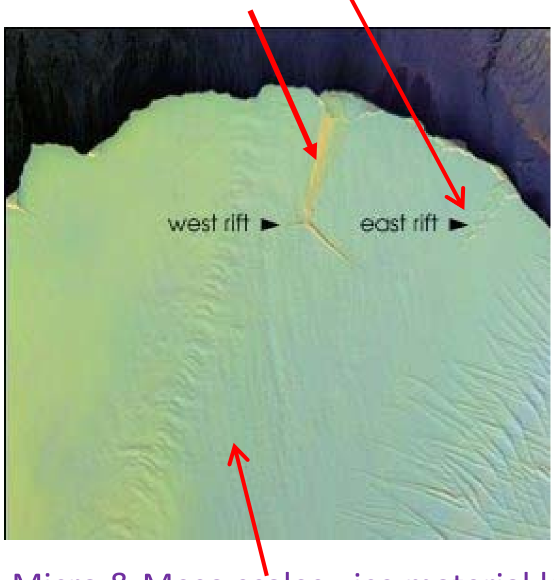


Larsen B diminishing shelf 1998-2002



Wilkins ice shelf
Recent 2008 collapse

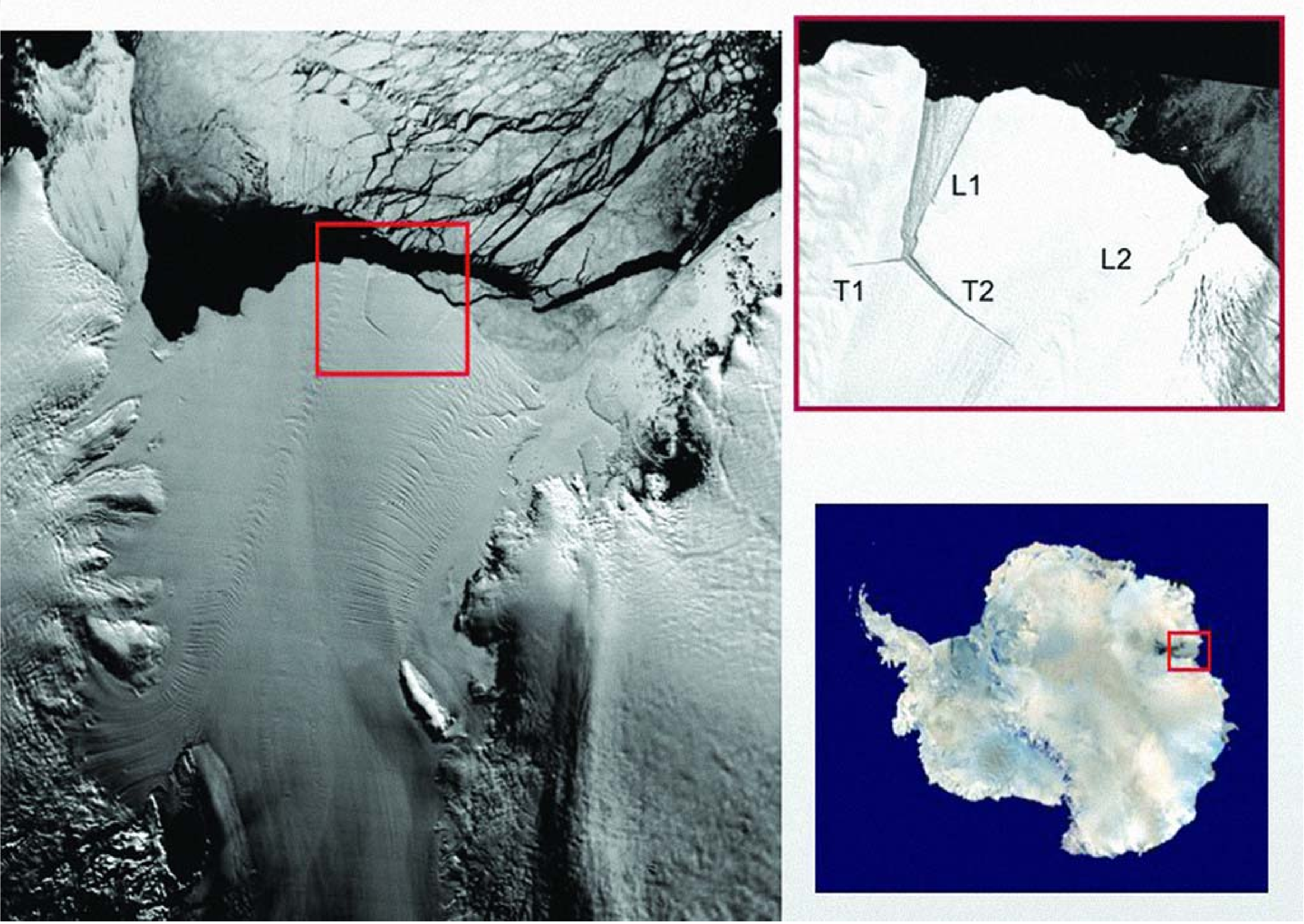
Macro scale - rifts will be represented by cracks (XFEM)



Micro & Meso scales - ice material law given by viscoelastic damage model

Amery ice shelf

Amery ice shelf



Computational Modeling of Fracture

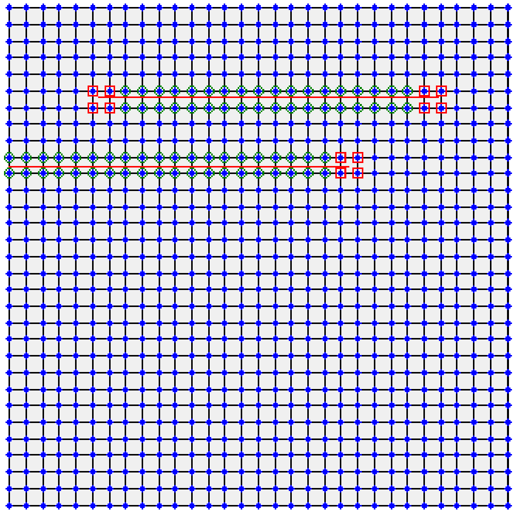
Classical FEM approach to fracture mechanics

