Experimental Based Models for Shock and Vibration Environments

Troy Jon Skousen
Sandia National Laboratories
1515 Eubank Blvd. SE
Albuquerque, NM 87123
(505) 284-9260

Jerome S Cap
Sandia National Laboratories
1515 Eubank Blvd. SE
Albuquerque, NM 87123
(505) 844-1213

Paul A Larkin
Sandia National Laboratories
1515 Eubank Blvd. SE
Albuquerque, NM 87123
(505) 845-9967

Response data measured during system level field and laboratory tests are routinely used to determine component test specifications. However, it is often the case that the system inputs change after the testing is completed and repeating the test is either impossible and/or too costly. In such a situation it is desirable to develop an analytical model of the system that can then be used to simulate the response to the new inputs. The purpose of this paper is to describe an approach that was used to develop a model directly from the experimental data.

INTRODUCTION

The purpose of this paper is to describe an approach that was used to develop a "transfer function model" directly from the experimental data for a laboratory resonant fixture test designed to simulate a pyrotechnic event for a payload.

The payload being tested had three mounting feet. Due to the size of the payload, the pyrotechnic event was simulated by applying a transient excitation separately in each of three mutually perpendicular directions at each of the three feet (9 unique test configurations total). Each test was performed twice, thereby making for a total of 18 individual tests.

BASIS FOR ANALYTICAL MODEL

It was decided to develop the analytical model using Transmissibility Response Functions (TRFs) in order to provide rapid turnaround. The simplest means of developing a TRF model would be to divide the Fast Fourier

Sandia National Laboratories is a multi-program laboratory operated and managed by Sandia Corporation, a wholly owned subsidiary of Lockheed Martin Corporation, for the U.S. Department of Energy's National Nuclear Security Administration under contract DE-AC04-94AL85000.

Transforms (FFTs) for all of the response points by the FFT for the corresponding in-axis input. This approach represents a Single-Input- Multiple-Output (SIMO) model.

However, based on our experience with the testing of large structures, it was decided that the off-axis excitation associated with each input might be a significant contributor to the actual test. With that as our working hypothesis, we chose to develop a Multiple-Input-Multiple-Output (MIMO) TRF model. It was assumed that if the off-axis inputs turned out to be linearly dependent versions of the in-axis input, then the results for the SIMO and MIMO models would collapse to the same values. However, if the off-axis responses turn out to be independent then the MIMO model should prove to be a better predictor of the true response.

DERIVATION OF TRF MODEL

Equation (1) describes the most generalized TRF model.

$$R_{jk} = H_{ji} I_{ik} \tag{1}$$

Where *R* is the matrix of response FFTs, *I* is the matrix of input FFTs, *H* is the matrix of TRFs, *i*=input number, *j*=response number, and *k*=test number. As a matter of implementation, it was originally decided to use the 3-axis inputs from all 6 tests for a single foot to develop the MIMO model. Equation (2) shows how the equation (1) was implemented.

$$\begin{bmatrix} R_{1x_1} & R_{1x_2} & R_{1y_1} & R_{1y_2} & R_{1z_1} & R_{1z_2} \\ R_{2x_1} & R_{2x_2} & R_{2y_1} & R_{2y_2} & R_{2z_1} & R_{2z_2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ R_{nx_1} & R_{nx_2} & R_{ny_1} & R_{ny_2} & R_{nz_1} & R_{nz_2} \end{bmatrix} = \begin{bmatrix} H_{1x} & H_{1y} & H_{1z} \\ H_{2x} & H_{2y} & H_{2z} \\ \vdots & \vdots & \vdots \\ H_{nx} & H_{ny} & H_{nz} \end{bmatrix} \times \begin{bmatrix} I_{xx_1} & I_{xx_2} & I_{xy_1} & I_{xy_2} & I_{xz_1} & I_{xz_2} \\ I_{yx_1} & I_{yx_2} & I_{yy_1} & I_{yy_2} & I_{yz_1} & I_{yz_2} \\ I_{zx_1} & I_{zx_2} & I_{zy_1} & I_{zy_2} & I_{zz_1} & I_{zz_2} \end{bmatrix} \tag{2}$$

The number of columns in *I* and *R* correspond to the number of tests used to construct the transfer function matrix. The rows in *I* correspond to the in-axis and off-axis inputs from a given test and the rows in *R* correspond to the "N" response locations. Equation (3) shows how the matrices are rearranged to solve for *H* using the Moore-Penrose Pseudo Inverse [1].

$$H_{ji} = R_{jk} \times pinv(I_{ik}) \tag{3}$$

The net effect of using all of the inputs to derive the MIMO model is that the resulting model reflects the least squares average of how the payload responded to the different tests.

