



An implementation of the X-FEM for eulerian solid-mechanics

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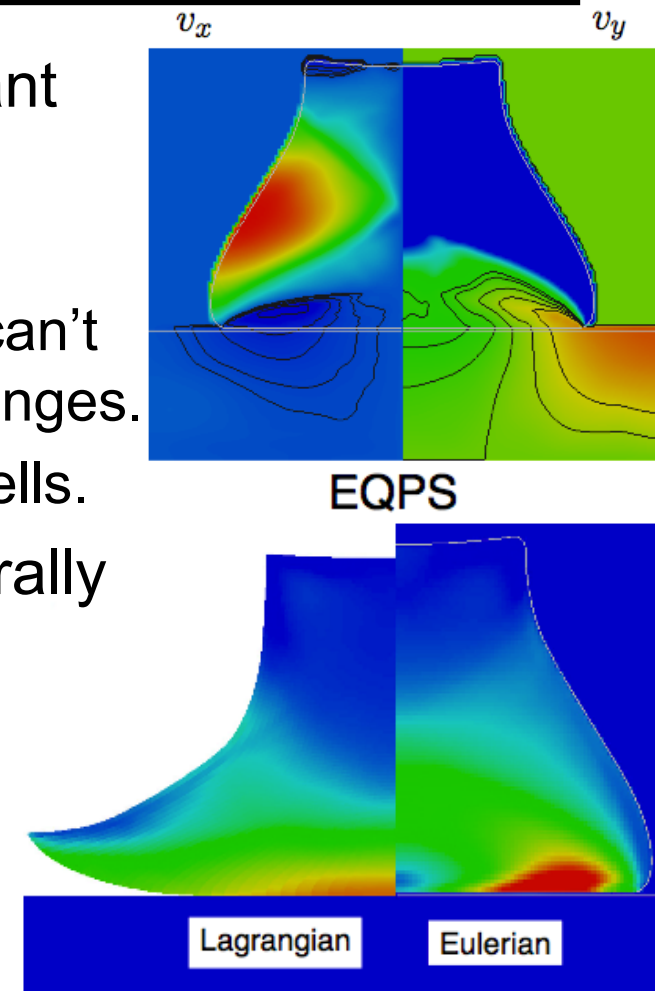
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Previous approaches for mixed material cells are problematic for some physics:

- Multi-material problems with significant vorticity/distortion:
 - Lagrangian approaches tangle.
 - Arbitrary Lagrangian Eulerian (ALE) can't merge materials without topology changes.
 - Eulerian approach produces mixed-cells.
- Current mixed-cell approaches generally assume materials are “well” mixed:
 - Assume “equilibrated” state
 - Single velocity/displacement field
 - Lack of intra-element interfaces





Lagrangian step requires closure model(s) for mixed-cell properties:

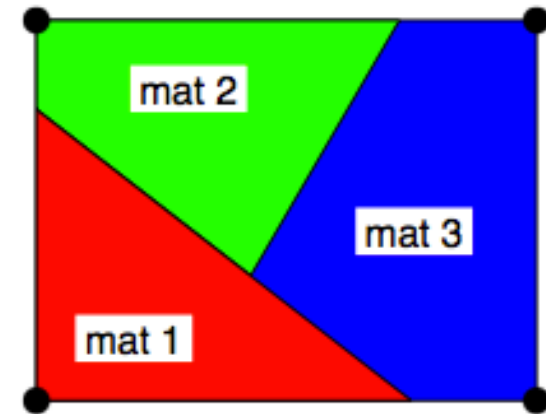
$$\mathbf{a}^n = \mathbf{M}^{-1} \left[\mathbf{f}_{hg}^n + \mathbf{f}_{ext}^n - \int_{\Omega} \mathbf{B}^t \boldsymbol{\sigma}^n \right]$$

$$\mathbf{v}^{n+1/2} = \mathbf{v}^{n-1/2} + \overline{\Delta t} \mathbf{a}^n \quad \mathbf{x}^{n+1} = \mathbf{x}^n + \overline{\Delta t} \mathbf{v}^{n+1/2}$$

$$\mathbf{D}^{n+1/2} = \frac{1}{2} (\mathbf{L}^t + \mathbf{L})^{n+1/2} \quad \mathbf{D}_m = \mathcal{F}(\mathbf{D}, etc...)$$

$$\boldsymbol{\sigma}_m^{n+1} = \mathcal{M}_m(\boldsymbol{\sigma}_m^n, \mathbf{D}_m^{n+1/2}, etc...) \quad e^{n+1} = e^n + \overline{\Delta t} m^{-1} \int_{\Omega} \boldsymbol{\sigma}_m^n : \mathbf{D}_m^{n+1/2}$$

$$\boldsymbol{\sigma} = \mathcal{G}(\boldsymbol{\sigma}_m, etc...)$$



Problem	Expected	Predicted



Why X-FEM:

- Mechanism for intra-element material interfaces.
- Retains base FEM convergence properties.
- Large literature base for X-FEM in context of large deformation, explicit lagrangian mechanics.
- Beginning to be adapted to “operator-split” multi-material eulerian solid-mechanics [VB06; DLZRM10]:
 - explicit (central difference) lagrangian solve,
 - followed by data transfer “remap” to “better” mesh.



Goal is to develop capability for (unmixed) intra-element material physics...

- as a surface phenomenon
- with merging interfaces and fixed mesh topologies
- distinct velocity/displacement fields per material
- while maintaining advantages of current explicit-dynamics code-base
- and capitalizing on existing infrastructure.