- Mesh conforms to crack boundaries
- Crack propagation → remeshing at each step
  - Requires double-nodes for crack opening and fine mesh for tip singularities

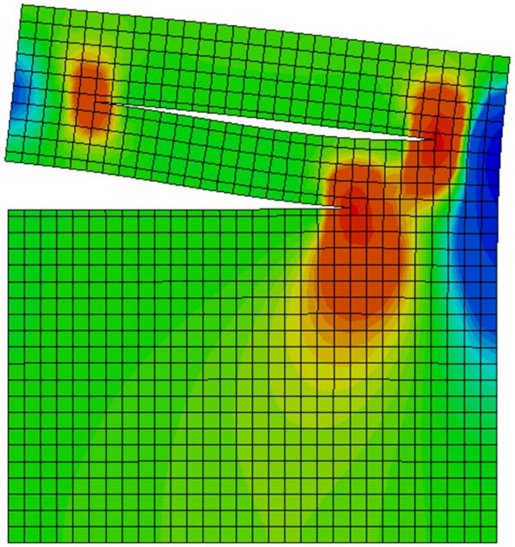
eXtended Finite Element Method (XFEM)*

- Base mesh independent of crack geometry
- Crack propagation → adding "enriched" DOF with special basis functions to existing nodes
  - Crack geometry defined through levelsets
  - Discontinuities and singularities captured through special basis functions (enrichments)
  - Enrichments have local support

XFEM mesh



Stresses in *y* direction when bottom edge fixed and uniform traction applied on top edge in *y* direction



* Belytschko & Black (1999), Moes et al. (1999)

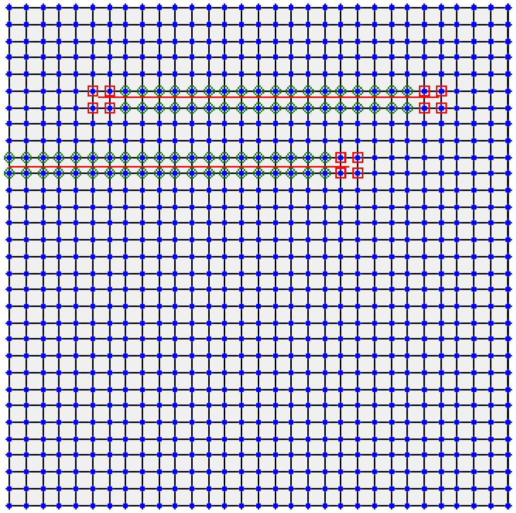
XFEM Formulation

Displacement trial function (shifted basis form.):

$$u^h(\mathbf{x}) = \sum_{I=1}^{n} N_I(\mathbf{x})u_I$$

$$🟩\ + \sum_{i=1}^{n_h} N_{I_i}(\mathbf{x})\left(H(\mathbf{x}) - H\left(\mathbf{x}_{I_i}\right)\right)a_{I_i}$$

$$🟥\ + \sum_{i=1}^{n_f} N_{\hat{I}_i}(\mathbf{x}) \sum_{J=1}^{n_J} \left(F_J(\mathbf{x}) - F_J\left(\mathbf{x}_{\hat{I}_i}\right)\right) b_{\hat{I}_i J}$$



- 🟩 **Jump enrichment:**

$$H(\mathbf{x}) = \begin{cases} 1 & \text{above } \Gamma_{c+} \\ -1 & \text{below } \Gamma_{c-} \end{cases}$$

- 🟥 **Tip enrichments:**

$$F_J(r,\theta) = \left\{ \overbrace{\sqrt{r}\sin\left(\frac{\theta}{2}\right)}^{J=1}, \overbrace{\sqrt{r}\cos\left(\frac{\theta}{2}\right)}^{J=2}, \overbrace{\sqrt{r}\sin\left(\frac{\theta}{2}\right)\sin(\theta)}^{J=3}, \overbrace{\sqrt{r}\cos\left(\frac{\theta}{2}\right)\sin(\theta)}^{J=4} \right\}$$

XFEM Linear system

Strain-displacement relations:

$$\mathbf{B}^e_{enr} = \nabla_{sym}\mathbf{N}^e_{enr}$$

- Symmetric gradient operator applied to enriched basis-function matrix

Weak form

Stiffness matrix:

$$\mathbf{A}_e = \int_{\Omega_e} \left(\mathbf{B}^e_{enr}\right)^T \mathbf{D}\mathbf{B}^e_{enr} d\Omega_e$$

- Numerical quadrature for stiffness matrix

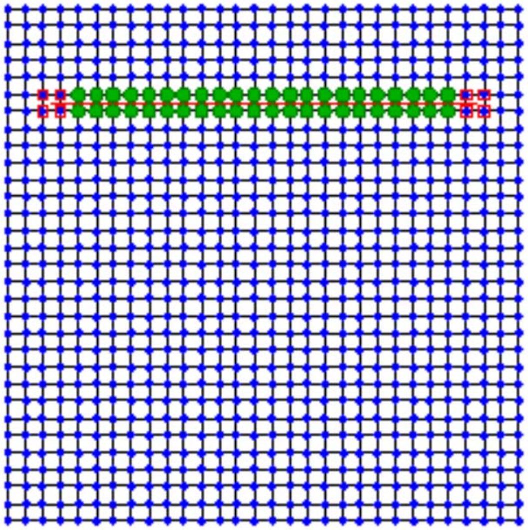
Assembly

XFEM Linear System:

