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Artificial Viscosity Flux Limiting in Lagrangian Shock Hydrodynamic Finite Element Computations

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Outline of Presentation

- Introduction and motivation
- Continuum equations
  - Kinematics
  - Balance Laws
- Artificial viscosity
  - Hyperbolic PDE concepts
  - Flux limiting
  - Numerical simulations
- Hyperviscosity
  - Algorithm
  - Numerical simulations
- Concluding remarks

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In search of a better artificial viscosity

- Typical artificial viscosity methods for Lagrangian hydrodynamic calculations are only first-order accurate.
- The shock-capturing viscosity is active in compression, regardless of whether the compression is adiabatic or a shock.
- Ideally the viscosity should vanish (go to zero) if the fluid flow is smooth (adiabatic/isentropic).
- The goal is to construct a second-order accurate artificial viscosity method, one which can “tell the difference” between shocked and smooth flows.

Previous Work

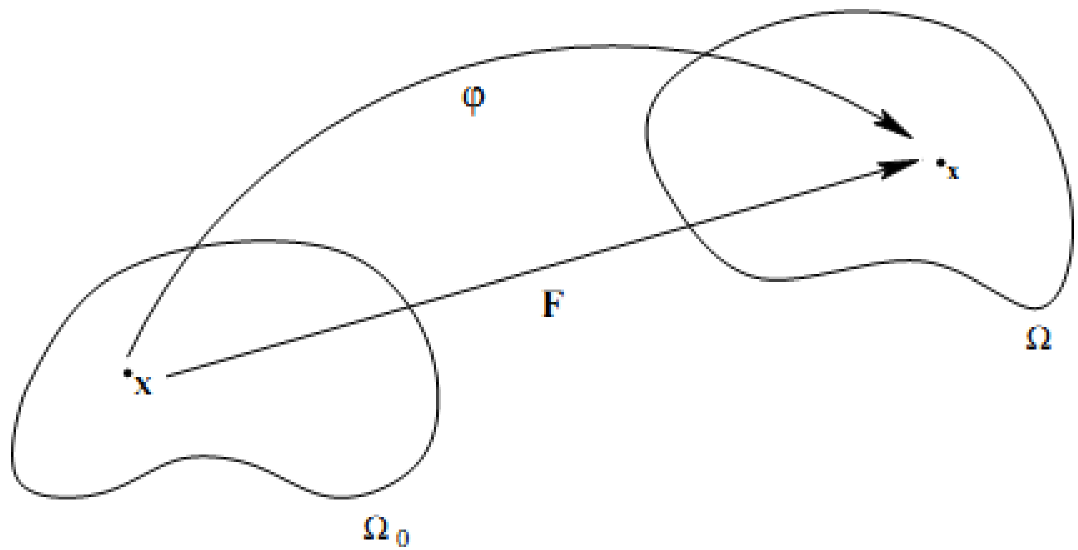
[1] J. VonNeumann and R.D. Richtmyer, "A Method for the Numerical Calculation of Hydrodynamic Shocks", *Journal of Applied Physics*, **21**(3), March 1950, 232-237.

[2] Mark L. Wilkins, "Use of artificial viscosity in multidimensional fluid dynamic calculations", *Journal of Computational Physics*, **36**(3), July 1980, 281-303.

[3] Tz.V. Kolev and R.N. Rieben, "A tensor artificial viscosity using a finite element approach", *Journal of Computational Physics*, **228**(1), December 2009, 8336-8366.

[4] Culbert B. Laney, *Computational Gasdynamics*, Cambridge University Press, 1998.

Basic kinematics underlying hyperbolic flow



$$\boldsymbol{\varphi} : \Omega_0 \times [0, T] \to \Omega \subset \mathbb{R}^3$$

$$\mathbf{F} := D\boldsymbol{\varphi} = \mathrm{GRAD}[\boldsymbol{\varphi}]$$

$$J = \det[\mathbf{F}]$$

$$\mathbf{v} = \dot{\boldsymbol{\varphi}}$$

$$\mathrm{grad}[\cdot] = \mathbf{F}^{-T}\mathrm{GRAD}[\cdot] \Longrightarrow \mathrm{grad}[\mathbf{v}] = \dot{\mathbf{F}}\mathbf{F}^{-1}$$

Integral Lagrangian form of the conservation laws

- **Conservation of mass**

$$\rho_0 - \rho J = 0$$

- **Conservation of linear momentum**

$$\int_{\Omega_0} \boldsymbol{\eta} \bullet \rho_0 \dot{\mathbf{v}} \, \mathrm{d}\Omega_0 + \int_{\Omega} \mathrm{grad}^s[\boldsymbol{\eta}] \bullet (-p\mathbf{I} + \boldsymbol{\sigma}_{art}) \, \mathrm{d}\Omega \; = 0 \quad \forall \boldsymbol{\eta}$$

- **Conservation of energy**

$$\int_{\Omega_0} \phi \cdot \rho_0 \dot{\epsilon} \, \mathrm{d}\Omega_0 - \int_{\Omega} \phi \cdot \mathrm{grad}^s[\mathbf{v}] \bullet (-p\mathbf{I} + \boldsymbol{\sigma}_{art}) \, \mathrm{d}\Omega \; = 0 \quad \forall \phi$$

- **This is a global system of hyperbolic conservation laws written in weak form.**