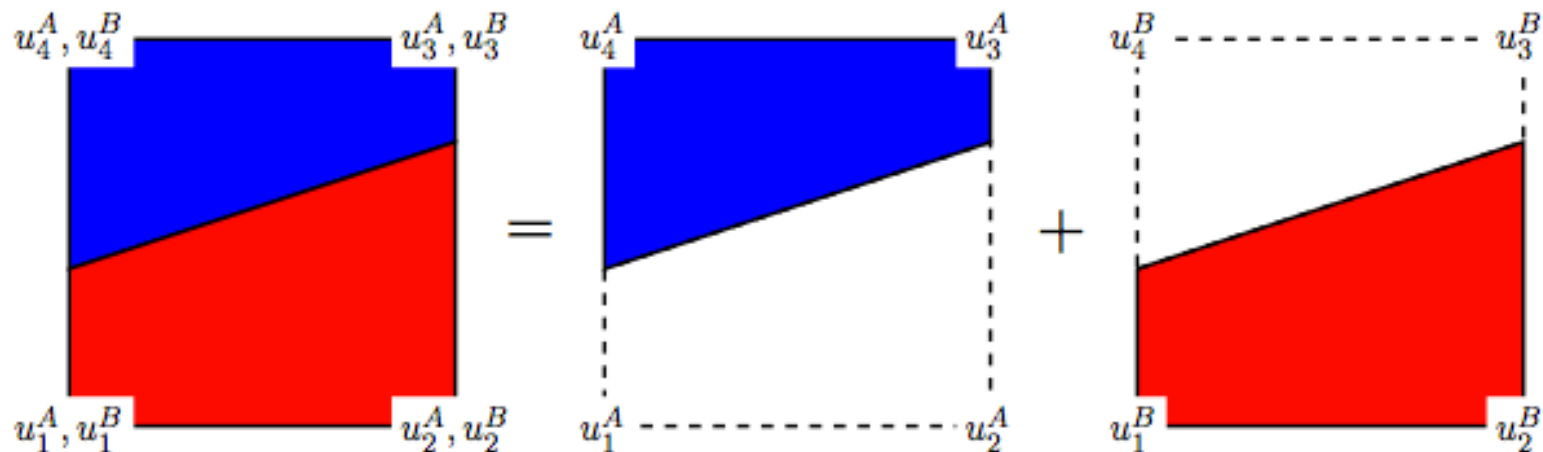
We follow the XFEM decomposition approach [HH04, SAB06] ...

Multi-material enriched element equivalent to multiple single-material elements:

$$u^h(\mathbf{x}) = \sum_I u_I^0(\mathbf{x}) N_I(\mathbf{x}) + \sum_m \sum_J u_J^m N_J(\mathbf{x}) H_m(\mathbf{x})$$

$$u^h(\mathbf{x}) = u_A^h(\mathbf{x}) + u_B^h(\mathbf{x})$$

$$u^h(\mathbf{x}) = \sum_J u_J^m N_J(\mathbf{x}) H_m(\mathbf{x})$$





Data transfer can be accomplished in a number of ways:

- With the goal of preserving some key features:
 - Conserve mass, momentum and internal energy.
 - Do not create new minima and maxima (TVD).
 - Volume fractions sum to one after remap.
- Options include:
 - Interpolation (violates conservation) [DLZRM10]
 - Projection methods (violates conservation and TVD)
 - Geometric intersection with Van Leer limiting
 - conservation is built in.
 - limiting controls production of minima and maxima.

Geometric intersection with Van Leer limiting in two dimensions (1):

- Taylor Series provides functional form on donor mesh:

$$f(\mathbf{x}) = \underbrace{\bar{f}_e}_{\text{first-}} + \underbrace{(\mathbf{x} - \mathbf{x}_e)^t \mathbf{G}_e}_{\text{second-}} + \underbrace{\frac{1}{2}(\mathbf{x} - \mathbf{x}_e)^t \mathbf{H}_e (\mathbf{x} - \mathbf{x}_e)^t - \chi}_{\text{third-order}}$$

χ provides conservation:

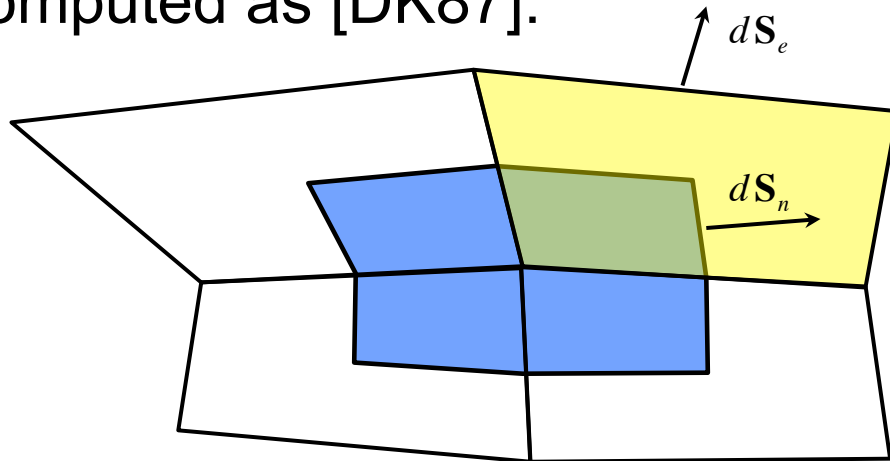
$$\bar{f}_e A_e = \int f(\mathbf{x}) d\Omega_e \Rightarrow \chi = \frac{1}{2A_{e \cap m}} \int (\mathbf{x} - \mathbf{x}_e)^t \mathbf{H}_e (\mathbf{x} - \mathbf{x}_e^t) d\Omega_e$$

- Gradients/hessians computed as [DK87]:

$$\mathbf{G}_n = \frac{1}{A_n} \oint \bar{f}_e d\mathbf{S}_n$$

$$\mathbf{G}_e = \frac{1}{A_e} \sum_n A_{n \cap e} \mathbf{G}_n$$

$$\mathbf{H}_e = \frac{1}{A_e} \oint \mathbf{G}_n d\mathbf{S}_e$$





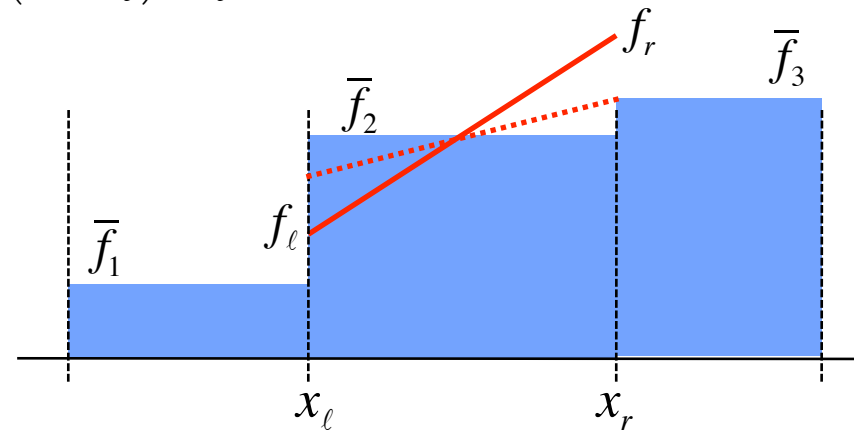
Geometric intersection with Van Leer limiting in two dimensions (2):