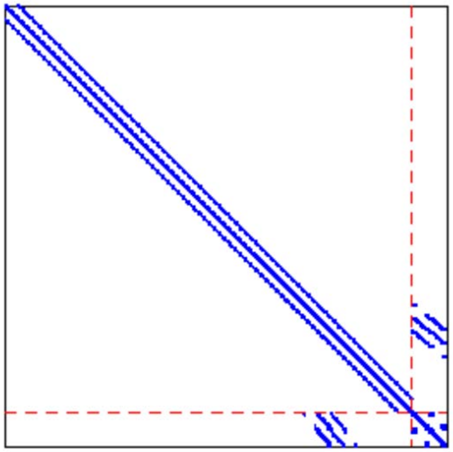
$$\begin{bmatrix} A_{rr} & A_{rx} \\ A_{xr} & A_{xx} \end{bmatrix} \begin{bmatrix} u_r \\ u_x \end{bmatrix} = \begin{bmatrix} \tilde{f}_r \\ \tilde{f}_x \end{bmatrix}$$

- Enriched DOF grouped together at the end in u_x
- A_{xx} small compared to A_{rr} for relatively small number of cracks
- Dense blocks in A_{xx} correspond to tip functions

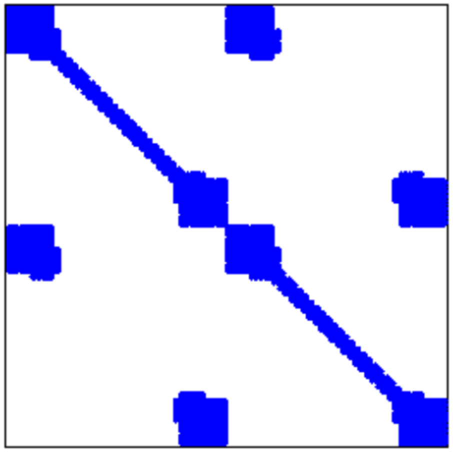
XFEM mesh



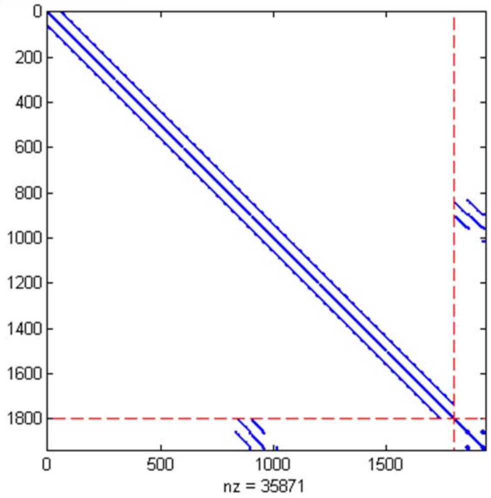
Sparsity pattern of A



Sparsity pattern of A_{xx}

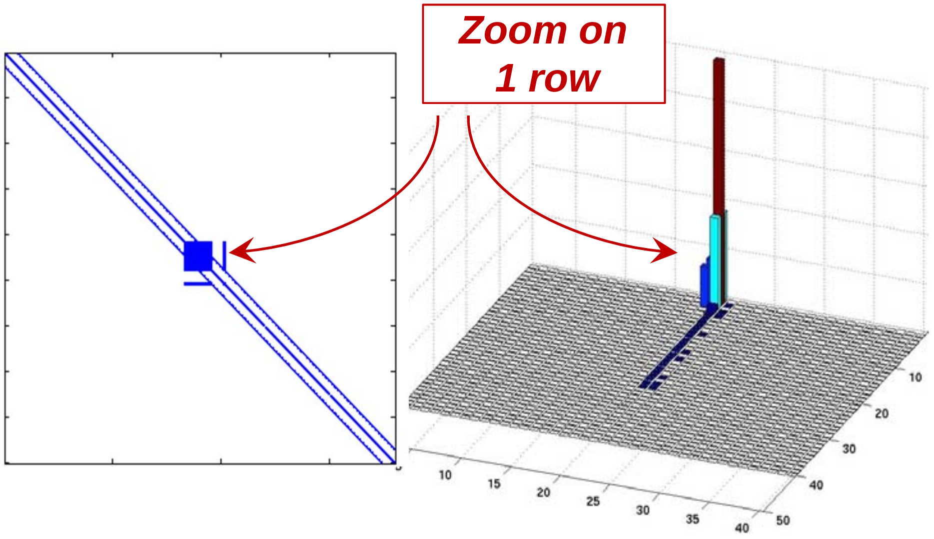


Schur Complement



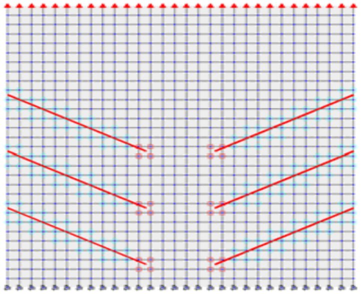
$$A = \begin{pmatrix} A_{rr} & A_{rx} \\ A_{xr} & A_{xx} \end{pmatrix}$$

$$S = A_{rr} - A_{rx}A_{xx}^{-1}A_{xr}$$



AMG Applied to S

- $\hat{S}$: small entries dropped from S
- Aggregation & SparsityPattern(A_{rr} .* $\hat{S}$)
- Standard AMG (energy min.)



Mesh	Scalar AMG	Variable Block AMG	AMG on Schur Complement
30×30	180	89	10
60×60	-	103	11
90×90	-	114	11
120×120	-	126	12

Implicit Schur complement

Lemma: Schur complement/projection commutativity. Let P be interpolation associated with a 2-level AMG method applied to S. Define a 2-level AMG method for the full 2x2 system by

$$\bar{P} = \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix}$$

Then, projected Schur complement & Schur complement of projected full system are equivalent.