Spatial discretization of hyperbolic conservation laws

- **Basic form**

$$\partial_t \mathbf{u}^h + \partial_x^h \left[\mathbf{F}^h(\mathbf{u}^h) \right] = \mathbf{0}$$

- **Decompose the numerical flux $\mathbf{F}^h$ into high-order and low-order contributions. (Flux-corrected transport, self-adjusting hybrid schemes, TVD*,…)**
- **Self-adjusting hybrid scheme**

$$\mathbf{F}^h(\mathbf{u}^h) = (1-\theta)\mathbf{F}^{HO}(\mathbf{u}^h) + \theta\mathbf{F}^{LO}(\mathbf{u}^h)$$

$$\theta_i = \frac{||u_{i+1} - 2u_i + u_{i-1}||}{||u_{i+1} - u_i|| + ||u_i - u_{i-1}||} \qquad \text{1-D}$$

- **The limiter θ_i looks like a normalized Laplacian.**

*P.K. Sweby, "High Resolution Schemes Using Flux Limiters for Hyperbolic Conservation Laws", *SIAM Journal on Numerical Analysis*, **21**, pp. 995-1011, 1984

Maybe one can use the velocity Laplacian to limit the artificial viscosity?

- **A linear velocity field is smooth, does not represent a shocked flow, and also has zero Laplacian.**
- **Computation of the velocity Laplacian**

$$(\nabla^2 \mathbf{v}) = \nabla(\nabla \cdot \mathbf{v}) - \nabla \times (\nabla \times \mathbf{v}) = \mathrm{div}[\mathrm{grad}[\mathbf{v}]]$$

$$\int_\Omega \boldsymbol{\eta} \bullet (\nabla^2 \mathbf{v})\, \mathrm{d}\Omega \;=\; -\int_\Omega \mathrm{grad}[\boldsymbol{\eta}] \bullet \mathrm{grad}[\mathbf{v}]\, \mathrm{d}\Omega \;+\; \int_{\partial\Omega} \boldsymbol{\eta} \bullet \mathrm{grad}[\mathbf{v}]\, \mathbf{n}\, \mathrm{d}\Gamma \quad \forall \boldsymbol{\eta}$$

- **Normalize using the triangle inequality**

$$\theta_A \;=\; \frac{\left\| -\int_{\Omega_A} \mathrm{grad}[\mathbf{v}]\mathrm{grad}[N^A] + \int_{\partial\Omega_A} \mathrm{grad}[\mathbf{v}] N^A \mathbf{n} \right\|}{\int_{\Omega_A} \|\mathrm{grad}[\mathbf{v}]\mathrm{grad}[N^A]\| + \int_{\partial\Omega_A} \|\mathrm{grad}[\mathbf{v}] N^A \mathbf{n}\|} \;\leq\; 1$$

General structure of an improved artificial viscosity has several elements

- High-order "flux" is "zero artificial viscosity".
- Low-order "flux" is "standard artificial viscosity".
- Limited artificial viscosity if $\mathrm{trace}[\mathbf{d}] < 0.0$

$$\mathbf{d} = \mathrm{grad}^s[\mathbf{v}]$$

$$\boldsymbol{\sigma}_{art}^{LO} = \rho \left[c_1 \; c \; h \; + \; c_2 \; \|\mathrm{trace}[\mathbf{d}]\| \; h^2 \; \right] \mathbf{d}$$

$$\boldsymbol{\sigma}_{art} = \theta \boldsymbol{\sigma}_{art}^{LO}$$

- If the velocity field is linear, then the artificial viscosity is zero on both the interior and the boundary of arbitrary unstructured meshes.
- Important to include boundary terms (red boxed terms on previous slide).

What happens asymptotically with mesh refinement?

- **Assume the flow is smooth.**
  - **The standard non-limited artificial viscosity is O(h).**
  - **The limiter itself is O(h). In one dimension,**

$$\theta_i = \frac{|u_{i+1} - 2u_i + u_{i-1}|}{|u_{i+1} - u_i| + |u_i - u_{i-1}|} = \frac{h\,|u''(x_i)|}{2\,|u'(x_i)|} \; + \; \mathcal{O}(h^3)$$

  - **When the artificial viscosity is multiplied by the limiter, the result is $O(h^2)$.**
  - **The final limited viscosity is $O(h^2)$, and goes to zero one order faster than the standard artificial viscosity.**
- **Assume the flow is shocked, with a finite jump as $h \to 0$.**

$$\{u_{i+1} = 1, u_i = 0, u_{i-1} = 0\} \quad \Longrightarrow \quad \theta_i = 1$$