IMPLEMENTATION OF MODEL

Figure 1 shows a typical acceleration response. For the Pseudo-Inverse model each FFT was derived using a single signal zero padded to produce the desired frequency resolution.

As a matter of comparison the corresponding SIMO analyses were also performed. Each SIMO analysis still utilized the two unique tests for each axis, but by definition only used the data from one configuration at a time. This reduced the order of the *R* and *I* matrices from a 3x6 to 1x2. Similarly, the size of *H* was reduced from Nx3 to Nx1.

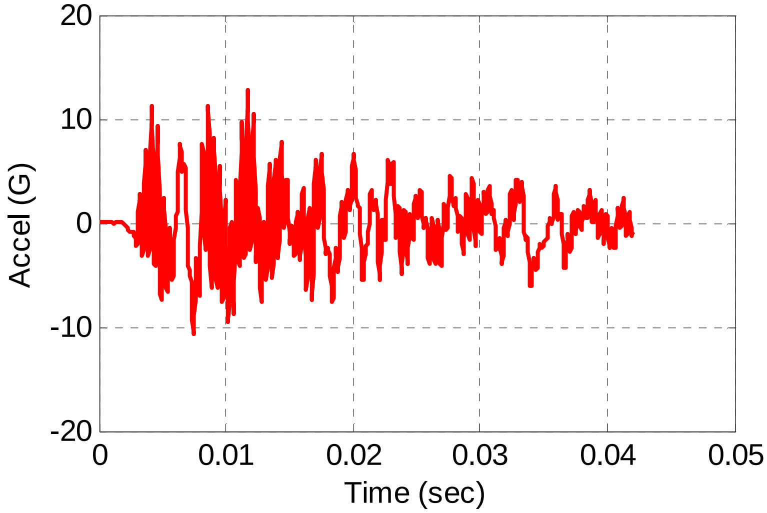


Figure 1: Typical Test Response

Figure 2 shows the measured Shock Response Spectra (SRS) for a single point on the structure along with the SRS for the corresponding Pseudo-Inverse MIMO and SIMO predictions (denoted PAMIMO and PISIMO respectively, where the "P" denotes the pseudo-inverse analysis method and the "A" and "I" denote whether all three axes worth of test data were used or just the data from one test direction.).

There were too many response locations to permit an objective assessment of the different methods just using SRS plots. Therefore, the SRS for the TRF models were normalized by dividing them by the SRS of the measured data and then converted to units of deciBels (dB) using the formula shown in equation (4).

$$E = 20\log\left(S_C / S_M\right) \tag{4}$$

S_C and S_M are the computed and measured SRS respectively. The channel specific normalized errors, E, were then averaged together to create a test specific error spectra, E_A. Figure 3 shows a comparison of E_A for the PAMIMO and PISIMO simulations.

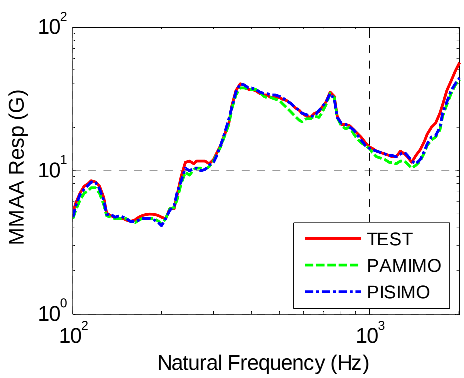


Figure 2: SRS for the Measured and Predicted Responses

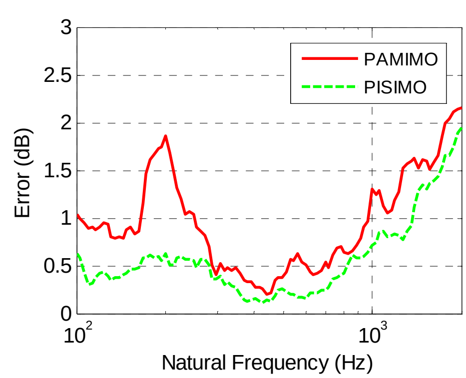


Figure 3: Normalized Error Spectra for the Predicted Responses

In what was a rather unexpected result, the SIMO model generally did a better job in predicting the measured response (i.e., a smaller dB error) than the MIMO model using all 6 sets of tests.