Coarsening of Schur Complement

$$\begin{aligned} S &= A_{rr} - A_{rx} A_{xx}^{-1} A_{xr} \\ \Longrightarrow P^T S P &= P^T \left(A_{rr} - A_{rx} A_{xx}^{-1} A_{xr} \right) P \\ &= P^T A_{rr} P - P^T A_{rx} A_{xx}^{-1} A_{xr} P \end{aligned}$$

Schur Complement of Coarse-level 2x2 matrix

$$\begin{aligned} \bar{P} &= \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix} \\ \Longrightarrow \bar{P}^T A \bar{P} &= \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} A_{rr} & A_{rx} \\ A_{xr} & A_{xx} \end{bmatrix} \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix} \\ &= \begin{bmatrix} P^T A_{rr} P & P^T A_{rx} \\ A_{xr} P & A_{xx} \end{bmatrix} \\ \Longrightarrow \bar{S} &= P^T A_{rr} P - P^T A_{rx} A_{xx}^{-1} A_{xr} P \end{aligned}$$

Can be generalized for multilevel via recursion.

Implicit Schur complement

<u>Lemma</u>: Let relaxation on Schur complement & on full 2x2 system be defined by

reduced system:

$$u_r \leftarrow u_r + M_{rr}^{-1} r_r$$

full system:

$$\tilde{u}_x \quad \leftarrow \quad A_{xx}^{-1}(\tilde{f}_x - A_{xr}\tilde{u}_r)$$

$$\begin{bmatrix} \tilde{u}_r \\ \tilde{u}_x \end{bmatrix} \leftarrow \begin{bmatrix} \tilde{u}_r \\ \tilde{u}_x \end{bmatrix} + \begin{bmatrix} M_{rr} & 0 \\ A_{xr} & A_{xx} \end{bmatrix}^{-1} \begin{bmatrix} \tilde{r}_r \\ \tilde{r}_x \end{bmatrix}$$

Then, iterates are equivalent if initial guesses and rhs chosen consistently.

<u>Thm</u>: Recursively define interpolation & relaxation as in 2 previous Lemma to construct two AMG methods: one applied to S and one applied to the full 2x2 system.

Then, AMG iterates are equivalent if fine level initial guess and rhs chosen consistently.

→ Multigrid solver can use implicit Schur complement, and never form *S* explicitly.

Recall $\bar{P} = \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix}$

<u>Caveat</u>: P & M_{rr} must be computable without an explicit form of S !!

Approximation & Implicit Schur Complements

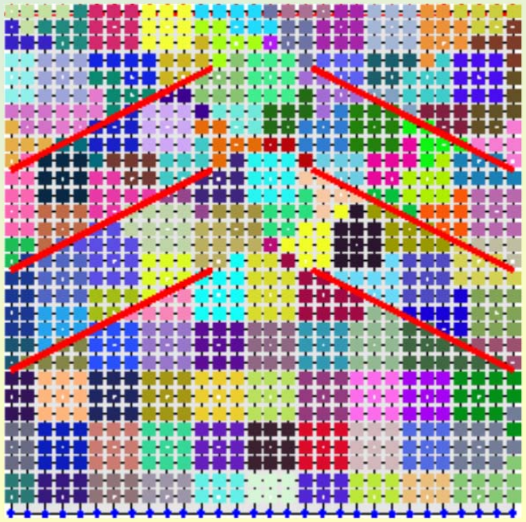
Avoid explict S when building *P*

- using levelsets &
- coordinates define $\hat{A}_{rr}(i,j) = \begin{cases} 0 & \text{if crack crossing} \\ A_{rr}(i,j) & \text{otherwise} \end{cases}$

$\Rightarrow$ run standard energy minimization, but

Use $\hat{A}_{rr}$ for aggregation & sparsity pattern

Use A_{rr} for energy definition

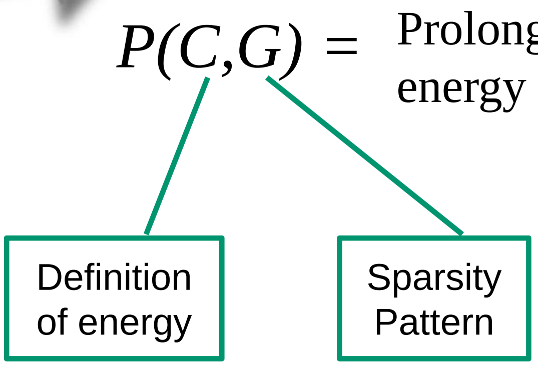


Less expensive smoother

- Approximate $(A_{xx}^{[k]})^{-1}$ via Gauss-Seidel
- Approximate $(M_{rr}^{[k]})^{-1}$ via Gauss-Seidel

$\Rightarrow$
- One GS sweep on $A_{xx}u_x = f_x - A_{xr}u_r$
- One GS sweep on $A_{rr}u_r = f_r - A_{rx}u_x$
- One GS sweep on $A_{xx}u_x = f_x - A_{xr}u_r$

Approximating AMG on S



$P(C,G)$ = Prolongator generated via energy minimization

Definition of energy

Sparsity Pattern

$\hat{G}$: G with crack crossings removed

M_{rr} : Relaxation smoother for Schur complement

R_r : Smoother for regular dof

R_r : Smoother for enriched dof

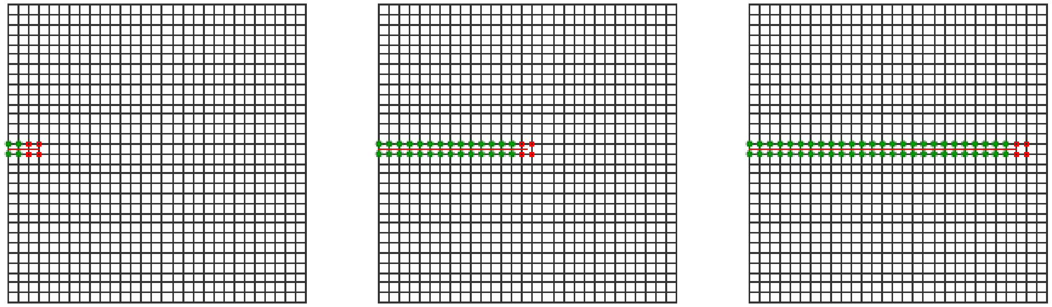
	Schur		Hybrid		
	$P(S,\hat{S})$	$P(\underline{A_{rr}},\hat{S})$	$P(A_{rr},\underline{\hat{A}_{rr}})$	$P(A_{rr},\hat{A}_{rr})$	$P(A_{rr},\hat{A}_{rr})$
Mesh	M_{rr} = GS on S		R_r =GS on S	R_r =GS on $\underline{A_{rr}}$	R_r =GS on A_{rr}
			R_x =Direct Solve	R_x =Direct Solve	R_r =GS on $\underline{A_{xx}}$
30×30	10	11	14	18	18
60×60	11	11	13	17	17
90×90	11	11	13	17	17
120×120	12	11	13	17	17