  - **In this simple example situation, the limiter is one.**

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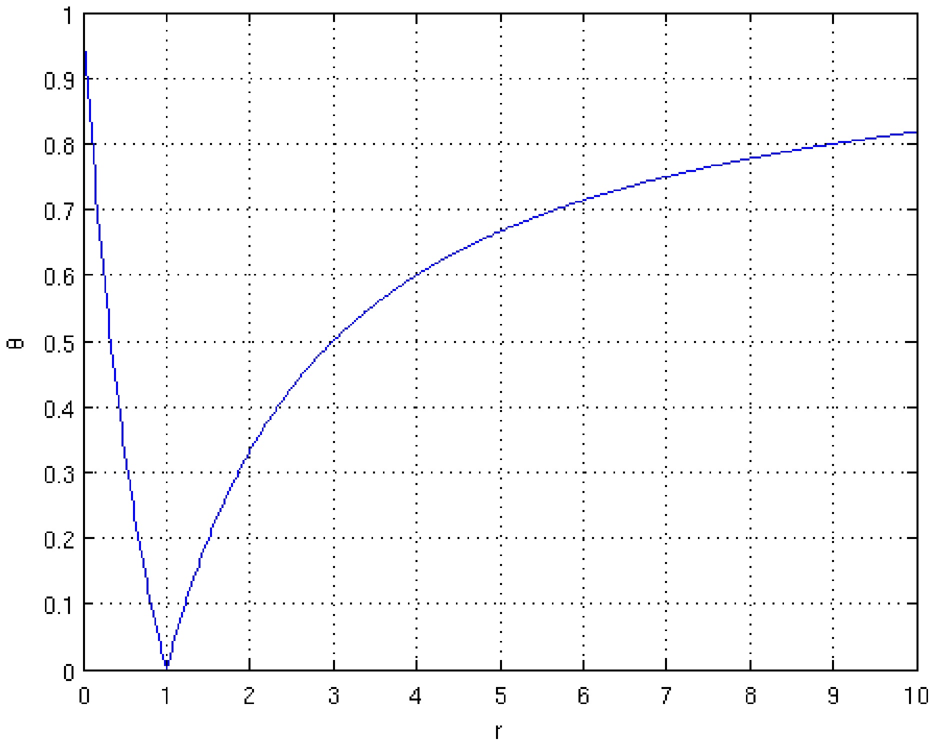
In general the limiter is a highly non-linear function of the discrete gradients

- **More generally,**

$$\Delta_i u := (u_i - u_{i-1}) \Longrightarrow \theta_i = \frac{|\Delta_{i+1}u - \Delta_i u|}{|\Delta_{i+1}u| + |\Delta_i u|} = \frac{|1-r|}{1+|r|} \quad (\text{for } \Delta_{i+1}u > 0)$$

$$r = \frac{\Delta_i u}{\Delta_{i+1} u}$$



Shock:

$$\lim_{r \to 0} \theta_i = 1$$

$$\lim_{r \to \infty} \theta_i = 1$$

Smooth:

$$\lim_{r \to 1} \theta_i = 0$$

Details of the numerical implementation

- **Use standard single-point integration (Q1/P0) four-node finite elements to discretize the weak form.**
- **Flanagan-Belytschko viscous hourglass control (scales linearly with sound speed) with parameter 0.05**
- **Second-order accurate (in time) predictor-corrector time integration algorithm.**
- **Artificial viscosity limiting based on Laplacian of velocity field.**
- **Gamma-law ideal gas equation-of-state.**
- **Constants** $c_1 = 1.0 \quad \text{and} \quad c_2 = 1.5$ **.**

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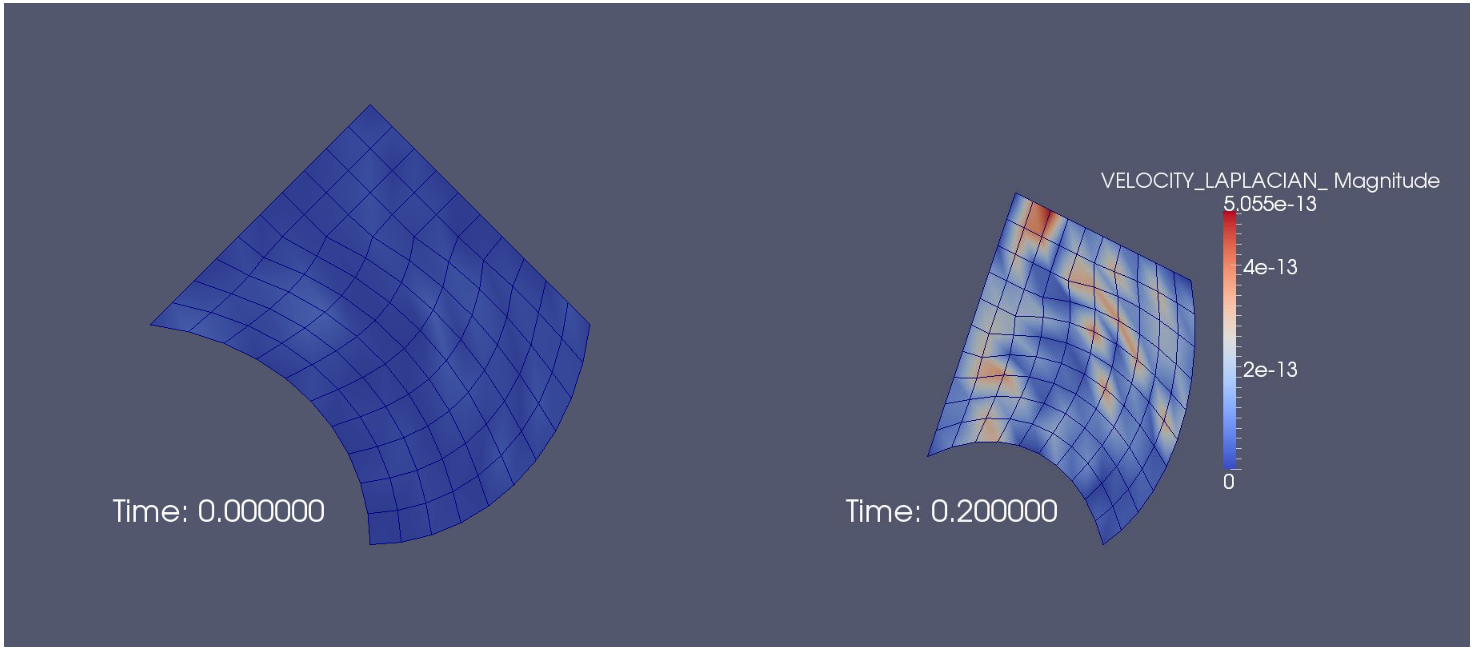
Zero Laplacian velocity field patch test