EXAMINATION OF EXPERIMENTAL MODEL

Two possible explanations were put forth for this result. The first explanation was that the Pseudo-Inverse approach might be introducing spurious noise due to the inclusion of low level off-axis inputs. In order to investigate this possibility, a MIMO model was generated using only the inputs from a single test configuration (denoted PIMIMO).

The second possible explanation was that the variance errors might be higher than necessary due to the fact that only one test was included in the calculation of each FFT used in equation (3). In order to investigate this possibility, the TRF equations were reformulated to compute the FFTs using each "input" in a given direction (whether it was from an in-axis or an off-axis test). This formulation was solved using Smallwood's method [2] by concatenating the N inputs together and then computing the resulting FFT in a manner that is analogous to how FFTs are computed for random vibration signals (i.e., FFTs for several blocks of data are averaged together). We ended up evaluating 6 distinct model/analysis techniques, which are summarized in Table 1.

Table 1: Analysis Techniques

Case ID	Description	Input DOF	Number of tests in FFT	Analytical Method
PAMIMO	MIMO model using all 6 tests for a given foot.	3x6	1	Pseudo-Inverse
PISIMO	SIMO model using 2 in-axis tests	1x2	1	Pseudo-Inverse
PIMIMO	MIMO model in-axis and off-axis inputs from a single configuration	3x2	1	Pseudo-Inverse
FAMIMO	MIMO model using all 6 tests for a given foot.	3x1	6	Smallwood
FISIMO	SIMO model using 2 in-axis tests	1x1	2	Smallwood
FIMIMO	MIMO model in-axis and off-axis inputs from a single configuration	3x1	2	Smallwood

In order to better study how each of the analysis methods affects the accuracy of the MIMO and SIMO models, it was easier to look at the differences between the MIMO and SIMO normalized error spectra. Therefore, Figure 4 presents the differential error spectra, E_D=$E_{A(MIMO)}$-$E_{A(SIMO)}$, for the X, Y, and Z axis tests using the six different methods.

The PINV MIMO model using only the data from a single test configuration (PIMIMO) produced better results than the corresponding SIMO model. In fact, as can be seen in Figure 5, the PIMIMO model actually produced an exact match to the test data.

This study clearly showed that the root cause of the higher than expected MIMO response predictions was the decision to use all of the tests to generate *H*. However, the fact that the PIMIMO model produced an exact reproduction of the test results was almost too good to be true. At first we suspected that there was an error in our calculations. Since the initial simulations were performed using the first of the two tests performed in each configuration, we generated the responses to the second test using the calculated *H* matrix and the results were also an identical match for the corresponding test data. We then modeled the input at another foot and again predicted an exact match for the test data.