- Scale gradient to enforce monotonicity:

$$f(\mathbf{x}) = \bar{f}_e + s_G (\mathbf{x} - \mathbf{x}_e)^t \mathbf{G}_e$$

$$s_{G,r} = \begin{cases} 1 & \text{if } (f_r - \bar{f}_2)(\bar{f}_3 - f_r) \geq 0 \\ (\bar{f}_3 - \bar{f}_2) / (f_r - \bar{f}_2) & \text{otherwise} \end{cases}$$

$$s_{G,\ell} = \begin{cases} 1 & \text{if } (f_\ell - \bar{f}_2)(\bar{f}_1 - f_\ell) \geq 0 \\ (f_1 - \bar{f}_2) / (f_\ell - \bar{f}_2) & \text{otherwise} \end{cases}$$



$$s_G = \max(\min(s_{G,\ell}, s_{G,r}), 0)$$

- If third-order also scale hessian terms:

$$f(\mathbf{x}) = \bar{f}_e + s_G (\mathbf{x} - \mathbf{x}_e)^t \mathbf{G}_e + s_H \left[(\mathbf{x} - \mathbf{x}_e)^t \mathbf{H}_e (\mathbf{x} - \mathbf{x}_e)^t - \chi \right]$$



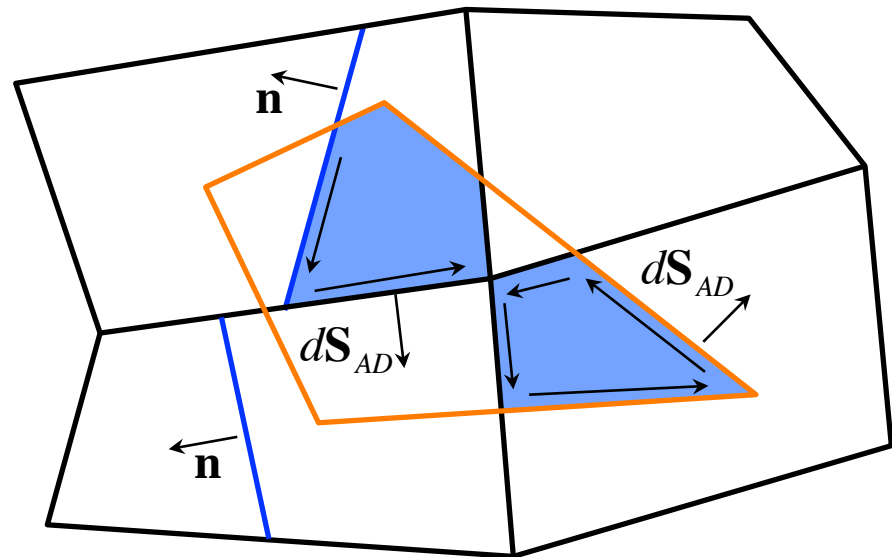
Geometric intersection with Van Leer limiting in two dimensions (3):

- Integrate function over donor-acceptor intersection element and accumulate to acceptor [D83].

$$\bar{f}_A = \frac{1}{A_A} \sum_D \int_{A_D \cap A_A} f_D(\mathbf{x}) dA = \frac{1}{A_A} \sum_D \oint_{\Gamma_{AD}} \mathbf{g}_D(\mathbf{x}) d\mathbf{S}_{AD}$$

$\nabla^t \mathbf{g}_D(\mathbf{x}) \equiv f_D(\mathbf{x})$

- Further restrict integral to filled region of donor mesh.



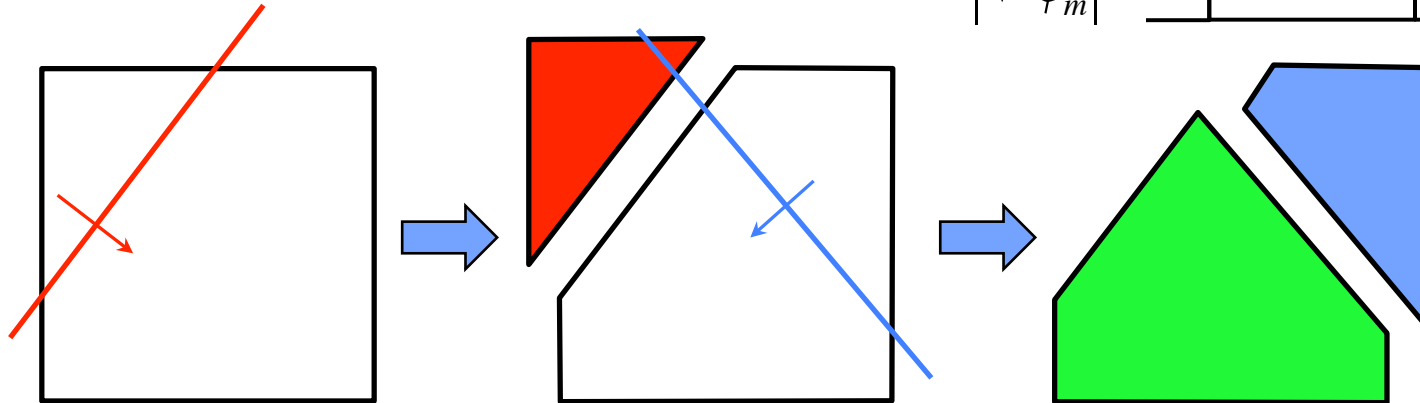


Use interface reconstruction rather than level-set approach: [DVMR08]

- Interfaces rebuilt after remap step.
- Using VOF approach:
 - Compute material volume-fraction gradients.
 - Reposition interface along normal to match volume.
 - Remove material from cell.

$$\mathbf{n}_m = \frac{\nabla^t \phi_m}{|\nabla^t \phi_m|}$$

$\phi_1 = 0.7$ $\phi_2 = 0.3$ $\phi_3 = 0$		$\phi_1 = 0$ $\phi_2 = 1$ $\phi_3 = 0$
$\phi_1 = 1$ $\phi_2 = 0$ $\phi_3 = 0$		
	$\phi_1 = 0$ $\phi_2 = 0$ $\phi_3 = 1$	





Explicit Central Difference discretization requires care for stability ...

$$\mathbf{M}\mathbf{a}^n = \mathbf{f}_{int}^n + \mathbf{f}_c$$

- lumped mass matrix with uniform partitioning of element mass to nodes [MRMCB08]

$$\mathbf{M}^e = \left(\rho^e A^e V_f^e / 4 \right) \mathbf{I}_{8 \times 8}$$

- matched with gradient operator mean quadrature [SAB06]

$$\bar{\mathbf{B}} = \int_{\Omega_e} \mathbf{B} d\Omega / A_e$$

- and constraint enforcement between XFEM interfaces:

$$\mathbf{f}_c = ?$$



We understand the issues but use “node-segment-like” lagrange multipliers...