- **Linear velocity field**

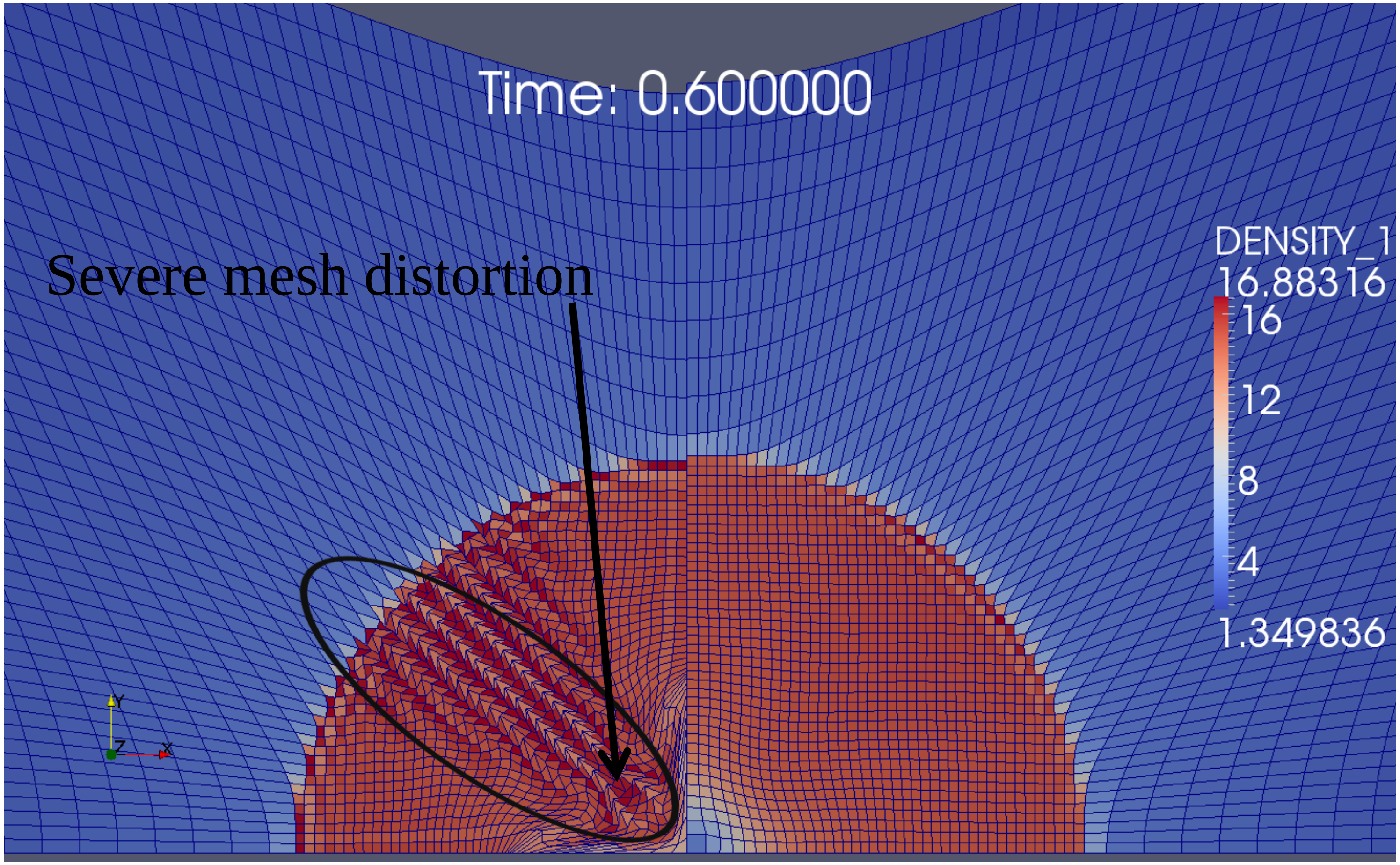
$$\mathbf{v} = (-2x_1 - x_2)\mathbf{e}_1 + (2x_1 - x_2)\mathbf{e}_2$$

- **Test on an initially distorted mesh.**
- **The velocity Laplacian is zero everywhere (test passes).**
- **Inclusion of the boundary terms is critical.**