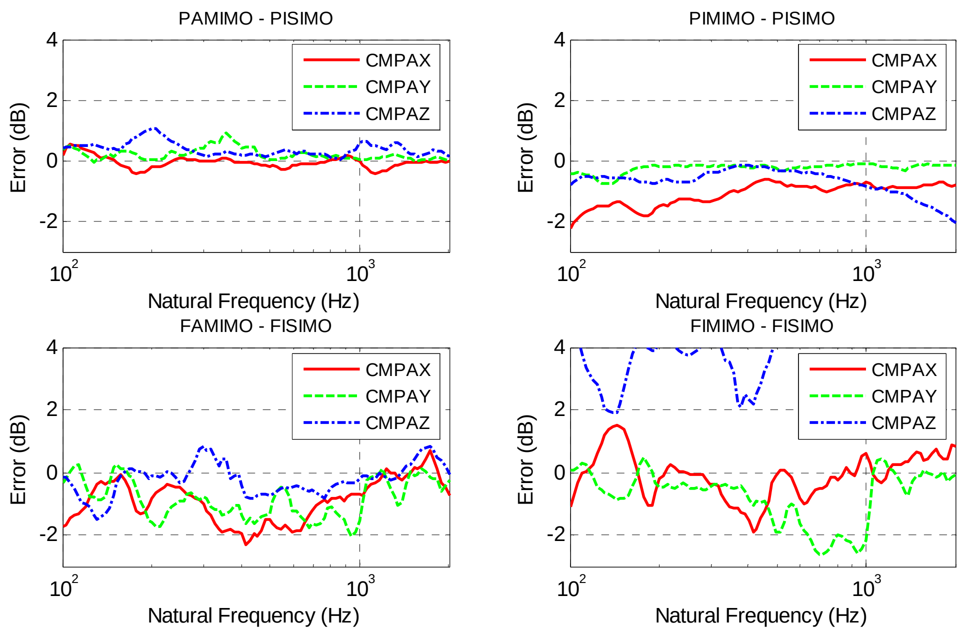


Figure 4: Difference in Normalized Error Spectra

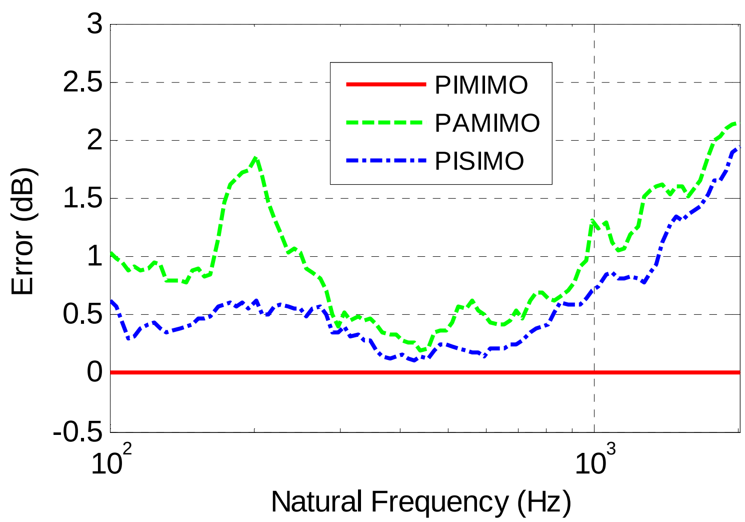


Figure 5: Normalized Error Spectra for Measured and Predicted PINV Responses

ANALYTICAL CASE STUDY

It was decided to take a step back and study a simple, contrived structure. The model consisted of a two input, single output (TISO) model. For the purposes of this study, the two TRFs can be thought of as the in-axis and an off-axis TRFs. The TRFs between each input to the response location were comprised of the TRFs for four Single-Degree-of-Freedom (SDOF) oscillators added together and normalized so as to have unity gain at 0 Hz. Table 2 identifies the SDOF resonant frequencies. Figure 6 shows the TRF magnitudes.