... in an attempt to:

- minimize interpenetration of XFEM interfaces,
 - and retain a finite stable time-step.
 - Other options for explicit XFEM:
 - Merge (small time step) [VB06]
 - Mortar lagrange multiplier (not LBB for XFEM)
 - Penalty (overlap, mass modifications) [DLZRM10]
 - Nitsche’s (overlap, mass modifications) [AHD11]
 - Vital Vertices LM (quad robustness) [BMW09, HAD11]
- ... so we use it anyway for it’s practicality and economy.



Forward Increment Lagrange Multiplier approach [CTK91] ...

$$\begin{bmatrix} \mathbf{M} & \Delta t \mathbf{G}^t \\ \mathbf{G} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{v}^{n+1/2} \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{f}^n \\ \mathbf{0} \end{bmatrix}$$

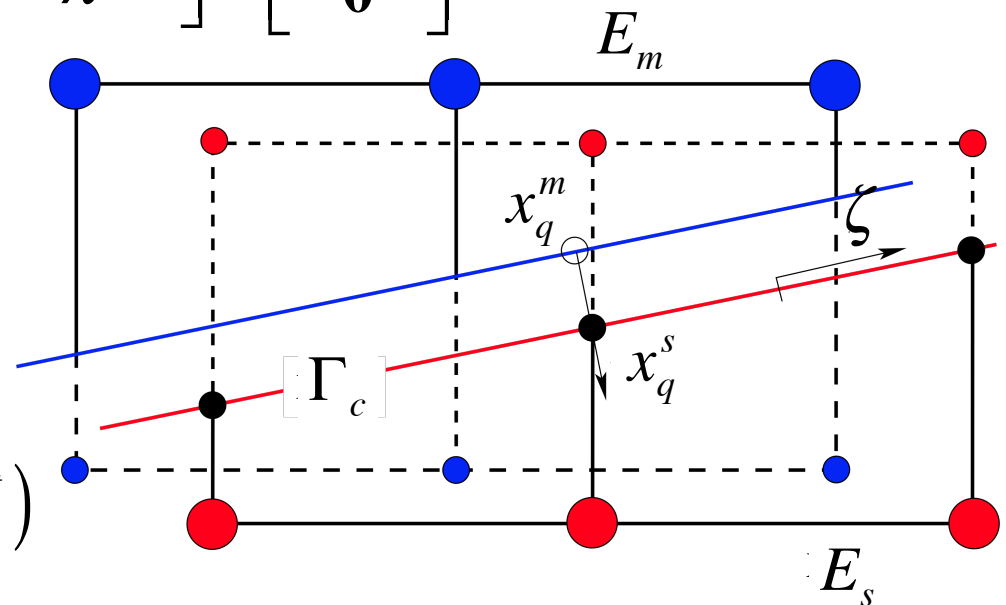
Algorithm:

$$\mathbf{v}_0^{n+1/2} = \mathbf{M}^{-1} \mathbf{f}^n$$

$$\mathbf{r}_i = \mathbf{G} \mathbf{v}_i^{n+1/2}$$

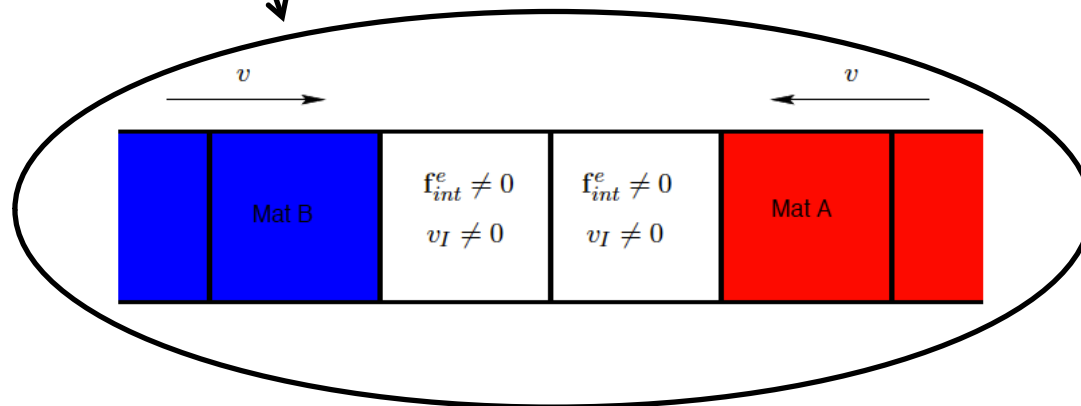
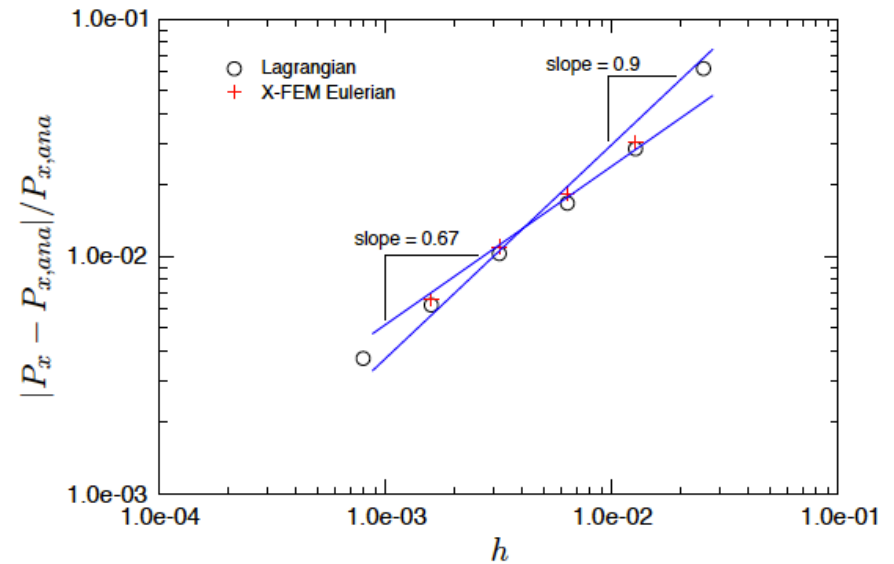
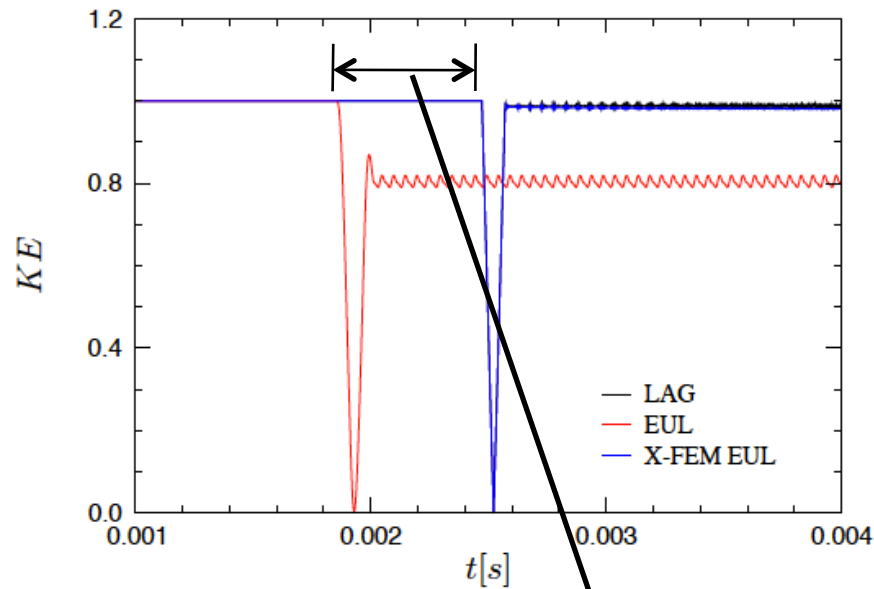
$$\lambda_{i+1} = \lambda_i + \mathbf{H} \mathbf{r}_i \quad \mathbf{H} \approx (\mathbf{G} \mathbf{M}^{-1} \mathbf{G}^t)$$