Numerical Results…

Test Cases:

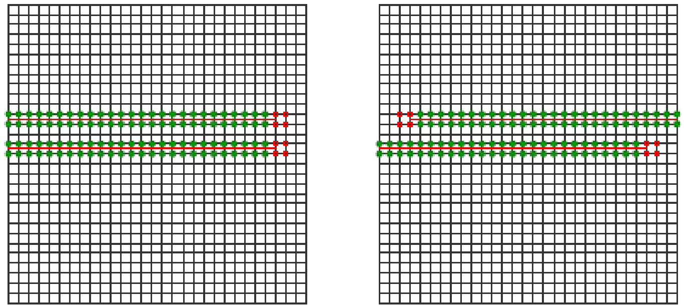
- Both edge cracks and interior cracks considered
- CG preconditioned with AMG
- VBlk AMG: block form of standard AMG with 1 pre + 1 post **block** sym(GS)
- Hybrid Standard AMG: $P(A_{rr}, A_{rr})$ with 1 pre + 1 post sym(GS) on 2x2 system
- Quasi-AMG: $P(A_{rr}, \hat{A}_{rr})$ with 1 pre + 1 post sym(GS) on 2x2 system

Single Propagating Crack



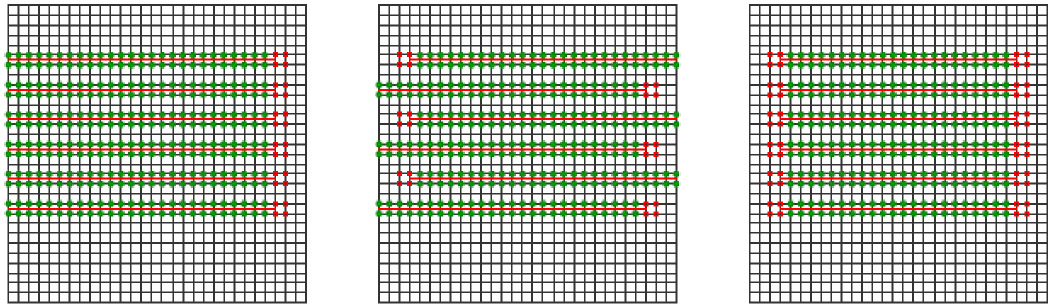
(a) Case 1a (b) Case 1b (c) Case 1c

Two Cracks



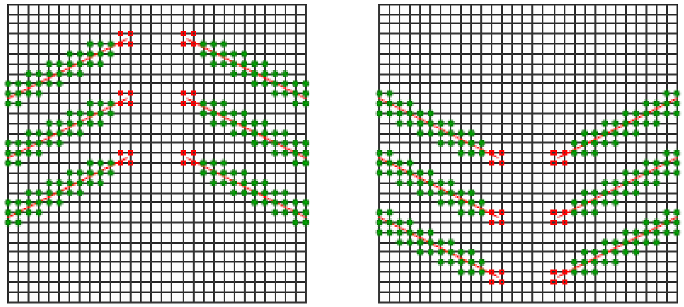
(d) Case 2a (e) Case 2b

Six Cracks



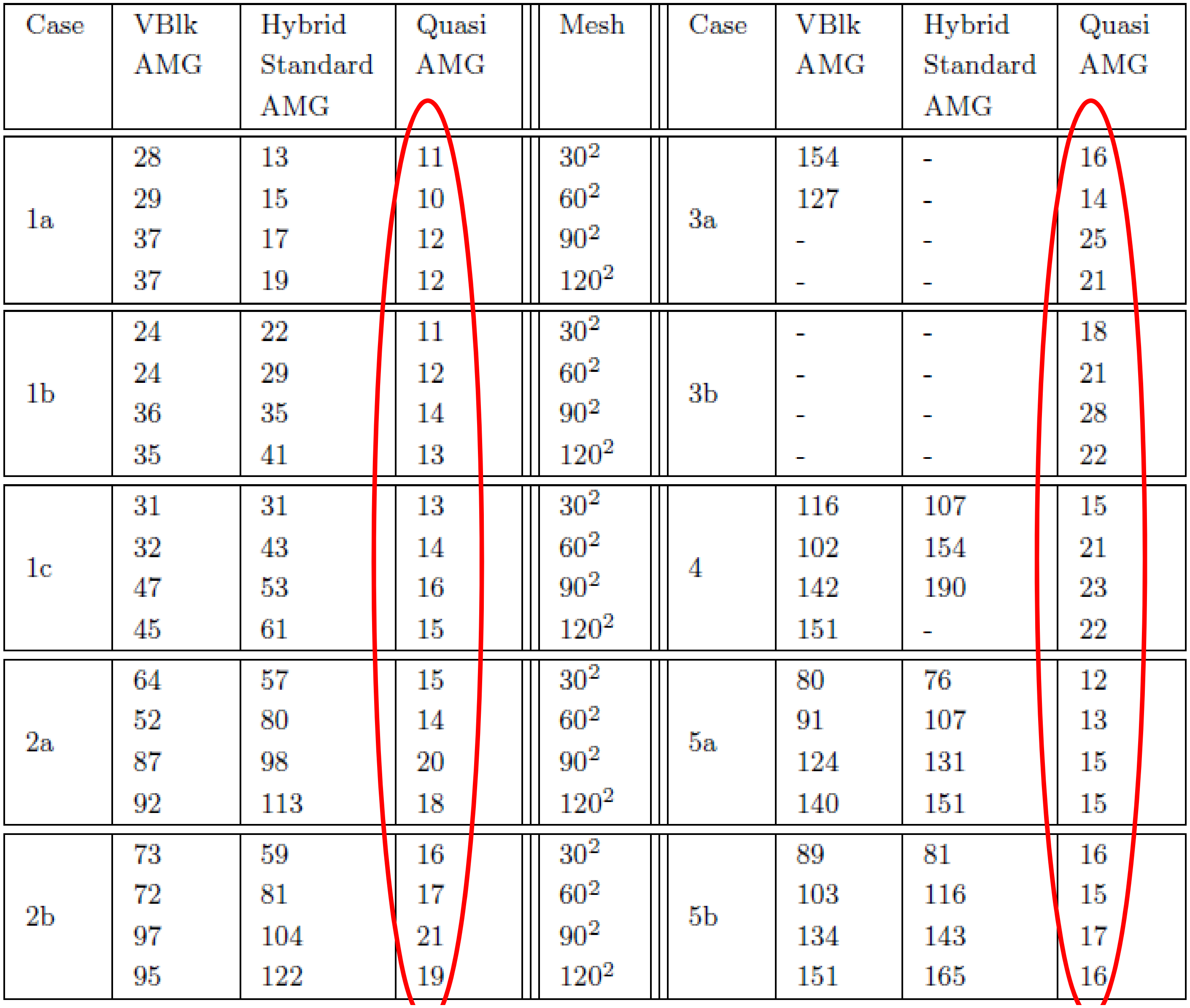
(f) Case 3a (g) Case 3b (h) Case 4

Inclined Cracks



(i) Case 5a (j) Case 5b

Numerical Results…



Case	VBlk AMG	Hybrid Standard AMG	Quasi AMG	Mesh	Case	VBlk AMG	Hybrid Standard AMG	Quasi AMG
1a	28	13	11	30^2	3a	154	-	16
	29	15	10	60^2		127	-	14
	37	17	12	90^2		-	-	25
	37	19	12	120^2		-	-	21
1b	24	22	11	30^2	3b	-	-	18
	24	29	12	60^2		-	-	21
	36	35	14	90^2		-	-	28
	35	41	13	120^2		-	-	22
1c	31	31	13	30^2	4	116	107	15
	32	43	14	60^2		102	154	21
	47	53	16	90^2		142	190	23
	45	61	15	120^2		151	-	22
2a	64	57	15	30^2	5a	80	76	12
	52	80	14	60^2		91	107	13