Numerical Simulations I

- **Noh Implosion Test**



Limited Not Limited

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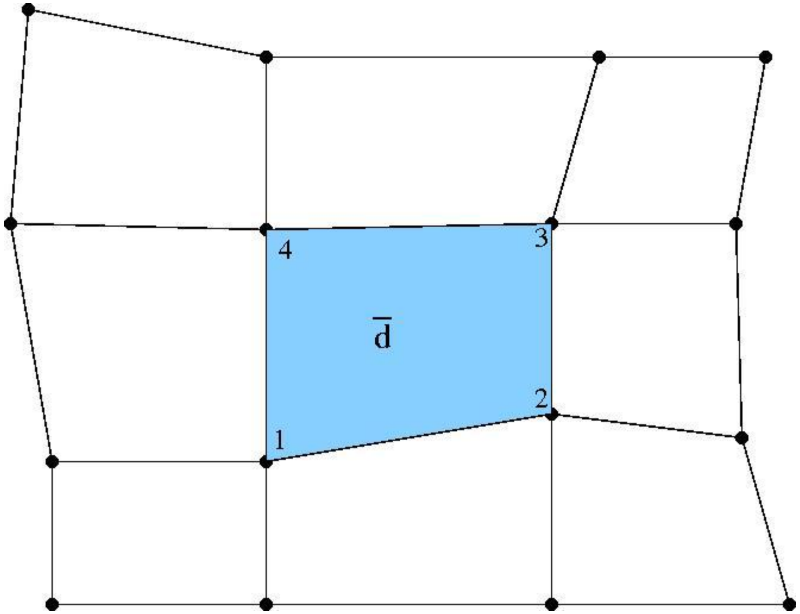
HyperViscosity

- **Define $\bar{\mathbf{d}}$ as the mean rate of deformation over a patch of elements.**

$$\Omega_{patch} = \bigcup_{A=1}^{4} \operatorname{supp}(N^A)$$

$$\bar{\mathbf{d}} = \frac{1}{\operatorname{meas}(\Omega_{patch})} \int_{\Omega_{patch}} \mathbf{d} \, \mathrm{d}\Omega$$



- **Add additional viscosity**

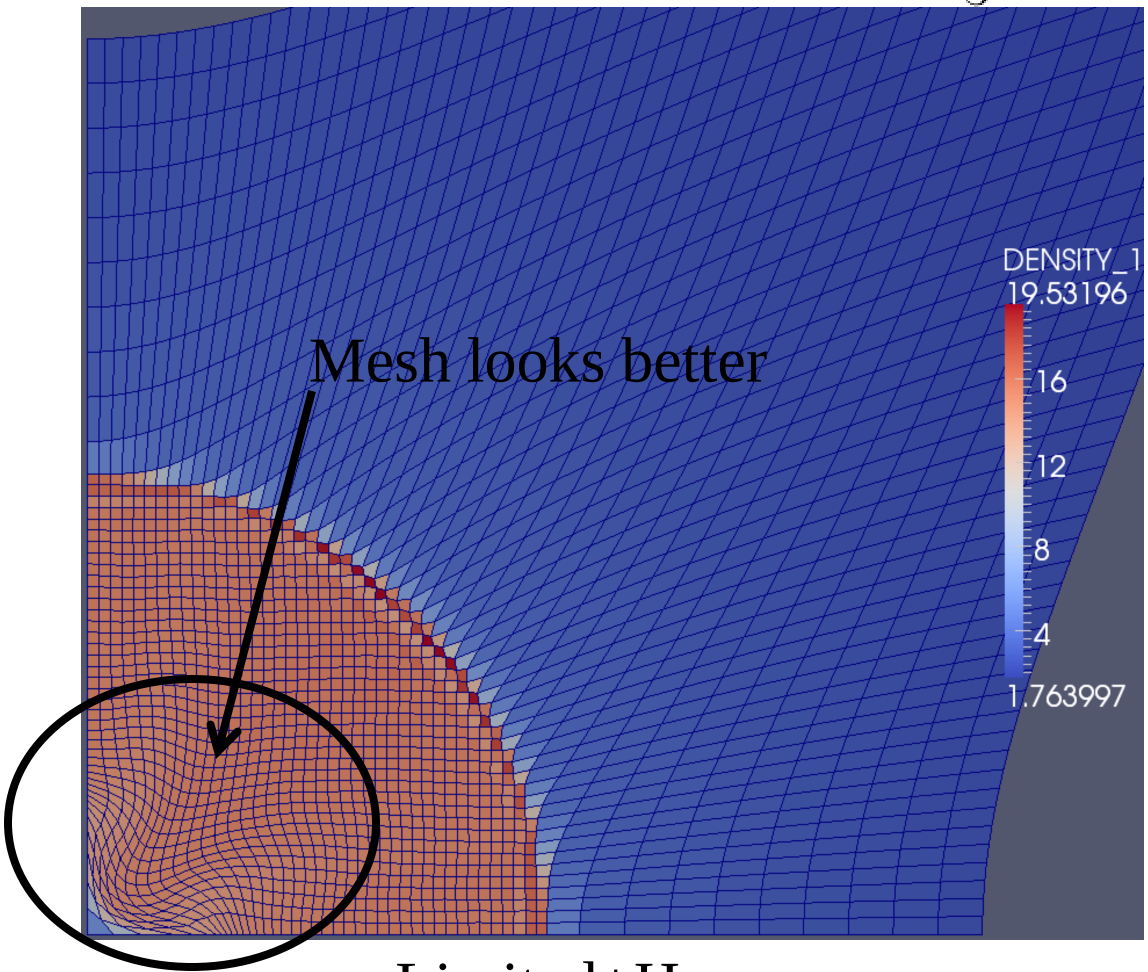
$$\boldsymbol{\sigma}_{hyper} = c_3 \left[\boldsymbol{\sigma}^{LO}_{art}(\mathbf{d}) - \boldsymbol{\sigma}^{LO}_{art}(\bar{\mathbf{d}}) \right]$$

$$\boldsymbol{\sigma}_{art} = \theta \boldsymbol{\sigma}^{LO}_{art}(\mathbf{d}) + (1-\theta)\boldsymbol{\sigma}_{hyper}$$