$$\mathbf{v}_{i+1}^{n+1/2} = \mathbf{M}^{-1} (\mathbf{f}^n - \Delta t \mathbf{G}^t \lambda_{i+1})$$



No additional limitations to stable time step [DLZRM10,VMR10]

Provides enhanced results for simple one-dimensional problem [CTK91]...

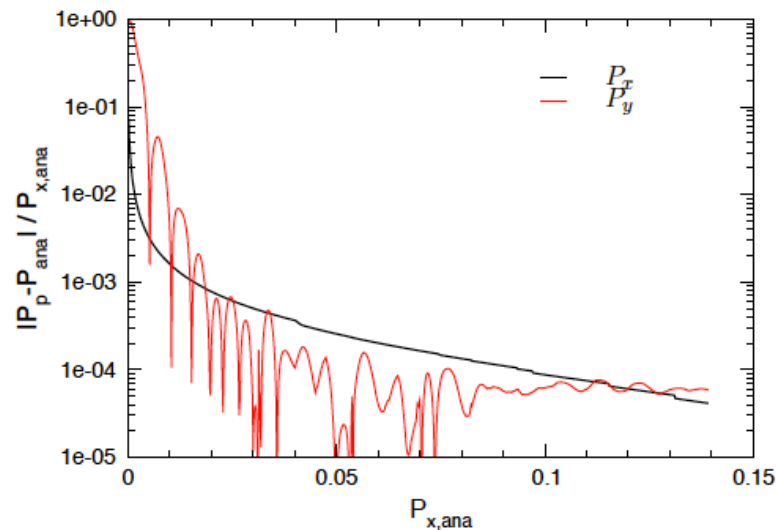
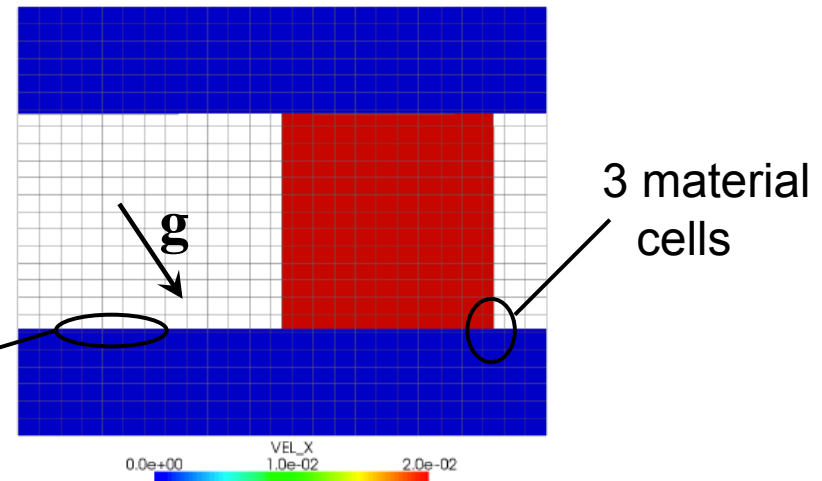




... and simple two-dimensional problems ...

“Rigid” block sliding
frictionlessly between
“rigid” platens.