Table 2: Resonant Frequencies for TISO Model

Mode	In-Axis TRF	Off-Axis TRF
1	50	50
2	100	125
3	150	180
4	200	250

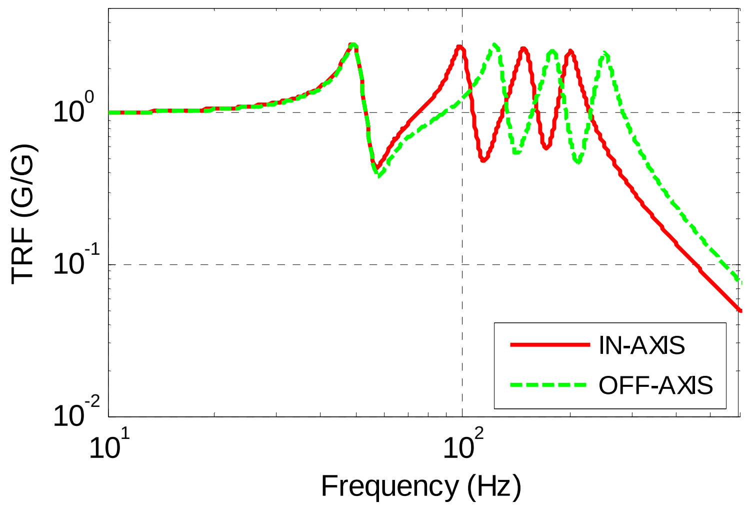


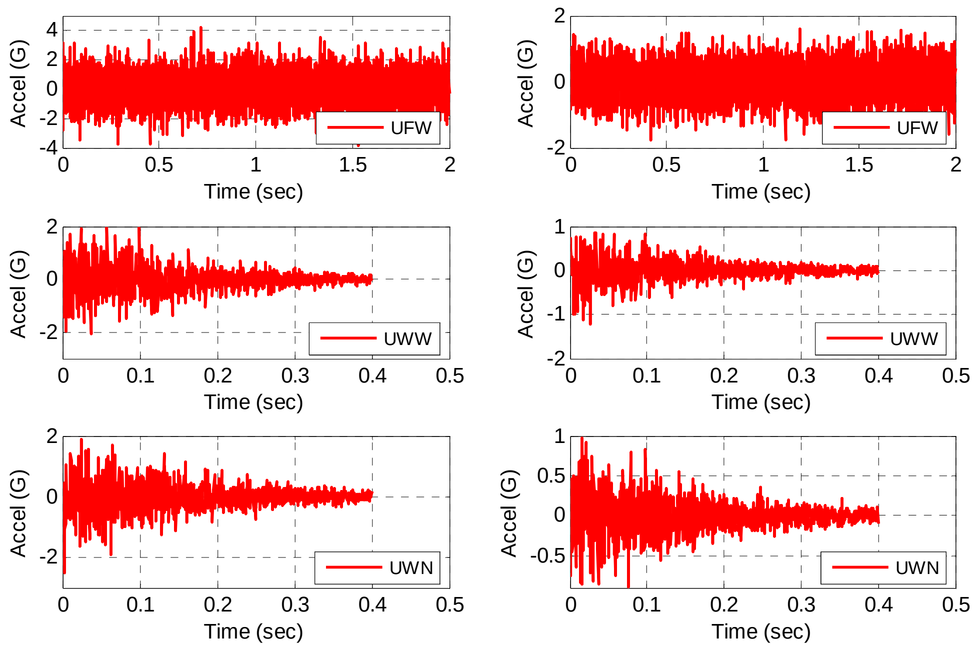
Figure 6: TRFs for TISO Model

The TISO model was excited using the six different loading cases identified in Table 3. The amplitude of the off-axis input was always one-half of the corresponding in-axis input. For the "windowed" inputs the windowing function was an exponential that decayed to one-tenth of the initial peak value. For the cases where additional uncorrelated white noise was added to the inputs and outputs, that noise was scaled to equal one-tenth of the amplitude of the in-axis input (and hence was equal in amplitude to the primary pulse at the end of the pulse).

Table 3: TISO Load Cases

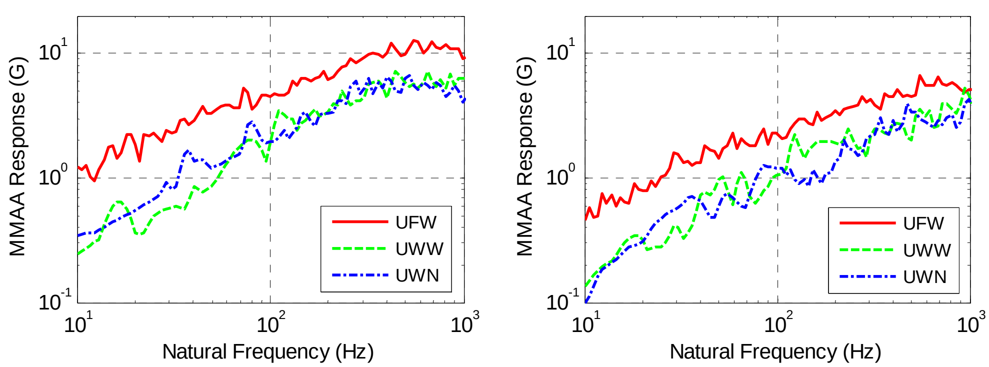
Case	In-Axis Input	Off-Axis Input	Noise
1) CFW	Flat White	Correlated Flat White	----
2) UFW	Flat White	Uncorrelated Flat White	----
3) CWW	Windowed White	Correlated Windowed White	----
4) UWW	Windowed White	Uncorrelated Windowed White	----
5) CWN	Windowed White	Correlated Windowed White	Uncorrelated White
6) UWN	Windowed White	Uncorrelated Windowed White	Uncorrelated White