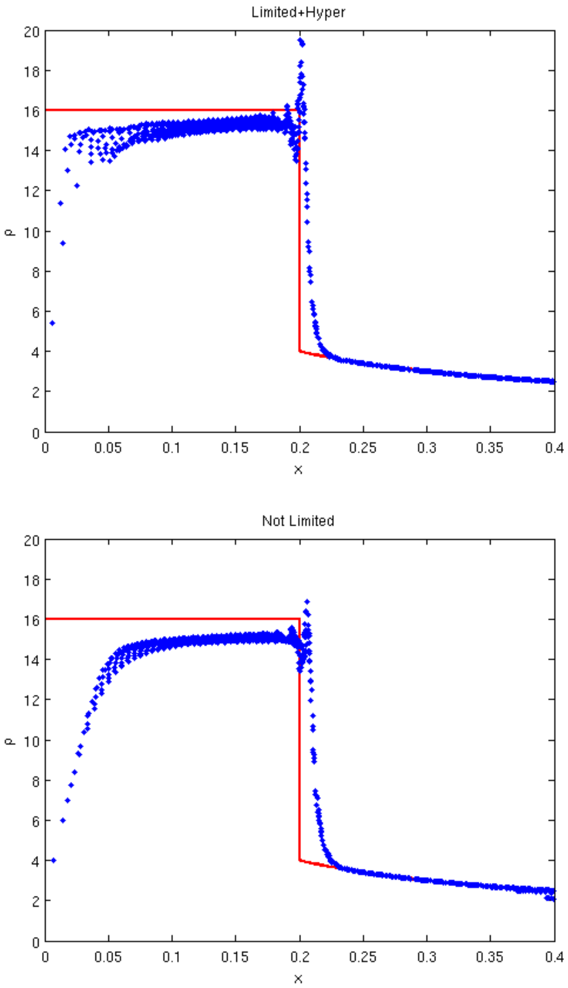
- **The hyperviscosity also vanishes for a linear velocity field since in that case $\mathbf{d} = \bar{\mathbf{d}}$.**

Numerical Simulations II

- **Noh Implosion Test** $c_3 = 1.0$



Limited+Hyper



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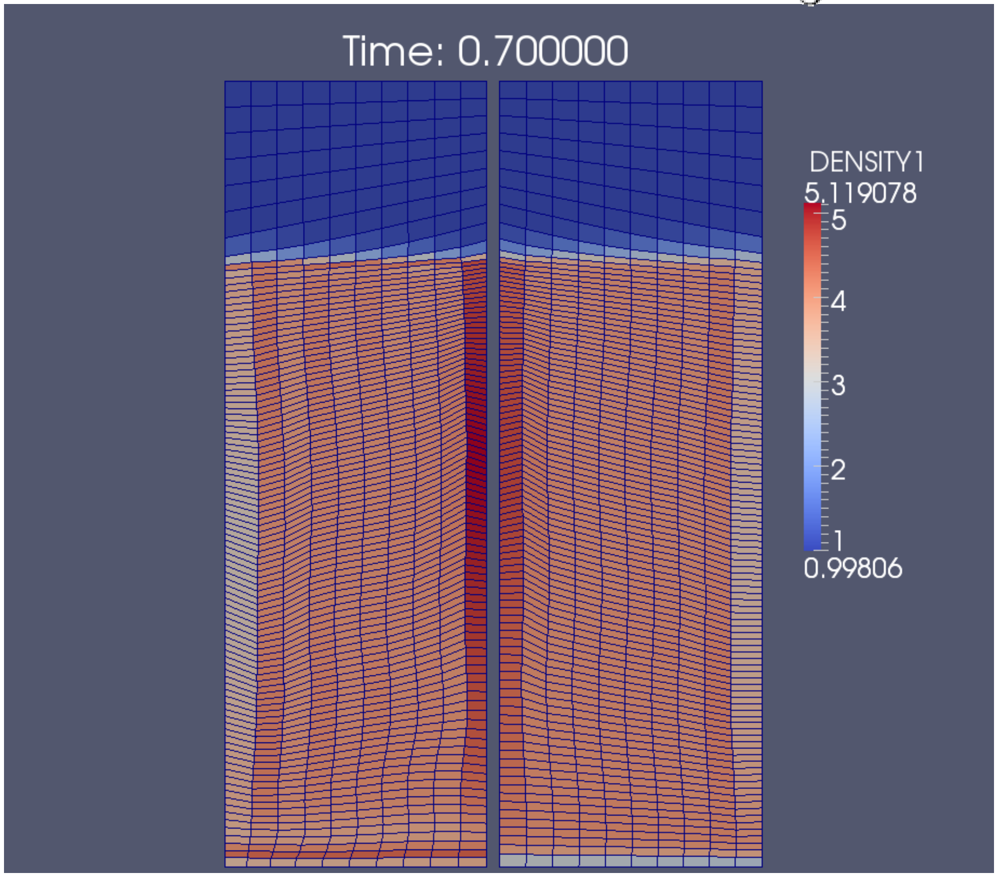
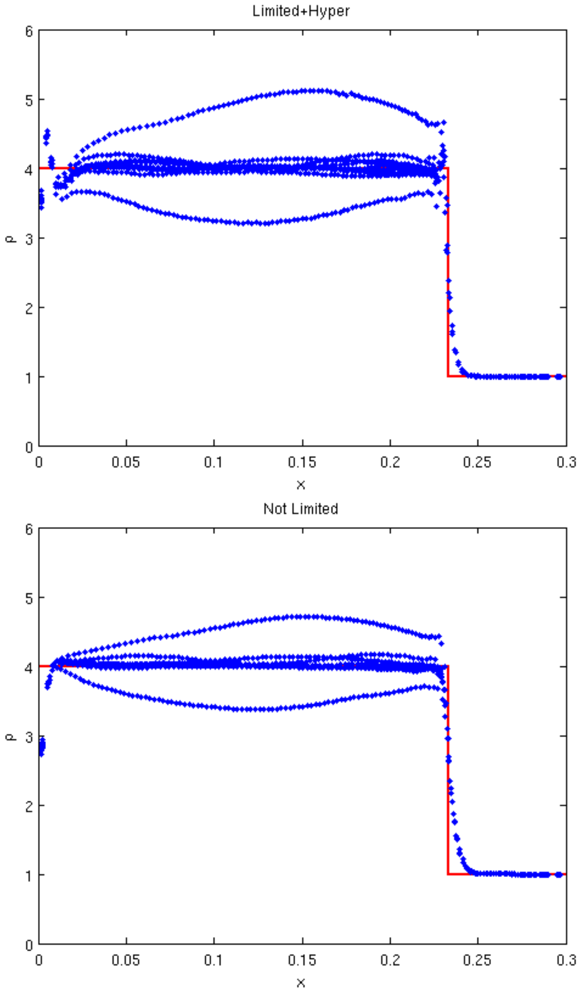
Numerical Simulations II