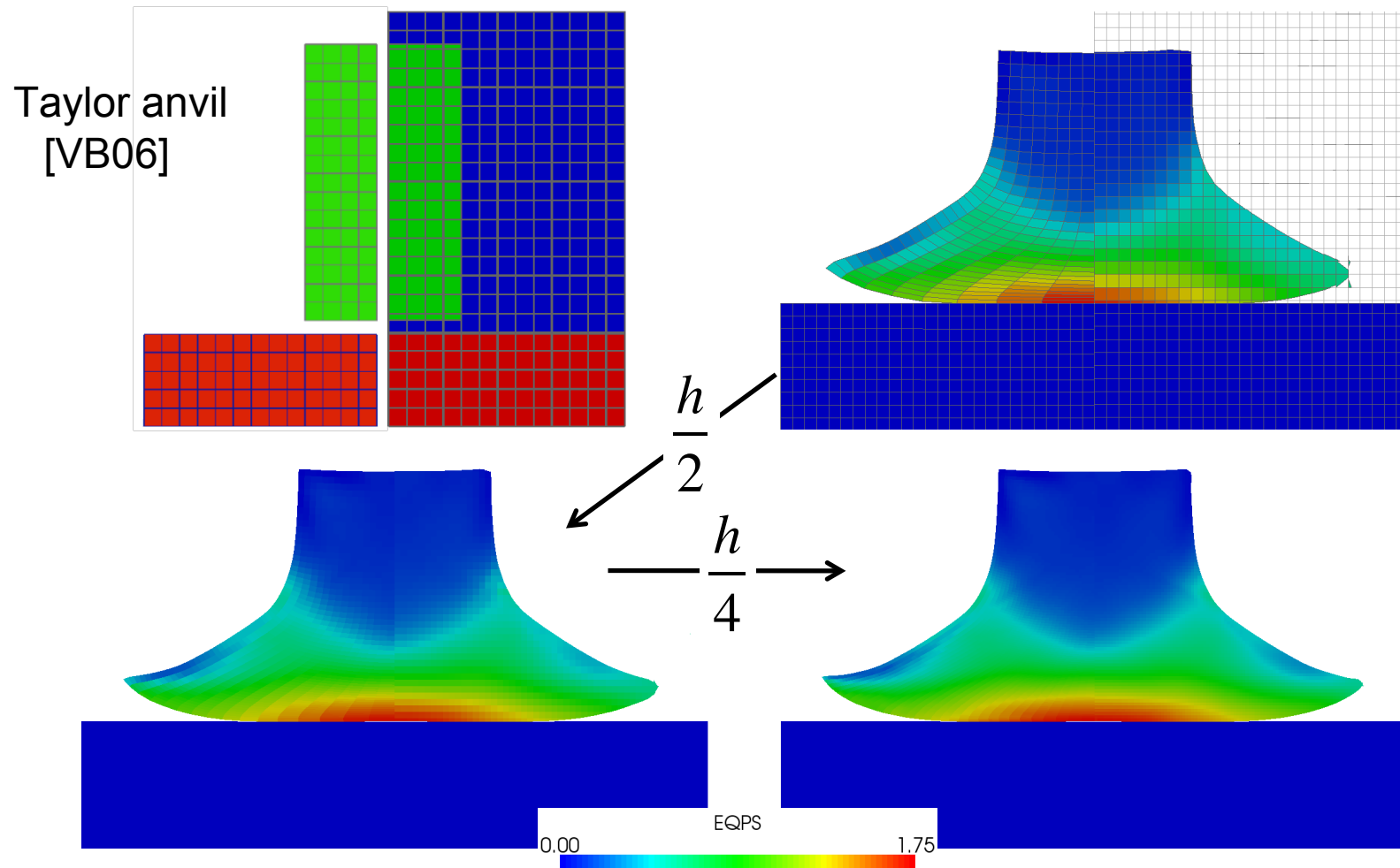
material not
mesh-aligned



Excellent agreement for
momentum compared
to analytical solution.

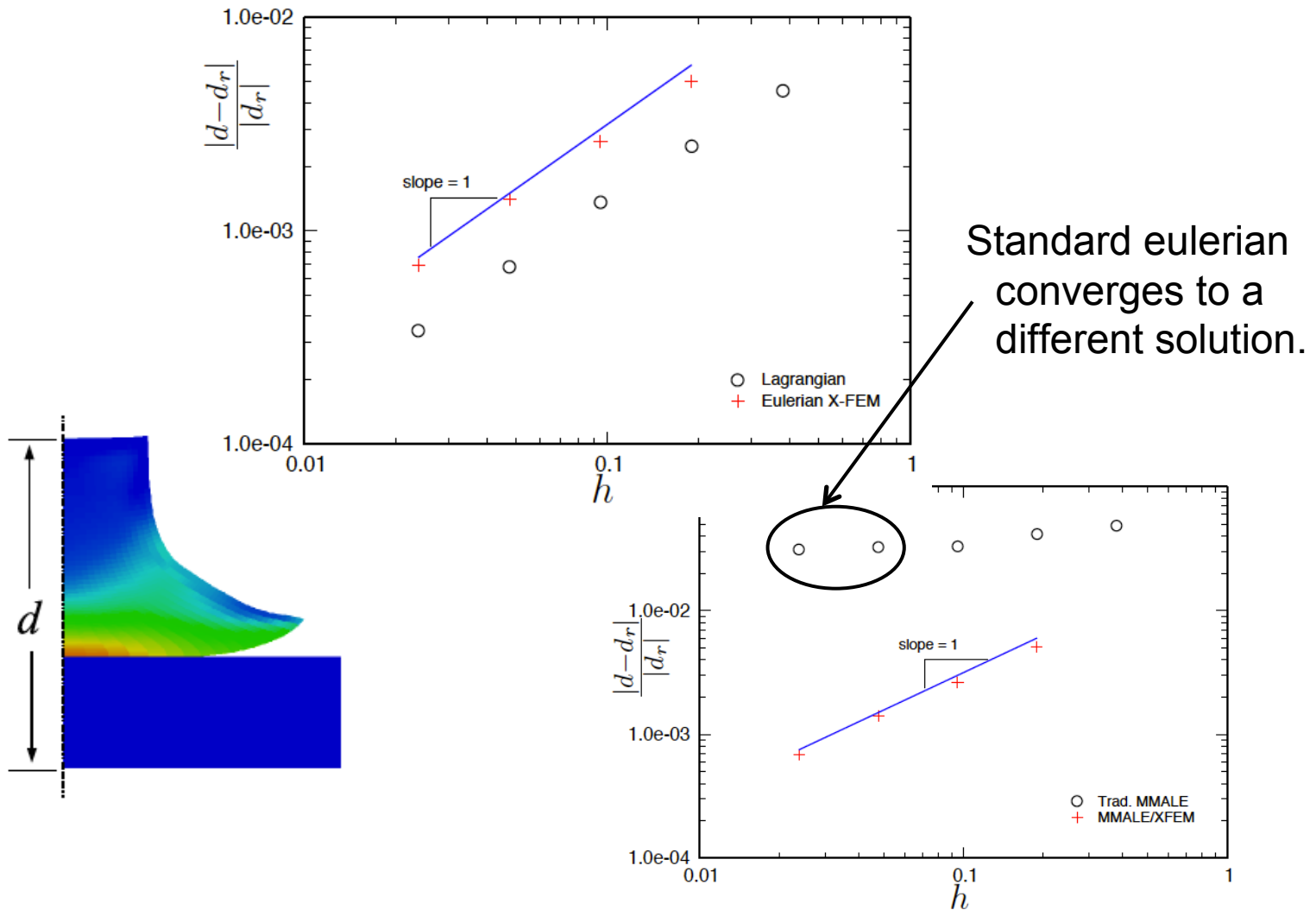


... and accuracy comparable to Lagrangian
for more complex problems ...