	87	98	20	90^2		124	131	15
	92	113	18	120^2		140	151	15
2b	73	59	16	30^2	5b	89	81	16
	72	81	17	60^2		103	116	15
	97	104	21	90^2		134	143	17
	95	122	19	120^2		151	165	16

# of levels	full complexity	(1,1) complexity
2	1.673	1.607
3	1.815	1.716
3	1.65	1.583
4	1.699	1.621

Exploiting Flexibility

Coarsening, sparsity pattern, p_{ij} choice are often tied together within many **AMG methods**

Example: smoothed aggregation

1) **Aggregate:** $\mathcal{A}_i \cup \mathcal{A}_j = \{1, \dots, |V|\}, \mathcal{A}_i \cap \mathcal{A}_j = \emptyset, diam(\mathcal{A}_i) \approx 3$

2) $P_0 = BlkDiag(\mathcal{R}_i \; B) \quad =$

ColDim(P) = 3 $N_{\mathcal{A}}$
⇒ smaller search space
⇒ lower $nnz(A_H)$

Graph of $\tilde{A}$

3) $P = P_0 + \omega D^{-1} \tilde{A} P_0$

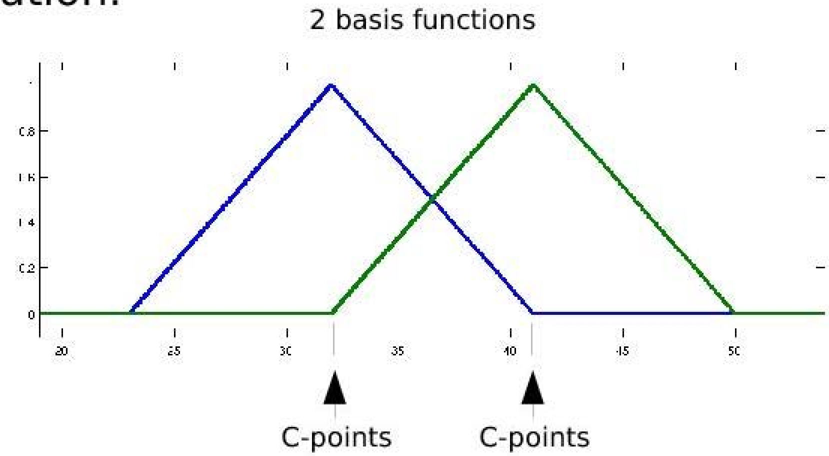
- ~~ColDim(B) = ColDim($\mathcal{B}_i$) ⇒ ColDim(P) = ColDim(B) * $N_{\mathcal{A}}$ = 6 * $N_{\mathcal{A}}$~~
- sparsity pattern is $|\tilde{A}| \, P_0$
- Should have $\tilde{A} P_0 B_c = 0$ where B_c is coarse representation of B (modes requiring accurate interpolation)

$$\Rightarrow \tilde{A} B = 0 \text{ (as } P_0 B_c = B)$$

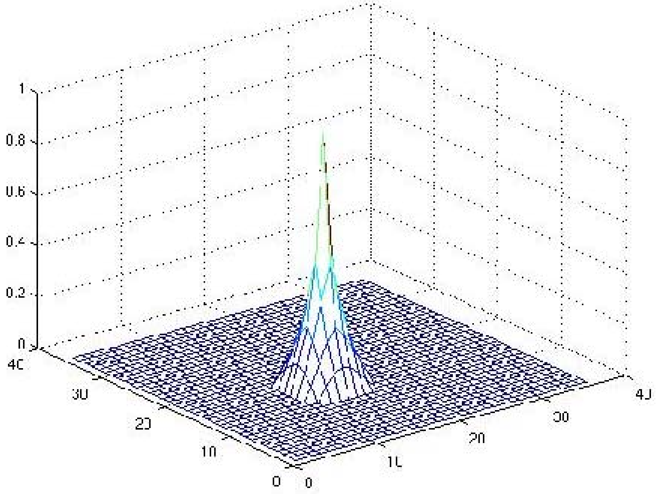
We will also exploit ability to change norm.