Figure 7 shows the examples of the different input acceleration signals for the uncorrelated load cases: 1) Flat White, 2) Windowed White, and 3) Windowed White with Uncorrelated Noise. Figure 8 shows the corresponding SRS.



In-Axis Inputs Off-Axis Inputs

Figure 7: Correlated Input Acceleration Histories for TISO Study



In-Axis Inputs Off-Axis Inputs

Figure 8: SRS for Input Acceleration Histories for TISO Study

For each load case, the responses for two independent inputs were generated using the theoretical TRFs and inputs that were uncorrelated relative to each other. The SISO and MISO 'pinv' models were then used to predict the TRFs in a manner that was analogous to how the experimental test data were analyzed. The predicted TRFs were then used to generate the response for the first of the two simulated tests. As expected, Cases 1 and 3 are trivial because both the SISO and MISO models can predict the exact result. Figure 9 presents a comparison of the SRS for the MISO and SISO predictions versus the theoretical responses for the other four cases. Figure 10 presents the comparison of the corresponding normalized error spectra.

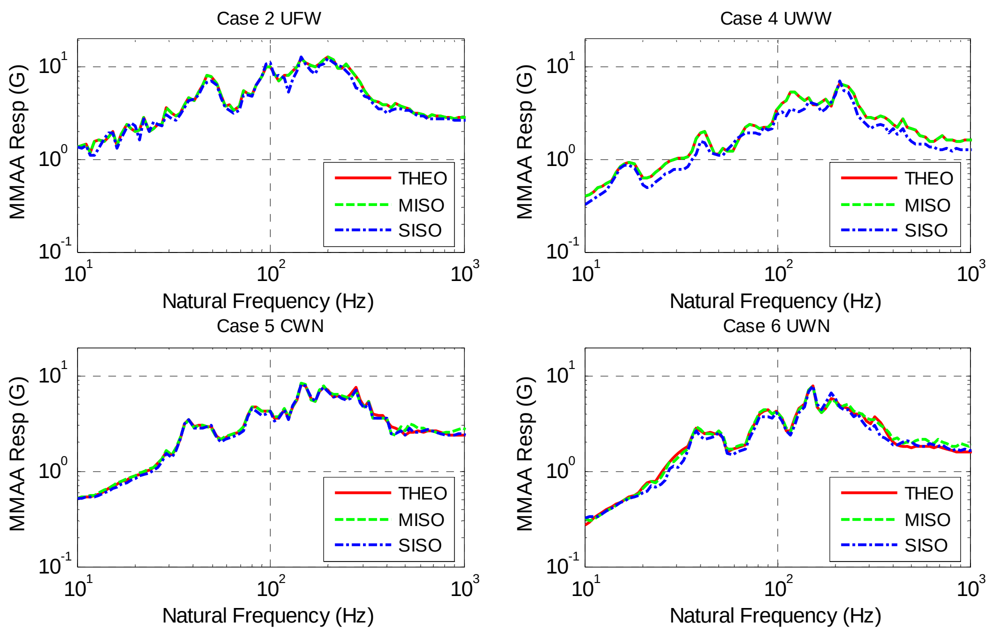


Figure 9: Comparison of SRS for MISO and SISO Predictions versus Theoretical Response