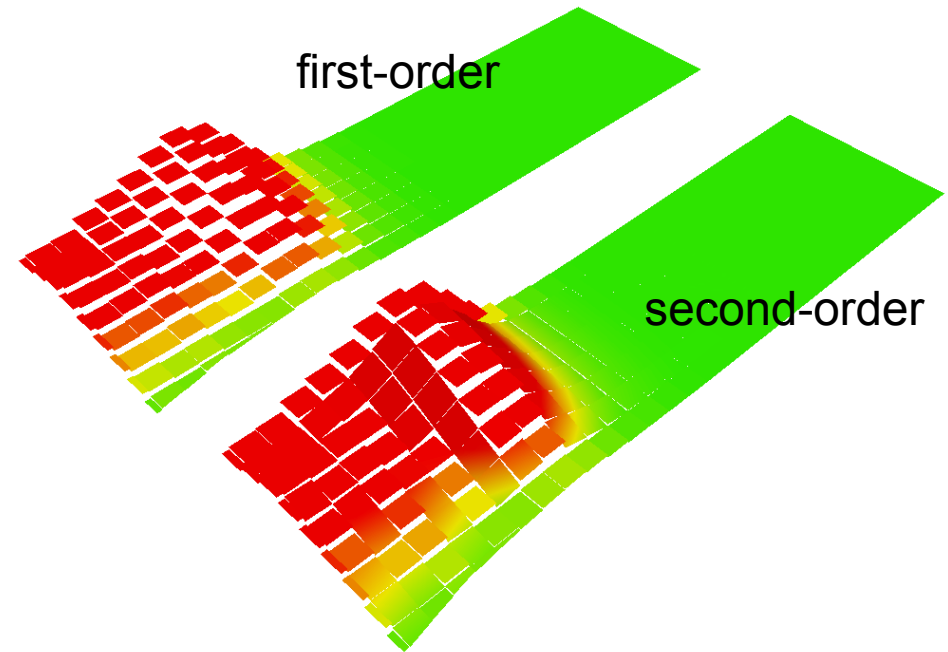
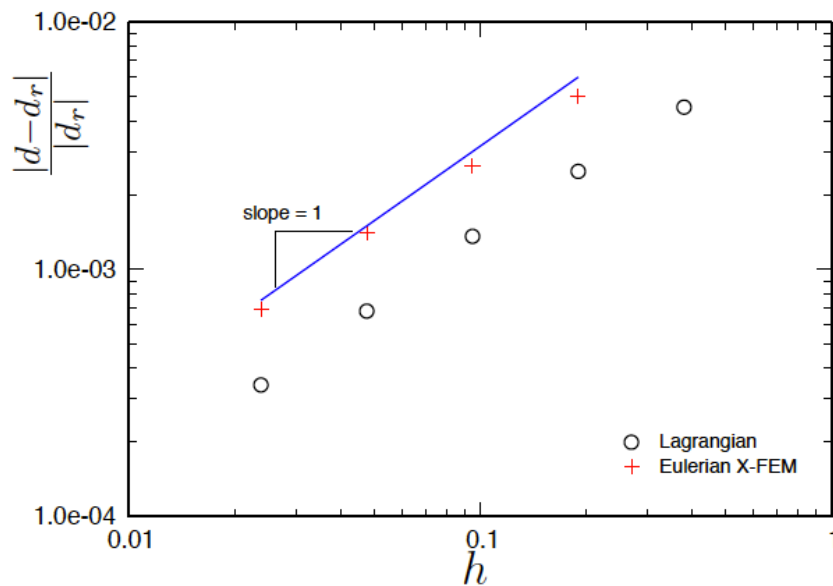


... as well as comparable rates of convergence ...



For this problem remap order has little effect on accuracy/rate-of-convergence ...

Convergence in height for first- and second-order remap.

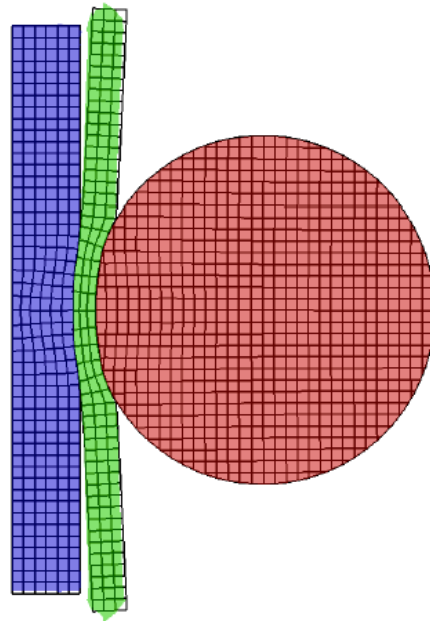


Density reconstructions at early time.