Weakly Constrained Case

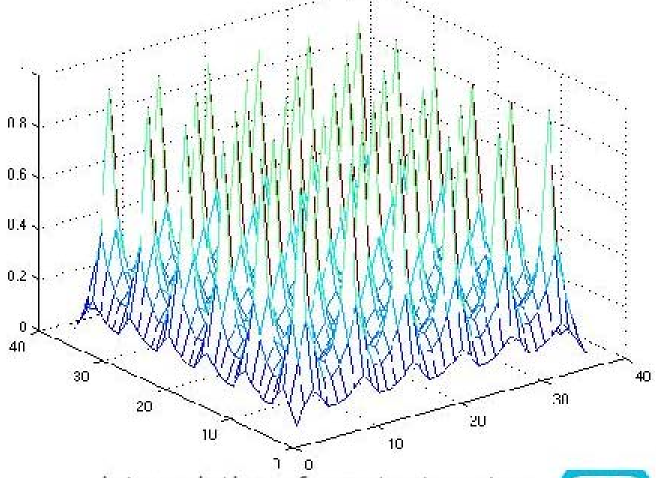
- For 1D Laplace problems, constraining only at root points corresponds to linear interpolation:



- For 2D Laplace problems, constant vector is not accurately interpolated:



basis function



interpolation of constant vector
(plotting of P.**1**)

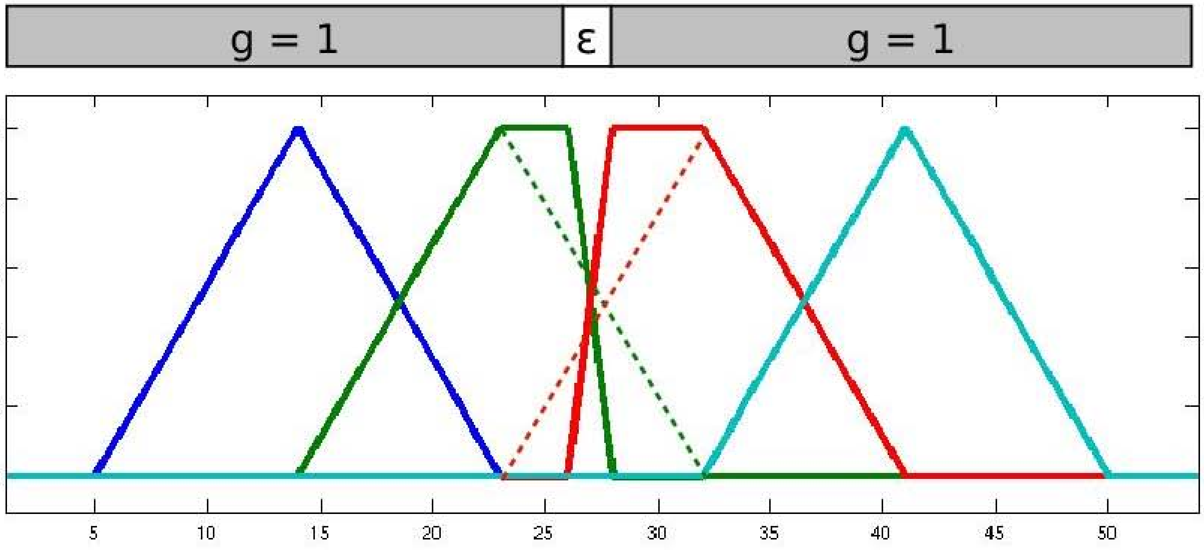
Fully constrained

Problematic if constraints do not capture physics

Example: $(g(x)\ u_x)_x$ = f with two constraints. If $B_{.,1}=1, B_{.,2}=x$,
then constants and linears are exactly interpolated

exact
interpolation
vectors

$$nnz(P_i) \leqslant 2 \Rightarrow fully\ constrained \Rightarrow linear\ interpolation\ independent\ of\ g(x)$$



Problem: fully constraining ignores energy:

- Jumps are not captured in P
- Left and right regions connected within coarse level due to interpolation across epsilon region

Smoothed Aggregation & General Energy Minimization

Smoothed Aggregation

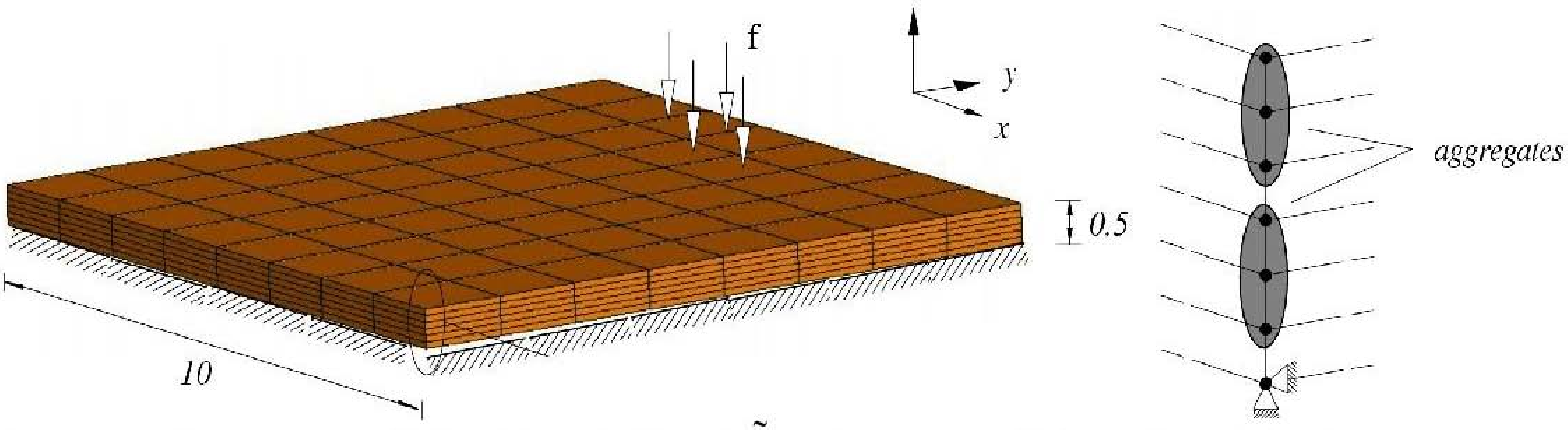
- 1 step minimization
- No need to apply Q ...