- **Saltzmann Test**

$c_3 = 1.0$



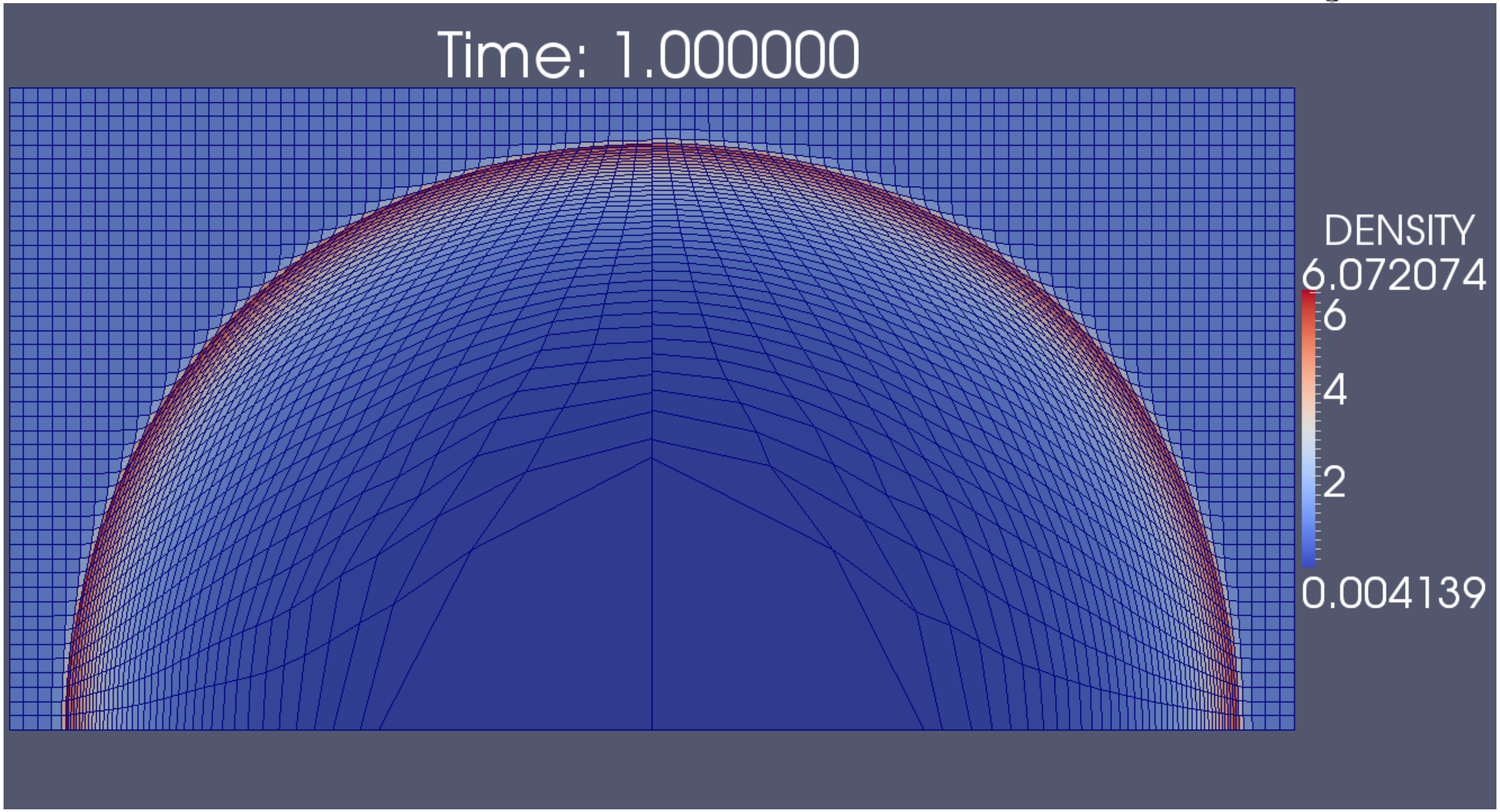
Limited+Hyper   Not Limited



Numerical Simulations III

- **Sedov Blast Wave Test**

$c_3 = 1.0$

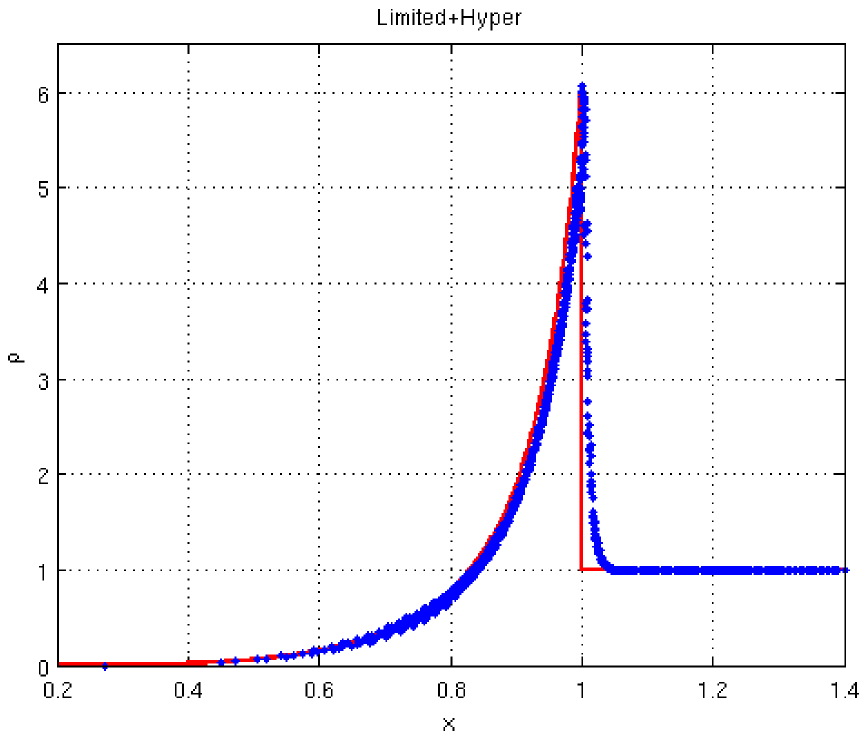


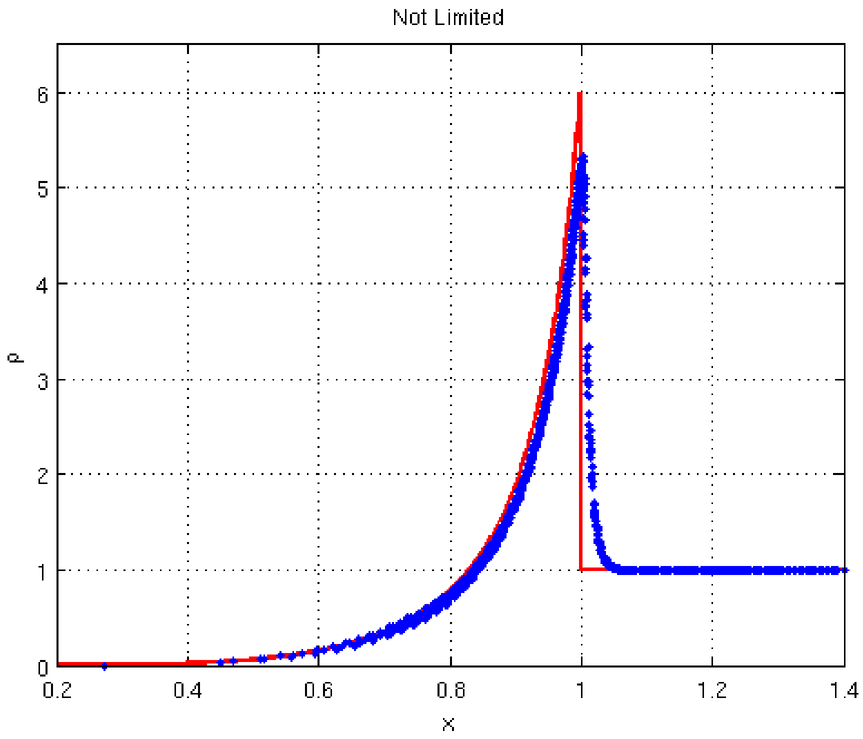
Limited+Hyper

Not Limited

Numerical Simulations III

- **Sedov Blast Wave Test**





- **These results look promising…**

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Artificial viscosity limiting is a work in progress

- Limiter shows potential, but is strongly coupled to the hourglass control algorithm.
- What's the issue? Hypothesis…
  - Artificial viscosity causes heating (for an ideal gas), which increases sound speed, which increases hourglass control scaling?
  - Reducing artificial viscosity reduces heating (for an ideal gas), which reduces sound speed, which reduces hourglass control scaling?
- Good results for Eulerian (Lagrange+remap) simulations (where hourglassing is less of a problem).
- Much more work, including formal verification, is needed.