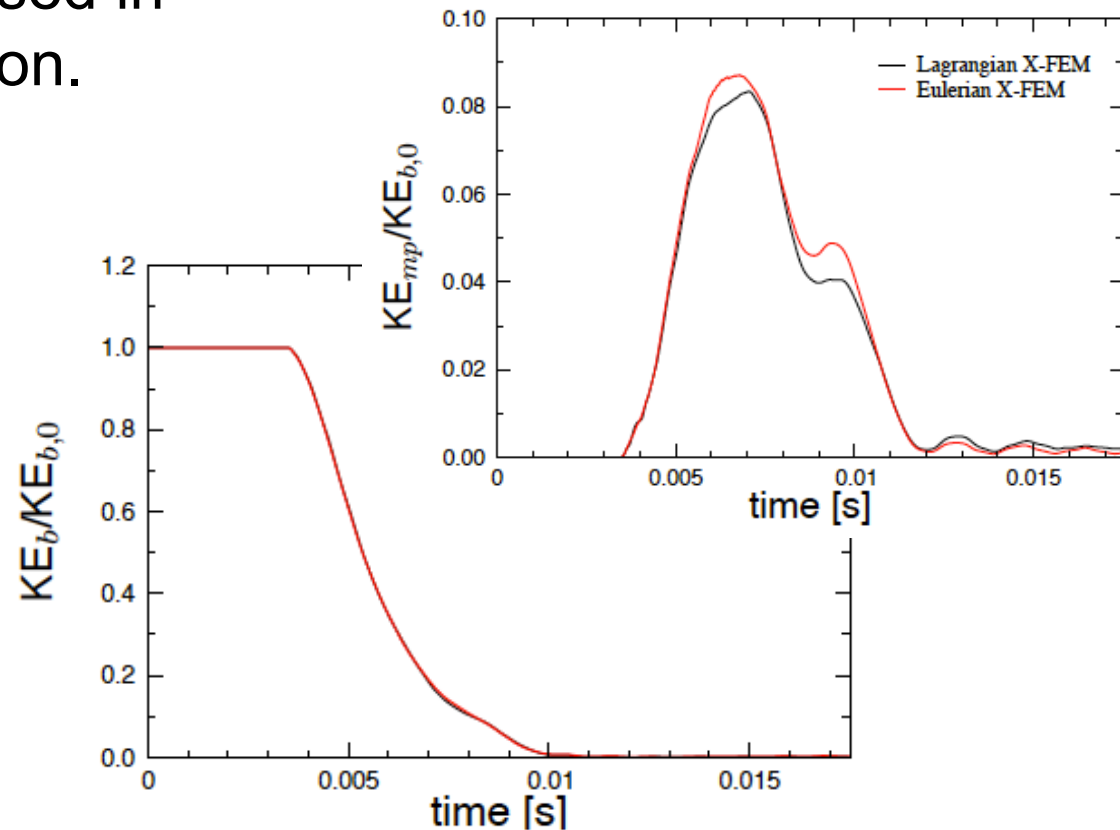


More complicated problems demonstrate the utility/advantages of the approach ...

Whipple Shield used in satellite protection.



Lagrangian: black mesh
Eulerian X-FEM: painted

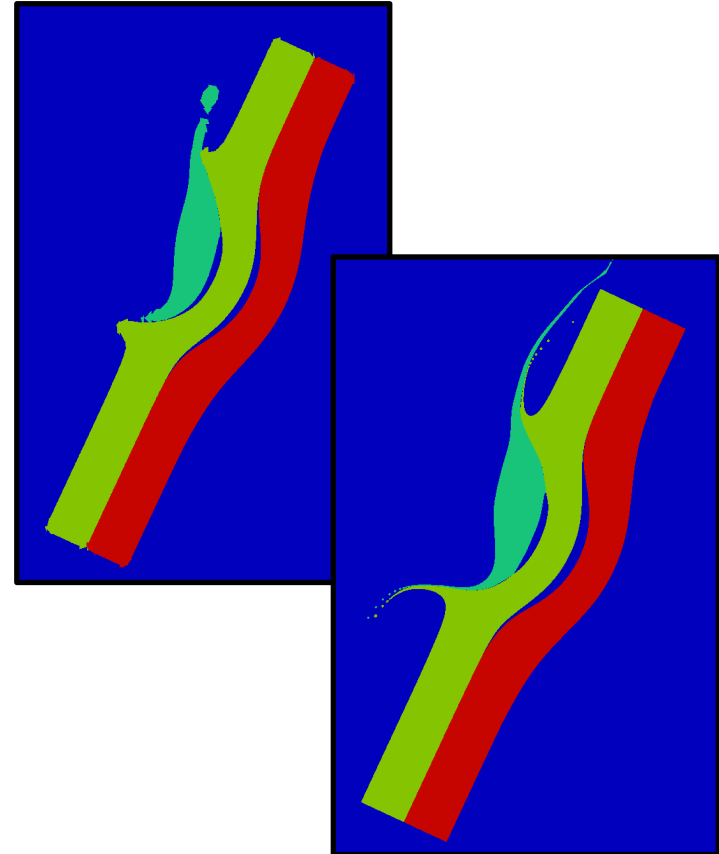
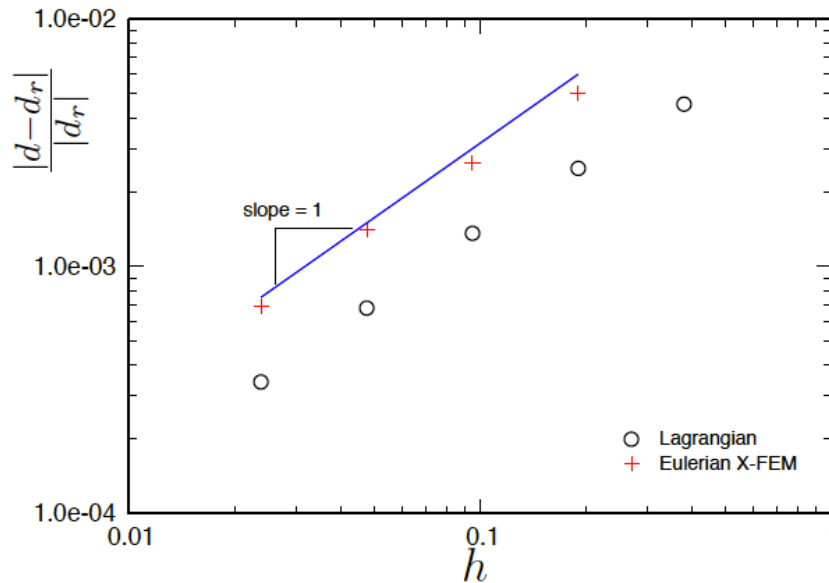


Low-velocity impact predictions compare well.



More complicated problems demonstrate the utility/advantages of the approach ...

Whipple Shield used in satellite protection.



High-velocity impact difficult for lagrangian and unrealistic for eulerian are possible with XFEM.



Conclusions:

- Developing capability to more accurately treat multi-material cells in an “operator-split” ALE context.
- Capability builds on existing ALE infrastructure.
- Uses XFEM ideas to provide provide unique kinematics for each material in a cell.
- Uses interface reconstruction rather than level-set ideas to address conservation and complex interface intersections.
- Employs higher-order, conservative remapping algorithms. Advantages are unclear at this point.
- Demonstrates good convergence/accuracy for problems investigated here.



(Incomplete) References:

- AHD11: Annavarapu et al., IJNME, submitted.
- BMW09: Bechet et al., IJNME 78, 931.
- CKT91: Carpenter et al., IJNME 32, 103.
- D83: Dukowicz, JCP 54, 411.
- DK87: Dukowicz and Kodis, SIAM J. Sci Stat Comput 8, 305.
- DLZRM10: Dubois et al., Comp Mech 46, 329.
- DMRV08: Dolbow et al., CMAME 197, 439.
- HAD11: Hautefeuille et al., IJNME, in revision.
- HH04: Hansbro and Hansbro, CMAME 193, 3523.
- MRMCB08: Menouillard et al., IJNME 74, 447.
- SAB06: Song et al., IJNME 67, 868.
- VB06: Vitali and Benson, IJNME 67, 1420.
- VMR10: Voth et al., USNCCM10, Columbus.