  ... but $A\ P_0\ B_c$ must equal 0.
- *almost* fixed sparsity pattern governed by $|\ A\ |\ |\ P_0\ |$
- # DOFs per node = dim{N}

General Energy Minimization

- multi-step minimization
- No restriction on $A\ P_0$.
- Any sparsity pattern
- Arbitrary # DOFs per node

- Anisotropic problems require anisotropic coarsening and sparsity pattern to mimic semi-coarsening



This is normally accomplished by defined $\tilde{A}$ where small (weak entries) are dropped during prolongator construction

Smoothed Aggregation & General Energy Minimization

Smoothed Aggregation

- 1 step minimization
- No need to apply Q ...
  ... but $\tilde{A}$ $P_0 B_c$ must equal 0.
- *almost* fixed sparsity pattern governed by $|\tilde{A}|\,|P_0|$
- # DOFs per node = dim{N}

General Energy Minimization

- multi-step minimization
- No restriction on $\tilde{A}$ P_0.
- Any sparsity pattern
- Arbitrary # DOFs per node

For elasticity:

- Proper $\tilde{A}$ such that $\tilde{A} P_0 B_c = 0$ with weak connections dropped is non-obvious
- Anisotropic coarsening gives small aggregates ⇒ high operator complexities
- 3 DOFs/node on finest level ...
  ... but dim{N} = 6 for 3D elasticity ⇒ 6 DOFs/node ⇒ even higher complexities

Energy-minimization - Elasticity

Lots of choices. We focus on:

3 DOFs/nodes on the coarse grid ➡ Does smaller search space limit quality of interpolation?

- N: 6 rigid body modes (3 translations & 3 rotations)
- CG to solve A P = 0 (effectively defines energy)
- P_0 & sparsity pattern are smoothed aggregation inspired
  - Coarse nullspace defined by injection of fine nullspace @ root nodes
  - Initial Guess: $P_{sa0} + X^T (X X^T)^{-1} (\hat{B} - X P_{sa0})$ where P_{sa0} is smoothed aggregation P_0 for just translations
  - Sparsity Pattern: | $\tilde{A}$ | | P_{sa0} | except @ root points which are constrained to have only 1 nnz/row associated with injection
    - Avoids linear dependency issues
- $\tilde{A}$ defined using distance Laplacian + dropping for sparsity pattern
- No need for $\tilde{A} B = 0$
- A is still used to define energy (as opposed to $\tilde{A}$)

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Experiments

- 3D Linear Elasticity, plan stress
- AMG accelerate by CG, stopping criterion $||r_{relative}||<10^{-10}$
- Influence of the number of energy-minimization steps to the convergence.

  Problem size: 30^3. Stretch factor: ε

# iter.	$\varepsilon = 1$	$\varepsilon = 10$	$\varepsilon = 100$
0	17	19	21
1	12	12	14
2	11	12	13
5	11	12	13
20	11	12	13

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Experiments

- Comparison with Smoothed-Aggregation:
  - SA: 6 DOFs/node
  - Energy-Minimization: 3 DOFs/node, 6 nullspace vectors

Mesh	$\epsilon = 1$				$\epsilon = 10$				$\epsilon = 100$			
	SA		Emin		SA		Emin		SA		Emin	
10^3	6	*1.30*	7	*1.07*	8	*2.81*	8	*1.22*	9	*3.21*	8	*1.24*
15^3	8	*1.19*	9	*1.05*	10	*2.32*	10	*1.15*	12	*2.54*	12	*1.16*
20^3	8	*1.24*	9	*1.06*	10	*2.59*	9	*1.18*	13	*3.05*	10	*1.20*
25^3	9	*1.26*	8	*1.07*	11	*2.76*	9	*1.20*	14	*3.04*	10	*1.20*
30^3	10	*1.22*	11	*1.05*	12	*2.52*	12	*1.17*	15	*3.06*	13	*1.19*
35^3	10	*1.24*	10	*1.06*	12	*2.66*	12	*1.18*	16	*3.03*	13	*1.19*
40^3	10	*1.26*	9	*1.06*	12	*2.77*	12	*1.19*	16	*3.21*	11	*1.21*

Tab. : Iteration count and complexity (lower complexity = faster run time) for SA and energy minimization for various mesh sizes and stretch factors.

complexity: $\dfrac{\sum_i nnz(A_i)}{nnz(A)}$

Conclusions

- Krylov minimization can be used to generate “energy” minimizing prolongators/restrictors for symmetric & non-symmetric systems
  - CG, GMRES
- Some linear algebra issues …
- Flexibility
  - Coarsening, e.g. F/C aggregation, irregular
  - Grid transfer sparsity patterns
  - Norms defining energy

* sparsity pattern & energy norm flexibility used to XFEM
ice fracture problem

* sparsity pattern and dimension of P flexibility used for anisotropic elasticity

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