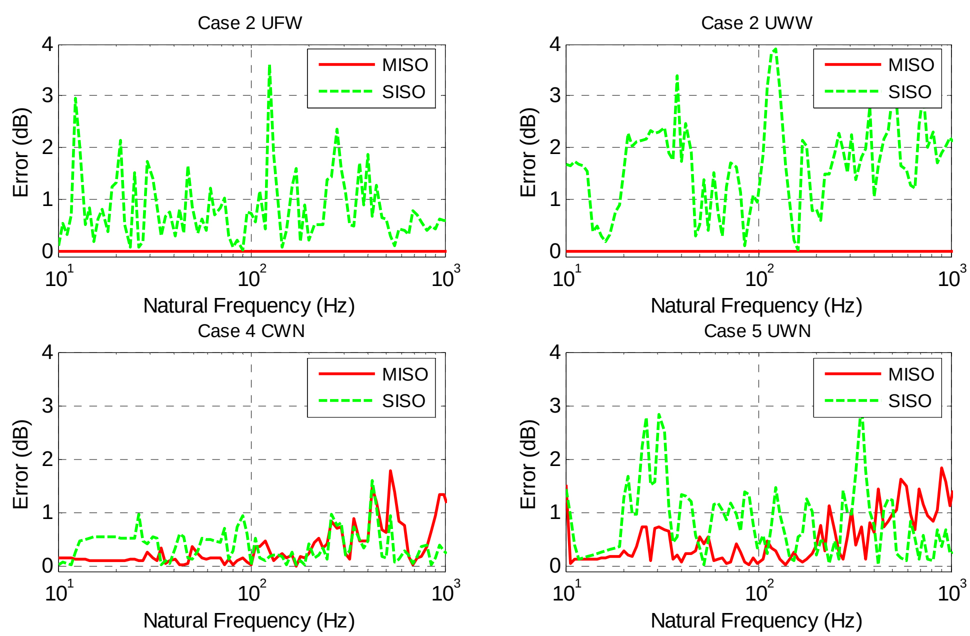


Figure 10: Comparison of Normalized Error Spectra for MISO and SISO Predictions versus Theoretical Response

Based on the results shown in these plots, the MISO model matches or exceeds the accuracy of the SISO model for every case except where the uncorrelated noise is significant (i.e., for frequencies > 300 Hz in Cases 5 and 6). Therefore, the most probable explanation that we can come up with for why the PIMIMO model exactly predicted the test results while the PISIMO model did not was that the inputs were sufficiently uncorrelated and the data did not contain any measurable uncorrelated noise.

CRITERIA FOR CHOOSING A METHOD

While the PIMIMO model did reproduce the test results, there is still no guarantee that it would be able to predict the responses to a significantly different input (nonlinearities in the structure will cause the TRFs to change in ways that cannot be anticipated). Conversely, the results for the SIMO model were not that bad, so it is plausible that a SIMO model may produce results that are sufficiently accurate for one's needs and requires much less effort to implement.

Therefore, the final step in this study was to establish criteria by which the analyst might determine when the effort needed to develop a MIMO model was warranted.

Based on the results of the TISO study, it occurred to us that the coherence between the in-axis and off-axis inputs combined with the coherence between the in-axis input and the responses might serve as a good measure of whether or not a MIMO model was appropriate. Figure 11 presents the difference between the MISO and SISO normalized error spectra along with the Off-axis / In-axis and Response / In-axis coherence spectra for the TISO study. The coherence spectra were re-averaged using 1/3rd octave bandwidths in order to permit the reader to distinguish between the different spectra.

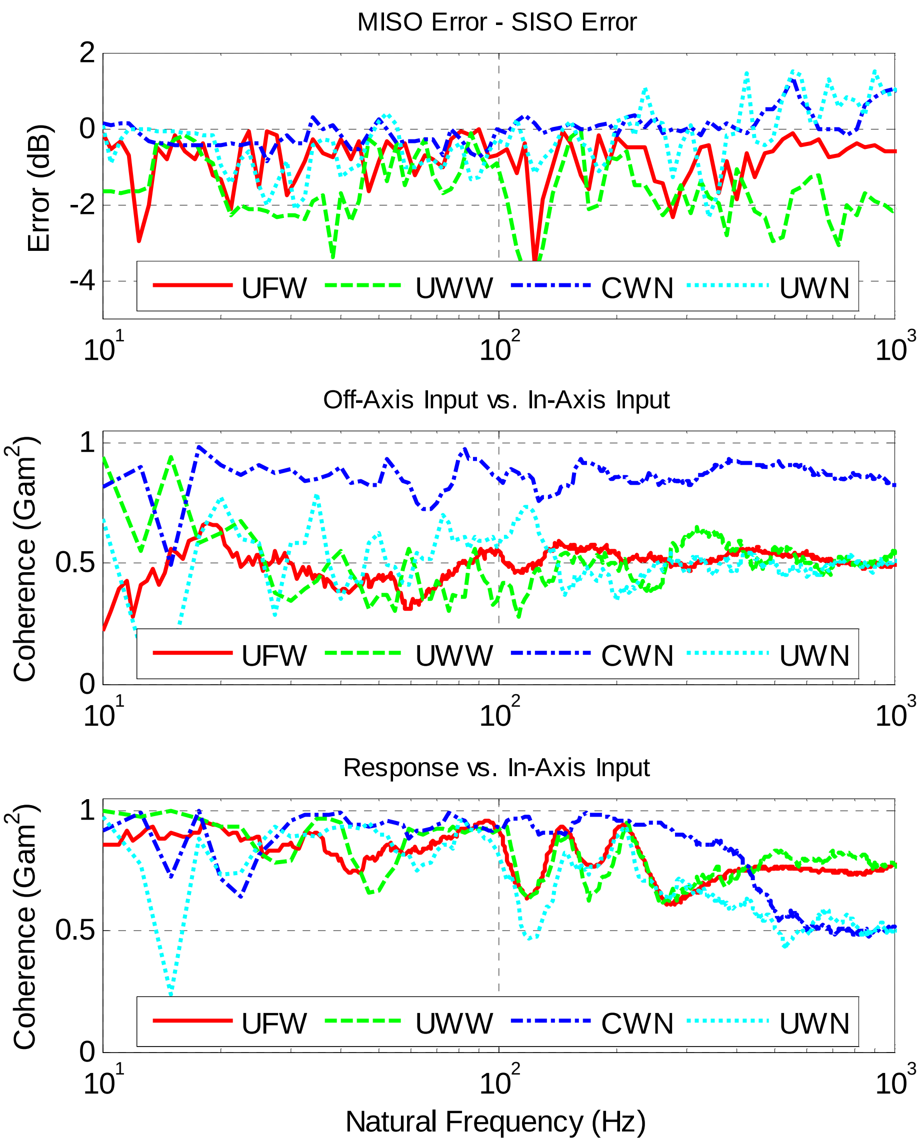


Figure 11: Criteria for Choosing Between SIMO and MIMO Analysis Model

As one would expect for the contrived TISO cases, the coherence between the off-axis and in-axis inputs was high for the correlated cases (although it was < 1) and low for the uncorrelated cases (generally around 0.5). Similarly, the coherence between the response and the in-axis input is generally high except for the cases where the uncorrelated noise is a significant contributor (i.e., in the high frequency response for cases 5 and 6 – the CWN and UWN simulations).

Therefore, if the in-axis and off-axis inputs are highly coherent and/or the response versus in-axis input coherence is low then a SIMO model will probably do as well as a MIMO model. Otherwise, a MIMO model will be more accurate.

CONCLUSIONS

We have shown that a MIMO TRF model can more accurately predict the response of a structure than a traditional SIMO TRF model if there are off-axis inputs that are uncorrelated with respect to the corresponding in-axis inputs. However, since the likelihood of test data containing absolutely no uncorrelated noise is highly unlikely, it is still not clear why the PIMIMO model exactly predicted the test responses.

The fact that the PAMIMO model produced less accurate predictions than the PISIMO model is seen as a sign of the need to pay attention to the choice of inputs to the MIMO TRF model. Apparently the data used to generate the model should be compiled from homogenous data sets to produce the best results (in this case, from one test direction on one foot).

In conclusion, while a SIMO model is simpler to implement, we would recommend generating both a SIMO and an MIMO model when the data infer the existence of multiple uncorrelated inputs and then choosing whichever model produces the more accurate responses.

REFERENCES

[1] Golub, Gene H., 1989, *Matrix Computations*, The Johns Hopkins University Press, Baltimore.

[2] Smallwood, David O., “Multiple-Input Multiple-Output (MIMO) linear systems extreme inputs/outputs,” *Shock and Vibration*, 2007, Vol. 14, No. 2, pp107-132.